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Randomized Linear Search for a Drifted Brownian Target with Direction-Reversal Costs

Mohamed Abd Allah El-Hadidy ^{1,*}  and Alaa Awad Alzulaibani ² ¹ Mathematics Department, Faculty of Science, Tanta University, Tanta 31527, Egypt² Department of Mathematics and Statistics, College of Science in Yanbu, Taibah University, Yanbu Governorate 46477, Saudi Arabia; alzulaibani@taibahu.edu.sa* Correspondence: melhadidi@science.tanta.edu.eg

Abstract

This paper introduces a randomized linear-search model for a target moving on the real line according to Brownian motion with drift. The searcher alternates between the two directions through expanding random excursions, with a fixed cost incurred at each change in direction. The performance criterion combines the expected encounter time with the expected number of reversals. A survival-probability representation is derived for the total expected cost, together with sufficient and necessary integrability conditions for its finiteness. The existence of an optimal randomized strategy is also established within a compact parametric class. The symmetric zero-drift case is characterized by a reflection property, whereas nonzero drift may lead to asymmetric search behavior. In addition, the optimal cost is shown to be non-decreasing and concave in the reversal cost. The numerical study illustrates the effects of drift, the initial search direction, the expansion factor, and the reversal cost. Overall, the model provides a unified framework for randomized search for a moving target under operational direction-reversal costs.

Keywords: Brownian target; randomized linear search; direction-reversal cost; first-encounter time; optimal search strategy

MSC: 90B40; 60J65; 93E20; 60G40

1. Introduction

Search theory provides a mathematical framework for locating an object whose position is either unknown or only partially described by probabilistic information. Its applications arise in search-and-rescue operations, fault detection along communication and power networks, inspection of pipelines, mineral exploration, robotics, and the tracking of mobile particles. A typical search problem requires the construction of a trajectory that detects the target while minimizing a suitable measure of cost, such as the expected detection time, the travelled distance, or the search effort. General treatments of search games and optimal search for stationary and moving targets can be found in Refs. [1–4].

One of the most fundamental models in this area is the linear search problem. In its classical form, a searcher starts from the origin of the real line and alternates between the two directions until a hidden target is encountered. The target may be placed according to a known probability distribution, or it may be treated as an adversarially located object. Early studies established the main structural features of oscillating search paths and showed that expanding excursions are necessary when the target may lie arbitrarily far from the



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starting point [5–8]. These ideas later became closely related to the cow-path problem and to competitive search on rays. Randomization was also introduced to reduce the sensitivity of deterministic strategies to the unknown side and distance of the target [1,2].

Most classical formulations assume that the target remains stationary and that a change in direction can be made instantaneously and without cost. These assumptions simplify the analysis, but they are restrictive in applications. A target may continue to move during the search, and each reversal of the searcher may involve braking, turning, energy consumption, communication, or operational delay. Consequently, a search path with many short oscillations may have a small idealized detection time while being inefficient once the physical cost of changing direction is included.

Recent developments in trajectory and control optimization further illustrate that changes in motion and control actions are subject to realistic dynamical and operational constraints. Guan et al. [9] considered three-dimensional trajectory optimization for fixed-wing UAVs while explicitly accounting for full vehicle dynamics, and Meng et al. [10] incorporated state and obstacle constraints directly into a model predictive control framework for real-time USV trajectory planning. More generally, constrained control interventions have also been studied in flexible manipulator systems using event-based input–output constrained control [11]. Although these studies address different systems from the Brownian-search problem considered here, they reinforce the practical motivation for treating direction reversals as non-negligible operations rather than cost-free changes in motion.

Moving-target search problems have therefore received considerable attention. McCabe [12] studied the search for a one-dimensional random walker and established the existence of search plans with finite crossing-time moments under suitable assumptions. For a target moving continuously according to Brownian motion, El-Rayes et al. [13] formulated a linear search model in which the searcher moves continuously on the real line with bounded speed. They obtained conditions under which the expected first meeting time is finite and proved the existence of an optimal search path. Their analysis showed that the integrability of the target’s initial position plays a central role in the finiteness of the expected encounter time.

Several extensions of the Brownian-target model have subsequently been considered. Mohamed et al. [14] developed a multiplicative search scheme for a Brownian target located on one of several intersecting real lines and examined both the finiteness of the first meeting time and the existence of an optimal plan. El-Hadidy [15] studied the search for a multidimensional Brownian target using multiple sensors, while cooperative and coordinated search mechanisms for Brownian and random-walk targets were investigated in [16–18]. These contributions broadened the geometry of the search domain and the number of participating searchers, but their principal performance criterion remained the expected meeting or detection time.

Randomized motion parameters provide another extension of linear search theory. Instead of prescribing every excursion deterministically, the distances or velocities of the searcher may be generated from probability distributions. Such a formulation can represent uncertainty in motion, randomized exploration, or variability in the operating environment. El-Hadidy [19] considered a one-dimensional Brownian target when both the search distances and velocities are independent random variables with known probability density functions. A related multidimensional multiplicative model with random distances and velocities was developed in [20]. These works established the feasibility of randomized search plans and studied conditions related to the finiteness of the search process. However, these works did not optimize a criterion that explicitly penalizes the number of direction reversals.

The cost of turning has been examined in a different branch of the literature. De-mainie et al. [21] studied online search for a stationary object on the line when a fixed charge is incurred whenever the searcher changes direction. They derived competitive guarantees and extended the analysis to star-search problems and randomized strategies. Angelopoulos et al. [22] later revisited search with turning costs on concurrent rays through infinite-dimensional linear programming and clarified the optimality of round-robin-type strategies. More general cost structures for movement away from and toward the origin were also considered in [23]. These models demonstrate that reversal costs can substantially alter the preferred search pattern. Nevertheless, they are primarily formulated for stationary hidden targets and are evaluated through worst-case competitive ratios rather than through an expected first-encounter criterion for a continuously moving stochastic target.

The present paper connects these previously separate directions. We consider a target whose position evolves on the real line according to a Brownian motion with drift. The searcher starts at the origin and alternates between the positive and negative directions. The distance covered during each successive excursion is random and is generated by an expanding probabilistic rule, ensuring that increasingly distant regions can be explored. The target is not observed before detection, and the randomized search plan is therefore specified independently of the unobserved Brownian sample path. The performance criterion combines the expected first-encounter time with a fixed penalty for every direction reversal made before detection. This formulation captures two competing objectives. Longer excursions reduce the number of costly reversals but may delay the exploration of the opposite side. Shorter excursions provide more frequent alternation but may accumulate a large switching cost. The presence of drift introduces an additional source of asymmetry, since the target has a systematic tendency to move in one direction. As a result, the distributions governing rightward and leftward excursions need not have the same parameters.

The main contribution of this paper is the development of a unified stochastic first-encounter framework that combines three features that have previously been studied largely in separate settings: Brownian target motion, randomized expanding search distances, and an explicit direction-reversal cost. Randomized search distances have been considered in Brownian-target models [19,20], while fixed turning costs have been studied mainly for stationary targets within competitive-search formulations [21–23]. The present model brings these elements together by allowing the target to follow a Brownian motion with drift and the searcher to use alternating random excursions whose scales increase over time. The expected cost combines the first-encounter time with the number of reversals completed before detection. The analysis derives a survival-probability representation for both cost components, establishes conditions for the finiteness of the total expected cost, and proves the existence of an optimal randomized strategy within a compact parametric class. It also characterizes the reflection structure of the symmetric zero-drift case, explains the directional asymmetry induced by nonzero drift, and identifies how the optimal cost and reversal behavior vary with the reversal penalty. The accompanying numerical study illustrates the effects of drift, the initial-direction probability, the expansion factor, and the reversal cost on search performance.

The proposed model offers a probabilistic extension of search with turning costs and an operational extension of Brownian-target search. It is relevant whenever repeated changes in direction consume non-negligible time, energy, or resources. The framework also provides a basis for further models involving random velocities, delayed reversals, multiple searchers, partial observations, or multidimensional target motion.

The remainder of the paper is organized as follows. Section 2 introduces the Brownian target process, the randomized expanding search plan, the first-encounter time, and the reversal-counting process. Section 3 derives a representation of the expected total cost

and establishes sufficient conditions for its finiteness. Section 4 studies the structural properties of admissible strategies and proves the existence of an optimal randomized plan. Section 5 examines the influence of drift and reversal cost and discusses the symmetric and asymmetric cases. Numerical illustrations and sensitivity analyses are presented in Section 6. Section 7 concludes the paper and outlines possible extensions.

2. Model Formulation and Randomized Search Plan

Let $(\Omega, \mathcal{E}, P)$ be a complete probability space supporting all random elements considered below. The target moves on the real line according to a Brownian motion with drift, while the searcher starts from the origin and follows an alternating randomized path. The target is not observed before detection; therefore, the random variables governing the search plan are assumed to be independent of the target process.

2.1. Target Motion

The position of the target at time $t \geq 0$ is given by

$$X(t) = X_0 + \mu t + \sigma B(t), \quad (1)$$

where $B(t)$ is a standard Brownian motion, $\mu \in \mathbf{R}$ is the drift coefficient, and $\sigma > 0$ is the diffusion coefficient. The initial position X_0 is a real-valued random variable with probability distribution v , independent of $B(t)$.

The drift determines the preferred direction of the target motion. In particular, $\mu > 0$ corresponds to a tendency toward the positive half-line, whereas $\mu < 0$ corresponds to a tendency toward the negative half-line. The case $\mu = 0$ reduces to the symmetric Brownian target considered in classical linear-search models. The present construction extends the bounded-speed search framework previously studied in [13].

