

Article

Integral of an Alpha Unpredictable Function

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Abstract

The present paper is devoted to the study of integral properties of alpha unpredictable functions. The problem of invariance of recurrence under integration is one of the most challenging and interesting in the theory of functions. The problem was solved for quasiperiodic, almost periodic, and Poisson stable functions. All of the solutions were subjected to the condition of bounded integrals. The alpha-unpredictable functions are the endpoint in the row of recurrent functions. The new class is the cross-border point from regularity to chaos in the dynamics presented through the elements of the row. In the research, we finalized the proof that any theoretical functional recurrence is invariant with respect to integration, provided the result is bounded.

Keywords: alpha-unpredictable function; ultra Poincaré chaos; Poisson stability; method of included intervals; integral of alpha unpredictable function; divergence and convergence sequences

MSC: 26A24; 34H10

1. Introduction

This paper investigates the invariance of alpha-unpredictable functions under the operation of integration, and establishes a key conclusion: provided that the integral remains bounded, the integral of an alpha-unpredictable function preserves alpha-unpredictability. Methodologically, the paper introduces the “method of included intervals,” which, in comparison with traditional approaches based on functional analysis and the method of comparability, renders the proof more concise and intuitive, while also demonstrating good operability in practical applications. The theoretical proof is structurally sound, with rigorous derivations of the core inequalities, a clear demonstration of sufficiency, and a correct application of previously established algebraic results in the corollary. The introduction accurately traces the historical development of recurrence theory, and the bibliography appropriately covers both classical and contemporary works in the field.

The theory of recurrent functions holds a significant position in modern analysis, particularly in the fields of differential equations, and began in celestial mechanics [1,2]. It provides a framework for describing a broad class of nonlinear and nonautonomous processes that exhibit regular, yet not strictly periodic, dynamics [3,4].

Beginning with the classical works of P. Bohl, E. Esclangon [5,6], and culminating with H. Bohr [7], the foundational concepts of quasi-periodic functions and almost periodicity were established [8]. These concepts have found numerous applications in mathematical physics, control theory, biology, and neural networks. The first instance of non-closed



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recurrent dynamics, known as Poisson stability, was introduced by H. Poincaré in 1892. This area of study has become increasingly sophisticated in relation to recurrent functions. Subsequent research built upon Poincaré's ideas, filling the gap between periodicity and Poisson stability. Our current research pushes this understanding further.

In exploring chaotic problems, we discovered that previous studies portrayed chaos primarily as a “collective” phenomenon. By extending H. Poincaré's definition to include the property of alpha unpredictability in “individual” dynamics, we proved that chaos occurs in the closure of an alpha unpredictable trajectory [9]. We have termed this new class of processes “ultra Poincaré chaos.” The functions examined in this study represent initial points in the dynamics of shifts [10] within functional spaces. Consequently, we can assert that our research is focused on investigating chaos, particularly concerning conditions under which chaos is preserved in the presence of an integral operator.

On methodological aspects of the research, it is worth noting that verification of the convergence sequence in the Poisson stability have traditionally been approached through methods of functional analysis [8] or the method of comparability [11–13]. In contrast, our study proposes a new, simple, and effective *method of included intervals* [9]. It suggests that to prove convergence on a bounded interval, one must identify a larger one such that convergence on this including set implies stability in the included interval. In this article, we have developed this method and expanded the number of structures that can be analyzed within this framework. To demonstrate its efficacy and potential, we have verified a known auxiliary result in the main part of our discussion. It effectively works for various types of differential equations and real-world models [9,14], but the present realisation of the method is a specific one, and can be useful for further analysis of the theory of functions.

Notably, alpha unpredictability, which encompasses Poisson stability, serves as a constructive condition for verification and is weaker than other conditions typically used for synchronization. We have written several papers demonstrating that conservative criteria are ineffective, while synchronization of divergent and convergent sequences of alpha unpredictability is indeed useful [15]. This finding may hold significant interest for research into chaos.

2. Preliminaries

In the present research, we consider continuous real-valued functions that are determined and uniformly bounded on the real axis.

