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Highly Accurate Numerical Method for First-Kind Singular Integral Equations

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Abstract

In this paper, we present a highly accurate numerical scheme for solving first-kind Cauchy-type singular integral equations (CSIEs). The method is based on approximating the unknown function using Chebyshev series and applying modified Gauss–Chebyshev quadrature formulas with Gauss–Lobatto nodes to evaluate weighted kernel integrals. The proposed approach is versatile and can effectively handle bounded, unbounded, and semi-bounded solution domains. The invertibility of the associated operator is established for both bounded and unbounded cases, ensuring the uniqueness of the solution. Furthermore, convergence rates are rigorously proven in Hilbert space. Several numerical examples are provided to demonstrate the performance of the method, along with detailed comparisons to existing numerical techniques. These results confirm the simplicity, efficiency, and reliability of the proposed scheme, making it a powerful tool for solving first-kind CSIEs in a wide range of applications.

Keywords: Cauchy-type singular integral equation; Chebyshev polynomials; Hilbert space; Gauss–Chebyshev quadrature formula; boundary conditions

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1. Introduction: General Statement of the Singular Problems

Let us consider Cauchy-type singular integral equations (CSIEs) of the first and second kind in the following forms:

$$\frac{1}{\pi}\fint_{-1}^{1}\varphi(t)\left[\frac{K(x,t)}{t-x}+L_1(x,t)\right]dt=f(x),\quad -1<x<1, \tag{1}$$

$$a(x)\varphi(x)+\frac{b(x)}{\pi}\fint_{-1}^{1}\frac{\varphi(t)}{t-x}dt+\frac{1}{\pi}\int_{-1}^{1}L(x,t)dt=g(x),\quad -1<x<1, \tag{2}$$

where x is the singular point, $K(x,t), L_1(x,t), L(x,t)$ and $f(x)$ are given real-valued continues functions and $\varphi(t)$ is an unknown function to be determined. Notation $\fint_{-1}^{1} \frac{\varphi(t)}{t-x} dt$ denotes Cauchy singular integrals on the interval $[-1,1]$.

A substantial body of research has been devoted to the approximate solution of singular integral equations (SIEs) of both the first and second kind (1) and (2) on finite as well as infinite intervals. Solving SIEs of the first kind (1) is related to eigenvalues and eigenfunction problems; therefore, it is a more complicated problem than the SIEs of the second kind (2). Analytical solution of Equations (1) and (2) can be found in rare cases; therefore, there is a need for numerical or semi-analytical methods. In the last few decades, many numerical methods have been imposed to solve both Equations (1) and (2). For example, during 1981–1982, Ioakimidis [1,2] introduced collocation and Galerkin methods combined with Gauss–Radau and Lobatto–Jacobi quadrature schemes to approximate solutions of SIEs with index equal to -1.

As an application, the problem of a straight crack subjected to an exponentially distributed normal load was analyzed, and the corresponding stress intensity factors at the crack tips were computed. In 1985, Belotserkovskii and Lifanov [3] proposed a direct collocation approach, widely known as the discrete vortices method (MDV), for solving SIEs of both the first and second kind (1) and (2). Their method is based on selecting the singular point x either as the midpoint of the subinterval $[t_j, t_{j+1}]$ or at the nodal points $x = t_j$ for a fixed index j when dealing with characteristic SIEs (i.e., Equation (1) with $L_1(x,t) = 0$). This framework was later extended to higher-dimensional SIEs (Lifanov (1996) [4]). The principal advantage of this approach lies in its simplicity and ease of implementation. Later, Belotserkovskii et al. (1987) [5] applied the MDV method to plane problems involving non-smooth boundaries. Chakrabarti (1989) [6] proposed a direct method for solving two SIEs with logarithmic singularities arising in the analysis of surface water wave scattering by vertical barriers. In 1990, Israilov (1990) [7] developed quadrature formulas based on linear spline approximations for solving characteristic SIEs in both bounded and unbounded domains. It was shown that the approximation error decreases progressively as the number of nodes increases. Berthold et al. (1992) [8] introduced a fast algorithm for solving the generalized airfoil equation and identified the optimal order of approximation. Golberg and Chen (1997) [9] provided a comprehensive treatment of various types of integral equations in their book *Discrete Projection Methods for Integral Equations*, where discrete projection techniques are analyzed for a broad class of problems originating from boundary value formulations of partial differential equations. Their work covers several categories of integral equations, including Fredholm, Volterra, Cauchy singular, hypersingular, and boundary integral equations. Later, Capobianco et al. (2000) [10] developed collocation and quadrature-based collocation methods for the approximate solution of Prandtl-type singular integro-differential equations in weighted spaces of continuous functions. They also established theoretical convergence results along with corresponding convergence rates. Chan et al. (2003) [11] investigated HSIs of the form

$$I_\alpha(T_n, m, r) = \int_{-1}^{1} \frac{T_n(s)\left(1-s^2\right)^{m-1/2}}{(s-r)^\alpha} \, |r| < 1, \; m \in \mathbb{N},$$
$$I_\alpha(U_n, m, r) = \int_{-1}^{1} \frac{U_n(s)\left(1-s^2\right)^{m-1/2}}{(s-r)^\alpha} ds, \quad |r| < 1, \tag{3}$$

where $T_n(x)$ and $U_n(x)$ are the Chebyshev polynomials of the first and second kinds, respectively. Exact formulas are derived for the cases $\alpha = \{1,2,3,4\}$ and $m = \{0,1,2,3\}$. Moreover, the integrals are also evaluated when $r > 1$ in order to calculate the stress intensity factors.

The main properties and applications of all types of Chebyshev polynomials are given in Mason and Handscomb (2003) [12]. Chakrabarti and Berghe (2004) [13] developed an approximate method for Equation (1) based on polynomial approximation and using the zeros $x_j\,(j = 0, 1, 2, ..., n)$ of the Chebyshev polynomials $T_{n+l}(x) = cos[(n+1)arccos(x)]$ for all four types of solutions. Mandal and Bhattacharya (2007) [14] developed an approximate method for the classes of integral equations (Fredholm SIEs of the second kind, characteristic hyper SIEs and hyper SIEs of the second kind) based on Bernstein polynomials. Eshkuvatov et al. (2009) [15] proposed Chebyshev polynomial approximation for SIEs of the first kind and successfully implemented it to the solution of all solutions of characteristic SIEs. In the approximation, Chebyshev polynomials of the first kind, $T_n(x)$, second kind, $U_n(x)$, third kind, $V_n(x)$, and fourth kind, $W_n(x)$, corresponding to their respective weight functions, were utilized. It is shown that for a linear force function, the proposed method gives an exact solution. Several numerical results for different force functions are given to illustrate the efficiency and accuracy of the method.

Chakrabarti and Martha (2012) [16] proposed a method named "Analysis of functions of variables" to evaluate different types of real improper integrals, then solved various types of Cauchy-type SIEs of the first kind. Dardery and Allan (2014) [17] considered Equation (1) solved the SIEs of the first kind by using Chebyshev polynomials of the first–fourth kind to obtain systems of linear algebraic equations; these systems are solved numerically. The method is illustrated by solving some examples. In 2016, Mandal and Chakrabarti [18] published the book *Applied SIEs*, which covers a variety of Abel IEs, linear IEs and SIEs, with special emphasis on their methods of solution. After describing the various forms of integral equations in the introductory chapter, they consider the singular integral equations of the first kind which involve both logarithmic as well as Cauchy-type singularities in their kernels and have been taken up for their complete solutions. Particularly, simple examples are examined in detail to explain the underlying major mathematical ideas. Some very special methods of solution of singular integral equations are described in detail.

For the characteristic SIE of Cauchy type, Eshkuvatov et al. (2009) [19] constructed a new quadrature formula based on a combination of the discrete vortex method and linear spline approximation for the solution of characteristic SIEs of the first kind in the case of a bounded solution. Estimation of errors were obtained in the classes of functions $H^{([1,1],A)}$ and $C^1([1,1])$. In 2011, Eshkuvatov et al. [20] developed the hybrid method for Cauchy-type singular integrals, over a finite interval in all cases. Shadimetov and Akhmedov, (2022) [21] and Akhmedov et al. (2024) [22] developed the optimal quadrature formulas with derivatives in the $L_2^{(m)}[-1,1]$ for an approximate solution of a characteristic singular integral equation of the first kind (in Equation (1), $L_1(x,t) = 0$) with the Cauchy kernel. An approximate solution for the characteristic singular integral equation is obtained, applying the optimal quadrature formulas (OQF). Explicit forms of coefficients for the optimal quadrature formulas are obtained. Some numerical results are presented. Thus, they show the possibility of solving characteristic singular integral equations with higher accuracy using the optimal quadrature formula based on the Sobolev method.

In the work of Hamzah et al. (2021) [23], a new mathematical model is developed for the analytical study of two cracks in the upper plane of dissimilar materials under various mechanical loadings (shear, normal, tearing and mixed stresses) with different geometry conditions. It leads to the problem of a new mathematical model of hypersingular integral equations (HSIEs). This HSIEs is solved by using the modified complex potentials (MCPs) function and the continuity conditions of the resultant force together with the crack opening displacement (COD) function as a unknown function. Eshkuvatov et al. (2016) [24] developed the modified homotopy perturbation method (HPM) to solve the hypersingular integral equations (HSIEs) of the first kind on the interval [1,1] with the assumption that the kernel of the hypersingular integral is constant on the diagonal of the domain. The existence of the inverse of a hypersingular integral operator leads to the convergence of HPM in certain cases. Norm convergence of HPM is obtained in Hilbert space.

Mamatova et al. (2023) [25] solved systems of Cauchy-type SIEs of the first-kind with constant coefficients by the hybrid method (combination of Gauss elimination technique and homotopy perturbation method). They first used the Gauss elimination technique to reduce the system of Cauchy SIEs of the first kind to a diagonal triangular system of algebraic equations then solved it using the HPM. Several numerical examples and comparisons with the Chebyshev collocation and Galerkin methods are presented. They demonstrated that the HPM provides highly accurate and exact solutions for characteristic systems of singular integral equations, including complex-valued systems, regardless of the choice of initial guesses in the Hölder class of functions. In 2026, Eshkuvatov et al. [26] solved one-dimensional Cauchy-type SIEs of the first kind by implementing HPM for obtaining bounded, unbounded and semi-bounded solutions. Theoretical convergence of HPM to the exact solution in the class of Holder was proved.

Integral equations can arise in modeling processes involving cumulative effects, transport, reaction kinetics, and inverse determination of model parameters. Such formulations may also be relevant to biological production systems (Zhuang et al. (2022) [27]), including microbial biosynthesis processes, where the underlying dynamics can involve coupled reaction, transport, and accumulation mechanisms. The study of the microbial biosynthesis of straight-chain aliphatic carboxylic acids provides an example of the broader modeling context in which mathematical descriptions of biological production systems can be developed.

In this section, general SIEs (1) of the first kind are considered for the cases of bounded, unbounded and semi-bounded solutions, and the collocation method is outlined together with kernel expansions. For the unique solution of the unbounded and bounded cases, the following conditions

$$\begin{aligned} &\frac{1}{\pi}\int_{-1}^{1}\varphi(x)dx = C, \\ &\varphi(-1) = \varphi(1) = 0. \end{aligned} \tag{4}$$

are imposed respectively.

