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Weighted Stein–Weiss and John–Nirenberg Inequalities for Fractional Singular Integral Operators on Fofana Spaces

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Abstract

The Stein–Weiss weighted inequality and the John–Nirenberg inequality are among the most important and widely applied results in harmonic analysis, underlying a broad range of developments in weighted norm estimates and the theory of bounded mean oscillation. Motivated by their central role, in this article we investigate new refinements of these two classical inequalities in the setting of variable exponent Fofana spaces. These results are established for the first time in this class of generalized function spaces, in which the exponent $\mathrm{p}(\cdot)$ appearing in the norm is variable and the exponent η is constant. A further novelty is that, whereas such inequalities have previously been developed in other function spaces using the Riesz kernel, in the present study we instead use the more general Bessel–Riesz-type kernel, which additionally incorporates a decay parameter β controlling behavior at infinity and so yields sharper, more flexible estimates than the Riesz kernel alone. To verify the correctness of our results, we construct several nontrivial examples, establish some associated results, and give remarks recovering existing results under different settings; moreover, several of these results are new even in the constant-exponent case. In particular, we establish John–Nirenberg inequalities for functions of bounded mean oscillation in terms of the norm of variable exponent Fofana spaces on $\mathbb{R}^{\nu}$, via an atomic Hardy space argument, recovering the classical and variable-Lebesgue-space John–Nirenberg inequalities as special cases.

Keywords: Stein–Weiss inequality; John–Nirenberg inequality; generalized Fofana spaces; variable exponent Hardy spaces; mathematical operators; Harmonic analysis

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1. Introduction

The Stein–Weiss inequality and the John–Nirenberg inequality are two cornerstones of modern harmonic analysis, each having generated an extensive and still-growing body of research. The Stein–Weiss inequality, originally concerned with weighted norm estimates for the Riesz potential, has since been extended along several directions: to systems

of integral equations and sub-Laplacian settings on the Heisenberg group [1], to sharp constants and the existence of extremal functions [2], to reverse-type inequalities on the upper half space [3], and to settings incorporating angular integrability in combination with Caffarelli–Kohn–Nirenberg-type weights [4]. In parallel, the John–Nirenberg inequality, which characterizes functions of bounded mean oscillation through exponential integrability of their oscillation, has been substantially generalized to noncommutative martingale settings via atomic decomposition [5], to critical Sobolev spaces [6], to Musielak–Orlicz Campanato spaces on spaces of homogeneous type [7], and, more recently, to broader extensions and applications through refined covering and extrapolation arguments [8].

Given $0 < \gamma < \nu$, the fractional integral operator (Riesz potential) is defined as the convolution operator

$$I_\gamma \varphi(\xi) = \int_{\mathbb{R}^\nu} \frac{\varphi(\zeta)}{|\xi - \zeta|^{\nu-\gamma}}\, d\zeta. \tag{1}$$

The boundedness for this operator on Lebesgue spaces traces back to the foundational work of Hardy and Littlewood [9], who first established such estimates, and was subsequently generalized to the higher-dimensional setting by Sobolev [10]. Their result asserts that, whenever $0 < \gamma < \nu$, $1 < \mathrm{p} < \nu/\gamma$, and the exponent q is chosen so as to satisfy the scaling relation $\frac{1}{\mathrm{p}} - \frac{1}{\mathrm{q}} = \frac{\gamma}{\nu}$,

$$\|I_\gamma \varphi\|_{L^{\mathrm{q}}(\mathbb{R}^\nu)} \le C \|\varphi\|_{L^{\mathrm{p}}(\mathbb{R}^\nu)}. \tag{2}$$

Stein and Weiss [11] proved the following weighted refinement.

Theorem 1 (Stein–Weiss [11]). *Consider $0 < \gamma < \nu$, $1 < \mathrm{p} \le \mathrm{q} < \infty$, and constants κ, τ subject to the conditions*

(i) $-\frac{\nu}{\mathrm{q}} < \kappa \le \tau < \frac{\nu}{\mathrm{p}'}$;

(ii) $\frac{1}{\mathrm{p}} - \frac{1}{\mathrm{q}} = \frac{\gamma}{\nu} + \frac{\kappa - \tau}{\nu}$.

Then there is $C > 0$ such that $\big\| |\cdot|^\kappa I_\gamma \varphi \big\|_{L^{\mathrm{q}}(\mathbb{R}^\nu)} \le C \big\| |\cdot|^\tau \varphi \big\|_{L^{\mathrm{p}}(\mathbb{R}^\nu)}$ for every φ.

Theorem 1 has since been generalized along several lines. A Morrey-space version, in the Morrey space $M^\lambda_{\mathrm{p}}(\mathbb{R}^\nu)$ [12] introduced by Peetre [13] and Adams [14], was recently obtained by Kassymov, Ragusa, Ruzhansky, and Suragan [15].

