

Article

Fixed Point Theorems for Orbital Contractions on Directed Graphs in JS-Metric Spaces with Applications

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Abstract

This paper introduces a novel fixed-point framework for orbital contractions within the setting of JS-metric spaces endowed with directed graphs. By relaxing conventional metric conditions and incorporating graph-theoretical structures, we define a new class of operators known as orbitally $\mathcal{O}$ -transitive edge-preserving mappings. Utilizing an adaptive control function class $H(X)$, we establish sufficient conditions for the existence of fixed points that are specifically tailored to the iterative paths generated by the mappings. The theoretical results are supported by concrete examples demonstrating that our proposed framework strictly generalizes traditional edge-preserving properties. Furthermore, we illustrate the broad applicability and practical significance of our abstract theorems by establishing existence criteria for two major problems: nonlinear integral equations and nonlinear fractional differential equations involving Riemann–Liouville fractional derivatives with nonlocal integral boundary conditions. By reformulating these boundary value problems into equivalent Volterra-type integral equations, we show that our graph-based contraction approach is highly effective in both classical and fractional analytical settings.

Keywords: fixed point theorem; JS-metric space; directed graph; orbital contraction; integral equation; fractional differential equation

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1. Introduction

Fixed point theory is one of the most powerful and widely used tools in nonlinear functional analysis, providing a unified framework for solving various problems in differential equations, integral equations, and optimization. The cornerstone of this theory is the classical Banach Contraction Principle, which has inspired numerous generalizations concerning both contractive conditions and underlying abstract spaces.

A pivotal advancement in the generalization of metric spaces was introduced in 2015 by Jleli and Samet [1]. They proposed the concept of generalized metric spaces, commonly referred to as JS-metric spaces, which gracefully recovers and unifies various classes of abstract topological spaces, including standard metric, b -metric, dislocated metric, and modular spaces. Following this breakthrough, numerous researchers have

explored fixed point results within this generalized setting. For instance, El Kouch and El Marhrani [2] successfully extended classical fixed point theorems (e.g., Kannan and Chatterjea contraction mappings) to JS-metric spaces. Subsequently, Dumitrescu and Pitea [3] established fixed point theorems for almost (φ, θ) -contractions within these spaces, while Atiponrat et al. [4] significantly enriched the framework by incorporating graph-theoretic structures, investigating coincidence point theorems for KC-contraction mappings in JS-metric spaces endowed with a directed graph.

In a parallel effort to generalize contraction conditions, foundational concepts utilizing auxiliary functions were established by Jleli and Samet [5]. Their approaches introduced new classes of contractive mappings that substantially broadened the scope of traditional fixed point principles. Furthermore, the practical utility of these abstract theorems has been widely demonstrated in applied mathematics; classical works by Agarwal et al. [6], Harjani and Sadarangani [7], and Samet et al. [8] have successfully applied generalized contraction mappings to guarantee the existence of solutions for nonlinear integral equations.

Building upon this trajectory, recent contributions have shifted towards constructive and numerical approximations of these solutions. For example, Din [9] established fixed point results on ordered intervals with state-dependent memory, translating abstract integral operators into concrete convergent iterations with explicit constructive error control. Concurrently, Wang et al. [10] developed an improved high-order uniform accuracy numerical scheme for solving 2D general fractional Volterra integral equations by utilizing fixed point theory alongside dimensionality reduction techniques. While these recent advancements [9,10] beautifully highlight the computational and numerical aspects of fixed point theory, our current study focuses on fortifying the theoretical foundation. Specifically, we aim to establish robust existence criteria via a graph-based orbital contraction framework in generalized JS-metric spaces, which can subsequently motivate and underpin future numerical explorations and real-world applications.

In an independent yet equally significant direction, Jachymski [11] revolutionized the field by replacing standard partial orderings with directed graphs. He introduced the concepts of G-contractions and edge-preserving mappings, providing a highly flexible geometric framework for establishing the existence of fixed points. Since this seminal contribution, the study of fixed point theory on graph-endowed spaces has grown rapidly. Nicolae et al. [12] expanded Jachymski's approach to encompass multivalued and generalized contractive mappings, while Bojor [13] established fixed point theorems for Reich and Kannan-type contractions by exploiting the graphical structure. More recently, Poochinapan et al. [14] successfully utilized graph-based contractions to establish the existence of solutions and iterative approximations for nonlinear third-order differential equations. Collectively, these studies demonstrate that embedding a graph structure provides a powerful mathematical tool for relaxing conventional contractive conditions.

Recently, the synthesis of generalized metric spaces and graph theory has attracted considerable attention. Motivated by this growing interest, the present paper aims to significantly advance this line of research. Specifically, we introduce the novel concepts of $\mathcal{G}$ -compatible and orbital contractions within the framework of JS-metric spaces endowed with directed graphs. By utilizing orbitally $\mathcal{O}$ -transitive and edge-preserving mappings, we establish new sufficient conditions for the existence of fixed points. Finally, to demonstrate the practical utility of our abstract theoretical results, we successfully apply them to establish the existence of solutions for both nonlinear integral equations and fractional differential equations.

2. Preliminaries

In this section, we recall the foundational concepts of JS-metric spaces and directed graphs, which are essential for developing our main results. We begin by establishing the fundamental metric structure.

2.1. JS-Metric Spaces

Let X be a nonempty set and $J_s : X \times X \rightarrow [0, \infty]$ be a mapping. For each $x \in X$, we define the set of sequences converging to x with respect to J_s as

$$C(J_s, X, x) = \left\{ \{x_n\} \text{ is a sequence in } X : \lim_{n \rightarrow \infty} J_s(x_n, x) = 0 \right\}.$$

Utilizing this set, the generalized metric space is formally defined as follows.

Definition 1 ([1]). A function $J_s : X \times X \rightarrow [0, \infty]$ on a nonempty set X is called a **generalized metric** if it satisfies the following conditions:

- (J_{s1}) $J_s(x, y) = 0$ implies $x = y$ for all $x, y \in X$;
- (J_{s2}) $J_s(x, y) = J_s(y, x)$ for all $x, y \in X$;
- (J_{s3}) There exists a constant $s > 0$ such that

$$J_s(x, y) \leq s \limsup_{n \rightarrow \infty} J_s(x_n, y)$$

for all $x, y \in X$ and $\{x_n\} \in C(J_s, X, x)$.

In this case, the pair (X, J_s) is called a **JS-metric space**.

With the metric structure in place, we naturally extend the classical topological notions such as convergence, Cauchy sequences, and completeness to this generalized setting.

Definition 2 ([1]). Let (X, J_s) be a JS-metric space and $\{x_n\}$ be a sequence in X .

- 1. $\{x_n\}$ **J_s -converges** to $x \in X$ if $\{x_n\} \in C(J_s, X, x)$.
- 2. $\{x_n\}$ is a **J_s -Cauchy sequence** if $\lim_{m, n \rightarrow \infty} J_s(x_n, x_m) = 0$.
- 3. (X, J_s) is **J_s -complete** if every J_s -Cauchy sequence in X J_s -converges to a point in X .

In the following example, we construct a JS-metric space that will play a crucial role in our application sections later in the paper.

Example 1. Let $X = C[0, L]$ be the space of continuous real-valued functions on $[0, L]$. Define $J_s : X \times X \rightarrow [0, \infty)$ by

$$J_s(x, y) = \left(\max_{t \in [0, L]} |x(t) - y(t)| \right)^p,$$

where $p > 1$ is a fixed real number. Then (X, J_s) is a **J_s -complete JS-metric space** with the constant $s = 2^{p-1}$.

Proof. To show that (X, J_s) is a JS-metric space, we must verify the three axioms (J_{s1}), (J_{s2}), and (J_{s3}) of Definition 1. Let $d_\infty(x, y) = \max_{t \in [0, L]} |x(t) - y(t)|$ denote the standard supremum metric on $C[0, L]$. Thus, we can write $J_s(x, y) = (d_\infty(x, y))^p$.

Step 1: Verification of (J_{s1}). Suppose $J_s(x, y) = 0$ for $x, y \in X$. Then

$$(d_\infty(x, y))^p = 0 \implies d_\infty(x, y) = 0.$$

This means $\max_{t \in [0, L]} |x(t) - y(t)| = 0$, which implies $|x(t) - y(t)| = 0$ for all $t \in [0, L]$. Hence, $x(t) = y(t)$ for all t , yielding $x = y$. Thus, (J_{s1}) holds.

Step 2: Verification of (J_{s2}) . For any $x, y \in X$, since $|x(t) - y(t)| = |y(t) - x(t)|$ for all t , we have

$$J_s(x, y) = \left(\max_{t \in [0, L]} |x(t) - y(t)| \right)^p = \left(\max_{t \in [0, L]} |y(t) - x(t)| \right)^p = J_s(y, x).$$

Thus, (J_{s2}) holds.

Step 3: Verification of (J_{s3}) . Let $x, y \in X$ and let $\{x_n\}$ be a sequence in $C(J_s, X, x)$, meaning $\lim_{n \rightarrow \infty} J_s(x_n, x) = 0$. Consequently, $\lim_{n \rightarrow \infty} d_\infty(x_n, x) = 0$. By the standard triangle inequality for the supremum metric, we have

$$d_\infty(x, y) \leq d_\infty(x, x_n) + d_\infty(x_n, y).$$

Recall the well-known algebraic inequality: for any $A, B \geq 0$ and $p > 1$, $(A + B)^p \leq 2^{p-1}(A^p + B^p)$. Applying this inequality, we obtain

$$\begin{aligned} J_s(x, y) &= (d_\infty(x, y))^p \\ &\leq (d_\infty(x, x_n) + d_\infty(x_n, y))^p \\ &\leq 2^{p-1}((d_\infty(x, x_n))^p + (d_\infty(x_n, y))^p) \\ &= 2^{p-1}(J_s(x, x_n) + J_s(x_n, y)). \end{aligned}$$

Taking the limit superior on both sides as $n \rightarrow \infty$, we get

$$\begin{aligned} J_s(x, y) &\leq 2^{p-1} \limsup_{n \rightarrow \infty} (J_s(x, x_n) + J_s(x_n, y)) \\ &\leq 2^{p-1} \left(\lim_{n \rightarrow \infty} J_s(x, x_n) + \limsup_{n \rightarrow \infty} J_s(x_n, y) \right). \end{aligned}$$

Since $\lim_{n \rightarrow \infty} J_s(x, x_n) = 0$, this reduces to

$$J_s(x, y) \leq 2^{p-1} \limsup_{n \rightarrow \infty} J_s(x_n, y).$$

This demonstrates that (J_{s3}) is satisfied with the constant $s = 2^{p-1} > 0$.