The novelty of this research is that to solve SIEs of the first kind (1), we use a hybrid method (combination of the Chebyshev collocation method together with the kernel expansion method). It gives us highly accurate and fast convergence to the solution of Equation (1).

The structure of the note is arranged as follows: In Section 2, all the necessary tools are outlined and the quadrature method based on Gauss–Chebyshev approximation is formulated in Section 3. In Section 4, the details of the derivation of the Chebyshev collocation method (ChCM) method is presented for each cases. Section 5 discusses the existence and uniqueness of the solution in Hilbert space. Finally in Section 6, many examples are provided to verify the validity and accuracy of the proposed method, followed by the conclusion.

2. Preliminaries

It is known [12] that the singular operator C_g defined by

$$C_g u = \frac{1}{\pi}\int_{-1}^{1}\frac{\omega_r(t)}{t-x}u(t)dt,\ r = \{1,2,3,4\} \tag{5}$$

where

$$\begin{aligned}
&\omega_1(x) = \sqrt{1-x^2},\ \omega_2(x) = \frac{1}{\sqrt{1-x^2}}, \\
&\omega_3(x) = \sqrt{\frac{1-x}{1+x}},\ \omega_4(x) = \sqrt{\frac{1+x}{1-x}},
\end{aligned} \tag{6}$$

has the following antiderivative functions

$$\begin{aligned}
C_g U_m(x) &= \frac{1}{\pi} \fint_{-1}^{1} \frac{\sqrt{1-t^2}U_m(t)}{(t-x)}dt = -T_{m+1}(x),\ \ m = 0,2,..., \\
C_g T_m(x) &= \frac{1}{\pi} \fint_{-1}^{1} \frac{T_m(t)}{\sqrt{1-t^2(t-x)}}dt = U_{m-1}(x),\ \ m = 0,2,..., \\
C_g W_m(x) &= \frac{1}{\pi} \fint_{-1}^{1} \sqrt{\frac{1-t}{1+t}}\frac{W_m(t)dt}{t-x} = -V_m(x),\ \ m = 0,2,..., \\
C_g V_m(x) &= \frac{1}{\pi} \fint_{-1}^{1} \sqrt{\frac{1+t}{1-t}}\frac{V_m(t)dt}{(t-x)} = W_m(x),\ \ m = 0,2,...,
\end{aligned} \tag{7}$$

where $U_{-1}(x) = 0$ and $C_g T_0(x) = 0$.

Moreover, Darbery and Allan [17] summarized three term relations of four kinds of Chebyshev polynomials, which are given by Mason and Handscomb [12]:

$$\begin{cases}
T_0(x) = 1,\ T_(x) = x, \\
T_n(x) = 2xT_{n-1} - T_{n-2}(x),\ n \geq 2, \\
U_0(x) = 1,\ U_(x) = 2x, \\
U_n(x) = 2xU_{n-1} - U_{n-2}(x),\ n \geq 2, \\
V_0(x) = 1,\ V_(x) = 2x - 1, \\
V_n(x) = 2xV_{n-1} - V_{n-2}(x),\ n \geq 2, \\
W_0(x) = 1,\ W_(x) = 2x + 1, \\
W_n(x) = 2xW_{n-1} - W_{n-2}(x),\ n \geq 2.
\end{cases} \tag{8}$$

In the work of Mason and Handscomb [12], the following relations are given:

$$\begin{aligned}
&T_{m+1}(x) = \frac{1}{2}[U_{m+1}(x) - U_{m-1}(x)],\ \ m = 0,1,2,..., \\
&xU_m(x) = \frac{1}{2}[U_{m+1}(x) + U_{m-1}(x)],\ \ m = 0,1,2,... \\
&(1-x^2)U_n(x) = \frac{1}{2}[T_n(x) - T_{n+1}(x)],\ n = 0,1,2,..., \\
&V_m(x) = U_m(x) - U_{m-1}(x),\ m = 0,1,2,\cdots, \\
&W_m(x) = (-1)^m V_m(-x) = U_m(x) + U_{m-1}(x)),\ m = 0,1,2,\cdots,
\end{aligned} \tag{9}$$

where $U_{-1}(x) = 0$ and $U_n(-x) = (-1)^n U_n(x)$.

Let orthonormal polynomials $\phi_{n,i}(t)$, $i = \{1,2,3,4\}$ be defined as

$$\begin{aligned}
&\phi_{m,1}(t) = \sqrt{\frac{2}{\pi}}U_m(t),\ \phi_{m,2}(t) = \sqrt{\frac{2}{\pi}}T_m(t). \\
&\phi_{m,3}(t) = W_m(t),\ \phi_{m,4}(t) = V_m(t).
\end{aligned} \tag{10}$$

Due to the properties of Chebyshev polynomials (9), it can easily be shown that (7) can be written as

$$\begin{aligned}
C_g\phi_{m,1}(x) &= -\frac{1}{2}\Big(\phi_{m+1,1}(x) - \phi_{m-1,1}(x)\Big), \;\; m = 0,1,\ldots, \\
C_g\phi_{m,2}(x) &= \phi_{m-1,1}(x), \;\; m = 0,1,\ldots \\
C_g\phi_{m,3}(x) &= -\Big(\phi_{m,1}(x) - \phi_{m-1,1}(x)\Big), \;\; m = 0,1,\cdots, \\
C_g\phi_{m,4}(x) &= \Big(\phi_{m,1}(x) + \phi_{m-1,1}(x)\Big), \;\; m = 0,1,\cdots,
\end{aligned} \tag{11}$$

where $\phi_{-1,1}(x) = 0$.

Equation (11) is crucial in the convergence analysis.

3. Quadrature Method

In this section, we develop the Gauss–Chebyshev quadrature formula with Gauss–Lobatto nodes for weighted kernel integrals. It is known that many weighted kernel integrals do not have an exact solution in many circumstances. So we need a suitable quadrature for the numerical computation of weighted kernel integrals. Kythe (2005) [28] states that the Gauss quadrature formula of the form

$$\int_a^b w(t)f(t)dt = \sum_{i=0}^{n} A_i f(t_i), \tag{12}$$

is exact for all $f \in P_{2n+1}$ if the weights A_i and the nodes t_i are found for different orthogonal polynomial approximations of $f(x)$ on the interval $[a,b]$. In particular, if $[a,b] = [-1,1]$ and $w_r(t), r = \{1,2,3,4\}$ are defined by (6), and orthogonal polynomials are the Chebyshev polynomials of the first–fourth kind respectively, then the resulting formulas of Equation (12) are known as the Gauss–Chebyshev rule. Let us define the nodes $\xi_{j,1}, \xi_{j,2}, \xi_{j,3}, \xi_{j,4}$ as the zeros of $U_{n+1}(x), T_{n+1}(x), W_{n+1}(x), V_{n+1}(x)$ respectively,

$$\begin{aligned}
\xi_{k,1} &= cos\left(\frac{k\pi}{n+2}\right), \;\; k = 1,2,...,n+1, \\
\xi_{k,2} &= cos\left(\frac{(2k-1)\pi}{2(n+1)}\right), \;\; k = 1,2,...,n+1, \\
\xi_{k,3} &= cos\left(\frac{2k\pi}{2n+3}\right), \;\; k = 1,2,...,n+1, \\
\xi_{k,4} &= cos\left(\frac{(2k-1)\pi}{2n+3}\right), \;\; k = 1,2,...,n+1.
\end{aligned} \tag{13}$$

Lemma 1 (Mason and Handscomb [12])**.** *Open Gauss–Chebyshev rules are given as*

$$\begin{aligned}
\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-t^2}f(t)dt &= \sum_{k=1}^{n+1} A_{k,1}f(\xi_{k,1}) + R_{n+1,1}(f), \; A_{k,1} = \frac{1}{n+2}(1-\xi_{k,1}^2), \\
\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}f(t)dt &= \sum_{k=1}^{n+1} A_{k,2}f(\xi_{k,2}) + R_{n+1,2}(f), \; A_{k,2} = \frac{1}{n+1}, \\
\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}f(t)dt &= \sum_{k=1}^{n+1} A_{k,3}f(\xi_{k,3}) + R_{n+1,3}(f), \; A_{k,3} = \frac{2}{2n+3}(1-\xi_{k,3}), \\
\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}f(t)dt &= \sum_{k=1}^{n+1} A_{k,4}f(\xi_{k,4}) + R_{n+1,4}(f), \; A_{k,4} = \frac{2}{2n+3}(1+\xi_{k,4}).
\end{aligned} \tag{14}$$

The word "open" is used for not including endpoints. We usually omit "open" since all Gaussian rules with positive weight function are of the open type.

Theorem 1 (Mason and Handscomb [12])**.** *If $f \in C^{2n+2}[-1,1]$, then the error of Gauss–Chebyshev QFs (14) are given by*

$$
\begin{aligned}
R_{n+1,1}(f) &= \frac{\pi}{2^{2n+1}} \frac{f^{2n+2}(\eta)}{(2n+2)!}, \ \eta \in [-1,1], \\
R_{n+1,2}(f) &= \frac{\pi}{2^{2n+3}} \frac{f^{2n+2}(\eta)}{(2n+2)!}, \ \eta \in [-1,1]. \\
R_{n+1,3}(f) &= \frac{\pi}{2^{2n+2}} \frac{f^{2n+2}(\eta)}{(2n+2)!}, \ \eta \in [-1,1], \\
R_{n+1,4}(f) &= \frac{\pi}{2^{2n+2}} \frac{f^{2n+2}(\eta)}{(2n+2)!}, \ \eta \in [-1,1].
\end{aligned} \tag{15}
$$

Kythe (2005) [28] and Israilov (2002) [29] proved the following theorems respectively.

Theorem 2 (Johnson and Riess 1977 [30])**.** *Gaussian QF has precision $2n+1$ only if the points $x_i, i = 0, 1, ..., n$ are the zeros of $\phi_{n+1}(x)$, where $\phi_{n+1}(x)$ are orthogonal polynomials.*

Theorem 3 (Israilov (2002) [29])**.** *If $f \in C^{2n+2}[a,b]$, then the error term of Gaussian QF (14) is given by*

$$
R_n(f) = I_a^b(f) - I_n(f) = \frac{f^{2n+2}(\xi)}{(2n+2)!} \int_a^b \rho(x) P_{n+1}^2(x) dx, \ \ \xi \in [a,b] \tag{16}
$$

where $P_{n+1}(x)$ is the orthogonal polynomials of degree $n+1$ with $n+1$ distinct zeros and $\rho(x)$ is a weight function.

In the work of Eshkuvatov et al. (2016) [31], Gauss QF (14) is extended for the kernel integral with nonlinear terms, then in 2019, Eshkuvatov et al. [32] again extended Gauss QF for kernel integrals and proved the convergence of the method. In this note, we extend Gauss–Chebyshev QF (14) for the weighted kernel integrals for two cases using Chebyshev polynomials of all kinds. In Alhawamda et al. [33], a new class of orthogonal polynomials named Bachok–Hasham polynomials (BHP) is proposed for the homogeneous second kind of logarithmic singular integral equations (LogSIEs), which is an extension of the Chebyshev polynomials. Eigenfunctions and corresponding eigenvalues are found for the homogeneous second kind of LogSIEs. It is found that first kind of Bachok–Hasham polynomials are orthogonal with weight $w_k(x) = \frac{x^{k-1}}{\sqrt{1-x^{2k}}}$, where k is a positive odd integer. Properties of first kind of Bachok–Hasham polynomials are also proved. Finally, two examples are presented to show the validity and accuracy of the proposed method.