Theorem 2 (Kassymov, Ragusa, Ruzhansky, Suragan [15]). *Fix $0 < \gamma < \nu$, and κ, τ such that $0 \le \tau - \kappa \le \gamma$, and p such that $1 < \mathrm{p} < \frac{\nu}{\gamma - (\tau - \kappa)}$; define q by $\frac{1}{\mathrm{p}} - \frac{1}{\mathrm{q}} = \frac{\gamma - (\tau - \kappa)}{\nu - \lambda}$. Suppose further that*

$$-\frac{\nu - \lambda}{\mathrm{q}} < \kappa \le \tau < \frac{\nu}{\mathrm{p}'}, \qquad 0 < \lambda < \nu - \big(\gamma - (\tau - \kappa)\big)\mathrm{p}.$$

Then there is $C > 0$ such that $\big\| |\cdot|^\kappa I_\gamma \varphi \big\|_{M^\lambda_{\mathrm{q}}(\mathbb{R}^\nu)} \le C \big\| |\cdot|^\tau \varphi \big\|_{M^\lambda_{\mathrm{p}}(\mathbb{R}^\nu)}$ for every φ, where $M^\lambda_{\mathrm{p}}(\mathbb{R}^\nu)$ is the (constant-exponent) Morrey space.

Building on the boundedness of the fractional maximal operator and fractional integrals on variable Lebesgue spaces established by Capone, Cruz-Uribe, and Fiorenza [16], Samko [17] proved an analogous inequality in variable Lebesgue spaces on a bounded domain.

Theorem 3 (Samko [17]). *Given $0 < \gamma < \nu$ and a bounded open set $\Omega \subset \mathbb{R}^\nu$, fix $\xi_0 \in \overline{\Omega}$. Suppose $p(\cdot) \in \mathcal{P}(\Omega) \cap LH_0(\Omega)$ is such that $1 < p^- \le p^+ < \nu/\gamma$. Define $q(\cdot)$ by $\frac{1}{q(\xi)} = \frac{1}{p(\xi)} - \frac{\gamma}{\nu}$. Fix constants μ, τ such that*

$$\gamma\, p(\xi_0) - \nu < \tau < \nu\big(p(\xi_0) - 1\big), \qquad \mu = \frac{q(\xi_0)}{p(\xi_0)}\tau.$$

Then there is $C > 0$ such that $\big\| |\cdot - \xi_0|^{\mu/q(\cdot)} I_\gamma \varphi \big\|_{L^{q(\cdot)}(\Omega)} \le C \big\| |\cdot - \xi_0|^{\tau/p(\cdot)} \varphi \big\|_{L^{p(\cdot)}(\Omega)}$ for every φ.

A further, more general refinement of the Stein–Weiss inequality in variable Lebesgue spaces on all of $\mathbb{R}^\nu$ is due to Cruz-Uribe and Suragan [18].

Theorem 4 (Cruz-Uribe, Suragan [18]). *Fix $0 < \gamma < \nu$. Given $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^\nu) \cap LH(\mathbb{R}^\nu)$, $1 < p^- \le p^+ < \infty$, $1 < q^- \le q^+ < \infty$, $p(\xi) \le q(\xi)$ for all $\xi \in \mathbb{R}^\nu$, and constants κ, τ satisfying $-\frac{\nu}{q^+} < \kappa \le \tau < \frac{\nu}{(p^-)'}$, suppose further that $\frac{1}{p(\xi)} - \frac{1}{q(\xi)} = \frac{\gamma}{\nu} + \frac{\kappa - \tau}{\nu}$ for all $\xi \in \mathbb{R}^\nu$. Then there is $C > 0$ such that $\big\| |\cdot|^\kappa I_\gamma \varphi \big\|_{q(\cdot)} \le C \big\| |\cdot|^\tau \varphi \big\|_{p(\cdot)}$ for every φ.*

Utilizing the idea of the variable-exponent Morrey space [19], Cruz-Uribe, Ghorbanalizadeh, and Suragan [20] recently further generalized their own previous result, developed in Lebesgue spaces, for the Stein–Weiss inequality, which is the main motivation for the present paper.

Theorem 5 (Cruz-Uribe, Ghorbanalizadeh, Suragan [20]). *Fix $0 < \gamma < \nu$, let $\Omega \subset \mathbb{R}^\nu$ be a bounded domain, and fix $\xi_0 \in \Omega$. Given $p(\cdot), q(\cdot) \in \mathcal{P}(\Omega) \cap LH_0(\Omega)$, $1 < p^- \le p^+ < \infty$, $1 < q^- \le q^+ < \infty$, $p(\xi) \le q(\xi)$ for all $\xi \in \Omega$, and $\lambda(\cdot) : \Omega \to [0, \nu]$ with $0 < \lambda^- \le \lambda^+ < \nu$, $\lambda(\cdot) \in LH_0(\Omega)$, let κ, τ be constants such that*

$$\begin{gathered} 0 \le \tau - \kappa \le \gamma, \\ \frac{\lambda^+ - \nu}{q^+} < \kappa \le \tau < \frac{\nu}{(p')^+}, \\ 0 < \lambda^- \le \lambda(\xi) < \nu - (\gamma - \tau + \kappa)\, p(\xi), \end{gathered}$$

and $\frac{\gamma + \kappa - \tau}{\nu - \lambda(\xi)} = \frac{1}{p(\xi)} - \frac{1}{q(\xi)}$. Then there is $C > 0$ such that, for every φ,

$$\big\| |\cdot - \xi_0|^\kappa I_\gamma \varphi \big\|_{M^{\lambda(\cdot)}_{q(\cdot)}(\Omega)} \le C \big\| |\cdot - \xi_0|^\tau \varphi \big\|_{M^{\lambda(\cdot)}_{p(\cdot)}(\Omega)}.$$