Since all three axioms are fulfilled, we conclude that (X, J_s) is indeed a JS-metric space.

Finally, it is straightforward to verify that (X, J_s) is J_s -complete. A sequence $\{x_n\}$ is J_s -Cauchy if and only if $\lim_{m, n \rightarrow \infty} \left(\max_{t \in [0, L]} |x_n(t) - x_m(t)| \right)^p = 0$. Since $p > 1$, this is strictly equivalent to $\{x_n\}$ being a Cauchy sequence with respect to the standard supremum norm d_∞ . Because $C[0, L]$ is a Banach space under this norm, $\{x_n\}$ must converge uniformly to some function $x \in C[0, L]$. This uniform convergence naturally implies $\lim_{n \rightarrow \infty} J_s(x_n, x) = 0$. Thus, every J_s -Cauchy sequence J_s -converges, ensuring the J_s -completeness of the space. $\square$

An essential feature of JS-metric spaces is that the limit of a convergent sequence is well-behaved. Specifically, the uniqueness of the limit is guaranteed by the following proposition.

Proposition 1 ([1]). Let (X, J_s) be a JS-metric space and $\{x_n\}$ be a sequence in X . If $\{x_n\} \in C(J_s, X, x) \cap C(J_s, X, y)$ for some $x, y \in X$, then $x = y$.

Building upon the concept of convergence, the notion of continuity in JS-metric spaces is naturally defined via sequential limits.

Definition 3 ([1]). Let (X, J_s) be a JS-metric space. A mapping $f : X \rightarrow X$ is **continuous at** $x_0 \in X$ if $\{x_n\} \in C(J_s, X, x_0)$ implies $\{fx_n\} \in C(J_s, X, fx_0)$. Furthermore, f is **continuous on** X if it is continuous at every point in X .

2.2. Directed Graphs in JS-Metric Spaces

To relax standard contractive conditions, we integrate a graphical structure into the metric space. This geometric approach allows us to restrict contraction inequalities only to pairs of points connected by an edge. Following Jachymski [11], a JS-metric space (X, J_s) is said to be *endowed with a directed graph* $G = (V(G), E(G))$ if its vertex set $V(G)$ coincides exactly with X , the edge set $E(G)$ contains all loops (i.e., $(x, x) \in E(G)$ for all $x \in X$), and G has no parallel edges.

Throughout this paper, the triplet (X, J_s, G) denotes a JS-metric space (X, J_s) endowed with such a graph G .

By incorporating this graph structure, the traditional concept of continuity can be weakened. We define a graph-based continuity where the preservation of limits is required only for sequences whose consecutive terms are connected by edges.

Definition 4 ([4]). A mapping $f : X \rightarrow X$ on (X, J_s, G) is **G-continuous** if, for any $x \in X$ and sequence $\{x_n\} \in C(J_s, X, x)$ satisfying $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, it follows that $\{fx_n\} \in C(J_s, X, fx)$.

For our fixed point results to hold, the mapping and the graph must interact harmoniously. We require the mapping to preserve the graphical structure, and the graph itself to exhibit a chaining property. These regularities are captured in the following definitions.

Definition 5 ([4]). A mapping $f : X \rightarrow X$ on (X, J_s, G) is called **edge-preserving** if

$$(x, y) \in E(G) \implies (fx, fy) \in E(G).$$

Definition 6 ([4]). The edge set $E(G)$ of (X, J_s, G) satisfies the **transitivity property** if, for all $x, y, z \in X$,

$$(x, z) \in E(G) \quad \text{and} \quad (z, y) \in E(G) \implies (x, y) \in E(G).$$

3. Main Results

In this section, we introduce the concept of $\mathfrak{S}$ -compatible contractions and investigate their fixed points within the framework of JS-metric spaces endowed with directed graphs. To achieve this, we first need to relax classical structural requirements, such as standard edge-preservation and full transitivity. By restricting these conditions solely to the paths generated by the mapping's iterations and its orbits, we obtain a more flexible and generalized setting. We begin by establishing these essential definitions that form the foundation of our main results.

Definition 7. Let $\mathfrak{J} : X \rightarrow X$ be a mapping on (X, J_s, G) .

- (i) $\mathfrak{J}$ is **orbitally edge-preserving** if $(x, \mathfrak{J}x) \in E(G) \implies (\mathfrak{J}x, \mathfrak{J}^2x) \in E(G)$ and $J_s(\mathfrak{J}x, \mathfrak{J}^2x) < \infty$ for all $x \in X$.
- (ii) $\mathfrak{J}$ is **$\mathcal{O}$ -transitive** if $(x, y) \in E(G)$ and $(y, \mathfrak{J}y) \in E(G) \implies (x, \mathfrak{J}y) \in E(G)$ and $J_s(x, \mathfrak{J}y) < +\infty$ for all $x, y \in X$.

Furthermore, if $\mathfrak{J}$ is both orbitally edge-preserving and $\mathcal{O}$ -transitive, it is called an **orbitally $\mathcal{O}$ -transitive edge-preserving mapping**.

The relationship between these newly defined orbital properties and their classical graphical counterparts is straightforward but crucial, as highlighted in the following remark.

Remark 1.

- (1) Every edge-preserving mapping is also an orbitally edge-preserving mapping.
- (2) An edge-preserving mapping that is transitive is also an orbitally $\mathcal{O}$ -transitive edge-preserving mapping.

It is important to emphasize that while classical edge-preserving mappings naturally satisfy the orbital conditions, the converse is generally false. To explicitly demonstrate that our orbital concepts are a strict generalization, we construct the following example where a mapping preserves edges along its orbits but fails to preserve all edges in the graph.

Example 2. Let $X = \{a, b, c\} \subseteq [0, 1]$ and define a mapping $\mathfrak{J} : X \rightarrow X$ by

$$\mathfrak{J}a = a, \quad \mathfrak{J}b = c, \quad \text{and} \quad \mathfrak{J}c = c,$$

equipped with the generalized metric $J_s : X \times X \rightarrow [0, \infty)$ defined by $J_s(x, y) = x + y$ for all $x, y \in X$.

Let the directed graph G be given by the edge set $E(G) = \{(a, a), (c, c), (a, b)\}$. This establishes the structure (X, J_s, G) .

We first verify that $\mathfrak{J}$ is orbitally edge-preserving. The only points $x \in X$ satisfying the initial edge condition $(x, \mathfrak{J}x) \in E(G)$ are $x = a$ and $x = c$. For $x = a$, we have $(\mathfrak{J}a, \mathfrak{J}^2a) = (a, a) \in E(G)$ with a finite distance $J_s(a, a) = 2a < +\infty$. Similarly, for $x = c$, we obtain $(\mathfrak{J}c, \mathfrak{J}^2c) = (c, c) \in E(G)$ with $J_s(c, c) = 2c < +\infty$. Thus, $\mathfrak{J}$ satisfies the orbital edge-preserving condition.

However, $\mathfrak{J}$ is not edge-preserving in the classical sense. Although the pair (a, b) belongs to $E(G)$, mapping both endpoints yields $(\mathfrak{J}a, \mathfrak{J}b) = (a, c) \notin E(G)$, which demonstrates that classical edge-preservation fails.

To formulate our fixed point theorems, we must identify the precise subsets of the space where the mapping behaves harmoniously with the graph structure. Specifically, we are interested in the points that can act as starting points for graphical iterations, as well as the ultimate fixed points of the mapping.

Definition 8. For a mapping $\mathfrak{J} : X \rightarrow X$ on (X, J_s, G) , we define the set of points whose initial transitions along $\mathfrak{J}$ correspond to valid, finite-distance edges in G as

$$\mathfrak{D}(\mathfrak{J}) := \{x \in X : (x, \mathfrak{J}x) \in E(G) \text{ and } J_s(x, \mathfrak{J}x) < +\infty\}.$$

Additionally, the set of fixed points of $\mathfrak{J}$ is denoted conventionally by

$$F(\mathfrak{J}) := \{x \in X : \mathfrak{J}x = x\}.$$

A natural relationship exists between these two sets. For any $x \in F(\mathfrak{J})$, we have $\mathfrak{J}x = x$. Since the graph G is assumed to contain all loops, it immediately follows that $(x, \mathfrak{J}x) = (x, x) \in E(G)$, and its self-distance is trivially zero (which is finite). Thus, any fixed point is always a valid starting point for graphical iterations, yielding the inclusion

$$F(\mathfrak{J}) \subseteq \mathfrak{D}(\mathfrak{J}). \quad (1)$$

Before introducing our main contraction condition, we establish a class of auxiliary control functions. These functions will serve as dynamic scaling factors, allowing us to generalize traditional constant-factor contractions (such as the Banach Contraction Principle) into a more adaptive framework.

Definition 9. Let $H(X)$ denote the class of all functions $\mathfrak{H} : X \times X \rightarrow [0, 1]$ satisfying the following limit condition: for any sequences $\{x_n\}$ and $\{y_n\}$ in X such that the corresponding real sequence $\{J_s(x_n, y_n)\}$ is non-increasing,

$$\lim_{n \rightarrow \infty} \mathfrak{H}(x_n, y_n) = 1 \implies \lim_{n \rightarrow \infty} J_s(x_n, y_n) = 0.$$

To illustrate the richness of the class $H(X)$, we provide the following practical examples, which encompass both logarithmic and exponential scaling behaviors.

Example 3. The following functions $\mathfrak{H}$ belong to the class $H(X)$ for a given JS-metric J_s :

$$(1) \quad \mathfrak{H}(x, y) = \begin{cases} \frac{1}{12}, & \text{if } J_s(x, y) = 0, \\ \frac{\ln(1 + J_s(x, y))}{J_s(x, y)}, & \text{if } J_s(x, y) \neq 0. \end{cases}$$

$$(2) \quad \mathfrak{H}(x, y) = e^{-J_s(x, y)}.$$

With the foundational graphical properties and the control function class $H(X)$ now established, we are in a position to introduce the central concept of this paper. By marrying the orbital regularity of the graph with a dynamic, adaptive contraction condition, we formulate a new type of mapping. This graph-based contraction serves as the primary engine for driving our main fixed point results.