The searcher moves with constant speed $v > 0$. Throughout the analysis, we impose $v > |\mu|$. This condition requires the searcher's speed to exceed the magnitude of the deterministic drift of the target. It is therefore applicable to both positive and negative drift and ensures that the searcher is not outpaced by the deterministic component of the target motion in either direction.

2.2. Randomized Expanding Distances

The searcher explores the two half-lines through successive alternating excursions. Let $\{U_n\}_{n \geq 1}$ and $\{V_n\}_{n \geq 1}$ be two sequences of positive random variables. The variable U_n determines the random scaling of the rightward excursions, whereas V_n determines the random scaling of the leftward excursions.

We assume that both sequences are independent of X_0 and $B(t)$. The two sequences need not have the same distribution, since a nonzero drift may make an asymmetric search plan preferable.

Let $\rho > 1$; the initial measurement factors for search distances are $a_+ > 0$ and $a_- > 0$ in the positive and negative directions, respectively. Define $R_n = a_+ \rho^{n-1} U_n$, $L_n = a_- \rho^{n-1} V_n$ and $n \geq 1$. Here, R_n is the distance of the n -th right turning point from the origin, and L_n is the absolute distance of the n -th left turning point. The factor ρ^{n-1} generates an expanding search scale, while U_n and V_n introduce random variation around that scale. This preserves the exploratory structure of classical linear search while avoiding a completely deterministic sequence of turning points. The idea is consistent with the randomized search framework described in the current manuscript. We impose the following standing assumption.

Assumption 1. *There exist constants m and M satisfying $0 < m \leq U_n, V_n \leq M < \infty$ almost surely for every $n \geq 1$.*

This condition prevents degenerate excursions and guarantees that the expanding deterministic factor is not destroyed by extreme random perturbations.

Proposition 1. *Under Assumption 1, $R_n \rightarrow \infty$ and $L_n \rightarrow \infty$ almost surely.*

Proof. Since $R_n = a_+\rho^{n-1}U_n$, $L_n = a_-\rho^{n-1}V_n$ and using assumption 1, we get $R_n \geq a_+m\rho^{n-1}$ and $L_n \geq a_-m\rho^{n-1}$. Also, since $\rho > 1$, then $\rho^{n-1} \rightarrow \infty$. Therefore, both R_n and L_n tend to infinity almost surely. $\square$

Proposition 1 guarantees that the searcher eventually explores arbitrarily distant regions on both sides of the origin.

2.3. Initial Direction and Turning Points

Let I be a random variable taking values in $\{+1, -1\}$, independent of all other random elements, such that $P(I = +1) = p$ and $P(I = -1) = 1 - p$, where $0 \leq p \leq 1$. If $I = +1$, the searcher begins in the positive direction, and the sequence of turning points is defined by $Y_{2n-1} = R_n$, $Y_{2n} = -L_n$ and $n \geq 1$. If $I = -1$, the searcher begins in the negative direction, and $Y_{2n-1} = -L_n$, $Y_{2n} = R_n$ and $n \geq 1$. In both cases, $Y_0 = 0$. Thus, the sign of the turning points alternates, while their magnitudes grow randomly according to $R_n = a_+\rho^{n-1}U_n$ and $L_n = a_-\rho^{n-1}V_n$.

Remark 1. *In the symmetric zero-drift case, a natural specialization is $a_+ = a_-$, $U_n \stackrel{d}{=} V_n$ and $p = \frac{1}{2}$.*

2.4. Construction of the Search Path

The searcher moves between two consecutive turning points with constant speed v . Define the arrival times recursively by,

$$T_0 = 0 \text{ and } T_n = T_{n-1} + \frac{|Y_n - Y_{n-1}|}{v}, n \geq 1. \quad (2)$$

Equivalently,

$$T_n = \frac{1}{v} \sum_{j=1}^n |Y_j - Y_{j-1}| \quad (3)$$

For $T_{n-1} < t < T_n$, the searcher position is defined by

$$S(t) = Y_{n-1} + \frac{t - T_{n-1}}{T_n - T_{n-1}} (Y_n - Y_{n-1}) \quad (4)$$

Hence $S(t)$, and the search path is continuous and piecewise linear.

Proposition 2. *The process $S(t)$ defined by Equation (4) satisfies*

$$|S(t) - S(s)| \leq v|t - s|, s, t \geq 0. \quad (5)$$

Moreover, $T_n \rightarrow \infty$ almost surely.

Proof. On each interval $[T_{n-1}, T_n)$, the searcher moves linearly from Y_{n-1} to Y_n . From Equations (2) and (4),

$$\left| \frac{S(t)}{dt} \right| = \frac{Y_n - Y_{n-1}}{T_n - T_{n-1}} = v \frac{Y_n - Y_{n-1}}{|Y_n - Y_{n-1}|}, T_{n-1} < t < T_n$$

Hence, depending on the direction of the n -th excursion, $\frac{dS(t)}{dt} = +v$ or $\frac{dS(t)}{dt} = -v$ and therefore $\left|\frac{dS(t)}{dt}\right| = v$, $s < t$ on every open search segment. Since S is continuous at the turning times, for any

$$|S(t) - S(s)| = \left| \int_s^t S'(u) du \right| \leq \int_s^t |S'(u)| du \leq v(t - s).$$

By symmetry in s and t , $|S(t) - S(s)| \leq v|t - s|$, which proves Equation (5). Furthermore, by Proposition 1, the absolute values of the turning points tend to infinity. Since consecutive turning points lie on opposite sides of the origin, $|Y_n - Y_{n-1}| = |Y_n| + |Y_{n-1}|$. In particular, these distances grow without bound along the sequence. Therefore, from Equation (3), the accumulated travel time diverges, and hence $T_n \rightarrow \infty$ almost surely. This also rules out infinitely many direction reversals in finite time. $\square$

The second conclusion excludes infinitely many direction changes in finite time.

2.5. Information Structure

The searcher does not observe $X(t)$ before detection. In particular, the searcher has no access to the current displacement $X(t) - S(t)$. The variables I , U_n , and V_n are sampled independently of the target process. Therefore, the search path is an open-loop randomized strategy: its randomness determines the successive excursions, but it does not react to the unobserved Brownian trajectory. This assumption is essential. A strategy based directly on $X(t) - S(t)$ would require continuous observation of the target and would no longer represent a genuine search problem.

2.6. First-Encounter Time

The target is detected when the searcher and target occupy the same position. Define

$$\tau = \inf\{t \geq 0 : S(t) = X(t)\}. \quad (6)$$

If the two processes never meet, then $\tau = \infty$. Since both $X(t)$ and $S(t)$ have continuous sample paths, detection occurs through an actual meeting rather than through a jump.

Definition 1. A randomized search plan is called *finite* if $E[\tau] < \infty$.

This condition implies $P(\tau < \infty) = 1$. The converse is not generally true, since almost sure detection does not necessarily imply a finite expected detection time.

2.7. Direction-Reversal Process

Let $N(t)$ denote the number of direction reversals completed by time t . It is defined by

$$N(t) = \sum_{n=1}^{\infty} \mathbf{1}_{\{T_n \leq t\}}. \quad (7)$$

The number of reversals occurring before detection is

$$N_\tau = \sum_{n=1}^{\infty} \mathbf{1}_{\{T_n \leq \tau\}}. \quad (8)$$

The strict inequality means that no reversal cost is charged when the target is detected exactly at a turning time.

Proposition 3. *The expected number of direction reversals before detection satisfies $E[N_\tau] = \sum_{n=1}^{\infty} P(\tau > T_n)$.*

Proof. From Equation (8), we get the summands are nonnegative. Therefore, the monotone convergence theorem gives $E[N_\tau] = \sum_{n=1}^{\infty} E[\mathbf{1}_{\{T_n \leq \tau\}}]$. Since $E[\mathbf{1}_{\{T_n \leq \tau\}}] = P(\tau > T_n)$, then the proof is completed. $\square$

This representation shows that the reversal term is determined by the survival probabilities of the search process at the successive turning times.

2.8. Total Search Cost

Let $\kappa > 0$ denote the fixed cost of one direction reversal. The total random cost is defined by,

$$C_\kappa = \tau + \kappa N_\tau \quad (9)$$

The corresponding expected cost is

$$J_\kappa = E[C_\kappa] = E[\tau] + \kappa E[N_\tau]. \quad (10)$$

The first component measures the delay until detection, whereas the second measures the operational cost associated with repeated changes in direction.

Proposition 4. *The expected total cost admits the representation*

$$J_\kappa = \int_0^\infty P(\tau > t) dt + \kappa \sum_{n=1}^{\infty} P(\tau > T_n). \quad (11)$$

Proof. Because τ is nonnegative, $E[\tau] = \int_0^\infty P(\tau > t) dt$. Combining this identity with Proposition 3 yields Equation (11). $\square$

Equation (11) separates the continuous waiting cost from the discrete reversal cost and will be the starting point for the finiteness analysis in the next section.

2.9. Admissible Strategies

A randomized strategy is characterized by $\Theta = (p, a_+, a_-, \rho, F_+, F_-)$, where F_+ and F_- are the probability distributions of U_n , and V_n , respectively.