Ho [21] recently established a variable-exponent Stein–Weiss inequality in Herz spaces $K^{\delta(\cdot),\kappa}_{p(\cdot),\vartheta}$ under some new assumptions.

Theorem 6 (Ho [21]). *Let $\gamma \in (0, \nu)$, $\vartheta \in (0, \infty)$, $\delta(\cdot) \in L^\infty(\mathbb{R}^\nu)$, $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^\nu)$ with $p(\xi) \le q(\xi)$ for all $\xi \in \mathbb{R}^\nu$, $1 < p^- \le p^+ < \infty$, $1 < q^- \le q^+ < \infty$, and let $\kappa, \tau, p(\cdot), q(\cdot)$ satisfy the appropriate scaling and weight conditions, with δ_∞ satisfying*

$$-\frac{\nu}{p^+} + \gamma - \tau < \delta_\infty < \nu - \frac{\nu}{p^-} - \tau.$$

Then there is $C > 0$ such that $\|I_\gamma \varphi\|_{K^{\delta(\cdot),\kappa}_{q(\cdot),\vartheta}} \le C \|\varphi\|_{K^{\delta(\cdot),\tau}_{p(\cdot),\vartheta}}$ for every $\varphi \in K^{\delta(\cdot),\tau}_{p(\cdot),\vartheta}$.

Stein–Weiss inequalities have also been proved for the Laplace–Bessel differential operator [22], on Carnot groups [23], on the Stratified group [24], with the fractional Poisson kernel [25], in extremal-function form on general upper half spaces [26], on homogeneous Lie groups [27], together with two-weight Poincaré and Sobolev estimates on Morrey spaces [28], in sharpened/stronger form [29], and, most recently, with nonstandard power weights [30].

Alongside these Riesz-potential results, the Bessel–Riesz operator $T_{\gamma,\beta}$ has attracted independent interest. Multiplying the Riesz kernel $|\xi-\zeta|^{\gamma-\nu}$ by the extra factor $(1+|\xi-\zeta|)^{\beta}$ in the denominator keeps the kernel explicit, matches I_γ near the singularity, and reduces to it exactly at $\beta=0$, but supplies genuine polynomial decay at infinity governed by the parameter β; this decay is what removes the integrability restrictions that confine the plain Riesz potential to bounded domains or to a narrow scaling-determined range of exponents. The classical Bessel potential achieves a similar decay through an exponentially decaying kernel, at the cost of losing an explicit algebraic form (see Gatto, Pineda, and Urbina [31] on Besov–Lipschitz spaces for the Gaussian measure, and Mao, Wu, and Yang [32] on Riesz transform commutators in the Bessel setting); the Bessel–Riesz operator instead keeps the algebraic form while gaining only polynomial decay. The operator has been studied on anisotropic, generalized, B-, and Dunkl-type Morrey spaces and on Lebesgue and Morrey spaces with general measures [33–39]. In the variable-exponent setting, Nasir, Alshammari, and Raza [40] studied the one-dimensional instance of $T_{\gamma,\beta}$ in Lebesgue spaces on $\mathbb{R}_+$. Under a log-Hölder continuity assumption on the exponent $\mathrm{p}(\cdot)$ together with the natural range conditions $0<\gamma<1$ and $\gamma-\frac{1}{\mathrm{p}^+}<\beta<\infty$, they showed that a suitable target exponent $\mathrm{q}(\cdot)$ can be constructed so that

$$T_{\gamma,\beta}\varphi(\xi):=\int_{\mathbb{R}^\nu}\frac{|\xi-\zeta|^{\gamma-\nu}\varphi(\zeta)}{(1+|\xi-\zeta|)^{\beta}}\,d\zeta,\qquad \gamma\in(0,\nu),\ \beta\geq 0, \tag{3}$$

maps $L^{\mathrm{p}(\cdot)}(\mathbb{R}_+)$ continuously to $L^{\mathrm{q}(\cdot)}(\mathbb{R}_+)$.

Afzal, Alqahtani, Abbas, and Breaz [41] studied fractional Bessel–Riesz operators in variable-exponent Herz spaces, with applications to the Ginzburg–Landau equations, and Afzal [42] proved the following refinement of $T_{\gamma,\beta}$ to Herz spaces by adding exponential damping, and the result reads as follows.