Definition 10. A mapping $\mathfrak{J} : X \rightarrow X$ on (X, J_s, G) is called an **orbital contraction** if it satisfies the following conditions:

- (i) $\mathfrak{J}$ is an orbitally $\mathcal{O}$ -transitive edge-preserving mapping.
- (ii) There exists a function $\mathfrak{H} \in H(X)$ such that for all $(x, y) \in E(G)$,

$$J_s(\mathfrak{J}x, \mathfrak{J}y) \leq \mathfrak{H}(x, y)K(x, y), \quad (2)$$

where the composite distance function $K : X \times X \rightarrow [0, \infty)$ is defined by

$$K(x, y) = J_s(x, y) + |J_s(x, \mathfrak{J}x) - J_s(y, \mathfrak{J}y)|.$$

Recall that the generalized metric J_s is defined as a mapping from $X \times X$ into $[0, \infty]$, meaning that the distance between certain points in the space can potentially be infinite. To meaningfully investigate the convergence properties of our graph-based contractions and ultimately guarantee the existence of a fixed point, it is crucial to ensure that the iterative sequences generated by the mapping do not diverge to infinity. To strictly manage this extended codomain and bound these generalized distances, we introduce a supremum-based notation.

For any elements $x, y \in X$, we denote the supremum of their distances over the space as

$$\beta(x, y) = \sup\{J_s(x, y) : x, y \in X\}.$$

When evaluating this bound along specific sequences $\{x_n\}$ and $\{y_n\}$ in X , we define

$$\beta(x_n, y_n) = \sup\{J_s(x_n, y_n) : n \in \mathbb{N}\}.$$

By this construction, it naturally follows that the sequence of bounds exhibits a non-increasing property, meaning we have $\beta(x_n, y_n) \geq \beta(x_{n+1}, y_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$.

With this finiteness parameter established, we can now formally define the specific iterative sequence that will serve as the primary object of study in our main fixed point theorems.

Definition 11. Let $\mathfrak{J} : X \rightarrow X$ be a mapping on (X, J_s, G) . A sequence $\{x_n\}$ in X is called a $(\mathfrak{J}, G, J_s)$ -sequence if there exists an initial point $x_0 \in X$ such that $x_n = \mathfrak{J}x_{n-1} = \mathfrak{J}^n x_0$ for all $n \geq 1$, and

$$(x_n, x_m) \in E(G) \quad \text{and} \quad \beta(x_n, x_m) < +\infty \quad \text{for all } m \geq n \geq 1. \quad (3)$$

We now present and prove a foundational lemma, which establishes that as long as the mapping has at least one valid starting point, its orbital properties guarantee the existence of a well-behaved sequence. This lemma is essential to the proof of our main results.

Lemma 1. Let $\mathfrak{J} : X \rightarrow X$ be a mapping on (X, J_s, G) . If $\mathfrak{D}(\mathfrak{J}) \neq \emptyset$ and $\mathfrak{J}$ is an orbitally $\mathcal{O}$ -transitive edge-preserving mapping, then there exists a $(\mathfrak{J}, G, J_s)$ -sequence in X . Moreover, if $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then $F(\mathfrak{J}) \neq \emptyset$.

Proof. Since $\mathfrak{D}(\mathfrak{J}) \neq \emptyset$, there exists an initial point $x_0 \in X$ such that $(x_0, \mathfrak{J}x_0) \in E(G)$ and $J_s(x_0, \mathfrak{J}x_0) < +\infty$. By defining $x_1 = \mathfrak{J}x_0$ and utilizing the fact that $\mathfrak{J}$ is orbitally edge-preserving, we deduce that $(x_1, \mathfrak{J}x_1) \in E(G)$ and $J_s(x_1, \mathfrak{J}x_1) < +\infty$.

Proceeding recursively, we construct a sequence $\{x_n\}$ in X defined by

$$x_n = \mathfrak{J}x_{n-1} = \mathfrak{J}^n x_0 \quad \text{for all } n \geq 1.$$

It then follows by mathematical induction that

$$(x_n, x_{n+1}) \in E(G) \quad \text{and} \quad \beta(x_n, x_{n+1}) < +\infty \quad \text{for all } n \geq 0.$$

Furthermore, applying the $\mathcal{O}$ -transitivity of $\mathfrak{J}$ inductively yields

$$(x_n, x_m) \in E(G) \quad \text{and} \quad \beta(x_n, x_m) < +\infty \quad \text{for all } m \geq n \geq 0.$$

Hence, $\{x_n\}$ satisfies all conditions to be a $(\mathfrak{J}, G, J_s)$ -sequence.

Finally, if $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then by the definition of the iterative sequence, we have $\mathfrak{J}x_{n_0} = x_{n_0+1} = x_{n_0}$. This immediately implies that $x_{n_0} \in F(\mathfrak{J})$, and therefore $F(\mathfrak{J}) \neq \emptyset$. $\square$

Having established the existence of a valid iterative sequence in Lemma 1, we now analyze the behavior of the distances between consecutive terms of this sequence. The following lemma demonstrates that the orbital contraction condition forces these distances to strictly decrease and ultimately vanish in the limit.

Lemma 2. Let $\mathfrak{J} : X \rightarrow X$ be an orbital contraction on (X, J_s, G) . If there exists a $(\mathfrak{J}, G, J_s)$ -sequence $\{x_n\}$ in X such that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, then the real sequence $\{\mathfrak{D}_n := J_s(x_n, x_{n+1})\}$ is strictly decreasing and converges to 0.

Proof. Let $\{x_n\}$ be a $(\mathfrak{J}, G, J_s)$ -sequence in X . For each $n \geq 0$, we denote the consecutive distance as $\mathfrak{D}_n = J_s(x_n, x_{n+1})$. By the finiteness condition of the sequence, we know that $\mathfrak{D}_n < +\infty$ for all n .

Since $\mathfrak{J}$ is an orbital contraction, for any $n \geq 0$, we have

$$\begin{aligned}\mathfrak{D}_{n+1} &= J_s(x_{n+1}, x_{n+2}) \\ &= J_s(\mathfrak{J}x_n, \mathfrak{J}x_{n+1}) \\ &\leq \mathfrak{H}(x_n, x_{n+1})K(x_n, x_{n+1}) \\ &\leq K(x_n, x_{n+1}),\end{aligned}\tag{4}$$

where the last inequality holds because $\mathfrak{H}(x_n, x_{n+1}) \leq 1$.

By the definition of the composite distance K , we observe that

$$\begin{aligned}K(x_n, x_{n+1}) &= J_s(x_n, x_{n+1}) + |J_s(x_n, \mathfrak{J}x_n) - J_s(x_{n+1}, \mathfrak{J}x_{n+1})| \\ &= \mathfrak{D}_n + |\mathfrak{D}_n - \mathfrak{D}_{n+1}|.\end{aligned}$$

To show that $\{\mathfrak{D}_n\}$ is strictly decreasing, suppose for contradiction that it is not. Then, there exists an index $C \in \mathbb{N} \cup \{0\}$ such that $\mathfrak{D}_C \leq \mathfrak{D}_{C+1}$. This implies that

$$|\mathfrak{D}_C - \mathfrak{D}_{C+1}| = \mathfrak{D}_{C+1} - \mathfrak{D}_C,$$

which yields

$$K(x_C, x_{C+1}) = \mathfrak{D}_C + (\mathfrak{D}_{C+1} - \mathfrak{D}_C) = \mathfrak{D}_{C+1}.$$

Substituting this result into inequality (4), we obtain

$$\mathfrak{D}_{C+1} \leq \mathfrak{H}(x_C, x_{C+1})\mathfrak{D}_{C+1} \leq \mathfrak{D}_{C+1}.\tag{5}$$

Since we assume $x_n \neq x_{n+1}$ for all $n \geq 0$, we know that $\mathfrak{D}_{C+1} = J_s(x_{C+1}, x_{C+2}) > 0$. Furthermore, $\mathfrak{D}_{C+1}$ is finite. We can therefore divide inequality (5) by $\mathfrak{D}_{C+1}$ to obtain $\mathfrak{H}(x_C, x_{C+1}) = 1$. By the defining property of the class $H(X)$, this limit condition forces

$$J_s(x_C, x_{C+1}) = 0 \quad \text{or} \quad J_s(\mathfrak{J}x_C, \mathfrak{J}x_{C+1}) = J_s(x_{C+1}, x_{C+2}) = 0,$$

which clearly contradicts the assumption that $x_n \neq x_{n+1}$. Therefore, the sequence $\{\mathfrak{D}_n\}$ must be strictly decreasing; that is, $\mathfrak{D}_n > \mathfrak{D}_{n+1}$ for all $n \geq 0$.

Next, we show that $\lim_{n \rightarrow \infty} \mathfrak{D}_n = 0$. Since $\{\mathfrak{D}_n\}$ is a strictly decreasing sequence of non-negative real numbers, it must converge to some limit $\mathcal{D}^* \geq 0$. Suppose for contradiction that $\mathcal{D}^* > 0$. Because $\{\mathfrak{D}_n\}$ is strictly decreasing, we can simplify the absolute value in K :

$$K(x_n, x_{n+1}) = \mathfrak{D}_n + (\mathfrak{D}_n - \mathfrak{D}_{n+1}) = 2\mathfrak{D}_n - \mathfrak{D}_{n+1}.$$

Applying inequality (4), it follows that

$$\mathfrak{D}_{n+1} \leq \mathfrak{H}(x_n, x_{n+1})(2\mathfrak{D}_n - \mathfrak{D}_{n+1}) \leq 2\mathfrak{D}_n - \mathfrak{D}_{n+1}.$$

Since $\mathfrak{D}_n$ and $\mathfrak{D}_{n+1}$ are strictly positive and finite, the term $(2\mathfrak{D}_n - \mathfrak{D}_{n+1})$ is also strictly positive and finite. Dividing the inequality by this term and taking the limit as $n \rightarrow \infty$, we obtain

$$1 = \lim_{n \rightarrow \infty} \frac{\mathfrak{D}_{n+1}}{2\mathfrak{D}_n - \mathfrak{D}_{n+1}} \leq \lim_{n \rightarrow \infty} \mathfrak{H}(x_n, x_{n+1}) \leq 1,$$

which implies that

$$\lim_{n \rightarrow \infty} \mathfrak{H}(x_n, x_{n+1}) = 1.$$

By the property of functions in $H(X)$, this limit forces the sequence of distances to vanish:

$$\lim_{n \rightarrow \infty} J_s(x_n, x_{n+1}) = \lim_{n \rightarrow \infty} \mathfrak{D}_n = 0,$$

which directly contradicts our assumption that $\mathcal{D}^* > 0$. Hence, we conclude that

$$\lim_{n \rightarrow \infty} \mathfrak{D}_n = 0.$$

This completes the proof. $\square$

Before presenting our main result, we must define the appropriate notion of continuity for our mappings. Since we are restricting our attention to the graphical orbits generated by the mapping, we do not require classical continuity over the entire space. Instead, we only need the mapping to preserve limits along these specific graphical paths.