Definition 2. *An admissible strategy Θ satisfies $v > |\mu|$, $\rho > 1$, the random multipliers are positive and independent of the target process, the turning times satisfy $T_n \rightarrow \infty$ almost surely, and the search path is continuous and satisfies Equation (5).*

The collection of all admissible randomized strategies is denoted by $\tilde{\mathcal{A}}$. The optimization problem is $J_\kappa^* = \inf_{\Theta \in \tilde{\mathcal{A}}} J_\kappa(\Theta)$.

Definition 3. *An admissible strategy $\Theta^* \in \mathcal{A}$ is called optimal if $J_\kappa(\Theta^*) = J_\kappa^*$.*

The optimization criterion reflects the central balance of the model. Large excursions may reduce the frequency of direction reversals but delay the examination of the opposite half-line. More frequent alternation improves directional coverage but increases the accu-

mulated reversal cost. In addition, a nonzero target drift may make different rightward and leftward distance distributions optimal.

3. Finiteness of the Expected Total Cost

The analysis in this section is based on the main idea used in linear search for a Brownian target: if the searcher reaches a turning point on one side of the origin without detecting the target, then the continuity of the two trajectories implies that the target must still lie beyond that turning point on the same side. In the present model, however, the turning points R_n and L_n , as well as the corresponding arrival times T_n , are random variables determined by U_n and V_n . Therefore, the survival probabilities are first evaluated conditionally on the realized randomized search plan, and are then averaged with respect to the distributions of the search distances. Throughout this section, assumption 1 is assumed to hold.

3.1. Conditional Survival Probabilities

Let $\Xi = \sigma(I, U_1, V_1, U_2, V_2, \dots)$ denote the sigma-algebra generated by the randomness of the search plan. Conditional on Ξ , the turning points Y_n and the arrival times T_n are fixed, while the Brownian motion remains independent of the search plan.

Let the upper tail of the standard normal distribution be denoted by,

$$\bar{\Phi}(z) = 1 - \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-\frac{u^2}{2}} du \quad (12)$$

Proposition 5. Suppose that $I = +1$. If $X_0 = x > 0$, then

$$P(\tau > T_{2n-1} | X_0 = x, \Xi) \leq \bar{\Phi}\left(\frac{R_n - x - \mu T_{2n-1}}{\sigma \sqrt{T_{2n-1}}}\right). \quad (13)$$

If $X_0 = x < 0$, then

$$P(\tau > T_{2n} | X_0 = x, \Xi) \leq \bar{\Phi}\left(\frac{L_n - |x| + \mu T_{2n}}{\sigma \sqrt{T_{2n}}}\right). \quad (14)$$

If $I = -1$, the same bounds hold after interchanging the odd and even indices.

Proof. Assume first that $x > 0$ and $I = +1$. The searcher reaches the positive turning point R_n at time T_{2n-1} . If no encounter has occurred before that time, then the target must still lie to the right of the searcher. Otherwise, the continuity of $X(t)$ and $S(t)$ would imply that the two trajectories had met at an earlier time. Hence, $X(T_{2n-1}) > R_n$. Using the target-motion equation,

$$x + \mu T_{2n-1} + \sigma B(T_{2n-1}) > R_n.$$

Therefore,

$$B(T_{2n-1}) > \frac{R_n - x - \mu T_{2n-1}}{\sigma}.$$

Conditional on Ξ , $\frac{B(T_{2n-1})}{\sqrt{T_{2n-1}}}$ has the standard normal distribution. This gives Equation (13).

Now suppose that $x < 0$. If the searcher reaches the negative turning point $-L_n$ at time T_{2n} without meeting the target, then the target must remain to the left of the searcher. Thus, $X(T_{2n}) < -L_n$. Hence,

$$x + \mu T_{2n} + \sigma B(T_{2n}) < -L_n.$$

Since $x = -|x|$, then

$$-\sigma B(T_{2n}) > L_n - |x| + \mu T_{2n}.$$

Because $\frac{-B(T_{2n})}{\sqrt{T_{2n}}}$ also has the standard normal distribution, inequality (14) follows. $\square$

Averaging over the randomness of the search plan gives the unconditional bounds

$$P(\tau > T_{2n-1} | X_0 = x, I = +1) \leq E \left[\Phi \left(\frac{R_n - x - \mu T_{2n-1}}{\sigma \sqrt{T_{2n-1}}} \right) \right], \quad (15)$$

for $x > 0$, and

$$P(\tau > T_{2n} | X_0 = x, I = -1) \leq E \left[\Phi \left(\frac{L_n - |x| + \mu T_{2n}}{\sigma \sqrt{T_{2n}}} \right) \right], \quad (16)$$

for $x < 0$.

Equations (15) and (16) identify the precise point at which the random search distances enter the analysis. The random variables $R_n = a_+ \rho^{n-1} U_n$ and $L_n = a_- \rho^{n-1} V_n$ appear explicitly inside the Gaussian tail function. Moreover, T_n depends on the realized values of all preceding search distances. Therefore, the survival probabilities depend on the full distributions of U_n and V_n , rather than only on the deterministic bounds m and M .

3.2. Representation of the Two Cost Components

Proposition 6. For every $x \in \mathbb{R}$,

$$E[\tau | X = x_0] \leq \sum_{n=0}^{\infty} E[(T_{n+1} - T_n) P(\tau > T_n | X = x_0, \Xi)], \quad (17)$$

moreover,

$$E[N_\tau | X = x_0] \leq \sum_{n=0}^{\infty} E[P(\tau > T_n | X = x_0, \Xi)]. \quad (18)$$

Proof. We first consider the expected encounter time. Conditional on Ξ , the random variables defining the search plan have already been realized. Consequently, the turning points Y_n and the corresponding arrival times $0 = T_0 < T_1 < T_2 < \dots$ are fixed. By Proposition 2, $T_n \rightarrow \infty$ almost surely.

Since τ is nonnegative, the tail-integral formula gives

$$E[\tau | X = x_0] \leq \int_0^{\infty} P(\tau > t | X = x_0, \Xi) dt. \quad (19)$$

The probability $P(\tau > T_n | X = x_0, \Xi)$ is the conditional survival probability of the search process at time t . It represents the probability that the target has not yet been detected by time t , after the initial target position and the realized search plan have been fixed.

We divide the time axis into the successive search stages $[T_n, T_{n+1})$, $n \geq 0$. Using $T_n \rightarrow \infty$, Equation (19) becomes

$$E[\tau | X = x_0] \leq \sum_{n=0}^{\infty} \int_{T_n}^{T_{n+1}} P(\tau > t | X = x_0, \Xi) dt. \quad (20)$$

The conditional survival probability is non-increasing in t . Indeed, if $s \leq t$, then $\{\tau > t\} \subseteq \{\tau > s\}$ and hence

$$P(\tau > t|X = x_0, \Xi) \leq P(\tau > s|X = x_0, \Xi)$$

Therefore, for every $t \in [T_n, T_{n+1})$,

$$P(\tau > t|X = x_0, \Xi) \leq P(\tau > T_n|X = x_0, \Xi).$$

Substituting this bound into each integral in (20) yields

$$\begin{aligned} \int_{T_n}^{T_{n+1}} P(\tau > t|X = x_0, \Xi) dt &\leq \int_{T_n}^{T_{n+1}} P(\tau > T_n|X = x_0, \Xi) dt \\ &= (T_{n+1} - T_n)P(\tau > T_n|X = x_0, \Xi). \end{aligned}$$

Thus,

$$E[\tau|X = x_0] \leq \sum_{n=1}^{\infty} (T_{n+1} - T_n)P(\tau > T_n|X = x_0, \Xi).$$

Each term on the right-hand side has a direct interpretation. The quantity $T_{n+1} - T_n$ is the duration of the next search stage, while $P(\tau > T_n|X = x_0, \Xi)$ is the probability that the searcher actually reaches that stage without having detected the target. Thus, their product represents an upper bound for the contribution of that stage to the expected encounter time.

Taking expectation with respect to the randomness of the search plan and using the tower property gives

$$E[\tau|X = x_0] = E[E[\tau|X = x_0, \Xi]] \leq E\left[\sum_{n=1}^{\infty} (T_{n+1} - T_n)P(\tau > T_n|X = x_0, \Xi)\right].$$

Since all terms are nonnegative, Tonelli's theorem [24] permits the interchange of expectation and summation. Hence, Equation (17) will be obtained.

We now consider the number of reversals before detection. From its definition in Equation (8), the event $\{T_n > \tau\}$ means that the searcher reaches the n -th turning time before detecting the target. Equivalently, the search survives beyond T_n . Therefore, $\mathbf{1}_{\{T_n < \tau\}} = \mathbf{1}_{\{\tau > T_n\}}$. Conditioning on $X = x_0$ and Ξ , and applying the monotone convergence theorem, we obtain

$$E[N_\tau|X = x_0] = E\left[\sum_{n=1}^{\infty} \mathbf{1}_{\{T_n \leq \tau\}}|X = x_0, \Xi\right] = \sum_{n=1}^{\infty} P(\tau > T_n|X = x_0, \Xi)$$

Taking expectation with respect to Ξ , and again using Tonelli's theorem, gives

$$E[N_\tau|X = x_0] = E[E[N_\tau|X = x_0, \Xi]] = \sum_{n=1}^{\infty} E[P(\tau > T_n|X = x_0, \Xi)].$$

This proves Equation (18). $\square$

The proposition shows that both components of the cost are determined by the same conditional survival probabilities. For the encounter-time component, each probability is weighted by the duration of the corresponding search stage. For the reversal component, each probability represents the chance that the associated reversal is actually performed before detection. This is the same survival-probability principle used in the deterministic Brownian-target model, adapted here to random turning times and random search distances.