Case 1: Let us consider regular kernel integrals of the form

$$
\psi_r(x) = \frac{1}{\pi} \int_{-1}^{1} w_r(t) L(x,t) h(t) dt, \ r = \{1,2,3,4\}, \tag{17}
$$

where $w_r(t)$ are defined by (6) and let regular kernel $L(x,t)$ in (17) be given as separable kernel

$$
L(x,t) = \sum_{i=1}^{M} c_i(x) d_i(t), \tag{18}
$$

and function $h(t)$ in (17) be approximated by

$$
h(t) \cong S_{N,r}^*(t) = \sum_{k=0}^{N}{}' a_{k,r} P_{r,k}^*(t), \quad -1 \le t \le 1, \ r = \{1,2,3,4\}, \tag{19}
$$

where prime $'$ means that the first coefficients $a_{k,2}$ in the summation is halved and $P_{r,k}^*(t)$ are

$$P^*_{j,r}(t) = \begin{cases} U_j(t) = \dfrac{\sin((j+1)\theta)}{\sin(\theta)}, & r = 1, \\ T_j(t) = \cos(j\theta), & r = 2, \\ W_j(t) = \dfrac{\sin((j+1/2)\theta)}{\sin(\theta/2)}, & r = 3, \\ V_j(t) = \dfrac{\cos((j+1/2)\theta)}{\cos(\theta/2)}, & r = 4, \end{cases} \tag{20}$$

where $t = \cos\theta$. Then the function $P^*_{r,j}(t)$ satisfies the following orthogonality conditions:

$$u^{(r)}_{i,j} = \frac{1}{\pi}\langle P_{i,r}, P_{j,r}\rangle = \begin{cases} 0, & i \neq j, \\ 1/2, & i = j, \quad r = 1, \\ 1, & i = j = 0, \quad r = 2, \\ 1/2, & i = j \neq 0, \quad r = 2, \\ 1, & i = j, \quad r = \{3,4\}, \end{cases} \tag{21}$$

with respect to inner product $\langle f, g\rangle_r = \int_{-1}^{1} w_r(t) f(t) g(t) dt, \;\; r = \{1,2,3,4\}$.

In the case of separable kernel (18), Gauss–Chebyshev QF has the form

$$\begin{aligned} \psi_r(x) &= \sum_{i=1}^{M} c_i(x) \frac{1}{\pi}\int_{1}^{1} w_r(t) d_i(t) h(t) dt \\ &= \begin{cases} \sum_{i=1}^{M} c_i(x) \sum_{k=1}^{n+1} A_{k,1}\cdot d_i(t_{k,1}) h(t_{k,1}),\; r = 1 \\ \sum_{i=1}^{M} c_i(x) \sum_{k=1}^{n+1} A_{k,2}\cdot d_i(t_{k,2}) h(t_{k,2}),\; r = 2, \\ \sum_{i=1}^{M} c_i(x) \sum_{k=1}^{n+1} A_{k,3}\cdot d_i(t_{k,3}) h(t_{k,3}),\; r = 3, \\ \sum_{i=1}^{M} c_i(x) \sum_{k=1}^{n+1} A_{k,4}\cdot d_i(t_{k,4}) h(t_{k,4}),\; r = 4. \end{cases} \end{aligned} \tag{22}$$

Case 2: For non-degenerate kernel $L(x,t)$, we approximate it as follows:

$$L(x,t) \approx L_r(x,t) = \sum_{k=0}^{n}{}' a_{k,r}(x) P^*_{k,r}(t),\; r = \{1,2,3,4\}, \tag{23}$$

where prime $'$ means that the first term $a_{k,2}$ in the summation is halved in the case of $r = 2$, and $P^*_{k,r}(t)$, $r \in \{1,2,3,4\}$ are the second, first, fourth and third kinds of Chebyshev polynomials defined by (20). To find unknown functions $a_{k,2}(x)$ we use orthogonality conditions (21)

$$a_{j,r}(x) = \frac{2}{\pi}\int_{-1}^{1} L_r(x,t) w_r(t) P^*_{j,r}(t) dt,\; r = \{1,2,3,4\},\; j = \{0,1,\cdots,N_1\}. \tag{24}$$

Thus, the following Gauss–Chebyshev QF are obtained for non-degenerate kernel

$$\begin{aligned} \psi_r(x) &= \frac{1}{\pi}\int_{1}^{1} w_r(t) L(x,t) h(t) dt = \frac{1}{\pi}\sum_{k=0}^{N_1}{}' a_{k,r}(x)\int_{1}^{1} w_r(t) P^*_{r,k}(t) h(t) dt \\ &= \begin{cases} \sum_{k=0}^{N_1} a_{k,r}(x) \sum_{i=0}^{n+1} A_{i,1} f_1(t_{i,1}),\; f_1(t_{i,1}) = U(t_{i,1}) h(t_{i,1})\; r = 1, \\ \sum_{k=0}^{N_1}{}' a_{k,r}(x) \sum_{i=1}^{n+1} A_{i,2} f_2(t_{i,2}),\; f_2(t_{i,2}) = T(t_{i,2}) h(t_{i,2})\; r = 2, \\ \sum_{k=0}^{N_1} a_{k,r}(x) \sum_{i=1}^{n+1} A_{i,3} f_3(t_{i,3}),\; f_3(t_{i,3}) = W_i(t_{i,3}) h(t_{i,3})\; r = 3, \\ \sum_{k=0}^{N_1} a_{k,r}(x) \sum_{i=1}^{n+1} A_{i,4} f_4(t_{i,4})\; f_4(t_{i,4}) = V_i(t_{i,4}) h(t_{i,4})\; r = 4. \end{cases} \end{aligned} \tag{25}$$

where $t_{k,r}, r = \{1,2,3,4\}$ are defined by (13).

4. General Scheme of Chebyshev Approximation for the SIEs of the First Kind

Let us rewrite Equation (1) in the form

$$\frac{c_0}{\pi} \fint_{-1}^{1} \frac{\varphi(t)}{t-x} dt + \frac{1}{\pi} \int_{-1}^{1} L(x,t)\varphi(t) dt = f(x), \tag{26}$$

where $c_0 = K(x,x)$, which means that the kernel $K(x,t)$ in Equation (1) is constant on the diagonal of the region $D = [-1,1] \times [-1,1]$, i.e.,

$$L(x,t) = Q_1(x,t) + L_1(x,t), \tag{27}$$

and

$$Q_1(x,t) = \frac{K(x,t) - K(x,x)}{t-x}. \tag{28}$$

where the kernel $Q_1(x,t)$ is a continuous function when $t \neq x$, and it is constant when $t \to x$.

The main aim is to find four types of solutions of Equation (26); hence, we search for the solution in the form

$$\varphi(x) = w_r(x)u(x), \;\; r = \{1,2,3,4\}, \tag{29}$$

where $w_i(x)$, $i = \{1,2,3,4\}$ are defined by (6). Substituting (29) into (26) yields

$$\frac{c_0}{\pi} \int_{-1}^{1} \frac{w_r(t)}{t-x} u(t)\, dt + \frac{1}{\pi} \int_{-1}^{1} w_r(x) L(x,t)\, u(t)\, dt = f(x),\; r = \{1,2,3,4\},\; -1 < x < 1, \tag{30}$$

To find an approximate solution of Equation (30), the unknown $u(t)$ is approximated by

$$u(t) \cong u_{n,r}(t) = \sum_{j=0}^{n} b_{j,r} P^*_{j,r}(t), \;\; r = \{1,2,3,4\}, \tag{31}$$

where $P^*_{j,r}(t)$ are defined by (20) and approximate solution of Equation (26) is computed in the form

$$\varphi(t) \approx \varphi_{n,r}(t) = \omega_r(t) \sum_{j=0}^{n} b_{j,r} P^*_{j,r}(t), \;\; r = \{1,2,3,4\}, \tag{32}$$

where unknown coefficients $b_{j,r}$ are needed to be determined. To this end, substitute (31) into (30) to obtain

$$\begin{aligned} &\sum_{j=1}^{n} b_{j,r} \left[\frac{c_0}{\pi} \int_{-1}^{1} \frac{w_r(t)}{(t-x)} P^*_{j,r}(t)\, dt + \frac{1}{\pi} \int_{-1}^{1} w_r(t) L(x,t) P^*_{j,r}(t)\, dt \right] \\ &\quad + b_{0,r} \left[\frac{c_0}{\pi} \int_{-1}^{1} \frac{w_r(t)}{(t-x)}\, dt + \frac{1}{\pi} \int_{-1}^{1} w_r(t) L(x,t)\, dt \right] = f(x), \end{aligned} \tag{33}$$

Exact evaluation of the weighted SIs in (33) yields

$$\sum_{j=1}^{n} b_{j,r} \left[c_0\, G^*_{j,r}(x) + \psi^*_{j,r}(x) \right] + b_{0,r} \left[c_0 h_r(x) + \psi^*_{0,r}(x) \right] = f(x),\; r = \{1,2,3,4\}, \tag{34}$$

where

$$G^*_{j,r}(x) = \int_{-1}^{1} \frac{w_r(t)}{(t-x)} P^*_{j,r}(t)\, dt,\; j = 1,2,\ldots n, \tag{35}$$

are calculated exactly by using (7) and

$$\psi^*_{j,r}(x) = \frac{1}{\pi}\int_{-1}^{1} w_r(x)L(x,t)P^*_{j,r}(t)\,dt,\ j = 0,1,\ldots n \tag{36}$$

as well as

$$h_r(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{w_r(t)}{(t-x)}\,dt = \begin{cases} -x, & r=1, \\ 0, & r=2, \\ -1, & r=3, \\ 1, & r=4, \end{cases} \quad h'_r(x) = \begin{cases} -1, & r=1, \\ 0, & r=2, \\ 0, & r=3, \\ 0, & r=4. \end{cases} \tag{37}$$

Approximate Solution for SIEs in Different Cases

Let us consider the four cases of solutions of SIEs (30).

Case 1: Let $r = 1$; in this case, we search for the bounded solution on the edge of the interval $[-1,1]$ by imposing the second condition in Equation (4) i.e., $\varphi(-1) = \varphi(1) = 0$. Searching for the bounded solution in the form

$$u(t) \cong u_{n,1}(t) = \sum_{j=0}^{n} b_{j,1}U_j(t), \tag{38}$$

and the exact calculation of $G^*_{j,1}(x) = -T_{j+1}(x)$ together with (37) leads to

$$\sum_{j=1}^{n} b_{j,1}\Big[-c_0\,T_{j+1}(x) + \psi^*_{j,1}(x)\Big] + b_{0,1}\big(-c_0x + \psi^*_{0,1}(x)\big) = f(x), \tag{39}$$

where functions $\psi^*_{j,1}(x)$ can be calculated in the case of the non-degenerate kernel expansion (23) case as follows

$$\begin{aligned}\psi^*_{j,1}(x) &\approx \frac{1}{\pi}\int_{1}^{1} L_1(x,t)\sqrt{1-t^2}U_j(t)dt \\ &= \sum_{k=0}^{n} a_{k,1}(x)\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-t^2}U_k(t)U_j(t)\,dt = \frac{1}{2}a_{j,1}(x).\end{aligned} \tag{40}$$

For the degenerate kernel case (see Equation (18)), the following Gauss–Chebyshev QFs are used

$$\begin{aligned}\psi^*_{j,1}(x) &= \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-t^2}L(x,t)U_j(t)\,dt = \sum_{i=1}^{M} c_i(x)\frac{1}{\pi}\int_{1}^{1}\sqrt{1-t^2}d_i(t)U_j(t)dt \\ &= \sum_{i=1}^{M} c_i(x)\sum_{k=1}^{n+1} A_{k,1}\cdot d_i(t_{k,1})U_j(t_{k,1}),\ A_{k,1} = \frac{1}{n+2}(1-t^2_{k,1}),\end{aligned} \tag{41}$$

where $t_{k,1}$ are defined as the zeros of $U_{n+1}(x)$, i.e., $t_{k,1} = cos\left(\frac{k\pi}{n+2}\right), k = \{1,\cdots,n+1\}$.