Definition 12. A mapping $\mathfrak{J} : X \rightarrow X$ on (X, J_s, G) is said to be **orbitally G-continuous** if, for any $x \in X$ and any sequence $\{x_n\}$ in X such that $\{x_n\} \in C(J_s, X, x)$ and $x_n \in \mathfrak{D}(\mathfrak{J})$ for all $n \in \mathbb{N}$, it follows that $\{\mathfrak{J}x_n\} \in C(J_s, X, \mathfrak{J}x)$.

We are now ready to state and prove our primary theorem. It establishes a powerful equivalence: for an orbitally G-continuous orbital contraction defined on a complete space, the existence of a valid starting graphical point is both necessary and sufficient for the existence of a fixed point.

Theorem 1. Let (X, J_s) be a J_s -complete JS-metric space endowed with a directed graph G , and let $\mathfrak{J} : X \rightarrow X$ be an orbital contraction on (X, J_s, G) . If $\mathfrak{J}$ is orbitally G-continuous, then

$$\mathfrak{D}(\mathfrak{J}) \neq \emptyset \iff F(\mathfrak{J}) \neq \emptyset.$$

Proof. We first prove the forward implication ($\implies$). Suppose that $\mathfrak{D}(\mathfrak{J}) \neq \emptyset$. By Lemma 1, there exists a well-behaved $(\mathfrak{J}, G, J_s)$ -sequence $\{x_n\}$ in X generated by an initial point $x_0 \in \mathfrak{D}(\mathfrak{J})$ such that

$$x_n = \mathfrak{J}x_{n-1} = \mathfrak{J}^n x_0 \quad \text{for all } n \geq 1.$$

By the definition of this sequence, it satisfies the critical property that all its forward terms remain connected and finitely bounded, meaning

$$(x_n, x_m) \in E(G) \quad \text{and} \quad \beta(x_n, x_m) < +\infty \quad \text{for all } m \geq n \geq 1. \quad (6)$$

Let $\mathfrak{D}_n = J_s(x_n, x_{n+1})$. We divide our investigation into two possible cases:

Case 1: $x_{n_0} = x_{n_0+1}$ for some $n_0 \geq 0$. By the latter part of Lemma 1, x_{n_0} is already a fixed point of $\mathfrak{J}$, which immediately implies $F(\mathfrak{J}) \neq \emptyset$.

Case 2: $x_n \neq x_{n+1}$ for all $n \geq 0$. It follows directly from Lemma 2 that the sequence of consecutive distances strictly decreases to zero:

$$\lim_{n \rightarrow \infty} \mathfrak{D}_n = \lim_{n \rightarrow \infty} J_s(x_n, x_{n+1}) = 0. \quad (7)$$

We claim that $\{x_n\}$ is a J_s -Cauchy sequence. Suppose, for contradiction, that it is not. Then, there exists $\epsilon > 0$ such that for each $k \in \mathbb{N}$, we can find integers $n_k \geq k$ and $m_k \geq 1$, where m_k is the smallest integer satisfying

$$J_s(x_{n_k}, x_{n_k+m_k}) \geq \epsilon.$$

By the transitivity property of $E(G)$, we have $(x_{n_k}, x_{n_k+m_k}) \in E(G)$ for all $k \in \mathbb{N}$. Since $\mathfrak{J}$ is an orbital contraction, applying the contraction condition yields

$$\begin{aligned} J_s(x_{n_k+1}, x_{n_k+m_k+1}) &= J_s(\mathfrak{J}x_{n_k}, \mathfrak{J}x_{n_k+m_k}) \\ &\leq \mathfrak{H}(x_{n_k}, x_{n_k+m_k})K(x_{n_k}, x_{n_k+m_k}), \end{aligned} \quad (8)$$

where the composite distance K evaluates to

$$\begin{aligned} K(x_{n_k}, x_{n_k+m_k}) &= J_s(x_{n_k}, x_{n_k+m_k}) + |J_s(x_{n_k}, \mathfrak{J}x_{n_k}) - J_s(x_{n_k+m_k}, \mathfrak{J}x_{n_k+m_k})| \\ &= J_s(x_{n_k}, x_{n_k+m_k}) + |\mathfrak{D}_{n_k} - \mathfrak{D}_{n_k+m_k}|. \end{aligned}$$

Substituting this expression into inequality (8) and repeatedly iterating the contraction condition backwards down to x_0 and x_{m_k} , we obtain

$$\begin{aligned} J_s(x_{n_k+1}, x_{n_k+m_k+1}) &\leq \left(\prod_{i=0}^{n_k} \mathfrak{H}(x_{n_k-i}, x_{n_k+m_k-i}) \right) J_s(x_0, x_{m_k}) \\ &\quad + \sum_{j=0}^{n_k} \left(\prod_{i=0}^j \mathfrak{H}(x_{n_k-i}, x_{n_k+m_k-i}) \right) |\mathfrak{D}_{n_k-j} - \mathfrak{D}_{n_k+m_k-j}|. \end{aligned}$$

Let $M_k = \max_{0 \leq j \leq n_k} |\mathfrak{D}_{n_k-j} - \mathfrak{D}_{n_k+m_k-j}|$. We can bound the inequality as follows:

$$\begin{aligned} J_s(x_{n_k+1}, x_{n_k+m_k+1}) &\leq \left(\prod_{i=0}^{n_k} \mathfrak{H}(x_{n_k-i}, x_{n_k+m_k-i}) \right) J_s(x_0, x_{m_k}) \\ &\quad + M_k \sum_{j=0}^{n_k} \prod_{i=0}^j \mathfrak{H}(x_{n_k-i}, x_{n_k+m_k-i}). \end{aligned} \quad (9)$$

For each $k \in \mathbb{N}$, we choose an index $0 \leq i_k \leq n_k$ that maximizes the control function:

$$\mathfrak{H}(x_{n_k-i_k}, x_{n_k+m_k-i_k}) = \max_{0 \leq i \leq n_k} \mathfrak{H}(x_{n_k-i}, x_{n_k+m_k-i}).$$

Define $\delta := \limsup_{k \rightarrow \infty} \mathfrak{H}(x_{n_k-i_k}, x_{n_k+m_k-i_k})$. Since $\mathfrak{H}$ maps into $[0, 1]$, we consider two cases based on the value of δ .

Case 2.1: $\delta < 1$.

By iterating the contraction condition n_k steps backward directly on $J_s(x_{n_k}, x_{n_k+m_k})$, we have

$$\begin{aligned} J_s(x_{n_k}, x_{n_k+m_k}) &\leq \left(\prod_{i=1}^{n_k} \mathfrak{H}(x_{n_k-i}, x_{n_k+m_k-i}) \right) J_s(x_0, x_{m_k}) \\ &\quad + \tilde{M}_k \sum_{j=1}^{n_k} \prod_{i=1}^j \mathfrak{H}(x_{n_k-i}, x_{n_k+m_k-i}), \end{aligned}$$

where $\tilde{M}_k = \max_{1 \leq j \leq n_k} |\mathfrak{D}_{n_k-j} - \mathfrak{D}_{n_k+m_k-j}|$.

Since $\delta < 1$, there exists $r \in (0, 1)$ such that $\mathfrak{H} \leq r$ for sufficiently large k . Thus, the product term is bounded by $r^{n_k} \rightarrow 0$, and the summation is bounded by a convergent geometric series $\sum_{j=1}^{\infty} r^j = \frac{r}{1-r}$.

Since $\lim_{n \rightarrow \infty} \mathfrak{D}_n = 0$, it follows that $\tilde{M}_k \rightarrow 0$ as $k \rightarrow \infty$. Taking the limit as $k \rightarrow \infty$ in the inequality yields

$$\lim_{k \rightarrow \infty} J_s(x_{n_k}, x_{n_k+m_k}) = 0.$$

However, this directly contradicts our assumption that $J_s(x_{n_k}, x_{n_k+m_k}) \geq \epsilon > 0$.

Case 2.2: $\delta = 1$.

By passing to a subsequence, we may assume that

$$\lim_{k \rightarrow \infty} \mathfrak{H}(x_{n_k-i_k}, x_{n_k+m_k-i_k}) = 1.$$

Since $\mathfrak{H} \in H(X)$, this implies that

$$\lim_{k \rightarrow \infty} J_s(x_{n_k-i_k}, x_{n_k+m_k-i_k}) = 0.$$

Let $M'_k = \max_{1 \leq l \leq i_k} |\mathfrak{D}_{n_k-l} - \mathfrak{D}_{n_k+m_k-l}|$. Because $\lim_{n \rightarrow \infty} \mathfrak{D}_n = 0$, it follows that $\lim_{k \rightarrow \infty} M'_k = 0$. By iterating the original inequality i_k steps backwards from $J_s(x_{n_k}, x_{n_k+m_k})$, we obtain

$$\begin{aligned} J_s(x_{n_k}, x_{n_k+m_k}) &\leq \left(\prod_{j=1}^{i_k} \mathfrak{H}(x_{n_k-j}, x_{n_k+m_k-j}) \right) J_s(x_{n_k-i_k}, x_{n_k+m_k-i_k}) \\ &\quad + M'_k \sum_{l=1}^{i_k} \prod_{j=1}^l \mathfrak{H}(x_{n_k-j}, x_{n_k+m_k-j}). \end{aligned}$$

Note that $\mathfrak{H} \leq 1$, which bounds all product terms by 1. Combining this with the limits established above, we can choose an integer k_0 sufficiently large such that both

$$J_s(x_{n_{k_0}-i_{k_0}}, x_{n_{k_0}+m_{k_0}-i_{k_0}}) < \frac{\epsilon}{2}$$

and

$$M'_{k_0} \sum_{l=1}^{i_{k_0}} \prod_{j=1}^l \mathfrak{H}(x_{n_{k_0}-j}, x_{n_{k_0}+m_{k_0}-j}) < \frac{\epsilon}{2}.$$

Substituting these upper bounds back into the inequality for $k = k_0$, we deduce

$$\begin{aligned} \epsilon &\leq J_s(x_{n_{k_0}}, x_{n_{k_0}+m_{k_0}}) \\ &\leq J_s(x_{n_{k_0}-i_{k_0}}, x_{n_{k_0}+m_{k_0}-i_{k_0}}) + M'_{k_0} \sum_{l=1}^{i_{k_0}} \prod_{j=1}^l \mathfrak{H}(x_{n_{k_0}-j}, x_{n_{k_0}+m_{k_0}-j}) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This yields the strict inequality $\epsilon < \epsilon$, which is a clear contradiction.

Since both cases lead to a contradiction, our assumption must be false. Therefore, $\{x_n\}$ is a J_s -Cauchy sequence.