Definition 1 ([4,10]). A uniformly continuous and bounded on the real axis function $f(t)$ is said to be Poisson stable if there exists a strictly increasing sequence t_n that approaches infinity, such that the sequence of shifts $f(t + t_n)$ converges uniformly to the function $f(t)$ on every bounded interval.

The set t_n is called a *convergence* or *Poisson sequence*. One can also use the phrase a *sequence of return*.

Definition 2 ([9]). A Poisson-stable function $f(t)$ is defined as alpha unpredictable if, in addition to having a convergent sequence, there exist a strictly increasing sequence s_n that tends to infinity, and positive numbers ϵ_0 and δ such that for each index n , the following inequality is satisfied:

$$|f(t + t_n) - f(t)| > \epsilon_0 \quad \text{for all } t \in [s_n, s_n + \delta].$$

The sequence s_n is referred to as a *divergence sequence*, while the intervals $[s_n - \delta, s_n + \delta]$ are known as *divergence intervals*.

The existence of a sequence t_n is called *Poisson stability*, while the existence of a sequence s_n and constants ϵ_0, δ is called the *alpha unpredictability property*.

It is important to note that despite the long and rich history of Poisson stability, many articles and books have been published on this type of recurrent motion without providing a single example. The first Poisson stable functions were constructed in our works ([9] and papers cited there) and they are both alpha unpredictable and Poisson stable due to the definitions. The method used for their construction is a hybrid approach. That is, we use both discrete and continuous dynamics to have a final result.

Let us outline the algorithm. First, we construct a alpha unpredictable sequence, which serves as a solution to a logistic equation. Using this sequence, we then determine a piecewise constant alpha unpredictable function defined across the entire real axis. While this type of function could suffice, but it is discontinuous. Therefore, to obtain a continuous function, which completely satisfies definitions, we utilize an improper indefinite integral with the piecewise constant function as a factor for the integrand. This integral, evaluated at the upper limit, provides the function we need. Now, let us demonstrate specifically how this construction process is carried out.

An example of alpha unpredictable function. Consider, first, a solution ψ_i with all integer indices, of the logistic discrete equation

$$\lambda_{i+1} = \mu \lambda_i (1 - \lambda_i) \quad (1)$$

with $\mu = 3.92$. It is proved that the solution is alpha unpredictable sequence [9]. Its members belong to the unit interval $[0, 1]$. There exist a positive number ϵ_0 and sequences ζ_n, η_n both of which diverge to infinity such that $|\psi_{i+\zeta_n} - \psi_i| \rightarrow 0$ as $n \rightarrow \infty$ for each i in bounded intervals of integers and $|\psi_{\zeta_n+\eta_n} - \psi_{\eta_n}| \geq \epsilon_0$ for each natural number n .

Now, consider the following integral

$$\Theta(t) = \int_{-\infty}^t e^{-2(t-s)} \Omega(s) ds, \quad (2)$$

where t is a real argument. The function $\Omega(t)$ is a piece-wise constant one, and defined on the real axis by the equation $\Omega(t) = \psi_i$ for $t \in [i, i+1)$. The function $\Theta(t)$ is bounded such that $\sup_{-\infty < t < \infty} |\Theta(t)| \leq 1/2$, and is an alpha unpredictable continuous solution of the differential equation

$$x' = -2x + \Omega(t). \quad (3)$$

Currently, we do not have any alternative algorithm to determine the alpha unpredictable function. Therefore, to find an example of a bounded integral, we should investigate the following expression,

$$\int_0^t \int_{-\infty}^s e^{-2(s-u)} \Omega(u) du ds.$$

The task is not easy, since the function $\Omega(t)$ is chaotic. One can consider it as an open problem.

Method of included intervals. In our previous research, we introduced a new method to find the convergence sequence for functions derived from integration. Let us briefly explain this approach with an example.

Consider a Poisson stable function $f(t)$ with returning sequence t_n , and “integrate” it such that a new function is

$$F(t) = \int_{-\infty}^t e^{-2(t-s)} f(s) ds.$$

This indefinite improper integral, appears as a bounded solution of the inhomogeneous linear differential equation

$$x' = -2x + f(t). \quad (4)$$

Next, we shall prove existence of the returning sequence for the function.