3.3. Effective Expansion Condition

To obtain simple sufficient conditions for convergence, the deterministic drift of the target must not dominate the outward expansion of the search path.

Assumption 2. *There exists a constant c such that $|\mu| < c < v$, $\delta = c - |\mu| > 0$, and the following inequalities hold almost surely for $n \geq 1$. If $I = +1$, then $R_n \geq cT_{2n-1}$, $L_n \geq cT_{2n}$, whereas if $I = -1$, then $L_n \geq cT_{2n-1}$, $R_n \geq cT_{2n}$.*

Assumption 2 does not remove the randomness of the search distances. It does not specify the values of R_n and L_n ; it only imposes a lower bound on their rate of expansion relative to the time required to reach them. The actual survival probability remains determined by the expectations in Equations (15) and (16). The use of $|\mu|$ provides a uniform drift margin for both search directions and allows the subsequent tail estimates to be applied without distinguishing between positive and negative drift.

Lemma 1. *There exist positive constants A_1 and A_2 such that $A_1\rho^{n-1} \leq T_{2n-1}$, $T_{2n} \leq A_2\rho^n$, $n \geq 1$ almost surely. Moreover, $T_{n+1} - T_n \leq A_2\rho^n$, $n \geq 1$.*

Proof. By Assumption 1, $R_n \geq a_+m\rho^{n-1}$ and $L_n \geq a_-m\rho^{n-1}$. To reach a turning point, the searcher must travel at least the distance of that point from the origin. Therefore, one may take $A_1 = \frac{m\min\{a_+, a_-\}}{v}$. The total distance travelled before the end of cycle n is at most $2 \sum_{k=1}^n (R_k + L_k)$. Using the upper bound in Assumption 1, $R_k + L_k \leq M(a_+ + a_-)\rho^{k-1}$. Thus, $T_{2n-1}, T_{2n} \leq \frac{2M(a_+ + a_-)}{v} \sum_{k=1}^n \rho^{k-1}$. Since $\sum_{k=1}^n \rho^{k-1} = \frac{\rho^n - 1}{\rho - 1} \leq \frac{\rho^n}{\rho - 1}$, then we may take $A_2 = \frac{2M(a_+ + a_-)}{v(\rho - 1)}$. Finally, $T_{n+1} - T_n = \frac{|Y_{n+1} - Y_n|}{v} = \frac{|Y_{n+1}| + |Y_n|}{v}$, and then $T_{n+1} - T_n \leq A_2\rho^n$, $n \geq 1$ follows directly from the upper bounds on the turning distances. $\square$

Lemma 2. *For every $x \geq 0$ and every t satisfying $t \geq \frac{2\delta}{x}$, we have $\overline{\Phi}\left(\frac{\delta t - x}{\sigma\sqrt{t}}\right) \leq \exp\left(-\frac{\delta^2 t}{8\sigma^2}\right)$.*

Proof. Since $\delta t - x \geq \frac{\delta t}{2}$, we have $\frac{\delta t - x}{\sigma\sqrt{t}} \geq \frac{\delta t}{2\sigma}$. Using the standard Gaussian-tail bound $\overline{\Phi}(z) \leq \exp\left(-\frac{z^2}{2}\right)$, $z \geq 0$ we obtain $\overline{\Phi}\left(\frac{\delta t - x}{\sigma\sqrt{t}}\right) \leq \exp\left(-\frac{\delta^2 t}{8\sigma^2}\right)$. $\square$

With this symmetric drift margin, the arguments below apply to both signs of μ . Hence, the finiteness results in Theorems 1–3 remain valid under the revised speed condition $v > |\mu|$.

3.4. Finiteness of the Expected Number of Reversals

Theorem 1. *Under assumptions 1 and 2, there exist constants $K_1, K_2 \geq 0$ such that $E[N_\tau | X_0 = x] \leq K_1 + K_2 \log(1 + |x|)$, $x \in \mathbf{R}$. Consequently, if $E[\log(1 + |X_0|)] < \infty$, then $E[N_\tau] < \infty$.*

Proof. Assume first that $x > 0$ and $I = +1$. The searcher reaches the positive side at the times T_{2n-1} . Moreover, $\{\tau > T_{2n}\} \subseteq \{\tau > T_{2n-1}\}$, and therefore, $P(\tau > T_{2n} | X_0 = x, \Xi) \leq P(\tau > T_{2n-1} | X_0 = x, \Xi)$. Using Equation (18), $E[N_\tau | X_0 = x, I = +1] \leq 2 \sum_{n=1}^{\infty} E[P(\tau > T_{2n-1} | X_0 = x, \Xi)]$. Let $n(x)$ be the smallest integer satisfying $A_1\rho^{n(x)-1} \geq \frac{2x}{\delta}$. For the terms preceding $n(x)$, we use the trivial bound $P(\tau > T_{2n-1} | X_0 = x, \Xi) \leq 1$. For $n \geq n(x)$, assumption 2 gives $R_n - x - \mu T_{2n-1} \geq \delta T_{2n-1} - x$. By Proposition 5 and Lemma 2,

$$\begin{aligned}
 P(\tau > T_{2n-1} | X_0 = x, \Xi) &\leq \overline{\Phi}\left(\frac{\delta T_{2n-1} - x}{\sigma \sqrt{T_{2n-1}}}\right) \\
 &\leq \exp\left(-\frac{\delta^2 T_{2n-1}}{8\sigma^2}\right) \\
 &\leq \exp\left(-\frac{\delta^2 A_1}{8\sigma^2} \rho^{n-1}\right).
 \end{aligned} \tag{21}$$

Therefore,

$$E[N_\tau | X_0 = x, I = +1] \leq 2n(x) + 2 \sum_{n=n(x)}^{\infty} \exp[-K\rho^{n-1}],$$

where $K = \frac{\delta^2 A_1}{8\sigma^2}$. The last series converges and is bounded independently of x . From $A_1 \rho^{n(x)-1} \geq \frac{2x}{\delta}$, we have

$$n(x) \leq 1 + \frac{\log\left(1 + \frac{2x}{\delta A_1}\right)}{\log \rho}.$$

Hence, a logarithmic bound in x follows.

If $x < 0$, the negative turning points and Equation (14) are used. If $I = -1$, only the odd and even indices are interchanged. Averaging over I gives $E[N_\tau | X_0 = x] \leq K_1 + K_2 \log(1 + |x|)$, $x \in \mathbf{R}$. Finally, integrating with respect to the distribution of X_0 yields $E[N_\tau] < \infty$. $\square$

3.5. Finiteness of the Expected Encounter Time

Theorem 2. Under Assumptions 1 and 2, there exist constants $K_3, K_4 > 0$ such that $E[\tau | X_0 = x] \leq K_3 + K_4|x|$, $x \in \mathbf{R}$. Consequently, if $E[|X_0|] < \infty$, then $E[\tau] < \infty$.

Proof. We use the representation in Equation (17) and first consider the case $x < 0$ and $I = +1$. The search process is divided into two phases. The first phase continues until the expanding random search distances reach a scale larger than the initial position x . The index $n(x)$ is defined in Theorem 1. By Lemma 1, we have $T_{2n(x)} \leq A_2 \rho^{n(x)}$, then from the definition of $n(x)$, there exists a constant $K_5 > 0$ such that $T_{2n(x)} \leq K_5(1 + x)$. This bound covers the time spent before the search reaches the scale of x , without requiring an estimate of the survival probabilities during the early stages.

After stage $n(x)$, inequality (21) applies. Since $T_{2n} - T_{2n-2} \leq K_6 \rho^n$ for some constant $K_6 > 0$, the contribution of the remaining stages to Equation (17) is bounded by $K_6 \sum_{n=n(x)}^{\infty} \rho^n \exp[-K\rho^{n-1}]$. This series converges and is bounded above by a constant independent of x . Therefore, $E[\tau | X_0 = x, I = +1] \leq K_3 + K_4|x|$. The cases $x < 0$ and $I = -1$ are handled in the same way. Hence, $E[\tau | X_0 = x] \leq K_3 + K_4|x|$. Integrating with respect to v , this completes the proof. $\square$

Remark 2. The preceding proofs use the randomness of the search distances at two different levels. First, at the exact probabilistic level, the random quantities R_n , L_n , and T_n appear inside the Gaussian-tail expectations in (15) and (16). These formulas make it possible to study the influence of F_+ and F_- and to compare different distributions of the search distances. Second, at the convergence level, Assumptions 1 and 2 provide uniform bounds that guarantee convergence of the relevant series. These bounds do not replace the randomized model by a deterministic one. Rather, they provide sufficient conditions that hold for every possible realization of the random multipliers within their prescribed range. When particular distributions are assigned to U_n and V_n , Equations (15) and (16) may be used directly to obtain sharper estimates than the uniform bounds employed in Theorems 1 and 2.

3.6. Finiteness of the Total Cost

Theorem 3. Suppose that Assumptions 1 and 2 hold. If $E[X_0] < \infty$ then $J_\kappa = E[\tau + \kappa N_\tau] < \infty$.

Proof. By Theorem 2, $E[\tau] < \infty$. Moreover, $\log(1 + |x|) \leq |x|$, $x \in \mathbf{R}$, then the condition $E[X_0] < \infty$ implies $E[\log(1 + |X_0|)] < \infty$. Theorem 1 then gives $E[\tau] < \infty$. Consequently, $J_\kappa = E[\tau + \kappa N_\tau] < \infty$. $\square$

3.7. A Necessary Condition

Theorem 4. If an admissible randomized search strategy satisfies $J_\kappa < \infty$ then $E[|X_0|] < \infty$.