Because the space (X, J_s) is J_s -complete, there exists a point $x \in X$ such that

$$\lim_{n \rightarrow \infty} J_s(x_n, x) = 0,$$

which means that $\{x_n\} \in C(J_s, X, x)$. Furthermore, by the construction of our iterative sequence, we have $(x_n, \mathfrak{J}x_n) = (x_n, x_{n+1}) \in E(G)$ with $J_s(x_n, \mathfrak{J}x_n) < +\infty$, which implies that $x_n \in \mathfrak{D}(\mathfrak{J})$ for all $n \in \mathbb{N}$. Consequently, the sequence $\{x_n\}$ satisfies the prerequisite conditions for the orbital G -continuity of $\mathfrak{J}$. Applying this continuity yields

$$\{\mathfrak{J}x_n\} \in C(J_s, X, \mathfrak{J}x).$$

Since $\mathfrak{J}x_n = x_{n+1}$, it follows that the sequence $\{x_{n+1}\}$ J_s -converges to both x and $\mathfrak{J}x$. By the uniqueness of limits in J_s -metric spaces (Proposition 1), we conclude that $\mathfrak{J}x = x$. Thus, $x \in F(\mathfrak{J})$, proving that $F(\mathfrak{J}) \neq \emptyset$.

Conversely, suppose that $F(\mathfrak{J}) \neq \emptyset$. As established in Equation (1), every fixed point inherently possesses a self-loop in the graph, meaning it belongs to $\mathfrak{D}(\mathfrak{J})$. It immediately follows that $\mathfrak{D}(\mathfrak{J}) \neq \emptyset$, which completes the proof. $\square$

In the following example, we illustrate a case that supports the validity of the theorems established above.

Example 4. Let $X = [0, 1]$, and let J_s be a generalized metric defined by

$$J_s(x, y) = \begin{cases} x + y, & \text{if } x \neq 0 \text{ and } y \neq 0, \\ \frac{x}{2}, & \text{if } y = 0, \\ \frac{y}{2}, & \text{if } x = 0. \end{cases}$$

Note that (X, J_s) is a J_s -complete metric space and $J_s(x, y) \leq 2 < +\infty$ for all $x, y \in X$. Suppose that the directed graph G is defined by its edge set:

$$E(G) = \{(x, y) \in X \times X : x \neq 0 \text{ or } y = 0\}.$$

In addition, we define a mapping $\mathfrak{J} : X \rightarrow X$ by

$$\mathfrak{J}x = \frac{x}{x+12}.$$

We will show that $\mathfrak{J}$ is an Orbital contraction and thus $F(\mathfrak{J}) \neq \emptyset$ by Theorem 1. We proceed by proving the following claims:

Claim 1: $\mathfrak{J}$ is an orbitally $\mathcal{O}$ -transitive edge-preserving mapping.

First, we show that $\mathfrak{J}$ is orbitally edge-preserving. Let $x \in X$ such that $(x, \mathfrak{J}x) \in E(G)$. If $x = 0$, then $\mathfrak{J}0 = 0$, which yields $(\mathfrak{J}0, \mathfrak{J}^2 0) = (0, 0) \in E(G)$. If $x \neq 0$, then $\mathfrak{J}x = \frac{x}{x+12} > 0$. Since $\mathfrak{J}x \neq 0$, the first coordinate of the pair $(\mathfrak{J}x, \mathfrak{J}^2 x)$ is non-zero, meaning $(\mathfrak{J}x, \mathfrak{J}^2 x) \in E(G)$. Thus, $\mathfrak{J}$ is orbitally edge-preserving.

Next, we show that $\mathfrak{J}$ is $\mathcal{O}$ -transitive. Let $(x, y) \in E(G)$ and $(y, \mathfrak{J}y) \in E(G)$. The condition $(x, y) \in E(G)$ implies that $x \neq 0$ or $y = 0$. If $x \neq 0$, then $(x, \mathfrak{J}y) \in E(G)$ holds automatically. If $x = 0$, we must have $y = 0$, which gives $\mathfrak{J}y = \mathfrak{J}0 = 0$. Thus, $(x, \mathfrak{J}y) = (0, 0) \in E(G)$. In all cases, $\mathcal{O}$ -transitivity holds. (Note that the finiteness condition $J_s(\cdot, \cdot) < +\infty$ is trivially satisfied everywhere on X .)

Claim 2: $\mathfrak{J}$ satisfies the contraction condition (2).

Define a function $\mathfrak{H} : X \times X \rightarrow [0, 1]$ by

$$\mathfrak{H}(x, y) = \begin{cases} \frac{1}{12}, & \text{if } J_s(x, y) = 0, \\ \frac{\ln(1 + J_s(x, y))}{J_s(x, y)}, & \text{if } J_s(x, y) \neq 0. \end{cases}$$

It is a well-known analytical result that this function $\mathfrak{H}$ belongs to the class $H(X)$.

Let $(x, y) \in E(\mathcal{G})$. Then, $x \neq 0$ or $y = 0$. We consider the following cases based on the value of the distance $J_s(x, y)$:

Case 1: $J_s(x, y) = 0$.

By the property of the metric, this implies $x = y = 0$. Consequently, $\mathfrak{J}0 = 0$ and $J_s(\mathfrak{J}0, \mathfrak{J}0) = 0$. We have

$$J_s(\mathfrak{J}0, \mathfrak{J}0) = 0 \leq \frac{1}{12}K(0, 0) = \mathfrak{H}(0, 0)K(0, 0).$$

Thus, the contraction condition holds.

Case 2: $J_s(x, y) > 0$.

Before analyzing the specific points, we note a fundamental inequality: for any $t \in (0, 2]$, we have $\frac{t}{12} \leq \ln(1 + t)$. We now divide this case into two subcases based on the graphical edges:

Subcase 2.1: $y = 0$.

Since $J_s(x, y) > 0$, we must have $x \neq 0$. Thus, $\mathfrak{J}x > 0$ and $\mathfrak{J}y = 0$. We obtain

$$\begin{aligned} J_s(\mathfrak{J}x, \mathfrak{J}y) &= J_s\left(\frac{x}{x+12}, 0\right) \\ &= \frac{1}{2}\left(\frac{x}{x+12}\right) \\ &\leq \frac{1}{2}\left(\frac{x}{12}\right) = \frac{1}{12}\left(\frac{x}{2}\right) \\ &= \frac{1}{12}J_s(x, 0). \end{aligned}$$

Subcase 2.2: $y \neq 0$.

Since $(x, y) \in E(\mathcal{G})$, the condition forces $x \neq 0$. Thus, both $\mathfrak{J}x \neq 0$ and $\mathfrak{J}y \neq 0$. We obtain

$$\begin{aligned} J_s(\mathfrak{J}x, \mathfrak{J}y) &= J_s\left(\frac{x}{x+12}, \frac{y}{y+12}\right) \\ &= \frac{x}{x+12} + \frac{y}{y+12} \\ &\leq \frac{x}{12} + \frac{y}{12} \\ &= \frac{1}{12}(x + y) \\ &= \frac{1}{12}J_s(x, y). \end{aligned}$$

In both subcases where $J_s(x, y) > 0$, we have established that $J_s(\mathfrak{J}x, \mathfrak{J}y) \leq \frac{1}{12}J_s(x, y)$. Applying the fundamental inequality $\frac{1}{12}t \leq \ln(1 + t)$ for $t = J_s(x, y)$, we deduce

$$\begin{aligned} J_s(\mathfrak{J}x, \mathfrak{J}y) &\leq \frac{1}{12}J_s(x, y) \\ &\leq \ln(1 + J_s(x, y)) \\ &= \left[\frac{\ln(1 + J_s(x, y))}{J_s(x, y)} \right] J_s(x, y) \\ &= \mathfrak{H}(x, y)J_s(x, y). \end{aligned}$$

Finally, since the definition of $K(x, y)$ ensures that $K(x, y) \geq J_s(x, y)$, and $\mathfrak{H}(x, y)$ is non-negative, it immediately follows that

$$J_s(\mathfrak{J}x, \mathfrak{J}y) \leq \mathfrak{H}(x, y)K(x, y).$$

Therefore, $\mathfrak{J}$ satisfies condition (2) and is an orbital contraction.

Claim 3: $\mathfrak{D}(\mathfrak{J}) \neq \emptyset$.

For $x = 0$, we have $\mathfrak{J}0 = 0$. Since $(0, \mathfrak{J}0) = (0, 0) \in E(\mathcal{G})$, it follows that $0 \in \mathfrak{D}(\mathfrak{J})$. Thus, $\mathfrak{D}(\mathfrak{J})$ is non-empty.

Since all hypotheses of Theorem 1 are satisfied, we conclude that there exists a fixed point of $\mathfrak{J}$ in X (which is $x = 0$).

4. Applications

To demonstrate the practical utility and broad applicability of our graph-based orbital contraction framework, we apply our main theoretical results to establish the existence of solutions for two significant problems in nonlinear analysis. The first focuses on a class of nonlinear integral equations, which constitutes a vital area of fixed-point applications. The second extends our results to the setting of fractional differential equations. Fractional models, which naturally arise in diverse scientific fields, are characterized by their nonlocal behavior, making classical analytical techniques difficult to apply directly. By reformulating these complex problems into equivalent Volterra-type integral equations and invoking our proposed fixed-point theorems, we establish sufficient conditions for the existence of solutions.

Before proceeding to the specific details in Sections 4.1 and 4.2, we introduce the common functional space and highlight the structural advantages of our approach.

We investigate both problems within the space $C(I)$ of all continuous real-valued functions on a closed interval I (specifically, $I = [0, T]$ for the integral equations and $I = [0, L]$ for the fractional differential equations). To capture specific nonlinear behaviors and overcome the limitations of classical frameworks, we equip $C(I)$ with the JS-metric:

$$J_s(x, y) = \left(\max_{t \in I} |x(t) - y(t)| \right)^p,$$

where $p > 1$ is a fixed real number. Because $p > 1$, this distance function intentionally violates the classical triangle inequality, rendering standard Banach-type contraction principles inapplicable. However, it perfectly fulfills the JS-metric axioms with the parameter $s = 2^{p-1}$ (as demonstrated in Example 1).

Under this correct metric formulation, the convergence $J_s(x_n, x) \rightarrow 0$ strictly corresponds to standard uniform convergence. This guarantees that the space is J_s -complete and that general non-trivial (nonzero) solutions can be successfully captured.

Ultimately, while our applications are formulated within the standard continuous function space $C(I, \mathbb{R})$, endowing this space with a JS-metric ($p > 1$) and a directed graph yields a profound advantage. It allows us to replace rigid global contraction requirements with localized, orbital contractive conditions. Consequently, we can successfully handle integral and fractional operators that fail to be global contractions under standard norms but still exhibit contractive behavior along specific iterative trajectories (represented by the edges of the graph).