To verify the definition, fix an interval $[a, b]$ and a positive number ϵ . We shall show that for sufficiently large n it is true that $|F(t + t_n) - F(t)| < \epsilon$ for moments of the interval $[a, b]$. Include the set inside of another section $[c, b]$, where c is to be determined. Now, we assume that $|f(t + t_n) - f(t)| < \epsilon$ on this interval. Denote $M_f = \sup_{-\infty < t < \infty} |f(t)|$. We have that for all moments from $[a, b]$ it is true that

$$\begin{aligned} |F(t + t_n) - F(t)| &= \left| \int_{-\infty}^t e^{-2(t-s)} (f(s + t_n) - f(s)) ds \right| \leq \left| \int_{-\infty}^c e^{-2(t-s)} (f(s + t_n) - f(s)) ds \right| + \\ &\quad \left| \int_c^t e^{-2(t-s)} (f(s + t_n) - f(s)) ds \right| \leq e^{-2(a-c)} M_f + \frac{\epsilon}{2}. \end{aligned}$$

Find that under the condition $c < a - \frac{1}{2} \ln(\frac{2M_f}{\epsilon})$ the request inequality is correct. One can observe that the number c tends to $-\infty$, if ϵ diminishes to zero. In other words, the including interval enlarges.

The idea appears effective and is now in use for differential equations and mathematical models in different areas [9,14].

In the present paper, we realize the method in a new circumstance of the integral

$$F(t) = \int_{t_0}^t f(s) ds. \quad (5)$$

Since the integrand is a bounded function, we utilize points t_1 and t_2 , where the function approaches the sup and inf values of it. Then, as the including interval for given section $[a, b]$ we consider the minimal-length interval $[c, d]$, which contains all the considered elements: points t_1 and t_2 and the section $[a, b]$. Again, we are in the circumstances of an *included* interval. This is why, it is reasonable to say about the *method of included intervals*.

Loosely speaking, we say that the method is considered when, for a certain integration, verification of sequence of convergence on a given set requires that the integrand admits returning converges on a longer interval.

Functional study of chaos through alpha unpredictability. There are two theories for the science of mathematical chaos. The first one relies on the *sensitivity* and another on the *alpha unpredictability*. The first direction is known by definitions of Devaney and Li-Yorke [16,17], while alternative one is ultra Poincaré chaos [9]. The sensitivity in its definitions requires at least two motions of dynamics and is considered a “collective” concept. However, in the case of alpha unpredictability, only a single trajectory is needed for its introduction, making it an “individual” concept. This trajectory, when represented in time and space, is an alpha unpredictable function. Therefore, we can assert that ultra Poincaré chaos is now more deeply integrated into functional research than in previous studies. Investigating chaos processes will be enhanced by analyzing alpha unpredictability. This paper demonstrates a step in that direction.

3. The Main Result

The object to analyze in the paper is the integral (5). It is easily seen that the function is defined for all real numbers. The uniform continuity of the function follows from the boundedness of the integrand.

Theorem 1. Assume that a function $f(t)$ is alpha unpredictable. The integral $F(t)$ is alpha unpredictable if and only if it is bounded.

Proof. The necessity of the assertion is an immediate consequence of Definitions 1 and 2.

The uniform continuity of the function follows from the boundedness of the integrand. In the following, we will apply the characteristics of the function $f(t)$. Specifically, we will consider sequences t_n and s_n , as well as the positive constants ε_0 and δ .

Firstly, we shall verify the sufficiency for existence of an convergence sequence, and for that we combine ideas of our *method of intervals* [9] with those of the *sup* and *inf* approach applied to a similar task for almost periodicity in the book [7].

Let us demonstrate that the convergence sequence t_n is common for the both functions.

Assume that $F(t)$ is bounded on the axis. That is, $\inf_{-\infty < t < \infty} F(t)$ and $\sup_{-\infty < t < \infty} F(t)$ are real numbers. Denote them m and M respectively. Obviously, $m < M$, since we plan to consider a non-trivial case.