Theorem 7 (See [41,42]). *Let* $\mathrm{s}\in(0,\infty]$, $\mathrm{r}(\cdot)\in P_\infty^{\log}(\mathbb{R}^\nu)$ *be such that*

(i) $1<\mathrm{r}^-\leq\mathrm{r}^+<\infty$;

(ii) $\delta(\cdot)\in L^\infty(\mathbb{R}^\nu)$ *is log-Hölder continuous at infinity, with value at infinity* δ_∞ *satisfying* $-\frac{\nu}{\mathrm{r}_\infty}<\delta_\infty<\frac{\nu}{\mathrm{r}'_\infty}$.

If $T_{\gamma,\beta}$ $(\gamma\in(0,\nu),\ \beta\geq 0)$ *is bounded on* $L^{\mathrm{r}(\cdot)}(\mathbb{R}^\nu)$, *then it extends to a bounded operator on the homogeneous Herz space* $K^{\delta(\cdot)}_{\mathrm{r}(\cdot),\mathrm{s}}(\mathbb{R}^\nu)$: *there is* $C>0$ *such that*

$$\|T_{\gamma,\beta}\varphi\|_{K^{\delta(\cdot)}_{\mathrm{r}(\cdot),\mathrm{s}}(\mathbb{R}^\nu)}\leq C\|\varphi\|_{K^{\delta(\cdot)}_{\mathrm{r}(\cdot),\mathrm{s}}(\mathbb{R}^\nu)},\qquad \varphi\in K^{\delta(\cdot)}_{\mathrm{r}(\cdot),\mathrm{s}}(\mathbb{R}^\nu)\cap L^{\mathrm{r}(\cdot)}(\mathbb{R}^\nu). \tag{4}$$

Among the fundamental results of harmonic analysis, the John–Nirenberg inequality [43] occupies a distinguished place: it quantifies how a function of bounded mean oscillation, though not itself necessarily bounded, cannot stray far from its local average on any ball, and shows that the measure of such deviations decays exponentially. This exponential self-improvement is what makes $BMO(\mathbb{R}^\nu)$ such a robust substitute for $L^\infty(\mathbb{R}^\nu)$ in the study of singular integral operators, elliptic PDEs, and quasiconformal mappings, and it underlies applications ranging from endpoint estimates for the Hilbert and Riesz transforms to

regularity theory for elliptic and parabolic equations with rough coefficients. The classical John–Nirenberg inequality is stated as follows: for $\varphi \in BMO(\mathbb{R}^\nu)$, $B \in \mathbb{B} = \{B(\xi_0, R) : \xi_0 \in \mathbb{R}^\nu, R > 0\}$, and $\varphi_B := |B|^{-1} \int_B \varphi(\zeta)\, d\zeta$,

$$|\{\xi \in B : |\varphi(\xi) - \varphi_B| \geq \mathfrak{t}\}| \leq C_1 e^{-C_2 \mathfrak{t}/\|\varphi\|_{BMO}} |B|, \qquad \mathfrak{t} > 0, \tag{5}$$

which, once rewritten in terms of the $L^1(\mathbb{R}^\nu)$ norm of the indicator function, is

$$\left\|\chi_{\{\xi \in B : |\varphi - \varphi_B| \geq \mathfrak{t}\}}\right\|_{L^1(\mathbb{R}^\nu)} \leq C_1 e^{-C_2 \mathfrak{t}/\|\varphi\|_{BMO}} \|\chi_B\|_{L^1(\mathbb{R}^\nu)}. \tag{6}$$

Ho [44] showed that if $\mathrm{p}(\cdot)$ is locally log-Hölder continuous and satisfies the log-Hölder decay condition at infinity, then (6) continues to hold with $L^1(\mathbb{R}^\nu)$ replaced by $L^{\mathrm{p}(\cdot)}(\mathbb{R}^\nu)$, that is,

$$\left\|\chi_{\{\xi \in B : |\varphi - \varphi_B| \geq \mathfrak{t}\}}\right\|_{L^{\mathrm{p}(\cdot)}(\mathbb{R}^\nu)} \leq C_1 e^{-C_2 \mathfrak{t}/\|\varphi\|_{BMO}} \|\chi_B\|_{L^{\mathrm{p}(\cdot)}(\mathbb{R}^\nu)}. \tag{7}$$

This inequality has since been pursued along several further lines: via sparse Brudnyi/John–Nirenberg spaces [45], in Gagliardo–Nirenberg-type form using weak Lebesgue spaces and *BMO* [46], for grand martingale Hardy spaces with variable exponents [47], on martingale variable Lorentz–Karamata spaces [48], for new *BMO*-type spaces with interpolation and applications [49], for non-doubling measures [50], on Besov–Bourgain–Morrey–Campanato spaces with sharp constants [51], together with weight-invariance results [52], for predual spaces of generalized *BMO*-martingale spaces [53], and in fractional-order and multidimensional operator settings on Campanato spaces [54].