4.1. Application to Integral Equations

We consider the following nonlinear integral equation:

$$x(t) = \int_0^T F(t, s, x(s)) ds \quad (10)$$

for $t \in [0, T]$, where $T > 0$ is a real number and $F : [0, T] \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a given continuous function.

Working within the framework of the J_s -complete JS-metric space (X, J_s) where $X = C([0, T], \mathbb{R})$, we further endow X with a directed graph G defined by its edge set:

$$E(G) = \{(x, y) \in X \times X : x(t) \geq y(t) \text{ for all } t \in [0, T]\}.$$

Theorem 2. Consider the integral Equation (10) and suppose that the following conditions hold:

- (i) $F : [0, T] \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function;
- (ii) F is non-decreasing in its third variable; i.e., for any $x, y \in \mathbb{R}$ with $x \geq y$, we have $F(t, s, x) \geq F(t, s, y)$ for all $t, s \in [0, T]$. Additionally, the following condition is satisfied:

$$|F(t, s, x) - F(t, s, y)| \leq \frac{e^{-J_s(x, y)}}{T} |x - y|$$

for all $t, s \in [0, T]$ and $x, y \in \mathbb{R}$ with $x \geq y$;

- (iii) There exists $x_0 \in X$ such that $x_0(t) \geq \int_0^T F(t, s, x_0(s)) ds$ for all $t \in [0, T]$.

Then, the integral Equation (10) has at least one solution in X .

Proof. Define a mapping $\mathfrak{J} : X \rightarrow X$ by

$$\mathfrak{J}x(t) = \int_0^T F(t, s, x(s)) ds$$

for all $x \in X$ and $t \in [0, T]$. Clearly, finding a solution to (10) is equivalent to finding a fixed point of $\mathfrak{J}$. Note that $\mathfrak{J}$ is continuous on X due to the continuity of F .

First, we show that $\mathfrak{J}$ is an orbitally $\mathcal{O}$ -transitive edge-preserving mapping. Let $(x, \mathfrak{J}x) \in E(G)$. Then $x(t) \geq \mathfrak{J}x(t)$ for all $t \in [0, T]$. By the monotonicity assumption (ii), we have $F(t, s, x(s)) \geq F(t, s, \mathfrak{J}x(s))$, which implies

$$\mathfrak{J}x(t) = \int_0^T F(t, s, x(s)) ds \geq \int_0^T F(t, s, \mathfrak{J}x(s)) ds = \mathfrak{J}^2x(t).$$

This means $(\mathfrak{J}x, \mathfrak{J}^2x) \in E(G)$. To verify $\mathcal{O}$ -transitivity, suppose $(x, y) \in E(G)$ and $(y, \mathfrak{J}y) \in E(G)$. This implies $x(t) \geq y(t) \geq \mathfrak{J}y(t)$ for all t , which trivially yields $x(t) \geq \mathfrak{J}y(t)$, meaning $(x, \mathfrak{J}y) \in E(G)$. Thus, $\mathfrak{J}$ possesses the required graph-preserving properties.

Next, we demonstrate that $\mathfrak{J}$ is an orbital contraction. Let $(x, y) \in E(G)$. For any $t \in [0, T]$, applying the contraction bound from assumption (ii) yields:

$$\begin{aligned} |\mathfrak{J}x(t) - \mathfrak{J}y(t)| &= \left| \int_0^T (F(t, s, x(s)) - F(t, s, y(s))) ds \right| \\ &\leq \int_0^T |F(t, s, x(s)) - F(t, s, y(s))| ds \\ &\leq \int_0^T \frac{e^{-J_s(x, y)}}{T} |x(s) - y(s)| ds \\ &\leq \frac{e^{-J_s(x, y)}}{T} \int_0^T \left(\max_{\tau \in [0, T]} |x(\tau) - y(\tau)| \right) ds \\ &= \frac{e^{-J_s(x, y)}}{T} \left(T \cdot \max_{\tau \in [0, T]} |x(\tau) - y(\tau)| \right) \\ &= e^{-J_s(x, y)} \max_{\tau \in [0, T]} |x(\tau) - y(\tau)|. \end{aligned}$$

Taking the maximum over all $t \in [0, T]$ and raising both sides to the power p (where $p > 1$), we obtain

$$\begin{aligned} J_s(\mathfrak{J}x, \mathfrak{J}y) &= \left(\max_{t \in [0, T]} |\mathfrak{J}x(t) - \mathfrak{J}y(t)| \right)^p \\ &\leq \left(e^{-J_s(x, y)} \max_{\tau \in [0, T]} |x(\tau) - y(\tau)| \right)^p \\ &= e^{-pJ_s(x, y)} \left(\max_{\tau \in [0, T]} |x(\tau) - y(\tau)| \right)^p \\ &= e^{-pJ_s(x, y)} J_s(x, y). \end{aligned}$$

Define the control function $\mathfrak{H}(x, y) = e^{-pJ_s(x, y)}$. We now verify that $\mathfrak{H}$ belongs to the class $H(X)$: for any sequences $\{x_n\}$ and $\{y_n\}$ such that $\{J_s(x_n, y_n)\}$ is non-increasing, if

$$\lim_{n \rightarrow \infty} \mathfrak{H}(x_n, y_n) = \lim_{n \rightarrow \infty} e^{-pJ_s(x_n, y_n)} = 1,$$

then taking the natural logarithm immediately implies $-p \lim_{n \rightarrow \infty} J_s(x_n, y_n) = 0$. Since $p > 1$, we get $\lim_{n \rightarrow \infty} J_s(x_n, y_n) = 0$. Hence, $\mathfrak{H}$ is a valid member of $H(X)$.

Recall that the composite distance is defined as

$$K(x, y) = J_s(x, y) + |J_s(x, \mathfrak{J}x) - J_s(y, \mathfrak{J}y)|,$$

which naturally implies $K(x, y) \geq J_s(x, y)$. Therefore, we deduce that

$$J_s(\mathfrak{J}x, \mathfrak{J}y) \leq \mathfrak{H}(x, y) J_s(x, y) \leq \mathfrak{H}(x, y) K(x, y).$$

Thus, the generalized contraction condition is rigorously satisfied.

Finally, assumption (iii) guarantees that there exists $x_0 \in X$ such that $x_0(t) \geq \mathfrak{J}x_0(t)$ for all $t \in [0, T]$, meaning $(x_0, \mathfrak{J}x_0) \in E(G)$. Hence, the orbital set is non-empty, i.e., $\mathfrak{O}(\mathfrak{J}) \neq \emptyset$.

Since all hypotheses of Theorem 1 are completely satisfied, we conclude that $\mathfrak{J}$ has a fixed point in X , which serves as a solution to the integral Equation (10). $\square$

4.2. Application to Fractional Differential Equations

To establish the existence of solutions for the fractional problem via our main framework, we define the corresponding operator equation and introduce its associated structural properties.

Definition 13. Let (X, J_s) be a JS-metric space, and let $\mathfrak{J} : X \rightarrow X$ be a given mapping. We define $P(\mathfrak{J})$ as the collection of all functions $p : X \times X \rightarrow \mathbb{R}$ satisfying the following conditions:

- For every $x \in X$, if $p(x, \mathfrak{J}x) \geq 0$, then $p(\mathfrak{J}x, \mathfrak{J}^2x) \geq 0$ and $J_s(\mathfrak{J}x, \mathfrak{J}^2x) < +\infty$.
- For every $x, y \in X$, if $p(x, y) \geq 0$ and $p(y, \mathfrak{J}y) \geq 0$, then $p(x, \mathfrak{J}y) \geq 0$ and $J_s(x, \mathfrak{J}y) < +\infty$.

For each $p \in P(\mathfrak{J})$, we construct a directed graph $G_p = (X, E(G_p))$, with the edge set defined by

$$E(G_p) = \{(u, v) \in X \times X : p(u, v) \geq 0\}.$$

Lemma 3. Consider the framework $(C[0, L], J_s, G_p)$ equipped with a function $p \in P(\mathfrak{J})$. Under these conditions, the operator $\mathfrak{J}$ is an orbitally $\mathcal{O}$ -transitive edge-preserving mapping.

Proof. To prove that $\mathfrak{J}$ is an orbitally $\mathcal{O}$ -transitive edge-preserving mapping with respect to the directed graph $G_p = (C[0, L], E(G_p))$, we must verify two fundamental properties:

orbital edge-preservation and $\mathcal{O}$ -transitivity. We will utilize the definition of the edge set $E(G_p) = \{(u, v) \in C[0, L] \times C[0, L] : p(u, v) \geq 0\}$ and the properties of $p \in P(\mathfrak{J})$.

Step 1: Verification of orbital edge-preservation.

Let $x \in C[0, L]$ be an arbitrary element such that $(x, \mathfrak{J}x) \in E(G_p)$. By the definition of the edge set, this membership implies that

$$p(x, \mathfrak{J}x) \geq 0.$$

Since p belongs to the collection $P(\mathfrak{J})$, the first condition of Definition 13 guarantees that

$$p(\mathfrak{J}x, \mathfrak{J}^2x) \geq 0 \quad \text{and} \quad J_s(\mathfrak{J}x, \mathfrak{J}^2x) < +\infty.$$

The inequality $p(\mathfrak{J}x, \mathfrak{J}^2x) \geq 0$ ensures that the pair $(\mathfrak{J}x, \mathfrak{J}^2x)$ belongs to $E(G_p)$. Consequently, we have $(\mathfrak{J}x, \mathfrak{J}^2x) \in E(G_p)$ and $J_s(\mathfrak{J}x, \mathfrak{J}^2x) < +\infty$, which proves that $\mathfrak{J}$ is orbitally edge-preserving.

Step 2: Verification of $\mathcal{O}$ -transitivity.

Let $x, y \in C[0, L]$ be any elements satisfying $(x, y) \in E(G_p)$ and $(y, \mathfrak{J}y) \in E(G_p)$. By the definition of $E(G_p)$, these conditions translate to the inequalities

$$p(x, y) \geq 0 \quad \text{and} \quad p(y, \mathfrak{J}y) \geq 0.$$

Since $p \in P(\mathfrak{J})$, the second condition of Definition 13 directly implies that

$$p(x, \mathfrak{J}y) \geq 0 \quad \text{and} \quad J_s(x, \mathfrak{J}y) < +\infty.$$

Invoking the definition of the edge set again, $p(x, \mathfrak{J}y) \geq 0$ means that $(x, \mathfrak{J}y) \in E(G_p)$. Thus, for all $x, y \in C[0, L]$,

$$(x, y) \in E(G_p) \text{ and } (y, \mathfrak{J}y) \in E(G_p) \implies (x, \mathfrak{J}y) \in E(G_p) \text{ and } J_s(x, \mathfrak{J}y) < +\infty,$$

which establishes that $\mathfrak{J}$ is $\mathcal{O}$ -transitive.