To solve Equation (39), for unknown coefficients $b_{j,1}$, we use the collocation method and choose the suitable node points $\{x_i\}_{i=1}^{n}$ such as roots of $U_{n+1}(x)$ or $(1-x^2)U_{n-1}(x)$ or Gauss–Lobatto nodes $T'_{n+2}(x)$. Then Equation (39) leads to a system of algebraic equations (SysAEs)

$$\sum_{j=0}^{n} b_{j,1}\Big[c_0\,T_{j+1}(x_i) + \psi^*_{j,1}(x_i)\Big] + b_{0,1}\big(-c_0\,x_i + \psi^*_{0,1}(x_i)\big) = f(x_i),\ \ i = 1,2,\ldots,n+1 \tag{42}$$

where $\psi^*_{j,1}(x)$ are calculated either (40) or (41).

Solving (42), we find the unknown coefficients $b_{j,1}$, $j = 0, 1, \ldots, n$, then substituting the values of $b_{j,1}$ into Equation (32) yields the semi-analytical solution of Equation (26) for $r = 1$.

Let $r = 2$. For this case, we search for the approximate solution in the form

$$u(t) \cong u_{n,2}(t) = \sum_{j=0}^{n} b_{j,2} T_j(t), \tag{43}$$

and impose the first condition of Equation (4), i.e.,

$$\frac{1}{\pi} \sum_{j=0}^{n} b_{j,2} \int_{-1}^{1} \frac{T_j(t)}{\sqrt{1-t^2}} dt = C, \tag{44}$$

which leads to $b_{0,2} = C$. On the other hand, non-degenerate kernel expansion (23) gives

$$\psi^*_{j,2}(x) \approx \frac{1}{\pi} \int_{-1}^{1} \frac{L_2(x,t)}{\sqrt{1-t^2}} T_j(t)\, dt = \sum_{k=0}^{N_1}{}' a_{k,2}(x) \frac{1}{\pi} \int_{-1}^{1} \frac{T_k(t) T_j(t)}{\sqrt{1-t^2}}\, dt = \frac{1}{2} a_{j,2}(x). \tag{45}$$

In the degenerate kernel case (look at (18)), the following Gauss–Chebyshev QFs are used:

$$\begin{aligned} \psi^*_{j,2}(x) &= \frac{1}{\pi} \int_{-1}^{1} \frac{L(x,t)}{\sqrt{1-t^2}} T_j(t)\, dt = \sum_{i=1}^{M} c_i(x) \frac{1}{\pi} \int_{1}^{1} \frac{1}{\sqrt{1-t^2}} d_i(t) T_j(t) dt \\ &= \sum_{i=1}^{M} c_i(x) \sum_{k=1}^{n+1} A_{k,2} \cdot d_i(t_{k,2}) T_j(t_{k,2}),\ A_{k,2} = \frac{1}{n+1}, \end{aligned} \tag{46}$$

where $t_{k,2}$ are determined as the zeros of $T_{n+1}(x)$, i.e., $t_{k,2} = cos\left(\frac{(2k-1)\pi}{2(n+1)}\right), k = \{1, \cdots, n+1\}$. Due to the values of $b_{0,2} = C$ and exact value of $G_{j,2}(x) = U_{j-1}(x)$, Equation (33) can be written as follows:

$$\sum_{j=1}^{n} b_{j,2} \left[c_0\, U_{j-1}(x) + \psi^*_{j,2}(x) \right] = f_1(x), \tag{47}$$

where $f_1(x) = f(x) - C\psi^*_{0,2}(x)$.

To find the unknown coefficients $b_{j,2}$ in Equation (47), node points $x_{k,2}$ can be chosen as the roots of $T_n(x), U_n(x)$ or $T'_{n+1}(x)$ and it leads to the SysAEs.

$$\sum_{j=1}^{n} b_{j,2} \left[c_0\, U_{j-1}(x_{k,2}) + \frac{1}{2} a_{j,2}(x_{k,2}) \right] = f_1(x_{k,2}),\ k = \{1, 2, \cdots, n\}. \tag{48}$$

Solving Equation (48) yields the values of $b_{j,2}$, and substituting it into (42) as well as (32) yields the semi-analytical solution of Equation (26) for $r = 2$.

The cases of $r = \{3, 4\}$. Let us derive algorithm for $r = 3$. Small changes can be made to obtain algorithms for the case $r = 4$. Thus, in the case $r = 3$, we search for the approximate solution of Equation (30) in the form

$$u(t) \cong u_{n,3}(t) = \sum_{j=0}^{n} b_{j,3} W_j(t), \tag{49}$$

Due to (49), (30) and taking into account (35), we obtain

$$\sum_{j=1}^{n} b_{j,3}\Big[-c_0\, V_j(x) + \psi^*_{j,3}(x)\Big] + b_{0,3}\big[-c_0 + \psi^*_{0,r}(x)\big] = f(x), \tag{50}$$

Due to non-degenerate expansion (23), Equation (36) has the form

$$\begin{aligned}\psi^*_{j,3}(x) &\approx \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}L_3(x,t)W_j(t)\,dt \\ &= \sum_{k=0}^{N_1} a_{k,3}(x)\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}W_k(t)W_j(t)\,dt = a_{j,3}(x).\end{aligned} \tag{51}$$

For the degenerate kernel case (look at (18)), the following Gauss–Chebyshev QFs are used:

$$\begin{aligned}\psi^*_{j,3}(x) &= \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}L_3(x,t)W_j(t)\,dt = \sum_{i=1}^{M} c_i(x)\frac{1}{\pi}\int_{1}^{1}\sqrt{\frac{1-t}{1+t}}d_i(t)W_j(t)dt \\ &= \sum_{i=1}^{M} c_i(x)\sum_{k=1}^{n+1} A_{k,3}\cdot d_i(t_{k,3})W_j(t_{k,3}),\ A_{k,3} = \frac{2}{2n+3}(1-t_{k,3}),\end{aligned} \tag{52}$$

where $t_{k,3}$ are determined as the zeros of $W_{n+1}(x)$, i.e., $t_{k,3} = cos\left(\frac{2k\pi}{2n+3}\right),\ \ k = 1,\cdots,n+1.$

To find unknown coefficients $b_{k,3}$, node points $x_{k,3}$ are chosen as the roots of $(1-x^2)U_{n-1}(x), W_{n+1}(x)$ or Gauss–Lobatto points $T'_{n+2}(x)$, which leads to

$$\sum_{j=0}^{n} b_{j,3}\Big[-c_0\, V_j(x_k) + \psi^*_{j,3}(x_k)\Big] = f(x_k),\ k = \{1,2,\cdots,n+1\}, \tag{53}$$

where $\psi^*_{j,3}(x)$ can be calculated either (40) or (41).

Once we find the value of $b_{j,3}$ from (53) and substitute it into (42), as well as (32), it yields the semi-analytical solution of Equation (28) for $r = 3$.

For the case $r = 4$, we search for the approximate solution of Equation (30) in the form

$$u(t) \cong u_{n,4}(t) = \sum_{j=0}^{n} b_{j,4}V_j(t), \tag{54}$$

and to find $b_{j,4}$, we obtain the following SysAEs:

$$\sum_{j=0}^{n} b_{j,4}\Big[c_0\, W_j(x_k) + \psi^*_{j,4}(x_k)\Big] = f(x_k),\ k = \{1,2,\cdots n+1\}, \tag{55}$$

where nodes x_k are chosen as the roots of $V_{n+1}(x), (1-x^2)U_{n-1}(x)$ or $T'_{n+1}(x)$.

The function $\psi^*_{j,4}(x)$ is calculated in the case of the non-degenerate kernel case

$$\begin{aligned}\psi^*_{j,4}(x) &\approx \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}L_4(x,t)V_j(t)\,dt \\ &= \sum_{k=0}^{N_1} a_{k,4}(x)\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}V_k(t)V_j(t)\,dt = a_{j,4}(x).\end{aligned} \tag{56}$$

For the degenerate kernel case, the following Gauss–Chebyshev QFs are used:

$$\psi^{*}_{j,4}(x) = \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}L_4(x,t)V_j(t)\,dt = \sum_{i=1}^{M}c_i(x)\frac{1}{\pi}\int_{1}^{1}\sqrt{\frac{1+t}{1-t}}d_i(t)V_j(t)dt$$
$$= \sum_{i=1}^{M}c_i(x)\sum_{k=1}^{n+1}A_{k,3}\cdot d_i(t_{k,4})V_j(t_{k,4}),\ A_{k,4} = \frac{2}{2n+3}(1+t_{k,4}), \tag{57}$$

where $t_{k,4}$ are computed as the zeros of $V_{n+1}(x)$, i.e., $t_{k,4} = cos\left(\frac{(2k-1)\pi}{2n+3}\right),\ \ k = 1,\cdots,n+1.$

Thus, obtaining the unknown coefficients $b_{k,4}$ from the above SysAEs (57) and substituting it (32) gives the semi-analytical solution of Equation (28) for $r = 4$.

5. Existence of the Unique Solution for the Bounded and Unbounded Cases, i.e., $r = \{1, 2\}$

Let us rewrite Equation (30) in the operator form

$$C_r u + L_r u = f,\ \ r = \{1,2,3,4\},\ \ f \in L_{2\rho},\ \ u \in L_{1\rho}, \tag{58}$$

where

$$C_r u = \frac{c_0}{\pi}\int_{-1}^{1}\frac{w_r(t)u(t)}{t-x}dt,$$
$$L_r u = \frac{1}{\pi}\int_{-1}^{1}w_r(t)L(x,t)u(t)dt, \tag{59}$$

where kernel $L(x,t)$ is defined by (27).