The Fofana spaces used throughout this paper originate in the theory of Wiener amalgam spaces. In 1926, Wiener [55] first introduced the amalgam spaces $(L^1, \ell^2)(\mathbb{R})$ and $(L^2, \ell^\infty)(\mathbb{R})$; Holland [56] subsequently carried out a systematic study of the general amalgam spaces $(L^{\mathrm{p}}, \ell^{\mathrm{q}})(\mathbb{R}^\nu)$ in 1975, and, in 2012, Aydin and Gürkanli [57] introduced the variable exponent Wiener amalgam spaces $(L^{\mathrm{p}(\cdot)}, L^{\mathrm{q}}_\omega)(\mathbb{R}^\nu)$ and $(L^{\mathrm{p}(\cdot)}_\omega, L^{\mathrm{q}}_v)(\mathbb{R}^\nu)$. For $1 \leq \mathrm{p}, \mathrm{q} \leq \infty$, the amalgam space $(L^{\mathrm{p}}, \ell^{\mathrm{q}})(\mathbb{R}^\nu)$ carries the norm $\|\varphi\|_{\mathrm{p},\mathrm{q}} = \left\|\{\|\varphi\chi_{\mathrm{I}_k}\|_{L^{\mathrm{p}}(\mathbb{R}^\nu)}\}\right\|_{\ell^{\mathrm{q}}(\mathbb{R}^\nu)}$, I_k ranging over the unit cubes indexed by $k \in \mathbb{Z}^\nu$. For every $r > 0$ the dilation $St_r^{(\mathrm{p})} : \varphi \mapsto r^{-\nu/\mathrm{p}} \varphi(r^{-1}\cdot)$ is an isometry of $L^{\mathrm{p}}(\mathbb{R}^\nu)$, but this fails for the amalgam norm: if $\mathrm{p} \neq \mathrm{q}$ there is no α for which $\sup_{r>0} \|St_r^{(\alpha)}\varphi\|_{\mathrm{p},\mathrm{q}} < \infty$, even though $St_r^{(\alpha)}\varphi \in (L^{\mathrm{p}}, \ell^{\mathrm{q}})(\mathbb{R}^\nu)$ for every $\varphi \in (L^{\mathrm{p}}, \ell^{\mathrm{q}})(\mathbb{R}^\nu)$, $r > 0$, and $\alpha > 0$. Fofana [58] repaired this scaling deficiency in 1988 by introducing the spaces $(L^{\mathrm{p}}, \ell^{\mathrm{s}})^\alpha(\mathbb{R}^\nu)$, consisting of those $\varphi \in (L^{\mathrm{p}}, \ell^{\mathrm{q}})(\mathbb{R}^\nu)$ with $\sup_{r>0} \|St_r^{(\alpha)}\varphi\|_{\mathrm{p},\mathrm{q}} < \infty$, together with the continuous version $(L^{\mathrm{p}}, L^{\mathrm{q}})^\alpha(\mathbb{R}^\nu)$ recalled in (15) above (with $\vartheta(\rho) = \rho^{-\nu/\alpha}$). Because the resulting norm is, by construction, invariant under the dilation $St_r^{(\alpha)}$, the classical (and, as developed here, the variable exponent) Fofana scale removes exactly the scaling obstruction that limits plain amalgam spaces, while still containing the Lebesgue, Lorentz, and Morrey scales as special or limiting cases which is the reason we adopt it, rather than a plain amalgam space, as the ambient scale for our Stein–Weiss and John–Nirenberg estimates. For some other relevant boundedness properties of different operators and related inequalities in various function spaces, we refer to [59–67].

1.1. Motivation and Significance

Motivated by the literature review above, our work is guided by the following considerations.

- We are mainly motivated by the results stated above: Theorems 2–5 give Stein–Weiss inequalities on constant- and variable-exponent Lebesgue and Morrey spaces [15,17,18,20], and Theorem 6 gives one on variable-exponent Herz spaces [21]; however, no such in-

equality is available on the two-index variable-exponent Fofana scale $(L^{p(\cdot)}, L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})$, even though this scale refines the variable-exponent Morrey spaces of [19] (Section 2.5) as a special case. The Fofana scale is a natural ambient setting for such estimates precisely because it repairs a scaling deficiency inherent in the classical Wiener amalgam spaces $(L^{p}, \ell^{q})(\mathbb{R}^{\nu})$ [55]: for $p \neq q$, no dilation-adjusted amalgam norm remains uniformly bounded under the isometric dilations that preserve the plain $L^{p}(\mathbb{R}^{\nu})$ norm, so the amalgam norm itself is not dilation invariant in any useful sense. Fofana's [58] construction restores exactly this invariance by taking a supremum over dilated norms, and it is this built-in dilation invariance that yields sharper, scale-consistent estimates for both the Stein–Weiss and John–Nirenberg inequalities.