Proof. Since both components of the cost are nonnegative, $J_\kappa < \infty$ implies $E[\tau] < \infty$. Let $\mathfrak{F}_t$ be the completed filtration generated by X_0 , the random variables defining the search plan, and $\{B(s) : 0 \leq s \leq t\}$. Since the search-plan randomness and X_0 are independent of the Brownian motion, $B(t)$ remains a standard Brownian motion with respect to $\mathfrak{F}_t$. Moreover, because $X(t) - S(t)$ is continuous and adapted, $\tau = \inf\{t \geq 0 : X(t) = S(t)\}$ is an $\mathfrak{F}_t$ stopping time. For $m \geq 1$, define the bounded stopping time $\tau_m = \tau \wedge m$. Applying the optional stopping theorem to the martingale $B(t)^2 - t$ at τ_m gives $E[B(\tau_m)^2] = E[\tau_m] \leq E[\tau]$. Since $\tau_m \uparrow \tau$ almost surely and Brownian paths are continuous, $B(\tau_m) \rightarrow B[\tau]$ almost surely. Therefore, by Fatou's lemma [25], $E[B(\tau)^2] \leq \liminf_{m \rightarrow \infty} E[B(\tau_m)^2] \leq E[\tau] < \infty$. Hence,

by the Cauchy-Schwarz [26] inequality, $E|B(\tau)| \leq \left(E[B(\tau)^2]\right)^{\frac{1}{2}} \leq \sqrt{E[\tau]} < \infty$. At the encounter time, $S(\tau) = X_0 + \mu\tau + \sigma B(\tau)$ and hence $|X_0| \leq |S(\tau)| + |\mu|\tau + \sigma|B(\tau)|$. By Proposition 2, with $S(0) = 0$, $|S(\tau)| \leq v\tau$. Consequently, $|X_0| \leq (v + |\mu|)\tau + \sigma|B(\tau)|$. Taking expectations and using the bound established above for the stopped Brownian term $E|B(\tau)| \leq \sqrt{E[\tau]}$, we obtain

$$E[|X_0|] \leq (v + |\mu|)E[\tau] + \sigma E|B(\tau)| \leq (v + |\mu|)E[\tau] + \sigma\sqrt{E[\tau]} < \infty$$

$\square$

Corollary 1. Under Assumptions 1 and 2, the randomized search plan has finite expected total cost if and only if $E[|X_0|] < \infty$.

Proof. The sufficient part follows from Theorem 3, whereas the necessary part follows from Theorem 4. $\square$

Corollary 1 extends the integrability condition known for deterministic Brownian-target search while preserving random search distances and incorporating the cost associated with direction reversals.

4. Existence of an Optimal Randomized Strategy

This section establishes conditions under which the optimization problem admits a minimizing strategy. The proof proceeds by considering a minimizing sequence, extracting a convergent subsequence, and showing that the limiting strategy remains admissible and attains the minimum total expected cost.

4.1. Compact Strategy Class

For the existence result, we restrict the optimization problem to a compact parametric subclass of $\tilde{\mathcal{A}}$ (see, Definition 3). Let $\tilde{\mathcal{A}}_0 = \{\Theta = (p, a_+, a_-, \rho, F_+, F_-)\} \subseteq \tilde{\mathcal{A}}$ be a nonempty

class satisfying the following conditions: $p \in [0, 1]$, $a_+, a_- \in [a_{\min}, a_{\max}]$, $0 < a_{\min} < a_{\max} < \infty$, $\rho \in [\rho_{\min}, \rho_{\max}]$ and $1 < \rho_{\min} < \rho_{\max} < \infty$.

The distributions F_+ and F_- are assumed to belong to compact parametric families supported on $[m, M]$. Thus, they may be written as $F_+ = (\cdot, \eta_+)$ and $F_- = (\cdot, \eta_-)$, where $\eta_+, \eta_- \in \Lambda$. Accordingly, a strategy in $\tilde{\mathcal{A}}_{\square}$ may be represented by, $\tilde{\mathcal{A}}_{\square} = \{\Theta = (p, a_+, a_-, \rho, \eta_+, \eta_-)\} \subseteq \tilde{\mathcal{A}}$. We assume that the random multipliers can be generated in the form $U_n = Q_+(\xi_n; \eta_+)$ and $V_n = Q_-(\zeta_n; \eta_-)$ where $\{\xi_n\}$ and $\{\zeta_n\}$ are fixed sequences of independent uniform random variables on $(0, 1)$. For almost every $u \in (0, 1)$, the functions $\eta_+ \mapsto Q_+(u; \eta_+)$ and $\eta_- \mapsto Q_-(u; \eta_-)$ are continuous.

This representation places all admissible randomized plans on the same probability space. It also ensures that convergence of the distribution parameters produces convergence of the realized search distances.

Lemma 3. Let $\{\Theta_r\} \subset \tilde{\mathcal{A}}_{\square}$ and suppose that $\Theta_r \rightarrow \Theta$ as $r \rightarrow \infty$, then for every fixed n , $R_{n,r} \rightarrow R_n$, $L_{n,r} \rightarrow L_n$ and $T_{n,r} \rightarrow T_n$ almost surely such that r indexes the strategies in the convergent sequence, whereas n indexes the search stages. Moreover, for every $H > 0$, $\sup_{0 \leq t \leq H} |S_r(t) - S(t)| \rightarrow 0$ almost surely.

Proof. Since $U_n = Q_+(\xi_n; \eta_+)$ and $V_n = Q_-(\zeta_n; \eta_-)$ and the assumed continuity of the quantile functions, $U_{n,r} \rightarrow U_n$, $V_{n,r} \rightarrow V_n$ almost surely for each fixed n . Also, since $R_{n,r} = a_{+,r} \rho_r^{n-1} U_{n,r}$ and $L_{n,r} = a_{-,r} \rho_r^{n-1} V_{n,r}$, the convergence of the parameters gives $R_{n,r} \rightarrow R_n$, $L_{n,r} \rightarrow L_n$. The turning points $Y_{n,r}$ are obtained from $R_{n,r}$, $L_{n,r}$, and the initial direction. Hence, $Y_{n,r} \rightarrow Y_n$ for each fixed n . From the definition $T_{n,r} = \frac{1}{v} \sum_{j=1}^n |Y_{j,r} - Y_{j-1,r}|$, we obtain $T_{n,r} \rightarrow T_n$.

Finally, only finitely many turning points occur before a fixed time H , because $T_n \rightarrow \infty$. Between consecutive turning times, the search paths are linear with slopes of magnitude v . The convergence of the turning points and their arrival times therefore implies uniform convergence on $[0, H]$. $\square$

4.2. Lower Semicontinuity of the Total Cost

Let τ_r and N_{τ_r} denote the encounter time and the number of reversals associated with Θ_r . Let τ and N_{τ} correspond to the limiting strategy Θ .

Proposition 7. If $\Theta_r \rightarrow \Theta$, then $J_{\kappa}(\Theta) \leq \liminf_{r \rightarrow \infty} J_{\kappa}(\Theta_r)$.

Proof. We first consider the encounter-time component. Fix $t > 0$, and define $B_r(t) = \left\{ \min_{0 \leq s \leq t} |X(s) - S_r(s)| > 0 \right\}$ and $B(t) = \left\{ \min_{0 \leq s \leq t} |X(s) - S(s)| > 0 \right\}$. Since the target and search paths are continuous, $B_r(t) = \{\tau_r > t\}$ and $B(t) = \{\tau > t\}$. Suppose that a sample point belongs to $B(t)$. Then, $\min_{0 \leq s \leq t} |X(s) - S(s)| > 0$. By the uniform convergence in $\sup_{0 \leq t \leq H} |S_r(t) - S(t)| \rightarrow 0$, the same strict separation holds for S_r when r is sufficiently large. Therefore, $\mathbf{1}_{\{\tau > t\}} \leq \liminf_{r \rightarrow \infty} \mathbf{1}_{\{\tau_r > t\}}$. Fatou's lemma gives $P(\tau > t) \leq \liminf_{r \rightarrow \infty} P(\tau_r > t)$. Integrating with respect to t and applying Fatou's lemma once more yields $E[\tau] = \int_0^{\infty} P(\tau > t) dt \leq \liminf_{r \rightarrow \infty} E[\tau_r]$. This is the same lower-semicontinuity argument used in the deterministic Brownian-target model [13]. $\square$

We now consider the reversal term. Fix $n \geq 1$. If $T_n < \tau$ then the searcher and target remain separated on the compact interval $[0, T_n]$. Hence, $\min_{0 \leq t \leq T_n} |X(t) - S(t)| > 0$. Using

$T_{n,r} \rightarrow T_n$ and the uniform convergence of S_r to S , we obtain $T_{n,r} < \tau_r$ for all sufficiently large r . Therefore, $\mathbf{1}_{\{T_n < \tau\}} \leq \liminf_{r \rightarrow \infty} \mathbf{1}_{\{T_{n,r} < \tau_r\}}$. For every positive integer q , summing $\mathbf{1}_{\{T_n < \tau\}} \leq \liminf_{r \rightarrow \infty} \mathbf{1}_{\{T_{n,r} < \tau_r\}}$ over $n = 1, 2, \dots, q$ gives $\sum_{n=1}^q \mathbf{1}_{\{T_n < \tau\}} \leq \liminf_{r \rightarrow \infty} \sum_{n=1}^q \mathbf{1}_{\{T_{n,r} < \tau_r\}}$. Taking expectations, $\sum_{n=1}^q P(T_n < \tau) \leq \liminf_{r \rightarrow \infty} E[N_{\tau_r}]$. Letting $q \rightarrow \infty$ and using the monotone convergence theorem, $E[N_\tau] \leq \liminf_{r \rightarrow \infty} E[N_{\tau_r}]$. To combine the two cost components, define $C_{\kappa,r} = \tau_r + \kappa N_{\tau_r}$ and $C_\kappa = \tau + \kappa N_\tau$. The preceding pathwise arguments imply $C_\kappa \leq \liminf_{r \rightarrow \infty} C_{\kappa,r}$ almost surely. Therefore, Fatou's lemma gives $J_\kappa(\Theta) \leq E[C_\kappa] \leq E[\liminf_{r \rightarrow \infty} C_{\kappa,r}] \leq \liminf_{r \rightarrow \infty} E[C_{\kappa,r}]$. Hence, $J_\kappa(\Theta) \leq \liminf_{r \rightarrow \infty} J_\kappa(\Theta_r)$.