Assumption 1. *Let $0 < \alpha < 1$ and Holder space be represented by*

$$C^{0,\alpha}[-1,1] = \left\{v \in C[-1,1] : \|v\|_{C^{0,\alpha}} = \sup_{x\neq t}\frac{|v(x)-v(t)|}{|x-t|^{\alpha}} < \infty,\ 0<\alpha<1\right\}. \tag{60}$$

We assume that the unknown function u belongs to the Holder space $C^{0,\alpha}[-1,1]$ after removing the prescribed endpoints in the case of unbounded weights, i.e.,

$$\phi(x) = \omega_r(x)u(x), \qquad u \in C^{0,\alpha}[-1,1],\ \ r = \{1,2,3,4\}, \tag{61}$$

where weights $\omega_r(x)$ are defined by (6). The kernel $K(x,t)$ in (1) is assumed to be sufficiently and uniformly regular in the neighborhood of the diagonal $x = t$, with

$$K(x,t) - K(x,x) = O(|t-x|^{\alpha}), \tag{62}$$

consequently, due to $L(x,t) = Q_1(x,t) + L_1(x,t)$ with given $L_1(x,t)$ in Equation (1), the kernel $Q_1(x,t)$ defined by

$$Q_1(x,t) = \frac{K(x,t)-K(x,x)}{t-x} \tag{63}$$

admits a bounded continuous extension at $t = x$ whenever the corresponding diagonal regularity condition holds. The regular kernel $L(x,t)$ in Equation (59) is assumed to be continuous on $[-1,1]^2$. Under these assumptions, the Cauchy principal value integral C_r, $r = \{1,2,3,4\}$ defined by (60) is well-defined for the admissible class of functions, while the resulting operator equation is subsequently considered in the weighted Hilbert spaces $C^{0,\alpha}$. Let us introduce operator $A_{r,\lambda}$ in the explicit form

$$A_{r,\lambda} := C_r + \lambda L_r : L_{1,\rho} \longrightarrow L_{2,\rho},\ \ r = \{1,2,3,4\} \tag{64}$$

and two more types of spaces $L_{2\rho}$ and $L_{1\rho}$.

- **Let** $L_{2\rho}(-1,1)$ denote the space of real-valued square integrable functions with respect to the weight ρ so that the space $L_{2\rho}$ endowed with an inner product

$$\langle u, v\rangle = \int_{-1}^{1} \rho(t)u(t)v(t)dt, \tag{65}$$

and a norm $||u||_2 = \sqrt{\langle u, u\rangle}$.
It is known that $\phi_{m,1}$, $m = 0, 1, \ldots,$ defined by (10) are orthonormal polynomials

$$||\phi_{m,1}||_2^2 = \int_{-1}^{1} \sqrt{1-t^2}\phi_{m,1}^2(t)dt = 1. \tag{66}$$

and the system $\{\phi_{m,1}\}_{k=0}^{\infty}$ is a complete orthonormal basis for $L_{2\rho}$ so that if $u \in L_{2\rho}$ then

$$u = \sum_{k=0}^{\infty}\langle u, \phi_{k,1}\rangle\phi_{k,1} = \frac{2}{\pi}\sum_{k=0}^{\infty}\langle u, U_k\rangle U_k, \tag{67}$$

where the sum converges in $L_{2\rho}$. Consequently, Parseval's equality holds, i.e.,

$$||u||_2^2 = \sum_{k=0}^{\infty}\langle u, \phi_{k,1}\rangle^2. \tag{68}$$

- **Second space** $L_{1\rho}$ is introduced according to the cases.
Case 1, $r = 1$ **is the bounded solution of Equation** (58)**:** In this case, the weight function $\rho(x) = \omega_1(x) = \sqrt{1-x^2}$ and second space $L_{1\rho}$ are bounded Hilbert space, and hence if $u \in L_{1\rho}$ then

$$u = \sum_{k-0}^{\infty}[\langle u, \phi_{k+1,1} - \phi_{k-1,1}\rangle]\phi_{k,1}. \tag{69}$$

Now we extend the operator C_1 defined by (59) as a bounded operator from $L_{1\rho}$ to $L_{2\rho}$. Using (11), we obtain

$$\begin{aligned} C_1 u &= c_0\sum_{k=0}^{\infty}\langle u, \phi_{k,1}\rangle C_1(\phi_{k,1}) \\ &= -\frac{c_0}{2}\sum_{k=0}^{\infty}\langle u, \phi_{k,1}\rangle(\phi_{k+1,1} - \phi_{k-1,1}) \\ &= -\frac{c_0}{2}[T_r(u) - T_l(u)], \end{aligned} \tag{70}$$

where

$$T_r(u) = \sum_{k=0}^{\infty}\langle u, \phi_{k,1}\rangle\phi_{k+1,1},\ T_l(u) = \sum_{k=0}^{\infty}\langle u, \phi_{k,1}\rangle\phi_{k-1,1}, \tag{71}$$

Observe that

$$\|C_1 u\|_2^2 = \left(\frac{c_0}{2}\right)^2\sum_{k=0}^{\infty}[\langle u, \phi_{k+1,1}\rangle - \langle u, \phi_{k-1,1}\rangle]^2 = \left(\frac{c_0}{2}\right)^2\|u\|_1^2.$$

It is not hard to show that $C_1^{-1} : L_{1\rho} \to L_{2\rho}$ exists and is given by

$$C_1^{-1}u = \frac{1}{c_0}\sum_{k=0}^{\infty}(-\langle u, \phi_{k,1}\rangle)\phi_{k,1}, \tag{72}$$

and hence, C_1 is invertible.

- **Case 1,** $r = 1$ **bounded solution of Equation** (58)**:** Now we prove that operators $(C_1 + L_1)$ defined by (59) are invertible. To this end, we revise some known facts.

Lemma 2. *Let A, B be operators acting in Hilbert space. If A is bounded and B is compact, then the products AB and BA are compact.*

Lemma 2 is proven by Reed and Simon [34] (Theorem VI.12, pp. 200).

Lemma 3. *Operator C_1 defined by* (70) *is compact.*

Proof. Operators $T_r(u)$ and $T_l(u)$ defined by (71) are bounded from $L_1(\rho) \to L_2(\rho)$. Moreover, boundedness and the compactly embeddability of T_r and T_l (Berthold et al. (1992) [8], Conclusion 2.3) implies the compactness of T_r and T_l. Since operators T_r and T_l are compact, their linear combinations are also compact, i.e., operator C_1 is also a compact operator. □

Lemma 4. *Operator $C_1 + L_1 : L_{1,\rho} \to L_{2,\rho}$ is invertible.*

To prove the invertibility of the operator $C_1 + L_1$, we consider a more general equation of the form

$$C_1 u + \lambda L_1 u = f, \;\; f \in L_{2\rho}, \;\; u \in L_{1\rho}, \tag{73}$$

Proof. Since operators C_1 and L_1 are compact, the linear combination $C_1 + \lambda L_1 : L_{1,\rho} \to L_{2,\rho}$ is also compact. Due to the Fredholm theorem (Reed and Simon [34] (1980, Theorem VI.14)), the inverse operator $C_1^{-1}(I + \lambda C_1 \cdot L_1)$ exists for all $\lambda \in \mathbb{C} \setminus C_1^*$, where C_1^* is a discrete subset of $\mathbb{C}$ (i.e., a set C_1^* has no limit points in $\mathbb{C}$) and for $\lambda \in C_1^*$ the null space $N(I + C_1^{-1} \cdot L_1))$ is finite, that is $z = \lambda^{-1}$ is the eigenvalue of $(I + C_1^{-1} \cdot L_1)$ with finite multiplicity. These facts show that the operator $A_{1,\lambda} = C_1 + \lambda L_1$ is invertible in the particular case that $\lambda = 1$ is also invertible. □

Assumption 2. $\lambda = 1$ *defined by* (73) *does not belong to C_1^*, i.e., it does not lie the null space of*

$$N(I + C_1^{-1} \cdot L) = 0.$$

These facts and above lemmas allows us to prove the following theorem.

Theorem 4. *Let the Assumption 2 hold, then the operator $C_1 + \lambda L_1$ is invertible, and the main Equation* (73) *for the case $r = 1$ has a unique solution.*

Proof. Since C_1 is invertible, we get the relation

$$C_1 + \lambda L_1 = C_1(I + \lambda C_1^{-1} L_1) \tag{74}$$

Since the right side of (74) exists and is invertible, then $(C_1 + L_1) : L_{1,\rho} \to L_{2,\rho}$ is also invertible according to the Lemma 4 and (74) exists and invertible. If it is so then from (73) it follows that

$$u = (C_1 + \lambda L_1)^{-1} f, \;\; f \in L_{2\rho}, \;\; u \in L_{1\rho}, \tag{75}$$

Thus, we have proved that (73) is solvable for the case $r = 1$. □

- **Case 2,** $r = 2$ **unbounded solution of Equation (58):** In this case weight function $\rho(x) = \omega_2(x) = 1/\sqrt{1-x^2}$ and second space $L_{1\rho}$ is also bounded Hilbert space and hence if $u \in L_{1\rho}$ then

$$u = \sum_{k-0}^{\infty} \langle u, \phi_{k-1,1} \rangle \phi_{k,2}. \tag{76}$$

Now we extend the operator C_2 defined by (59) as a bounded operator from $L_{1\rho}$ to $L_{2\rho}$. Using (11), we obtain

$$\begin{aligned} C_2 u &= c_0 \sum_{k=0}^{\infty} \langle u, \phi_{k-1,1} \rangle C_2(\phi_{k,2}) \\ &= -c_0 \sum_{k=1}^{\infty} \langle u, \phi_{k-1,1} \rangle \phi_{k-1,1} \end{aligned} \tag{77}$$

From (77), it follows that

$$\|C_2 u\|_2^2 = c_0^2 \sum_{k=1}^{\infty} [\langle u, \phi_{k-1,1} \rangle]^2 = c_0^2 \|u\|_1^2.$$

It can be shown that $C_2^{-1} : L_{1\rho} \to L_{2\rho}$ exists and is given by

$$C_2^{-1} u = \frac{1}{c_0} \sum_{k=0}^{\infty} (-\langle u, \phi_{k-1,1} \rangle) \phi_{k,2}, \tag{78}$$

and hence, C_1 is invertible.
From the above Lemmas and Assumption, we can prove the following theorem.

Theorem 5. *Let Assumption 2 hold, then the operator $A_{2,\lambda} = C_2 + \lambda L_2$ is invertible, and the main Equation (73) for the case $r = 2$ has a unique solution.*

Now we prove the convergence of the proposed method.

Theorem 6. *Let*

$$A_{r,\lambda} = C_r + \lambda L_r : L_{1,\rho} \to L_{2,\rho}, \quad r = \{1, 2, \} \tag{79}$$

be invertible, and let $u \in L_{1,\rho}$ be the unique solution of

$$A_{r,\lambda} u = f, \, r = \{1, 2\} \tag{80}$$

Let V_n denote the space of weighted Chebyshev polynomials of degree at most n, and let $u_n \in V_n$ be the numerical approximation obtained by the proposed Chebyshev collocation and quadrature scheme. Assume that the discrete operator $A_{\lambda,n}$ is consistent with A_λ and that

$$\|A_\lambda - A_{\lambda,n}\|_{L_{1,\rho} \to L_{2,\rho}} \le \varepsilon_n, \qquad \varepsilon_n \to 0. \tag{81}$$

Then, for sufficiently large n,

$$\|u - u_n\|_{L_{1,\rho}} \le C\Big(\|u - P_n u\|_{L_{1,\rho}} + \varepsilon_n\Big), \tag{82}$$

where $C > 0$ is constant, independent of n.