Since $\mathfrak{J}$ satisfies both orbital edge-preservation and $\mathcal{O}$ -transitivity on the graph G_p , we conclude that $\mathfrak{J}$ is an orbitally $\mathcal{O}$ -transitive edge-preserving mapping. $\square$

Building upon the graph-based framework established in previous sections, we now apply our fixed point theorems to investigate the existence of solutions for nonlinear fractional differential equations. By reformulating the fractional boundary value problem as an equivalent Volterra-type integral equation, we can elegantly apply our abstract results within the fractional calculus setting.

Here, we discuss the following nonlinear fractional differential equation with a nonlocal integral boundary condition of order $\alpha \in (n, n + 1]$:

$$\begin{cases} D_{RL}^\alpha u(t) + f(t, u(t)) = 0, & 0 < t < L, \\ u^{(k)}(0) = 0, & k = 0, 1, 2, \dots, n-1, \\ u(L) = \lambda \int_0^\eta u(\tau) d\tau, \end{cases} \quad (11)$$

where $L > 0$, $\lambda > 0$, $\lambda \neq \alpha$, and $0 < \eta < L$. The function $f : [0, L] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, and D_{RL}^α denotes the standard Riemann–Liouville fractional derivative of order α , defined by

$$D_{RL}^\alpha u(t) = \begin{cases} \frac{1}{\Gamma(n+1-\alpha)} \frac{d^{n+1}}{dt^{n+1}} \int_0^t \frac{u(\tau)}{(t-\tau)^{\alpha-n}} d\tau & \text{if } n < \alpha < n+1, \\ \frac{d^{n+1}}{dt^{n+1}} u(t) & \text{if } \alpha = n+1. \end{cases}$$

Next, we derive the Green's function associated with the linear counterpart of Equation (11). This representation is essential for establishing the existence of solutions.

Lemma 4. Let $n < \alpha \leq n+1$, $0 < \eta < L$, and $y \in C[0, L]$. Consider the linear fractional boundary value problem:

$$\begin{cases} D_{RL}^\alpha u(t) + y(t) = 0, & 0 < t < L, \\ u^{(k)}(0) = 0, & k = 0, 1, 2, \dots, n-1, \\ u(L) = \lambda \int_0^\eta u(\tau) d\tau, \end{cases} \quad (12)$$

with $\kappa := L^{\alpha-1} - \lambda \frac{\eta^\alpha}{\alpha} \neq 0$. Then, the problem has a unique solution given by

$$u(t) = \int_0^L G(t, \tau) y(\tau) d\tau,$$

where the Green's function $G(t, \tau)$ is defined as

$$G(t, \tau) = \frac{1}{\kappa \Gamma(\alpha)} \begin{cases} -\kappa(t-\tau)^{\alpha-1} + t^{\alpha-1} \left[(L-\tau)^{\alpha-1} - \frac{\lambda}{\alpha} (\eta-\tau)^\alpha \right], & \text{if } 0 \leq \tau \leq \min\{t, \eta\}, \\ t^{\alpha-1} \left[(L-\tau)^{\alpha-1} - \frac{\lambda}{\alpha} (\eta-\tau)^\alpha \right], & \text{if } t \leq \tau \leq \eta, \\ -\kappa(t-\tau)^{\alpha-1} + t^{\alpha-1} (L-\tau)^{\alpha-1}, & \text{if } \eta \leq \tau \leq t, \\ t^{\alpha-1} (L-\tau)^{\alpha-1}, & \text{if } \max\{t, \eta\} \leq \tau \leq L. \end{cases} \quad (13)$$

Proof. Since $n < \alpha \leq n+1$, it is well known (see [15]) that the general solution of $D_{RL}^\alpha u(t) = -y(t)$ is

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} y(\tau) d\tau + \sum_{k=1}^{n+1} C_k t^{\alpha-k}.$$

Using the conditions $u(0) = u'(0) = \dots = u^{(n-1)}(0) = 0$, we directly obtain $C_k = 0$, for $k = 2, 3, \dots, n+1$. Thus

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} y(\tau) d\tau + C_1 t^{\alpha-1}.$$

To compute C_1 , we note that

$$u(L) = -\frac{1}{\Gamma(\alpha)} \int_0^L (L-\tau)^{\alpha-1} y(\tau) d\tau + C_1 L^{\alpha-1},$$

and

$$\int_0^\eta u(\tau) d\tau = -\frac{1}{\Gamma(\alpha)} \int_0^\eta \int_0^\nu (\nu - \tau)^{\alpha-1} y(\tau) d\tau d\nu + C_1 \frac{\eta^\alpha}{\alpha}.$$

Reversing the order of integration gives

$$\frac{1}{\Gamma(\alpha)} \int_0^\eta \int_0^\nu (\nu - \tau)^{\alpha-1} y(\tau) d\tau d\nu = \frac{1}{\alpha\Gamma(\alpha)} \int_0^\eta (\eta - \tau)^\alpha y(\tau) d\tau.$$

Using the boundary condition $u(L) = \lambda \int_0^\eta u(\tau) d\tau$, we have

$$-\frac{1}{\Gamma(\alpha)} \int_0^L (L - \tau)^{\alpha-1} y(\tau) d\tau + C_1 L^{\alpha-1} = \lambda \left(-\frac{1}{\alpha\Gamma(\alpha)} \int_0^\eta (\eta - \tau)^\alpha y(\tau) d\tau + C_1 \frac{\eta^\alpha}{\alpha} \right),$$

which implies

$$C_1 = \frac{1}{\kappa\Gamma(\alpha)} \left[\int_0^L (L - \tau)^{\alpha-1} y(\tau) d\tau - \frac{\lambda}{\alpha} \int_0^\eta (\eta - \tau)^\alpha y(\tau) d\tau \right],$$

where $\kappa := L^{\alpha-1} - \lambda \frac{\eta^\alpha}{\alpha} \neq 0$. Substituting C_1 back into $u(t)$ and splitting according to the four possible orders of τ , t , and η produces the piecewise Green's function in (13). This completes the proof. $\square$

Lemma 5. Let $n < \alpha \leq n + 1$, $0 < \eta < L$, and $\lambda \in \mathbb{R}$ such that $\kappa := L^{\alpha-1} - \lambda \frac{\eta^\alpha}{\alpha} \neq 0$. The Green's function $G : [0, L] \times [0, L] \rightarrow \mathbb{R}$ defined in Equation (13) satisfies

$$\int_0^L |G(t, \tau)| d\tau \leq \frac{\sigma L^\alpha}{\alpha\Gamma(\alpha)},$$

where $\sigma := 1 + \frac{L^{\alpha-1}}{|\kappa|} + \frac{L^\alpha |\lambda|}{\alpha |\kappa|}$.

Proof. Let $t, \tau \in [0, L]$. We first consider the case where $0 \leq \tau \leq \min\{t, \eta\}$. By distributing the terms in Equation (13), we can express Green's function as

$$G(t, \tau) = \frac{1}{\Gamma(\alpha)} \left(-(t - \tau)^{\alpha-1} + \frac{t^{\alpha-1}}{\kappa} \left[(L - \tau)^{\alpha-1} - \frac{\lambda}{\alpha} (\eta - \tau)^\alpha \right] \right).$$

Using the triangle inequality and the bounds $0 \leq \tau, t, \eta \leq L$, we note that $t^{\alpha-1} \leq L^{\alpha-1}$. Furthermore, since $0 \leq \eta - \tau \leq L - \tau \leq L$, we have $(\eta - \tau)^\alpha \leq (L - \tau)^\alpha = (L - \tau)(L - \tau)^{\alpha-1} \leq L(L - \tau)^{\alpha-1}$. Applying these estimates, we obtain

$$\begin{aligned} |G(t, \tau)| &\leq \frac{1}{\Gamma(\alpha)} \left((t - \tau)^{\alpha-1} + \frac{t^{\alpha-1}}{|\kappa|} (L - \tau)^{\alpha-1} + \frac{t^{\alpha-1} |\lambda|}{\alpha |\kappa|} (\eta - \tau)^\alpha \right) \\ &\leq \frac{1}{\Gamma(\alpha)} \left((L - \tau)^{\alpha-1} + \frac{L^{\alpha-1}}{|\kappa|} (L - \tau)^{\alpha-1} + \frac{L^{\alpha-1} |\lambda|}{\alpha |\kappa|} L (L - \tau)^{\alpha-1} \right) \\ &= \frac{1}{\Gamma(\alpha)} \left(1 + \frac{L^{\alpha-1}}{|\kappa|} + \frac{L^\alpha |\lambda|}{\alpha |\kappa|} \right) (L - \tau)^{\alpha-1} \\ &= \frac{\sigma}{\Gamma(\alpha)} (L - \tau)^{\alpha-1}. \end{aligned}$$

The same bounding arguments can be applied to the other three cases in Equation (13), yielding the identical uniform bound for all $t, \tau \in [0, L]$:

$$|G(t, \tau)| \leq \frac{\sigma}{\Gamma(\alpha)} (L - \tau)^{\alpha-1}.$$

Integrating this inequality with respect to τ from 0 to L gives

$$\int_0^L |G(t, \tau)| d\tau \leq \frac{\sigma}{\Gamma(\alpha)} \int_0^L (L - \tau)^{\alpha-1} d\tau = \frac{\sigma L^\alpha}{\alpha \Gamma(\alpha)},$$

which completes the proof. $\square$

To establish the existence of solutions for the fractional boundary-value problem (11), we define the integral operator $\mathfrak{J} : C[0, L] \rightarrow C[0, L]$ by

$$(\mathfrak{J}u)(t) = \int_0^L G(t, \tau) f(\tau, u(\tau)) d\tau. \quad (14)$$

Here, the kernel $G(t, \tau)$ represents the Green's function derived in Lemma 4, and f characterizes the nonlinear effects. It is well known that a function $u \in C[0, L]$ is a solution to the differential Equation (11) if and only if it is a fixed point of the operator $\mathfrak{J}$ (i.e., $\mathfrak{J}u = u$). Consequently, the problem of finding a solution is equivalently reduced to locating a fixed point of $\mathfrak{J}$.

Working within the JS-metric space $(C([0, L]), J_s)$ introduced earlier, we impose the following structural assumption on the nonlinear term to apply our graph-based fixed point theorems.