4.3. Existence of an Optimal Strategy

Theorem 5. Assume that $\tilde{\mathcal{A}}_\square$ is closed and that the conditions of Section 3 hold for every strategy in $\tilde{\mathcal{A}}_\square$. If $E[|X_0|] < \infty$, then there exists a strategy $\Theta^* \in \tilde{\mathcal{A}}_\square$ such that $J_\kappa(\Theta^*) \leq \min_{\Theta \in \tilde{\mathcal{A}}_\square} J_\kappa(\Theta)$.

Proof. Let $J_{\kappa,0}^* = \inf_{\Theta \in \tilde{\mathcal{A}}_\square} J_\kappa(\Theta)$. Choose a minimizing sequence $\{\Theta^*\} \subset \tilde{\mathcal{A}}_\square$ satisfying $J_\kappa(\Theta_r) \rightarrow J_{\kappa,0}^*$. The parameter defined in Section 4.1 is compact. Therefore, there exists a subsequence, denoted again by $\{\Theta_r\}$, and a limit Θ^* such that $\Theta_r \rightarrow \Theta^*$. Since $\tilde{\mathcal{A}}_\square$ is closed, $\Theta^* \in \tilde{\mathcal{A}}_\square$. By Proposition 7, $J_\kappa(\Theta^*) \leq \liminf_{r \rightarrow \infty} J_\kappa(\Theta_r) = J_{\kappa,0}^*$. On the other hand, by the definition of the infimum, $J_{\kappa,0}^* \leq J_\kappa(\Theta^*)$. Thus, $J_\kappa(\Theta^*) = J_{\kappa,0}^*$. Therefore, Θ^* is an optimal randomized search strategy in $\tilde{\mathcal{A}}_\square$. $\square$

Remark 3. The compact restriction does not prescribe a particular distribution for the search distances. It only requires the distribution parameters to remain in a closed and bounded set. This prevents a minimizing sequence from escaping through increasingly small, increasingly large, or degenerate search distances.

Remark 4. The theorem establishes existence but does not guarantee uniqueness. Two strategies with different initial-direction probabilities or different excursion distributions may generate the same expected total cost.

The result extends the compactness and lower-semicontinuity method of the original Brownian-target model [13] to randomized turning distances and a cost criterion that includes direction reversals.

5. Symmetric and Asymmetric Search Regimes

The search structure is determined by the relation between the drift μ , the distribution of X_0 , and the reversal cost κ . The symmetric and asymmetric cases are treated separately below.

5.1. Symmetric Case

For a strategy $\Theta = (p, a_+, a_-, \rho, F_+, F_-)$, define its reflection by $\Theta_{ref} = (1-p, a_-, a_+, \rho, F_-, F_+)$. Let v_{ref} denote the distribution of $-X_0$. Reflection of both the target trajectory and the search path gives $J_\kappa(\Theta; \mu, v) = J_\kappa(\Theta_{ref}; -\mu, v_{ref})$. This identity follows because reflection preserves the encounter time and the number of reversals, while interchanging the positive and negative excursions.

Suppose now that $\mu = 0$ and that the initial distribution is symmetric, that is, $v_{ref} = v$. Then, $J_\kappa(\Theta; 0, v) = J_\kappa(\Theta_{ref}; 0, v)$. Hence, the set of optimal strategies is invariant under reflection. In particular, if the optimal strategy is unique, then $\Theta^* = \Theta_{ref}^*$ and therefore $p^* = 0.5$, $a_+^* = a_-^*$ and $F_+^* = F_-^*$. Without uniqueness, an optimal strategy and its reflection may be distinct while having the same total expected cost.

5.2. Asymmetric Case

If either $\mu \neq 0$ or the distribution of X_0 is not symmetric, the positive and negative directions are no longer equivalent. Then, $J_\kappa(\Theta; \mu, v) = J_\kappa(\Theta_{ref}; -\mu, v_{ref})$ relates the original problem to its reflected counterpart, rather than to the same optimization problem.

Accordingly, the optimal strategy may satisfy $p^* \neq 0.5$, $a_+^* \neq a_-^*$ and $F_+^* \neq F_-^*$. No universal ordering between a_+^* and a_-^* can be inferred from the sign of μ alone. The preferred asymmetry also depends on the initial distribution, the diffusion coefficient, the search speed, and the admissible distance distributions. Thus, directional inequalities should be determined from the complete cost function rather than imposed in advance.

5.3. Effect of the Reversal Cost

Let $J_\kappa^* = \min_{\Theta \in \mathcal{A}_\square} J_\kappa(\Theta)$ then for each fixed strategy, $J_\kappa(\Theta) = E[\tau_\Theta] + \kappa E[N_{\tau_\Theta}]$ is affine in κ . It follows that J_κ^* is non-decreasing and concave as a function of κ . Let $0 < \kappa_1 < \kappa_2$, and let Θ_1^* and Θ_2^* be optimal strategies for κ_1 and κ_2 , respectively. Optimality gives $E[\tau_{\Theta_1^*}] + \kappa_1 E[N_{\tau_{\Theta_1^*}}] \leq E[\tau_{\Theta_2^*}] + \kappa_1 E[N_{\tau_{\Theta_2^*}}]$, and $E[\tau_{\Theta_2^*}] + \kappa_2 E[N_{\tau_{\Theta_2^*}}] \leq E[\tau_{\Theta_1^*}] + \kappa_2 E[N_{\tau_{\Theta_1^*}}]$. Combining these inequalities yields $E[N_{\tau_{\Theta_2^*}}] \leq E[N_{\tau_{\Theta_1^*}}]$ and $E[\tau_{\Theta_2^*}] \geq E[\tau_{\Theta_1^*}]$.

Thus, increasing the reversal cost reduces the expected number of reversals selected by an optimal strategy, while the corresponding expected encounter time may increase. This identifies the precise trade-off between rapid alternation and the cost of changing direction.

6. Numerical Illustrations and Sensitivity Analysis

A Monte Carlo study was carried out to illustrate the main numerical features of the model. The Monte Carlo simulation follows standard sampling-based principles for stochastic models [27,28]. All numerical experiments were implemented in *Wolfram Mathematica, Version 7.0* [29]. Mathematica is proprietary commercial software developed by Wolfram Research, Inc.; software and licensing information are available from the official Wolfram website (<https://www.wolfram.com/mathematica/>) (accessed on 20 August 2026). Brownian sample paths were generated on a uniform time grid, and independent simulation runs were used to estimate the encounter time, the expected number of reversals, and the corresponding total cost. These implementation details are provided to facilitate reproduction of the numerical experiments.

The numerical experiments below are based on finite grids of candidate parameter values. Therefore, the reported numerical solutions should be interpreted as the best solutions within the tested grids, rather than as global optima over the full admissible strategy class. The baseline setting was $X_0 \sim N(0, 4)$, $v = 2$ and $\sigma = 0.7$ with $U_n, V_n \sim \text{uniform}[0.8, 1.2]$. Each reported estimate was obtained from 12,000 independent replications. The main simulations used a Brownian time step of $\Delta t = 0.02$, and the encounter time was estimated by linear interpolation whenever the sign of $X(t) - S(t)$ changed between two consecutive grid points. To assess the effect of time discretization, the same simulation procedure was also repeated with $\Delta t = 0.01$ and 0.005 for a representative parameter setting.

To check the robustness of the numerical estimates, we considered the setting $\mu = 0$, $p = 0.5$, $a_+ = a_- = 1$, $\rho = 5$, and $\kappa = 1$, while keeping the remaining baseline parameters unchanged. The same Monte Carlo procedure with 12,000 replications was used for each

value of Δt . The results are summarized in Table 1, which shows that the estimated encounter time, reversal count, and total cost remain close as the time step is refined.

Table 1. Discretization convergence of the Monte Carlo estimates.

$\hat{J}_1$	$\hat{E}[N_\tau]$	$\hat{E}[\tau]$	Δt
5.570	4.368	1.202	0.020
5.476	4.294	1.182	0.010
5.621	4.429	1.192	0.005

The estimates remain close as the time step is reduced from 0.02 to 0.005. The observed variation in the estimated encounter time and total cost is small relative to their overall magnitude, with no systematic change under further refinement. This indicates that the numerical conclusions are reasonably robust to the chosen time discretization.