- Moreover, all of Theorems 2–6 in the existing work are established using the Riesz potential I_{γ}, whose kernel $|\xi - \zeta|^{\gamma-\nu}$ decays too slowly at infinity to be integrable there. We work instead with the Bessel–Riesz operator $T_{\gamma,\beta}$ [33–39], whose second parameter β supplies genuine extra decay at infinity and, as shown in Theorem 12 and the remark following it, strictly enlarges the admissible range of Stein–Weiss weight exponents beyond the $\beta = 0$ (Riesz).
- Furthermore, the John–Nirenberg inequality is known on $L^{1}(\mathbb{R}^{\nu})$ [43] and on variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^{\nu})$ [44]; moreover, these existing results are established using the Riesz kernel, whereas here we work with the Bessel–Riesz kernel and with variable-exponent Fofana spaces, which refine and improve upon [43,44] in both respects the boundedness of the operator and the underlying function space.

Throughout the paper, $T_{\gamma,\beta}$ denotes the Bessel–Riesz operator (ν-dimensional, γ the singularity order, β the decay parameter), and we prove four groups of results: boundedness of $T_{\gamma,\beta}$ and its fractional maximal counterpart $M_{\gamma,\beta}$ on generalized Fofana spaces, with a norm equivalence between them; boundedness of the fractional maximal operator M_{σ} and the Riesz potential I_{σ}, together with $T_{\sigma,\beta}$, on variable exponent Fofana spaces on a bounded domain, a two-index generalization of the variable exponent Morrey spaces of [19]; a Stein–Weiss inequality for $T_{\gamma,\beta}$ in this variable exponent Fofana setting, generalizing Theorem 5; and John–Nirenberg inequalities for $BMO(\mathbb{R}^{\nu})$ in terms of variable exponent Fofana norms, generalizing the inequality (7).

1.2. Organization of the Article

The remainder of the paper is organized as follows. Section 2 collects the preliminary material: Section 2.1 fixes notation; Section 2.2 recalls variable exponent Lebesgue spaces and the log-Hölder classes LH_0; Section 2.3 recalls the constant-exponent weighted Fofana space $(L^{q}, L^{p})^{\vartheta}(\mathbb{R}^{\nu})$ and the boundedness of the Hardy–Littlewood maximal operator on it; Section 2.4 extends this to a genuinely variable inner exponent on all of $\mathbb{R}^{\nu}$, which is needed to make the decay parameter β of the Bessel–Riesz kernel non-trivial already at the level of unweighted estimates; Section 2.5 specializes to a bounded domain, giving the two-index generalization of the variable exponent Morrey space of [19]; and Section 2.6 develops the atomic Hardy-space and convexification needed for the John–Nirenberg inequality. Section 3 contains the main results, organized as follows: Section 3.1 proves the Stein–Weiss-free norm inequalities for $T_{\gamma,\beta}$ and $M_{\gamma,\beta}$ on the global variable exponent Fofana spaces of Section 2.4, together with a norm equivalence between them; Section 3.2 proves boundedness of M_{σ} and $T_{\sigma,\beta}$ on the bounded-domain variable exponent Fofana spaces of Section 2.5; Section 3.3 contains the paper's central result, the weighted Stein–Weiss inequality of Theorem 15 for $T_{\gamma,\beta}$ on variable exponent Fofana spaces, together with its corollaries, remarks, and a fully worked numerical example; and Section 3.4 establishes the John–Nirenberg inequality of Theorem 16 for $BMO(\mathbb{R}^{\nu})$ in terms of the variable exponent Fofana norm, together with the associated exponential-integrability corollary.

2. Preliminaries

2.1. Notation

Throughout the paper, we adhere to the following notational conventions, so that the same symbol always plays the same role.