Let us fix a positive ε and an interval $[a, b]$, as it is requested in Definition 1. Due to continuity of the function $F(t)$, and according to the definitions of *sup* and *inf*, there exist two points t_1 and t_2 such that

$$F(t_1) = M - \frac{3\varepsilon}{4}, \quad (6)$$

$$F(t_2) = m + \frac{3\varepsilon}{4}. \quad (7)$$

To apply the method of included intervals, let us fix the minimal-length interval $[c, d]$ that admits the points t_1, t_2, a, b ; consequently, we shall use it as the *including interval*. Next, consider a large index n such that

$$|f(t + t_n) - f(t)| < \frac{\varepsilon}{8(d - c)},$$

if t is between c and d . Then evaluate that

$$\begin{aligned} F(t_1 + t_n) - F(t_2 + t_n) &= F(t_1) + \int_{t_1}^{t_1+t_n} f(s)ds - F(t_2) - \int_{t_2}^{t_2+t_n} f(s)ds \geq \\ (M - \frac{3\varepsilon}{4}) - (m + \frac{3\varepsilon}{4}) - \int_{t_1}^{t_2} (f(s + t_n) - f(s))ds &> M - m - \frac{3\varepsilon}{2} - \frac{\varepsilon(d - c)}{8(d - c)} = \\ M - m - \frac{13\varepsilon}{8}. \end{aligned}$$

Rewrite the last inequality as

$$(M - F(t_1 + t_n)) + (F(t_2 + t_n) - m) < \frac{13\varepsilon}{8}.$$

Now, we have that for all arguments from the including interval, it is true that

$$\begin{aligned} F(t + t_n) - F(t) &= F(t_1 + t_n) + \int_{t_1+t_n}^t f(s)ds - F(t_1) - \int_{t_1}^t f(s)ds = F(t_1 + t_n) - F(t_1) + \\ &\int_{t_1}^t (f(s + t_n) - f(s))ds > M - \frac{13\varepsilon}{8} - M + \frac{3\varepsilon}{4} - \frac{\varepsilon(d - c)}{8(d - c)} = -\varepsilon. \end{aligned}$$

Similarly, one can find that

$$F(t + t_n) - F(t) = F(t_2 + t_n) + \int_{t_2+t_n}^{t+t_n} f(s)ds - F(t_2) - \int_{t_2}^t f(s)ds = F(t_2 + t_n) - F(t_2) + \int_{t_2}^t (f(s + t_n) - f(s))ds < m + \frac{13\varepsilon}{8} - m - \frac{3\varepsilon}{4} + \frac{\varepsilon(d-c)}{8(d-c)} = \varepsilon.$$

The last two inequalities imply that $|[F(t + t_n) - F(t)]| < \varepsilon$ for all arguments of the including interval, and consequently of the included interval $[a, b]$. It has been approved that t_n is a convergence sequence for the integral-function.

Now, we shall find a divergence sequence and other characteristics to finalize the proof. Fix an index n , and consider two alternative cases:

$$(i) |F(s_n + t_n) - F(s_n)| \geq \frac{\varepsilon_0 \delta}{2},$$

and

$$(ii) |F(s_n + t_n) - F(s_n)| < \frac{\varepsilon_0 \delta}{2}.$$

In what follows, we shall show that for the integral-function the amplitude of divergence can be taken $\frac{\varepsilon_0 \delta}{4}$, the length of the interval of divergence equals to $h = \frac{\varepsilon_0 \delta}{8M_f}$, where $M_f = \sup_{-\infty < t < \infty} |f(t)|$, and the element s'_n of the sequence for divergence moments can be chosen equal to s_n or $s_n + \delta - h$, depending on the two cases above.

(i). Assume that $|F(s_n + t_n) - F(s_n)| \geq \frac{\varepsilon_0 \delta}{2}$, and consider

$$F(t + t_n) - F(t) = F(s_n + t_n) - F(s_n) + \int_{s_n}^t (f(s + t_n) - f(s))ds.$$

Denote $h = \frac{\varepsilon_0 \delta}{8M_f}$, and evaluate the last difference on the interval $[s_n, s_n + h]$,

$$|F(t + t_n) - F(t)| > \frac{\varepsilon_0 \delta}{2} - 2M_f h = \frac{\varepsilon_0 \delta}{4}.$$

Thus, for the choice we have that the interval of divergence is $[s_n, s_n + h]$, and amplitude of the divergence is $\frac{\varepsilon_0 \delta}{4}$.