Definition 14. Consider the JS-metric space $(C[0, L], J_s, G_p)$, and let the operator $\mathfrak{J}$ be defined by Equation (14). We say that $\mathfrak{J}$ satisfies **Condition (C)** if there exists a function $\mathfrak{H} \in H(C[0, L])$ such that, for every $\tau \in [0, L]$ and all $(u, v) \in E(G_p)$, we have

$$|f(\tau, u(\tau)) - f(\tau, v(\tau))| \leq \frac{\alpha \Gamma(\alpha)}{\sigma L^\alpha} (\mathfrak{H}(u, v))^{\frac{1}{p}} |u(\tau) - v(\tau)|,$$

where σ is the bounding constant defined in Lemma 5.

Theorem 3. Consider the JS-metric space $(C[0, L], J_s)$ endowed with the directed graph G_p , where $p \in P(\mathfrak{J})$ and the operator $\mathfrak{J}$ is given by Equation (14). Assume that Condition (C) is fulfilled. Suppose further that there exists a function $u_0 \in C[0, L]$ such that $p(u_0(t), (\mathfrak{J}u_0)(t)) \geq 0$ for all $t \in [0, L]$.

Under these hypotheses, the operator $\mathfrak{J}$ possesses a fixed point $x^* \in C[0, L]$, which corresponds to a solution of the nonlinear fractional differential Equation (11).

Proof. By Lemma 3, $\mathfrak{J}$ is an orbitally $\mathcal{O}$ -transitive edge-preserving mapping. We now need to demonstrate that $\mathfrak{J}$ satisfies the contraction condition (2) to establish that it is an orbital contraction.

Let $(u, v) \in E(G_p)$. Using the definition of the operator $\mathfrak{J}$ and applying Condition (C), for any $t \in [0, L]$, we have

$$\begin{aligned}
|\mathfrak{J}u(t) - \mathfrak{J}v(t)| &= \left| \int_0^L G(t, \tau) (f(\tau, u(\tau)) - f(\tau, v(\tau))) d\tau \right| \\
&\leq \int_0^L |G(t, \tau)| |f(\tau, u(\tau)) - f(\tau, v(\tau))| d\tau \\
&\leq \int_0^L |G(t, \tau)| \left(\frac{\alpha \Gamma(\alpha)}{\sigma L^\alpha} (\mathfrak{H}(u, v))^{\frac{1}{p}} |u(\tau) - v(\tau)| \right) d\tau \\
&\leq \left(\frac{\alpha \Gamma(\alpha)}{\sigma L^\alpha} (\mathfrak{H}(u, v))^{\frac{1}{p}} \max_{\xi \in [0, L]} |u(\xi) - v(\xi)| \right) \int_0^L |G(t, \tau)| d\tau.
\end{aligned}$$

Applying the integral bound of Green's function from Lemma 5, which states that $\int_0^L |G(t, \tau)| d\tau \leq \frac{\sigma L^\alpha}{\alpha \Gamma(\alpha)}$, we obtain

$$\begin{aligned}
|\mathfrak{J}u(t) - \mathfrak{J}v(t)| &\leq \left(\frac{\alpha \Gamma(\alpha)}{\sigma L^\alpha} (\mathfrak{H}(u, v))^{\frac{1}{p}} \max_{\xi \in [0, L]} |u(\xi) - v(\xi)| \right) \left(\frac{\sigma L^\alpha}{\alpha \Gamma(\alpha)} \right) \\
&= (\mathfrak{H}(u, v))^{\frac{1}{p}} \max_{\xi \in [0, L]} |u(\xi) - v(\xi)|.
\end{aligned}$$

Taking the maximum over all $t \in [0, L]$ and raising both sides to the power p , we get

$$\begin{aligned}
J_s(\mathfrak{J}u, \mathfrak{J}v) &= \left(\max_{t \in [0, L]} |\mathfrak{J}u(t) - \mathfrak{J}v(t)| \right)^p \\
&\leq \left((\mathfrak{H}(u, v))^{\frac{1}{p}} \max_{\xi \in [0, L]} |u(\xi) - v(\xi)| \right)^p \\
&= \mathfrak{H}(u, v) \left(\max_{\xi \in [0, L]} |u(\xi) - v(\xi)| \right)^p \\
&= \mathfrak{H}(u, v) J_s(u, v).
\end{aligned}$$

Recall that the composite distance $K(u, v)$ is defined as

$$K(u, v) = J_s(u, v) + |J_s(u, \mathfrak{J}u) - J_s(v, \mathfrak{J}v)|.$$

Since $K(u, v) \geq J_s(u, v)$, we immediately deduce the contraction inequality as

$$J_s(\mathfrak{J}u, \mathfrak{J}v) \leq \mathfrak{H}(u, v) J_s(u, v) \leq \mathfrak{H}(u, v) K(u, v).$$

This shows that $\mathfrak{J}$ strictly satisfies the necessary conditions to be an orbital contraction.

Finally, by our initial assumption, there exists a function $u_0 \in C[0, L]$ for which $p(u_0(t), (\mathfrak{J}u_0)(t)) \geq 0$ for all $t \in [0, L]$. By the definition of our directed graph, this implies that $(u_0, \mathfrak{J}u_0) \in E(G_p)$, meaning the orbital set is non-empty ($\mathfrak{O}(\mathfrak{J}) \neq \emptyset$).

Since all hypotheses of Theorem 1 are successfully satisfied, we conclude that the operator $\mathfrak{J}$ possesses a fixed point $x^* \in C[0, L]$, which consequently serves as a solution to the nonlinear fractional differential problem (11). $\square$

5. Conclusions

In this paper, we have successfully established a novel and robust fixed-point framework for orbital contractions within the setting of JS-metric spaces endowed with directed graphs. By relaxing the conventional metric requirements and leveraging graphical structures, we introduced the concept of orbitally $\mathcal{O}$ -transitive edge-preserving mappings. Coupled with the adaptive control function class $H(X)$, this approach provides a highly

flexible generalization of classical fixed-point principles, specifically tailored to the iterative paths generated by the mappings.

To validate our theoretical findings, we presented concrete examples demonstrating that orbital edge-preservation strictly broadens the scope of traditional edge-preserving properties. Furthermore, we successfully illustrated the practical significance of our abstract theorems through two major applications. First, we proved the existence of solutions for a class of nonlinear integral equations. Second, we extended our approach to the fractional calculus setting, establishing existence criteria for nonlinear fractional differential equations involving Riemann–Liouville fractional derivatives and nonlocal integral boundary conditions. By reformulating these complex problems into equivalent Volterra-type integral equations via appropriate Green’s functions, we showcased the wide-ranging applicability of our graph-based contraction framework.

The concepts and analytical techniques developed in this study open several promising avenues for future research. While the current framework provides robust theoretical existence criteria, an immediate and compelling direction for future work inspired by recent computational advancements is to translate these abstract existence theorems into concrete numerical approximation schemes equipped with explicit constructive error controls. Furthermore, researchers may explore the extension of these graph-based orbital contractions to multivalued mappings or adapt the framework to other generalized topological structures, such as extended b -metric spaces, modular spaces, or fuzzy metric spaces. Additionally, applying these methodologies to investigate the existence and uniqueness of solutions for systems of fractional differential equations, impulsive differential equations, or problems involving other types of fractional derivatives (such as Caputo or Hadamard derivatives) remains an intriguing direction for further exploration.

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References

1. Jleli, M.; Samet, B. A generalized metric space and related fixed point theorems. *Fixed Point Theory Appl.* **2015**, *2015*, 61. [\[CrossRef\]](#)
2. El Kouch, Y.; El Marhrani, M. On some fixed point theorems in generalized metric spaces. *Fixed Point Theory Appl.* **2017**, *2017*, 23. [\[CrossRef\]](#)
3. Dumitrescu, D.; Pitea, A. Fixed point theorems on almost (φ, θ) -contractions in Jleli-Samet generalized metric spaces. *Mathematics* **2022**, *10*, 4239. [\[CrossRef\]](#)
4. Atiponrat, W.; Dangskul, S.; Khemphet, A. Coincidence point theorems for KC-contraction mappings in JS-metric spaces endowed with a directed graph. *Carpathian J. Math.* **2019**, *35*, 263–272. [\[CrossRef\]](#)
5. Jleli, M.; Samet, B. A new generalization of the Banach contraction principle. *J. Inequal. Appl.* **2014**, *2014*, 38. [\[CrossRef\]](#)
6. Agarwal, R.P.; Meehan, M.; O’Regan, D. *Fixed Point Theory and Applications*; Cambridge University Press: Cambridge, UK, 2001. [\[CrossRef\]](#)
7. Harjani, J.; Sadarangani, K. Fixed point theorems for weakly contractive mappings in partially ordered sets. *Nonlinear Anal.* **2009**, *71*, 3403–3410. [\[CrossRef\]](#)
8. Samet, B.; Vetro, C.; Vetro, P. Fixed point theorems for α - ψ -contractive type mappings. *Nonlinear Anal.* **2012**, *75*, 2154–2165. [\[CrossRef\]](#)

9. Din, Q. Fixed points on ordered intervals with state-dependent memory and constructive error control. *Rendiconti Circ. Mat. Palermo Ser. 2* **2026**, *75*, 161. [[CrossRef](#)]
10. Wang, Z.; Ma, J.; Cao, J. An improved high-order uniform accuracy scheme for nonlinear system of 2D general fractional Volterra integral equations. *Demonstr. Math.* **2026**, *59*, 20250240. [[CrossRef](#)]
11. Jachymski, J. The contraction principle for mappings on a metric space with a graph. *Proc. Am. Math. Soc.* **2008**, *136*, 1359–1373. [[CrossRef](#)]
12. Nicolae, A.; O'Regan, D.; Petruşel, A. Fixed point theorems for singlevalued and multivalued generalized contractions in metric spaces endowed with a graph. *Georgian Math. J.* **2011**, *18*, 307–327. [[CrossRef](#)]
13. Bojor, F. Fixed point theorems for Reich type contractions on metric spaces with a graph. *Nonlinear Anal.* **2012**, *75*, 3895–3901. [[CrossRef](#)]
14. Poochinapan, K.; Moonchai, S.; Chaobankoh, T.; Charoensawan, P. Existence of Solution to Nonlinear Third-Order Differential Equation and Iterative Method Utilization via Graph-Based Contraction. *Mathematics* **2025**, *13*, 1569. [[CrossRef](#)]
15. Diethelm, K. *The Analysis of Fractional Differential Equations: An Application Oriented Exposition Using Differential Operators of Caputo Type*; Springer: Berlin/Heidelberg, Germany, 2010.

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