- Ω denotes a bounded domain in the Euclidean space $\mathbb{R}^\nu$ of dimension $\nu \in \mathbb{N}$.
- $\xi, \zeta, \omega \in \Omega$ denote generic points; $\xi_0 \in \Omega$ denotes a fixed base point around which a weight is centered.
- $\rho > 0$ denotes a radius; $\rho_\iota := 2^\iota \rho$ denotes a dyadic multiple of ρ, ι being a dyadic summation index, with threshold index ι_0.
- φ, ϕ denote generic (test) functions.
- κ, τ denote real weight exponents attached to the power weight $|\xi - \xi_0|^\kappa$, respectively, $|\xi - \xi_0|^\tau$.
- $\gamma \in (0, \nu)$ and $\beta \geq 0$ denote, respectively, the singularity order and the decay parameter of the Bessel–Riesz operator $T_{\gamma,\beta}$.
- $\lambda(\cdot)$ denotes a variable Morrey exponent; $\alpha(\cdot)$, $\alpha_{\mathrm{p}}(\cdot)$, $\alpha_{\mathrm{q}}(\cdot)$ denote variable Fofana order parameters; $\eta \in [1, \infty)$ denotes a fixed outer Fofana exponent; $\sigma \in [0, \nu)$ denotes the order of the fractional maximal operator M_σ; $\mu := \lambda - \beta$.
- ϑ denotes a classical (constant-exponent) Fofana weight function, and $\Theta(\rho) := \vartheta(\rho)^{\mathrm{q}/\bar{\mathrm{q}}}$ its derived output weight.
- θ denotes a generic dummy positive scalar occurring inside an infimum defining a norm.
- $\mathrm{p}(\cdot), \mathrm{q}(\cdot), \mathrm{r}(\cdot), \mathrm{s}$ denote variable or constant Lebesgue-type exponents, always set in roman type, with $\mathrm{p}^\pm, \mathrm{q}^\pm$ their essential bounds and $\mathrm{p}'(\cdot)$ the conjugate exponent.
- $\delta(\cdot), \delta_\infty$ denote a Herz-space smoothness exponent (as in Theorem 7).
- $\mathfrak{r} \geq 1$ denotes a convexification power; $\mathfrak{t} > 0$ denotes a John–Nirenberg threshold parameter; $\mathbb{B} := \{B(\xi_0, R) : \xi_0 \in \mathbb{R}^\nu, R > 0\}$; $\varphi_B := |B|^{-1} \int_B \varphi(\zeta)\, d\zeta$ denotes the mean of φ over $B \in \mathbb{B}$.
- Ψ denotes an atom; Φ a Schwartz test function and M_Φ the associated Φ-maximal function; $\mathcal{X}$ a generic Banach function space; $\mathcal{K}, \mathcal{K}_{\mathrm{p}(\cdot)}$ a Hölder-type constant; $BMO(\mathbb{R}^\nu), H^1(\mathbb{R}^\nu), \mathcal{S}(\mathbb{R}^\nu), \mathcal{S}'(\mathbb{R}^\nu)$ the usual spaces of bounded mean oscillation, Hardy, Schwartz, and tempered distributions.
- $B(\xi, \rho)$ and $\widetilde{B}(\xi, \rho) := B(\xi, \rho) \cap \Omega$ denote balls, and χ_E the characteristic function of a set E. The operator names T, M, I, K and the generic constants C, C_ν retain their meaning throughout, their values possibly changing from line to line.

2.2. Variable Exponent Lebesgue Spaces

A variable exponent is a measurable function $\mathrm{p}(\cdot) : \Omega \to [1, \infty]$; $\mathcal{P}(\Omega)$ denotes the set of such functions, with $\mathrm{p}^+ = \operatorname{ess\,sup}_\Omega \mathrm{p}(\cdot)$, $\mathrm{p}^- = \operatorname{ess\,inf}_\Omega \mathrm{p}(\cdot)$. A function $\lambda(\cdot) : \Omega \to \mathbb{R}$ is locally log-Hölder continuous, $\lambda(\cdot) \in LH_0(\Omega)$, if there is C_0 such that

$$|\lambda(\xi) - \lambda(\zeta)| \leq \frac{C_0}{-\log|\xi - \zeta|}, \qquad \xi, \zeta \in \Omega,\ |\xi - \zeta| < \tfrac{1}{2}. \tag{8}$$

For $\mathrm{p}(\cdot) \in \mathcal{P}(\Omega)$, the modular is $\varrho_{\mathrm{p}(\cdot)}(\varphi) = \int_{\Omega \setminus \Omega_\infty} |\varphi(\xi)|^{\mathrm{p}(\xi)} d\xi + \|\varphi\|_{L^\infty(\Omega_\infty)}$, $\Omega_\infty = \{\mathrm{p}(\cdot) = \infty\}$, and

$$\|\varphi\|_{\mathrm{p}(\cdot)} = \|\varphi\|_{L^{\mathrm{p}(\cdot)}(\Omega)} = \inf\{\theta > 0 : \varrho_{\mathrm{p}(\cdot)}(\varphi/\theta) \leq 1\} \tag{9}$$

defines the variable Lebesgue space $L^{\mathrm{p}(\cdot)}(\Omega)$, a Banach function space (Cruz-Uribe and Fiorenza [68]). The conjugate $\mathrm{p}'(\cdot)$ satisfies $\frac{1}{\mathrm{p}(\xi)}+\frac{1}{\mathrm{p}'(\xi)}=1$ ($\frac{1}{\infty}=0$), and, for $\varphi\in L^{\mathrm{p}(\cdot)}(\Omega)$, $\phi\in L^{\mathrm{p}'(\cdot)}(\Omega)$,

$$\int_\Omega |\varphi\phi|\,d\xi \le 4\|\varphi\|_{\mathrm{p}(\cdot)}\|\phi\|_{\mathrm{p}'(\cdot)} \tag{10}$$

holds; if $\frac{1}{\mathrm{r}(\xi)}=\frac{1}{\mathrm{p}(\xi)}+\frac{1}{\mathrm{q}(\xi)}$ for all ξ, there is $K=K(\mathrm{p}(\cdot),\mathrm{q}(\cdot))\ge 1$ with

$$\|\varphi\phi\|_{\mathrm{r}(\cdot)} \le K\|\varphi\|_{\mathrm{p}(\cdot)}\|\phi\|_{\mathrm{q}(\cdot)} \tag{11}$$

(Cor. 2.28, [68]).