6.1. Reflection in the Symmetric Case

The first experiment examines the reflection property when $\mu = 0$ and the initial distribution is symmetric. The strategy $\Theta = (0.3, 1.2, 0.8, 5, F_+, F_-)$ was compared with its reflected strategy $\Theta_{ref} = (0.7, 0.8, 1.2, 5, F_-, F_+)$. For M_s independent simulation runs, the quantities $\hat{E}[\tau] = \frac{1}{M_s} \sum_{j=1}^{M_s} \tau^{(j)}$, $\hat{E}[N_\tau] = \frac{1}{M_s} \sum_{j=1}^{M_s} N_\tau^{(j)}$ denote the Monte Carlo estimates of the expected encounter time and the expected number of reversals, respectively. The estimated total cost is $\hat{J}_\kappa = \hat{E}[\tau] + \kappa \hat{E}[N_\tau]$. The notation $SE(\hat{J}_\kappa)$ denotes the corresponding Monte Carlo standard error. The reversal cost was fixed at $\kappa = 1$.

As shown in Table 2, the expected reversal counts are almost identical. The difference between the two estimated total costs is also modest relative to the Monte Carlo uncertainty. Thus, the numerical results are consistent with the reflection identity established for the symmetric regime. The symmetric case $p = 0.5$, with $a_+ = a_- = 1$, has also been included to permit a direct numerical comparison with the corresponding zero-drift case reported in Table 3.

Table 2. Numerical comparison of a strategy and its reflection.

Strategy	p	a_+	a_-	$\hat{E}[\tau]$	$\hat{E}[N_\tau]$	$\hat{J}_1$	$SE(\hat{J}_1)$
Θ	0.30	1.20	0.80	5.233	1.255	6.487	0.097
Symmetric case	0.50	1.00	1.00	4.368	1.202	5.570	0.075
Θ_{ref}	0.70	0.80	1.20	5.085	1.254	6.339	0.095

Table 3. Sensitivity to drift and the initial rightward probability.

μ	p	$\hat{E}[\tau]$	$\hat{E}[N_\tau]$	$\hat{J}_1$	$SE(\hat{J}_1)$
−0.2	0.30	5.851	1.289	7.139	0.115
−0.2	0.50	5.252	1.239	6.491	0.109
−0.2	0.70	4.530	1.199	5.729	0.090
0	0.30	4.217	1.185	5.402	0.073
0	0.50	4.282	1.191	5.474	0.074
0	0.70	4.377	1.197	5.574	0.075
0.2	0.30	4.471	1.187	5.659	0.088
0.2	0.50	5.070	1.224	6.293	0.102
0.2	0.70	5.850	1.278	7.128	0.116

6.2. Drift and the Initial Search Direction

The next experiment isolates the effect of the drift and the initial-direction probability. The excursion parameters were fixed at $a_+ = a_- = 1$, $\rho = 5$ and $\kappa = 1$ while μ and p were varied.

As shown in Table 3, for zero drift, changing p produces only a limited variation in the estimated cost. This agrees with the absence of a strong directional preference in the symmetric setting. Table 3 also shows that the situation changes when $\mu \neq 0$. For $\mu = -0.2$, the total cost decreases as p increases, whereas for $\mu = 0.2$, the opposite pattern is observed. The effect is visible in both the encounter-time component and the expected reversal count. Hence, the initial direction becomes an effective control parameter once the target motion is asymmetric.

To examine numerical consistency in the zero-drift case, absolute and relative differences between the corresponding estimates in Tables 2 and 3 were calculated for $p = 0.3$, 0.5 and 0.7. These comparisons are reported in Table A1 in Appendix A.

Figure 1 shows that the zero-drift curve remains nearly flat over the tested values of p , while the two nonzero-drift curves move in opposite directions. This numerical behavior supports the distinction between the symmetric and asymmetric regimes discussed in Section 5.

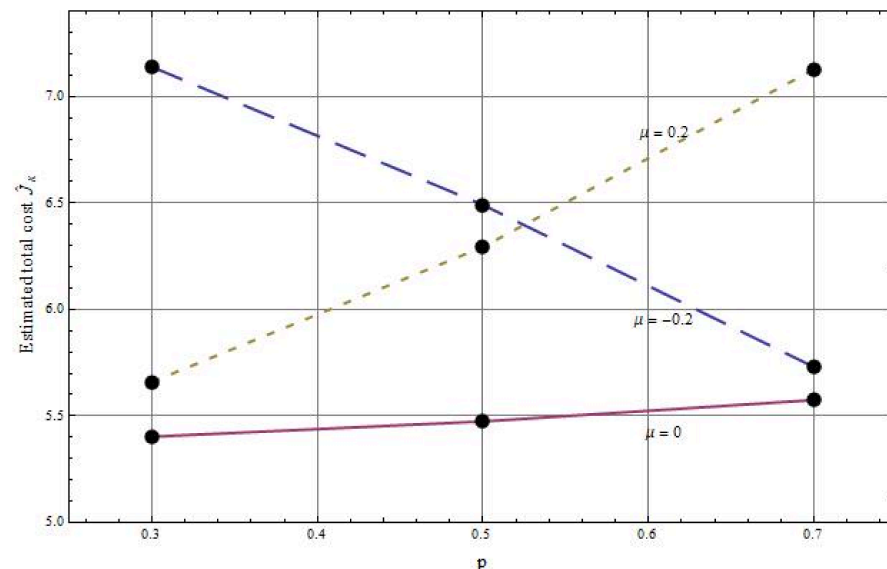


Figure 1. Estimated total cost as a function of p for different drift values.

6.3. Sensitivity to the Expansion Factor

To assess the role of the expansion factor ρ , the parameters were fixed at $\mu = 0$, $a_+ = a_- = 1$, $p = 0.5$, $\kappa = 1$ and ρ was varied over a finite grid.

Figure 2 shows that the estimated encounter time decreases sharply as ρ increases from 1.5 to 3, then remains relatively stable over the range $\rho = 4$ to 7, and finally increases at $\rho = 10$. The smallest estimated value is attained at $\rho = 7$, while the value at $\rho = 5$ is very close, indicating a broad near-optimal region rather than a sharply isolated minimum. This behavior reflects a balance between short excursions, which cause frequent alternation, and overly rapid expansion, which may delay the return to the opposite half-line.

The results in Table 4 reveal two competing effects. When ρ is small, the searcher performs many short excursions, which leads to a large expected number of reversals and a high total cost. Increasing ρ reduces repeated alternation and improves the cost over most of the tested range.

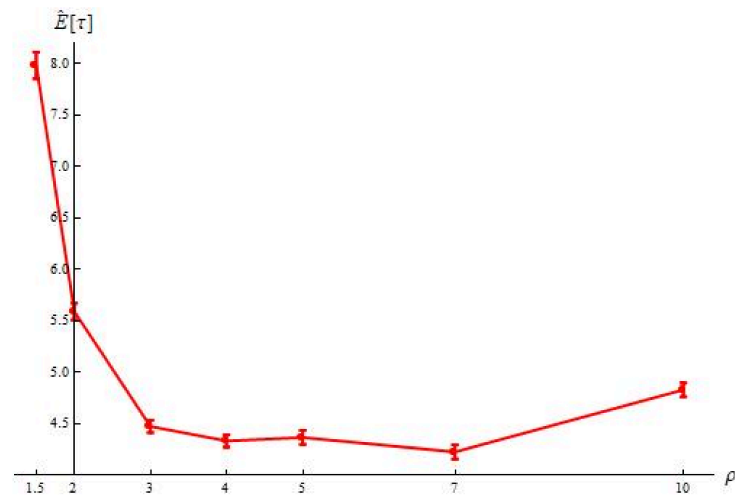


Figure 2. Estimated encounter time $\hat{E}[\tau]$ with Monte Carlo standard errors versus ρ .

Table 4. Sensitivity of the cost components to ρ .

ρ	$\hat{E}[\tau]$	$SE(\hat{E}[\tau])$	$\hat{E}[N_\tau]$	$\hat{J}_1$	$SE(\hat{J}_1)$
1.5	7.981	0.129	3.079	11.060	0.157
2.0	5.591	0.081	2.052	7.642	0.097
3.0	4.476	0.060	1.492	5.967	0.071
4.0	4.335	0.061	1.294	5.629	0.070
5.0	4.368	0.068	1.202	5.570	0.075
7.0	4.228	0.068	1.108	5.337	0.073
10.0	4.831	0.068	1.097	5.928	0.073

Within the tested grid, Table 4 shows that the lowest estimated value is attained at $\rho = 7$. Increasing the expansion factor further to $\rho = 10$ produces almost no additional reduction in the expected reversal count, while the encounter time rises from 4.228 to 4.831. Thus, excessively rapid expansion may delay the return to the opposite half-line.

Figure 3 displays the marked reduction in total cost between $\rho = 1.5$ and $\rho = 4$. Beyond this range, the improvement becomes smaller, and the curve eventually turns upward.