Proof. Let $P_n u \in V_n$ be the best polynomial approximation to u. Since $A_\lambda u = f$, we have

$$A_\lambda (u - P_n u) = f - A_\lambda P_n u. \tag{83}$$

By the invertibility of A_λ,

$$\|u - P_n u\|_{L_{1,\rho}} \leq \left\|A_\lambda^{-1}\right\| \|A_\lambda(u - P_n u)\|_{L_{2,\rho}}. \tag{84}$$

For the discrete approximation, the consistency assumption gives

$$\|(A_\lambda - A_{\lambda,n})P_n u\|_{L_{2,\rho}} \leq \varepsilon_n \|P_n u\|_{L_{1,\rho}}. \tag{85}$$

Combining the approximation and consistency errors and using the boundedness of A_λ^{-1}, we obtain

$$\|u - u_n\|_{L_{1,\rho}} \leq C\left[\|u - P_n u\|_{L_{1,\rho}} + \varepsilon_n\right]. \tag{86}$$

Since $P_n u$ is the best approximation,

$$\|u - P_n u\|_{L_{1,\rho}} = E_n(u)_{L_{1,\rho}}, \tag{87}$$

and hence

$$\boxed{\|u - u_n\|_{L_{1,\rho}} \leq C\left[E_n(u)_{L_{1,\rho}} + \varepsilon_n\right].} \tag{88}$$

This completes the proof. □

Consequences of Theorem 6 are given in Corollary 1.

Corollary 1. *If u has s derivatives satisfying the regularity assumptions required for algebraic Chebyshev approximation, and*

$$E_n(u)_{L_{1,\rho}} \leq C_u n^{-s}, \qquad \varepsilon_n \leq C_q n^{-s}, \tag{89}$$

then

$$\boxed{\|u - u_n\|_{L_{1,\rho}} \leq C^* n^{-s}.} \tag{90}$$

Thus the proposed method converges algebraically with order s. For analytic u, add: If u admits an analytic continuation to a Bernstein ellipse E_ρ, $\rho > 1$, and the quadrature error is compatible with the spectral approximation, then

$$\boxed{\|u - u_n\|_{L_{1,\rho}} \leq C_\rho \rho^{-n},} \tag{91}$$

which demonstrates spectral/geometric convergence.

6. Numerical Results for SIEs in the Cases of $r = \{1, 2, 3, 4\}$

Example 1 (Eshkuvatov et al. (2009) [15])**.** *Consider the following singular integral equation:*

$$\int_{-1}^{1} \frac{\varphi(t)}{t - x}\, dt = x^4 + 5x^3 + 2x^2 + x - \frac{11}{8}, \quad -1 < x < 1. \tag{92}$$

One can obtain the exact solutions of Equation (92), i.e.,

$$\begin{aligned}
&\textbf{Case(I)}: \quad \varphi(x) = -\frac{1}{\pi}\sqrt{1 - x^2}\left[x^3 + 5x^2 + \frac{5}{2}x + \frac{7}{2}\right], \\
&\textbf{Case(II)}: \quad \varphi(x) = \frac{1}{\pi\sqrt{1 - x^2}}\left[x^5 + 5x^4 + \frac{3}{2}\left(x^3 - x^2\right) - \frac{5}{2}x - \frac{9}{8}\right], \\
&\textbf{Case(III)}: \quad \varphi(x) = \frac{1}{\pi}\sqrt{\frac{1 + x}{1 - x}}\left[x^4 + 4x^3 - \frac{5}{2}x^2 + x - \frac{7}{2}\right], \\
&\textbf{Case(IV)}: \quad \varphi(x) = -\frac{1}{\pi}\sqrt{\frac{1 - x}{1 + x}}\left[x^4 + 6x^3 + \frac{15}{2}x^2 + 6x + \frac{7}{2}\right].
\end{aligned} \tag{93}$$

Remark 1. *Example 1 was discussed by Eshkuvatov et al. [15] in 2009 and by M. Abdulkawi (2015) [35], who applied the Differential Transform method and found only a semi-bounded solution, i.e., bounded at the right edge $x = 1$, and unbounded at the other edge $x = -1$. Here we are able to obtain an exact solution for every case of $r = \{1, 2, 3, 4\}$ with $N = 4, 5$. All four kinds of Chebyshev polynomials with corresponding weight functions are used to solve Equation (92). Approximate-analytical results show that the errors of the approximate solution with a small value of n for all cases are zero.*

Case 1, bounded solution $r = 1$: Let $N = 4$, then to find the bounded solution we use Equation (38),

$$u(x) \cong u_{n,1}(x) = \sum_{j=0}^{4} b_{j,1} U_j(x), \tag{94}$$

where unknowns $b_{j,1}$ are defined from Equation (42)

$$b_{0,1} = -\frac{19}{4}, \ b_{1,1} = -\frac{3}{2}, \ b_{2,1} = -\frac{5}{4}, \ b_{3,1} = -\frac{1}{8}, \ b_{4,1} = 0. \tag{95}$$

Substituting values of $b_{j,1}$ into (32), we obtain the approximate solution of Equation (92):

$$\varphi(x) = -\frac{1}{\pi}\sqrt{1-x^2}\left[x^3 + 5x^2 + \frac{5}{2}x + \frac{7}{2}\right]. \tag{96}$$

which is identical to the exact solution.
Case 2, unbounded solution $r = 2$: Let $N = 5$, then from Equation (43), we have

$$u(x) \cong u_{n,2}(x) = \sum_{j=0}^{5} b_{j,2} T_j(x), \tag{97}$$

where unknowns $b_{j,2}$ are defined from (44) and (47)

$$b_{0,2} = 0, \ b_{1,2} = -\frac{3}{4}, \ b_{2,2} = \frac{7}{4}, \ b_{3,2} = \frac{11}{16}, \ b_{4,2} = \frac{5}{8}, \ b_{5,2} = \frac{1}{16}. \tag{98}$$

Approximate solution of Equation (92) for the unbounded case is

$$\varphi(x) = \frac{1}{\pi\sqrt{1-x^2}}\left[x^5 + 5x^4 + \frac{3}{2}\left(x^3 - x^2\right) - \frac{5}{2}x - \frac{9}{8}\right]. \tag{99}$$

which coincides to the exact solution.
Case 3, semi-bounded solution $r = 3$: Let $N = 5$ then for unbounded solution on the right bound $x = 1$, and bounded solution on the left hand $x = -1$, we use Equation (49)

$$u(x) \cong u_{n,3}(x) = \sum_{j=0}^{4} b_{j,3} W_j(x), \tag{100}$$

where unknowns $b_{j,3}$ are defined from Equation (50)

$$b_{0.3} = -\frac{19}{8}, \ b_{1.3} = \frac{13}{8}, \ b_{2.3} = \frac{1}{8}, \ b_{3.3} = \frac{9}{16}, \ b_{4.3} = \frac{1}{16}, \ b_{5.3} = 0. \tag{101}$$

Using Equation (101), the approximate solution of Equation (92) for the case $r = 3$, is

$$\varphi(x) = \frac{1}{\pi}\sqrt{\frac{1+x}{1-x}}\left[x^4 + 4x^3 - \frac{5}{2}x^2 + x - \frac{7}{2}\right]. \tag{102}$$

which is identical to the exact solution.

Case 4, semi-bounded solution $r = 4$**:** Let $N = 5$, and for semi-bounded solution we can use Equation (41) to yield

$$u(x) \cong u_{n,4}(x) = \sum_{j=0}^{4} b_{j,4} V_j(x), \tag{103}$$

where unknowns $b_{j,4}$ are defined from Equation (49)

$$b_{0.4} = -\frac{19}{8}, \ b_{1.4} = -\frac{25}{8}, \ b_{2.4} = -\frac{11}{8}, \ b_{3.4} = -\frac{11}{16}, \ b_{4.4} = -\frac{1}{16}, \ b_{5.4} = 0. \tag{104}$$

Using Equation (104), the approximate solution of Equation (92) for $r = 4$ is

$$\varphi(x) = -\frac{1}{\pi}\sqrt{\frac{1-x}{1+x}}\left[x^4 + 6x^3 + \frac{15}{2}x^2 + 6x + \frac{7}{2}\right]. \tag{105}$$

Again, the exact solution can be obtained.

Example 2. *(Abdulkawi et al. (2009) [36]:) Consider the singular integral equation*

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\varphi(t)}{t-x}\,dt + \frac{1}{\pi}\int_{-1}^{1} x^5 t^5 \varphi(t)\,dt = x^5 + x^3 + x, \quad -1 < x < 1. \tag{106}$$

One can obtain the exact solutions of Equation (106)

$$\begin{aligned}
&\textbf{Case}(\textbf{I}): \quad \varphi(x) = -\sqrt{1-x^2}\left[x^4 + \frac{3}{2}x^2 + \frac{15}{8}\right], \\
&\textbf{Case}(\textbf{II}): \quad \varphi(x) = \frac{1}{\sqrt{1-x^2}}\left[x^6 + \frac{1}{2}x^4 + \frac{3}{8}x^2 - \frac{11}{6}\right], \\
&\textbf{Case}(\textbf{III}): \quad \varphi(x) = \sqrt{\frac{1+x}{1-x}}\left[x^5 - x^4 + \frac{3}{2}x^3 - \frac{3}{2}x^2 + \frac{15}{8}x - \frac{15}{8}\right], \\
&\textbf{Case}(\textbf{IV}): \quad \varphi(x) = -\sqrt{\frac{1-x}{1+x}}\left[x^5 + x^4 + \frac{3}{2}x^3 + \frac{3}{2}x^2 + \frac{15}{8}x + \frac{15}{8}\right].
\end{aligned} \tag{107}$$

Remark 2. *Abdulkawi et al. [36] investigated Example 2, using the Chebyshev polynomials of the first and second kind with the corresponding weight function to find bounded solution on the finite segment. They specifically obtained approximate solutions for $N = 5$. Here we solve Equation* (106) *for all cases and find the exact solution.*

Case 1, bounded solution $r = 1$**:** Let $N = 5$, then for bounded solution we use Equation (38)

$$u(x) \cong u_{n,1}(x) = \sum_{j=0}^{4} b_{j,1} U_j(x), \tag{108}$$

where unknowns $b_{j,1}$ are defined from Equation (42) which yields

$$b_{0,1} = -\frac{19}{8}, \ b_{1,1} = 0, \ b_{2,1} = -\frac{9}{16}, \ b_{3,1} = 0, \ b_{4,1} = -\frac{1}{16}, \ b_5 = 0. \tag{109}$$

Using Equation (109), the approximate solution of Equation (106) for the bounded case is

$$\varphi(x) = -\sqrt{1-x^2}\left[x^4 + \frac{3}{2}x^2 + \frac{15}{8}\right]. \tag{110}$$

which yields the exact solution.

Case 2, unbounded solution $r = 2$**:** Let $N = 6$, then from Equation (43) it follows that

$$u(x) \cong u_{n,2}(x) = \sum_{j=0}^{4} b_{j,2} T_j(x), \tag{111}$$

where unknowns $b_{j,2}$ are identified from (44) and (47)

$$b_{0,2} = 0, \;\; b_{1,2} = 0, \;\; b_{2,2} = \frac{29}{32}, \;\; b_{3,2} = 0, \;\; b_{4,2} = \frac{1}{4}, \;\; b_{5,2} = 0, \;\; b_{6,2} = \frac{1}{32}. \tag{112}$$

The approximate solution of Equation (106) can be obtained by substituting the values of $b_{j,2}$ into (32)

$$\varphi(x) = \frac{1}{\sqrt{1-x^2}}\left[x^6 + \frac{1}{2}x^4 + \frac{3}{8}x^2 - \frac{11}{6}\right]. \tag{113}$$

which coincides with the exact solution.