(ii). Let the following inequality be correct: $|F(s_n + t_n) - F(s_n)| < \frac{\varepsilon_0 \delta}{2}$ for the fixed index. First, we consider the equality

$$F(s_n + \delta + t_n) - F(s_n + \delta) = F(s_n + t_n) - F(s_n) + \int_{s_n}^{s_n + \delta} (f(s + t_n) - f(s))ds.$$

Apply the equality to obtain that

$$|F(s_n + \delta + t_n) - F(s_n + \delta)| > \varepsilon_0 \delta - \frac{\varepsilon_0 \delta}{2} = \frac{\varepsilon_0 \delta}{2}.$$

Now, use the equality,

$$F(t + t_n) - F(t) = F(s_n + \delta + t_n) - F(s_n + \delta) + \int_{s_n + \delta}^t (f(s + t_n) - f(s))ds,$$

to evaluate on the interval $[s_n + \delta - h, s_n + \delta]$,

$$|F(t + t_n) - F(t)| > \frac{\varepsilon_0 \delta}{2} - 2M_f h = \frac{\varepsilon_0 \delta}{4}.$$

It has been shown that the interval of divergence is $[s_n + \delta - h, s_n + \delta]$, and amplitude of the divergence is $\frac{\varepsilon_0 \delta}{4}$.

Thus, the characteristics for alpha unpredictability have been obtained for the fixed index. Since they do not depend on the index, we can confirm that the property is checked for the integral-function. The theorem is proved. $\square$

Corollary 1. Assume that $f(t)$ is an alpha unpredictable function, and

$$\int_{t_0}^t f(s)ds = ct + g(t), \quad (8)$$

where c is a real constant. If $g(t)$ is a bounded function, then it is alpha unpredictable one.

Proof. One can write that

$$\int_{t_0}^t (f(s) - c)ds = ct_0 + g(t).$$

By Theorem 4.3.1 in [9], the difference $f(t) - ct$ is an alpha unpredictable function, and the sum $ct_0 + g(t)$ is bounded. Thus, by Theorem 1, this sum is alpha unpredictable. Consequently, the function $g(t) = (ct_0 + g(t)) - ct_0$ is also alpha unpredictable by the same result in [9]. $\square$

Remark 1. We see that the constant c in (8)—paired with a bounded function $g(t)$ —exists if and only if it equals to the average of $f(t)$. Nevertheless, whether every alpha unpredictable function possesses an average is still an open question.

4. Discussion

The study of recurrent functions is an excellent source for formulating problems in function theory and functional analysis, predominantly focusing on periodic and almost periodic functions. In our research (see [9] and issues cited there), we introduced a new class of recurrency called *alpha unpredictable functions*. Consequently, we recognize the need to develop a new theoretical framework and are actively pursuing this direction.

We have already established the algebraic properties of alpha unpredictable functions in our published works and created algorithms to generate new members of this class with hybrid properties. These hybrid functions combine features of both regular and chaotic behaviors, and we refer to them as *compartmental alpha unpredictable functions*. The foundational basis for this class consists of periodic or almost periodic functions.

Additionally, we have explored extra conditions for periods and convergence sequences that ensure the alpha unpredictability of such composite functions. Although some elements of this emerging theory have been identified, numerous challenges still remain unresolved. In this article, we address one of them. We prove the theorem on integral of an unpredictable function, which is notable since a similar theorem already exists for periodic, quasi-almost-periodic, and Poisson stable functions. Thus, we continue to advance the development of the theory for recurrent functions. Moreover, we demonstrate a new technique of analysis for Poisson stability, more effective than those known in literature.

The next most relevant assertion in this series is the alpha unpredictability of the derivative. We also find the problem of Fourier series for the functions under examination to be particularly interesting, along with the study of the structure of convergence and divergence sequences, and the theory of functions on groups. Given that the alpha unpredictable function is a criterion for ultra-Poincare chaos, we expect new results that will contribute to the development of chaos research and its implications for functions and operators theory. Progress in this direction will open up new horizons in the development

of applied mathematics. Given complexity of the processes in our research, we understand that the proposed results can serve in the future for such applied research as chaos and also artificial neural networks [9].

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