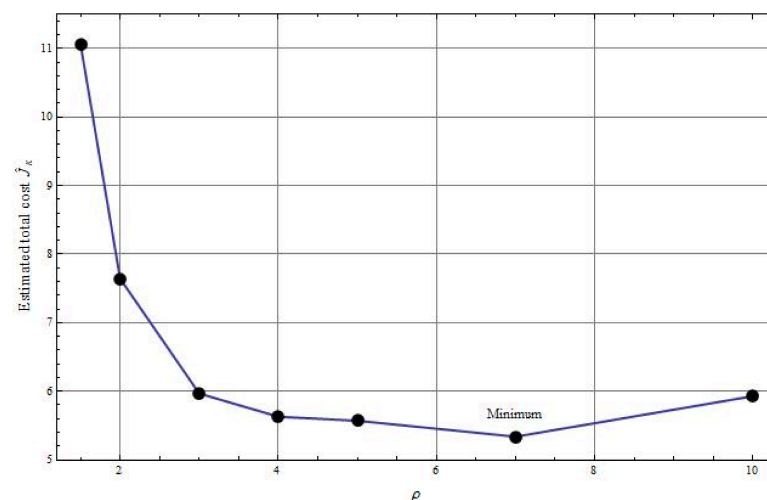


Figure 3. Estimated total cost versus the expansion factor.

Figure 4 shows that the expected reversal count decreases monotonically over the tested grid. However, the decline becomes very small for larger values of ρ . The comparison

with Figure 3 confirms that reducing reversals alone does not necessarily reduce the total cost.

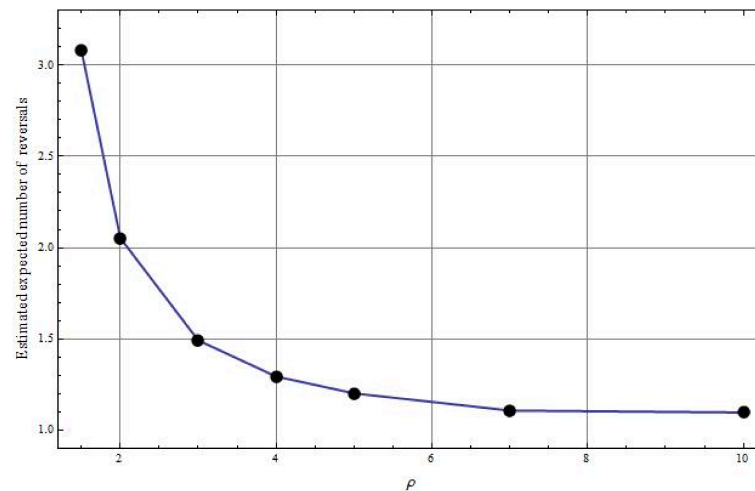


Figure 4. Estimated expected number of reversals versus the expansion factor.

Taken together, the three observables attain their most favorable values at different points of the tested grid. The estimated encounter time is smallest at $\rho = 7$, with $\rho = 5$ giving a very close value; this indicates that moderate expansion achieves a good balance between directional coverage and return frequency. The expected number of reversals is smallest at $\rho = 10$, since larger expansion reduces repeated alternation. However, the total cost is minimized at $\rho = 7$, because the additional reduction in reversals obtained at $\rho = 10$ is offset by the increase in the encounter time. Hence, the favorable values observed around $\rho = 5, 7$, and 10 reflect the trade-off between faster detection and fewer costly reversals, with the actual minima occurring at $\rho = 7$ for $E[\tau]$ and J_1 , and at $\rho = 10$ for $E[N_\tau]$.

6.4. Sensitivity to the Reversal Cost

The final experiment studies how the preferred excursion scale changes with κ . A finite family of symmetric strategies was considered by setting $a_+ = a_- = a$, $a = \{1, 2, 3, 4, 5, 6\}$ with $\rho = 3$. For each value of κ , the reported strategy is the one that produced the smallest estimated cost within this candidate set.

As reported in Table 5, the selected scale is $a = 3$ for $\kappa = 0$ and $\kappa = 1$. When the reversal penalty increases to 2 and 4, the preferred scale becomes $a = 4$, while $a = 6$ is selected at $\kappa = 8$.

Table 5. Grid-based sensitivity to the reversal cost.

κ	Selected a	$\hat{E}[\tau]$	$\hat{E}[N_\tau]$	Minimum Selected Cost
0	3	3.287	0.649	3.287
1	3	3.287	0.649	3.936
2	4	3.391	0.568	4.528
4	4	3.391	0.568	5.665
8	6	3.837	0.492	7.770

This adjustment reduces the expected reversal count from 0.649 to 0.492. At the same time, the expected encounter time increases from 3.287 to 3.837. The numerical results therefore display the expected trade-off: a larger reversal penalty favors longer excursions and fewer direction changes, but may delay detection.

Figure 5 shows the increase in the selected total cost as the reversal penalty becomes larger. The changes in the selected value of a indicate that the effect of κ is not lim-

ited to rescaling the objective function; it may also alter the preferred structure of the search strategy.

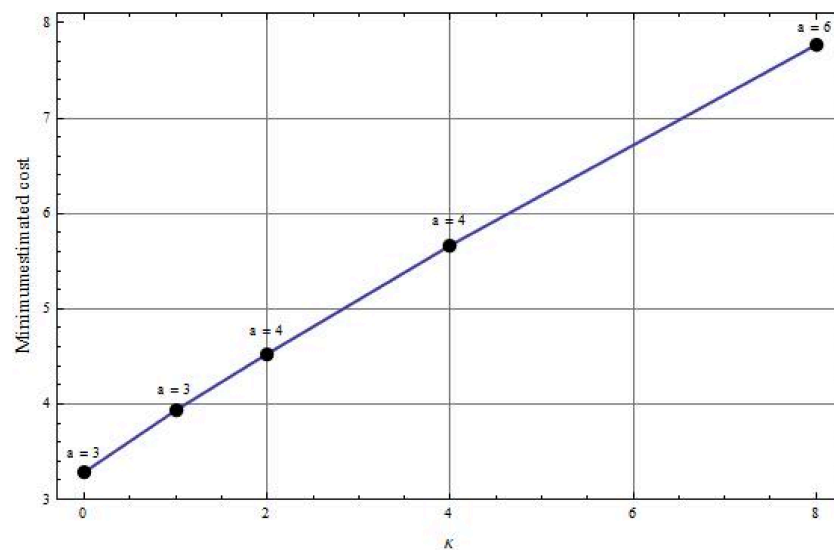


Figure 5. Minimum estimated cost over the candidate grid as κ varies.

The numerical values in Table 5 are minima over the stated finite grid. They are used to illustrate the sensitivity of the model and are not presented as global optima over the full admissible class.

The discretization check further confirms that the qualitative conclusions of the numerical study are not sensitive to moderate refinement of the Brownian time step.

7. Conclusions and Future Directions

This paper introduced a randomized linear-search model for a target moving on the real line according to a Brownian motion with drift. The searcher follows an alternating expanding path whose rightward and leftward excursion lengths are random, while each change in direction incurs a fixed penalty. This formulation combines target mobility, randomized exploration, and reversal costs within a single expected-cost criterion.

The analysis established a representation of the total expected cost in terms of the survival probabilities of the first-encounter time. Under the stated expansion and integrability assumptions, both the expected encounter time and the expected number of reversals are finite. A compact parametric strategy class was then used to prove the existence of an optimal randomized strategy. The results also clarified the role of symmetry: when the drift is zero and the initial target distribution is symmetric, reflected strategies have the same expected cost, whereas nonzero drift may produce different search behavior in the two directions.

The numerical study complemented the theoretical findings. It illustrated the reflection property in the symmetric regime, showed that the initial-direction probability becomes relevant when drift is present, and demonstrated that the expansion factor has a non-monotone effect on total cost. The simulations also confirmed the expected trade-off generated by the reversal penalty: larger values of the penalty favor strategies with fewer changes in direction, although this may increase the expected encounter time.

Several extensions arise naturally from the present framework. One direction is to allow the search speed to be random or state-dependent rather than constant. Another is to model reversals through explicit time delays or energy expenditures instead of a fixed additive cost. The same methodology could also be adapted to multiple searchers, partial observations, or search regions consisting of several rays or higher-dimensional domains.

A further question is whether sharper structural results can be obtained for the optimal excursion distributions, particularly under nonzero drift. These extensions would broaden the applicability of the model while preserving the probabilistic balance between rapid detection and operational switching costs.

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Appendix A. Numerical Consistency for the Zero-Drift Case

For $Q \in \{E[\tau], E[N_\tau], J_1\}$, the absolute and relative differences were calculated as $E_{abs}(Q) = |Q_2 - Q_1|$, $E_{rel}(Q) = \frac{|Q_2 - Q_1|}{\max(Q_2, Q_1)}$, where Q_2 and Q_1 denote the corresponding numerical estimates reported in Tables 2 and 3, respectively.

Table A1. Absolute and relative differences between the numerical estimates for $\mu = 0$.

p	Observable	E_{abs}	E_{rel}
0.3	$E[\tau]$	1.016	0.1942
0.3	$E[N_\tau]$	0.070	0.0558
0.3	J_1	1.085	0.1673
0.5	$E[\tau]$	0.086	0.0197
0.5	$E[N_\tau]$	0.011	0.0092
0.5	J_1	0.096	0.0172
0.7	$E[\tau]$	0.708	0.1392
0.7	$E[N_\tau]$	0.057	0.0455
0.7	J_1	0.765	0.1207

The differences in $E[\tau]$ are larger than those in $E[N_\tau]$, reflecting the greater sensitivity of the first-encounter time to Monte Carlo sampling and time discretization. For $p = 0.3$ and 0.7 , the larger discrepancies are also influenced by the different excursion-scale parameters used in Tables 2 and 3. In contrast, the $p = 0.5$ case uses the same symmetric parameter setting and therefore provides the most direct reproducibility check.

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