Case 3, semi-bounded solution $r = 3$**:** Let $N = 6$, then for finding the semi-bounded solution which is unbounded on the right edge $x = 1$, and bounded on the left edge $x = -1$, we use Equation (49)

$$u(t) \cong u_{n,3}(t) = \sum_{j=0}^{4} b_{j,3} W_j(x), \tag{114}$$

where unknowns $b_{j,3}$ are defined from Equation (49)

$$b_{0,3} = -\frac{19}{16}, \;\; b_{1,3} = \frac{19}{16}, \;\; b_{2,3} = -\frac{9}{32}, \;\; b_{3,3} = \frac{9}{32}, \;\; b_{4,3} = -\frac{1}{32}, \;\; b_{5,3} = \frac{1}{32}, \;\; b_{6,3} = 0. \tag{115}$$

Approximate-analytical solution of Equation (106) is determined by substituting (115) into (32)

$$\varphi(x) = \sqrt{\frac{1+x}{1-x}}\left[x^5 - x^4 + \frac{3}{2}x^3 - \frac{3}{2}x^2 + \frac{15}{8}x - \frac{15}{8}\right]. \tag{116}$$

This solution also coincides with the exact solution.

Case 4, semi-bounded solution $r = 4$**:** This is the case where the solution is unbounded on the left edge $x = -1$, and bounded on the right edge $x = 1$. Let $N = 6$, and an approximate-analytical solution for $r = 4$ is searched as

$$u(x) \cong u_{n,4}(x) = \sum_{j=0}^{4} b_{j,4} V_j(x), \tag{117}$$

where unknowns $b_{j,4}$ are defined from Equation (55)

$$b_{0,4} = -\frac{19}{16}, \;\; b_{1,4} = -\frac{19}{16}, \;\; b_{2,4} = -\frac{9}{32}, \;\; b_{3,4} = -\frac{9}{32}, \;\; b_{4,4} = -\frac{1}{32}, \;\; b_{5,4} = -\frac{1}{32}, \;\; b_{5,4} = 0. \tag{118}$$

These values is substitute into (32) to yield

$$\varphi(x) = -\sqrt{\frac{1-x}{1+x}}\left[x^5 + x^4 + \frac{3}{2}x^3 + \frac{3}{2}x^2 + \frac{15}{8}x + \frac{15}{8}\right]. \tag{119}$$

which yields the exact solution.

Example 3. *Okecha (2007) [37]: Consider the singular integral equation of the form*

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\varphi(t)}{t-x}\,dt + \frac{1}{\pi}\int_{-1}^{1} x^2\varphi(t)\,dt = -x^3 + \frac{1}{8}x^2 + \frac{1}{2}x, \quad -1 < x < 1. \tag{120}$$

It is not difficult to obtain the exact solutions of Equation (120) as follows:

$$\begin{aligned}
&\mathbf{Case(I)}: \quad \varphi(x) = -x^2\sqrt{1-x^2},\\
&\mathbf{Case(II)}: \quad \varphi(x) = -\frac{1}{\sqrt{1-x^2}}\left[x^4 - \frac{1}{8}x^3 - x^2 + \frac{1}{16}x + \frac{1}{8}\right],\\
&\mathbf{Case(III)}: \quad \varphi(x) = -\sqrt{\frac{1+x}{1-x}}\left[x^3 - x^2\right],\\
&\mathbf{Case(IV)}: \quad \varphi(x) = \sqrt{\frac{1-x}{1+x}}\left[x^3 + x^2\right].
\end{aligned} \tag{121}$$

Remark 3. *Let us compare the results between the method proposed by Okecha (2007) [37] and the current proposed method for Equation (120). In the work of Okecha [37], the bounded solution is considered only by applying a quadrature rule based on Lagrangian interpolation, with the zeros of Jacobi polynomials as nodes. They found a bounded solution with $n = 3$ which yields the maximum absolute error $\| u - u_n \|_\infty = 0.1665 \times 10^{-15}$. The current proposed method finds the exact solution of Equation (120) for all the cases.*

Case 1, bounded solution $r = 1$: Let $N = 3$, then the bounded solution of (120) is searched as follows:

$$u(x) \cong u_{n,1}(x) = \sum_{j=0}^{3} b_{j,1} U_j(x), \tag{122}$$

where unknowns $b_{j,1}$ are defined from Equation (42)

$$b_{0,1} = \frac{1}{4}, \; b_{1,1} = 0, \; b_{2,1} = \frac{1}{4}, \; b_{3,1} = 0. \tag{123}$$

Substituting these values into (32) yields an analytical-approximate solution of Equation (120)

$$\varphi(x) = -x^2\sqrt{1-x^2}. \tag{124}$$

which is identical to the exact solution.

Case 2, unbounded solution $r = 2$: Let $N = 4$, then the unbounded solution of Equation (120) is searched as follows:

$$u(x) \cong u_{n,2}(x) = \sum_{j=0}^{4} b_{j,2} T_j(x), \tag{125}$$

where unknowns $b_{j,2}$ are defined from Equation (49)

$$b_{0,2} = 0, \; b_{1,2} = \frac{1}{32}, \; b_{2,2} = 0, \; b_{3,2} = \frac{1}{32}, \; b_{4,2} = -\frac{1}{8}. \tag{126}$$

From (126) and (47), it follows that

$$\varphi(x) = -\frac{1}{\sqrt{1-x^2}}\left[x^4 - \frac{1}{8}x^3 - x^2 + \frac{1}{16}x + \frac{1}{8}\right]. \tag{127}$$

which yields an exact solution.

Case 3, semi-bounded solution $r = 3$: Let $N = 4$, and for the semi-bounded case (unbounded on the right edge and bounded on the left edge), we have

$$u(x) \cong u_{n,3}(x) = \sum_{j=0}^{4} b_{j,3} W_j(x), \tag{128}$$

here unknowns $b_{j,3}$ are defined from Equation (49)

$$b_{0,3} = \frac{1}{8}, \; b_{1,3} = -\frac{1}{8}, \; b_{2,3} = \frac{1}{8}, \; b_{3,3} = -\frac{1}{8}, \; b_4 = 0. \tag{129}$$

An approximate-analytical solution of Equation (120) for the semi-bounded case is

$$\varphi(x) = -\sqrt{\frac{1+x}{1-x}}\left[x^3 - x^2\right]. \tag{130}$$

again, an exact solution is obtained.

Case 4, semi-bounded solution $r = 4$: It is the case such that the solution is unbounded on the left edge $x = -1$ and bounded on the right edge $x = 1$. Let $N = 4$, and an analytical-approximate solution is searched as

$$u(x) \cong u_{n,4}(x) = \sum_{j=0}^{4} b_{j,4} W_j(x), \tag{131}$$

where unknowns $b_{j,4}$ are defined from Equation (55)

$$b_0 = \frac{1}{8},\ \ b_1 = \frac{1}{8},\ \ b_2 = \frac{1}{8},\ \ b_3 = -\frac{1}{8},\ \ b_4 = 0. \tag{132}$$

An analytical-approximate solution of Equation (120) for the semi-bounded case is

$$\varphi(x) = \sqrt{\frac{1-x}{1+x}}\left[x^3 + x^2\right]. \tag{133}$$

which coincides to the exact solution.

Example 4. *Consider the singular integral equation of the form*

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\varphi(t)}{t-x}\,dt + \frac{1}{\pi}\int_{-1}^{1} (x^2 t + x t^3)\varphi(t)\,dt = f(x), \quad -1 < x < 1. \tag{134}$$

where

$$f(x) = \frac{(80x^3 + 480x^2 + 640x + 40)\sqrt{3} - 140x^3 - 835x^2 - 1150x - 80}{8 + 4x}.$$

It is not obvious to find the exact solution of SIEs (134). Fortunately, by the trial method, the exact solutions of Equation (134) can be obtained as follows:

$$\begin{aligned}
&\textbf{Case(I)}: \quad \varphi(x) = \sqrt{1-x^2}\left[\frac{10x}{x+2}\right],\\
&\textbf{Case(II)}: \quad \varphi(x) = -\frac{1}{\sqrt{1-x^2}}\left[\frac{10x^2 + (20 - 10\sqrt{3})x - 20\sqrt{3} + 30}{x+2}\right],\\
&\textbf{Case(III)}: \quad \varphi(x) = -\sqrt{\frac{1+x}{1-x}}\left[\frac{10x - 10}{x+2}\right],\\
&\textbf{Case(IV)}: \quad \varphi(x) = \sqrt{\frac{1-x}{1+x}}\left[\frac{10x + 10}{x+2}\right].
\end{aligned} \tag{135}$$

Remark 4. *For these cases, exact solutions cannot be found but the error term of the approximate-analytical method can be calculated. The use of the Chebyshev collocation method (ChCM) in bounded, unbounded and semi-bounded cases is summarized in the following tables.*

All computations were carried out in Maple on a computer equipped with an Intel Celeron B820 processor (1.70 GHz) and 4 GB of RAM.

Table 1 shows the error ChCM for the bounded case. It reveals that the error term decreases drastically when number of nodes increases. Figure 1 presents the graphical representation for Table 1.

Table 2 demonstrates the error of ChCM for the unbounded scenario. The table illustrates a significant reduction in the error as the number of nodes increases. Figure 2 presents the graphical representation for Table 2.

Table 3 illustrates the ChCM error term in the semi-bounded situation (unbounded on the right edge $x = 1$ and bounded on the left edge $x = -1$). The table demonstrates a substantial decline in the error as the number of nodes increases. Figure 3 presents the graphical representation for Table 3.

Table 1. Bounded case $n = \{5, 12\}$.

x	**Exact Solution**	**Error ChCM $n = 5$**	**Error ChCM $n = 12$**
-0.950	2.9738085706659039075	$9.658726117828021 \times 10^{-3}$	$4.602983668400645 \times 10^{-7}$
-0.700	5.4934064834944999984	$2.133643972355125 \times 10^{-3}$	$3.684998667155761 \times 10^{-7}$
-0.300	5.6114070671585038185	$1.667922451341323 \times 10^{-3}$	$2.504236934267847 \times 10^{-7}$
-0.000	5.0000000000000000000	$1.985831287194382 \times 10^{-3}$	$1.343361581271742 \times 10^{-7}$
0.500	3.4641016151377545870	$1.376499922148058 \times 10^{-3}$	$5.897147964883273 \times 10^{-14}$
0.700	2.6449734920529074066	$9.159021915230111 \times 10^{-4}$	$1.964332260509540 \times 10^{-8}$
0.950	1.0584742370166776620	$1.013984246505038 \times 10^{-2}$	$1.188366749338977 \times 10^{-8}$
Max CPU time (s)	0.015	0.024	0.037

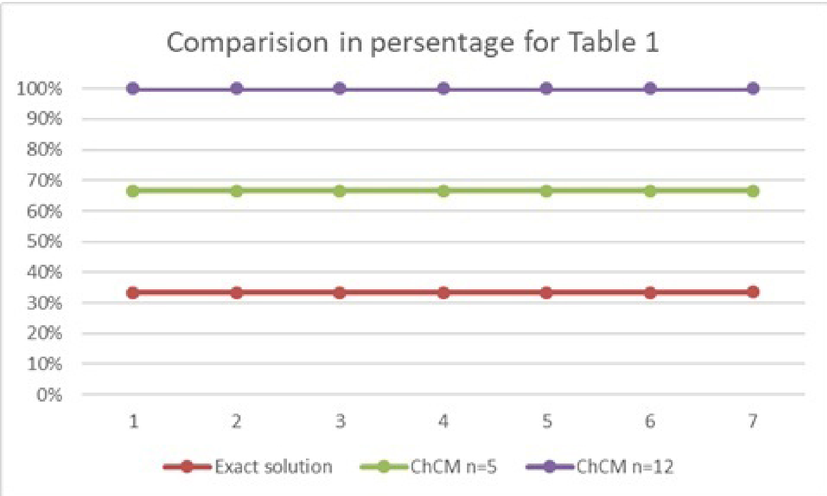

Figure 1. Graphical Representation for Table 1.

Table 2. Unbounded cases $n = \{5, 12\}$.

x	**Exact Solution**	**Error ChCM $n = 5$**	**Error ChCM $n = 12$**
-0.950	-5.6074333288846037284	$2.222696574074350 \times 10^{-2}$	$3.70381217297610 \times 10^{-6}$
-0.700	1.7413673065676012\78	$1.425070957360016 \times 10^{-3}$	$1.44562512751974 \times 10^{-6}$
-0.300	2.8025363127842104\99	$3.465606071134190 \times 10^{-3}$	$9.92268323176540 \times 10^{-7}$
-0.000	2.3205080756887729\35	$1.112503348109062 \times 10^{-2}$	$2.17763078763094 \times 10^{-13}$
0.500	0.3700908475527240\06	$1.024114773905721 \times 10^{-2}$	$5.08548789146919 \times 10^{-7}$
0.700	$-1.1070656848739913\12$	$6.857773074741913 \times 10^{-4}$	$6.96041636512220 \times 10^{-7}$
0.950	$-7.5227676624952635\31$	$7.869217672757339 \times 10^{-3}$	$1.31830652882355 \times 10^{-6}$
Max CPU time (s)	0.016	0.026	0.041

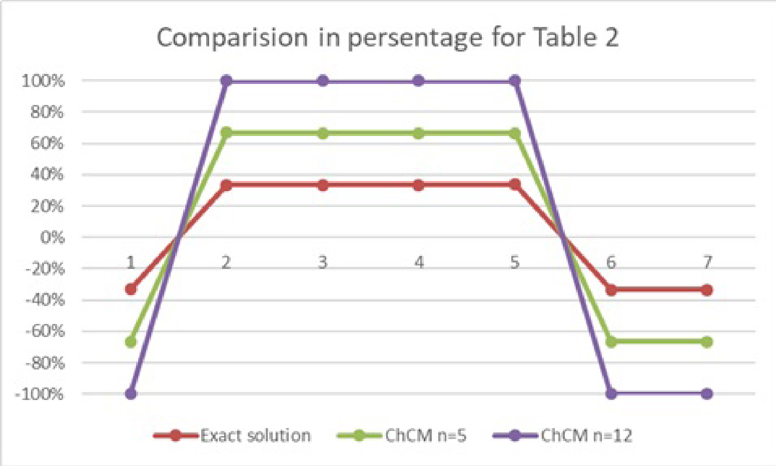

Figure 2. Graphical Representation for Table 2.

Table 3. Semi-bounded cases $n = \{5, 10\}$.

x	Exact Solution	Error ChCM $n = 5$	Error ChCM $n = 10$
-0.950	2.9738085706659039075	$3.006940851166090 \times 10^{-2}$	$4.598049818592453 \times 10^{-5}$
-0.700	5.4934064834944999984	$3.495245223967161 \times 10^{-3}$	$2.029644234956322 \times 10^{-5}$
-0.300	5.6114070671585038185	$9.343769424610154 \times 10^{-3}$	$1.302618167399078 \times 10^{-5}$
-0.000	5.0000000000000000000	$5.952052329775422 \times 10^{-3}$	$4.109141476154739 \times 10^{-6}$
0.500	3.4641016151377545870	$8.246196281947222 \times 10^{-3}$	$5.693791142590221 \times 10^{-6}$
0.700	2.6449734920529074066	$1.066194160946905 \times 10^{-2}$	$1.573941435694840 \times 10^{-5}$
0.950	1.0584742370166776620	$1.513330619647309 \times 10^{-2}$	$3.331750717165098 \times 10^{-5}$
Max CPU time (s)	0.016	0.031	0.047

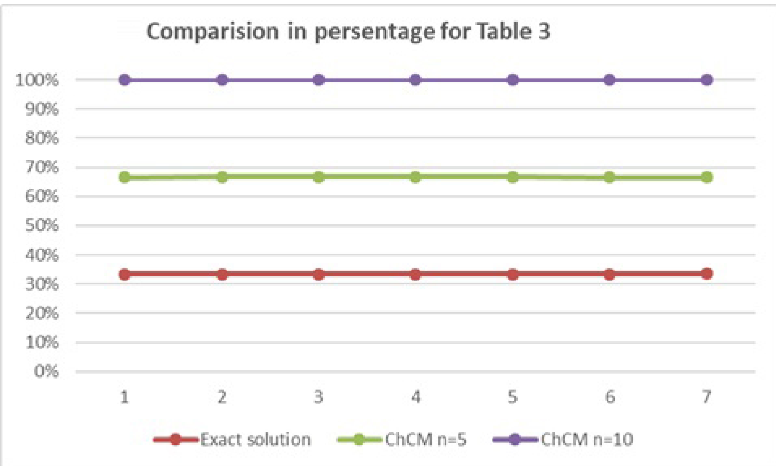

Figure 3. Graphical Representation for Table 3.

Table 4 provides a demonstration of the ChCM error in the semi-bounded scenario. The tables clearly shows a considerable reduction in the error term as the number of nodes increases. Figure 4 presents the graphical representation for Table 4.

Table 4. Semi-bounded cases $n = \{5, 10\}$.

x	Exact Solution	Error ChCM $n = 5$	Error ChCM $n = 10$
−0.950	2.9738085706659039075	$4.665262386405310 \times 10^{-3}$	$2.843414066211269 \times 10^{-5}$
−0.700	5.4934064834944999984	$7.612395445590916 \times 10^{-3}$	$1.262452007856274 \times 10^{-5}$
−0.300	5.6114070671585038185	$1.028310668559951 \times 10^{-3}$	$7.282787993731808 \times 10^{-6}$
−0.000	5.0000000000000000000	$1.980389755386657 \times 10^{-3}$	$1.369706904614641 \times 10^{-6}$
0.500	3.4641016151377545870	$9.133988644516633 \times 10^{-4}$	$6.326380323824735 \times 10^{-7}$
0.700	2.6449734920529074066	$4.456990600890250 \times 10^{-4}$	$8.067492085947929 \times 10^{-6}$
0.950	1.0584742370166776620	$1.027083992878249 \times 10^{-2}$	$1.577114964783914 \times 10^{-5}$
Max CPU time (s)	0.016	0.031	0.047

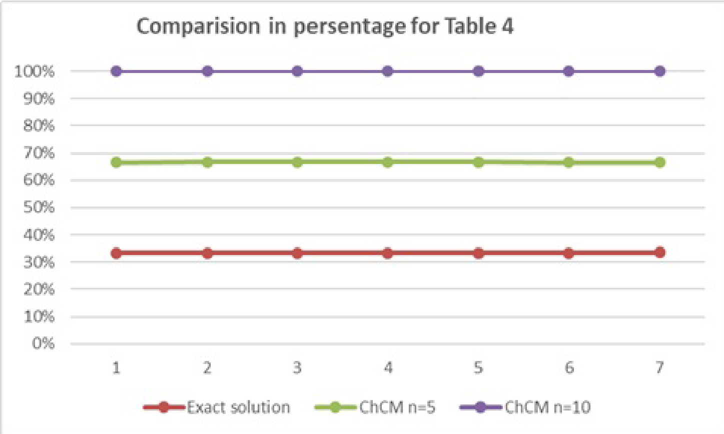

Figure 4. Graphical Representation for Table 4.

7. Discussion and Limitation

7.1. Discussion

The proposed Chebyshev collocation framework differs fundamentally from modern neural operator techniques such as FNO, MP-FVM, DRW, AFDONet, AFDONet-inv, and Adaptive Mamba Neural Operators. The present method is a deterministic approximation procedure in which the unknown solution is represented explicitly in a Chebyshev basis and the coefficients are obtained from the discretized singular integral equation. Consequently, no training dataset is required and the approximation is directly tied to the mathematical structure of the underlying Cauchy singular operator. The principal advantage of the present Chebyshev approach is therefore its deterministic and equation-specific character. Its accuracy can be analyzed through polynomial approximation, quadrature error, and the stability of the discretized operator without relying on statistical generalization from a training dataset. This is particularly important for first-kind Cauchy singular integral equations, where the ill-posed nature of the inverse problem requires careful treatment of singularities, endpoint behavior, and conditioning. On the other hand, neural operator approaches may have advantages for repeated solution of parameterized families of problems and may generalize across geometries or parameter ranges after suitable training. Their performance, however, depends on the quality and coverage of the training data and on the technique sand optimization procedure. Therefore, a direct claim that either approach is universally superior would not be justified. A systematic head-to-head comparison with these modern techniques on standardized CSIE benchmarks is an important direction for future research.

7.2. Limitations

The present study does not provide direct numerical benchmarks against all modern data-driven techniques discussed above. Implementing and training these models requires computational resources, GPU acceleration, training datasets, and substantial software development beyond the deterministic numerical experiments considered here. Therefore, we restrict the present comparison to the theoretically relevant discussion and established numerical methods. A systematic computational comparison with modern neural operators represents an important extension of the present work.

8. Conclusions

In the paper, we have developed a hybrid method (combining the Chebyshev collocation method (ChCM) and the kernel expansion method) for solving SIEs of the first kind, where the kernel $K(x,t)$ in Equation (1) is constant on the diagonal of the rectangle region $D = [-1,1] \times [-1,1]$. The collocation method is used to obtain a system of algebraic equations for the unknown coefficients $b_{j,r}, r = \{1,2,3,4\}$ in Equation (31). Examples 1–4 verify that the developed method is very high in accuracy and stable for SIEs of the first kind. From Tables 1–3, we can conclude that ChCM is exact if the solution of the considered problem is given as the product of weight function times polynomials, i.e., $w(t)t^n$. If the solution of SIEs is given as a product of weight function times rational function, i.e., $w(t)Q(t)$, then ChCM approaches the exact solution gradually when the number of nodes is increased. The numerical experiments indicate that the proposed Chebyshev collocation method provides competitive accuracy compared with the methods considered in Ref. [15] (2009), differential transform method in Ref. [35] (2015), and quadrature rule based on Lagrangian interpolation [37] (2007). A numerical solution is obtained with the help of Matlab 2023.2.1 software.

Future Outlook

An important direction for future research is to investigate the applicability of the proposed Chebyshev collocation framework to interdisciplinary inverse and nonlocal modeling problems arising in bioengineering, biochemical reaction systems, and chemical kinetics. In particular, extensions to integral formulations involving coupled reaction, transport, and memory effects may provide a useful computational framework for parameter identification and reconstruction problems in biological and chemical production systems. These investigations are beyond the scope of the present paper and will be considered in the future work.

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