Lemma 1 (Lemma 4, [19]). *Assume the following:*

- $\lambda(\cdot)\in LH_0(\Omega)$;
- $\rho>0$;
- $\xi,\zeta\in\Omega$ *satisfy* $|\xi-\zeta|\le\rho$.

Then there exists a constant $C=C(\lambda,\Omega)$, *depending only on* λ *and* Ω, *such that*

$$C^{-1}\rho^{-\lambda(\zeta)} \le \rho^{-\lambda(\xi)} \le C\rho^{-\lambda(\zeta)}. \tag{12}$$

Lemma 2 (Lemma 6, [19]). *Suppose* $\mathrm{p}(\cdot)\in\mathcal{P}(\Omega)\cap LH_0(\Omega)$. *There is* $C=C(\mathrm{p}(\cdot),\Omega)$ *such that*

$$\|\chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)} \le C|B(\xi,\rho)|^{1/\mathrm{p}(\xi)}, \qquad \xi\in\Omega,\ \rho>0. \tag{13}$$

Lemma 3 (Lemma 2.5, [20]). *Suppose* $\lambda(\cdot):\Omega\to[0,\nu]$ *satisfies* $0<\lambda^-\le\lambda^+<\nu$, $\lambda(\cdot)\in LH_0(\Omega)$. *Then for every* $\xi_0\in\Omega$, $\rho>0$,

$$\int_{\widetilde{B}(\xi_0,\rho)} |\xi-\xi_0|^{\lambda(\xi)-\nu}\,d\xi \le C\rho^{\lambda(\xi_0)}. \tag{14}$$

By (Lemma 2.7, [20]), if $\mathrm{p}(\cdot),\lambda(\cdot)\in LH_0(\Omega)$ and $\mathrm{p}^+<\infty$, then $\lambda(\cdot)/\mathrm{p}(\cdot)\in LH_0(\Omega)$; more generally:

- Any ratio of two $LH_0(\Omega)$ functions bounded away from 0 and ∞ is again in $LH_0(\Omega)$,

a fact used repeatedly below.

2.3. Weighted Fofana Spaces

Let $1\le\mathrm{q}\le\mathrm{p}\le\infty$. Following Nagacy, Kpata, and Diarra [69], $\mathcal{G}_{\mathrm{q},\mathrm{p}}$ denotes the set of $\vartheta:(0,\infty)\to(0,\infty)$ such that:

- $\rho\mapsto\vartheta(\rho)\rho^{\nu/\mathrm{p}}$ is almost decreasing;
- $\rho\mapsto\vartheta(\rho)\rho^{\nu/\mathrm{q}}$ is almost increasing.

For $\vartheta\in\mathcal{G}_{\mathrm{q},\mathrm{p}}$, the weighted Fofana space $(L^{\mathrm{q}},L^{\mathrm{p}})^{\vartheta}(\mathbb{R}^\nu)$ is

$$\|\varphi\|_{(L^{\mathrm{q}},L^{\mathrm{p}})^{\vartheta}(\mathbb{R}^\nu)} = \sup_{\rho>0}\frac{1}{\vartheta(\rho)}\rho^{\nu(-\frac{1}{\mathrm{q}}-\frac{1}{\mathrm{p}})}\,{}_\rho\|\varphi\|_{\mathrm{q},\mathrm{p}}, \qquad {}_\rho\|\varphi\|_{\mathrm{q},\mathrm{p}} = \Big\|\,\|\varphi\chi_{B(\zeta,\rho)}\|_{L^{\mathrm{q}}_\xi}\Big\|_{L^{\mathrm{p}}_\zeta}. \tag{15}$$

At $\vartheta(\rho)=\rho^{-\nu/\alpha}$, $\mathrm{q}\le\alpha\le\mathrm{p}$, this is the classical Fofana space $(L^{\mathrm{q}},L^{\mathrm{p}})^{\alpha}(\mathbb{R}^\nu)$ [58]; at $\mathrm{p}=\infty$ it is Nakai's generalized Morrey space $M^{\vartheta}_{\mathrm{q}}$ [70].

Proposition 1 (Section 2, [69]). *For* $\rho>0$, $1\le\mathrm{q},\mathrm{p}\le\infty$:

(i) $(L^{\mathrm{q}},l^{\mathrm{p}})_\rho(\mathbb{R}^\nu)$ *is a Banach space, and* $C^{-1}\rho^{\nu/\mathrm{p}}\,{}_\rho\widetilde{\|\varphi\|}_{\mathrm{q},\mathrm{p}} \le {}_\rho\|\varphi\|_{\mathrm{q},\mathrm{p}} \le C\rho^{\nu/\mathrm{p}}\,{}_\rho\widetilde{\|\varphi\|}_{\mathrm{q},\mathrm{p}}$;

(ii) if $\frac{1}{\mathrm{q}_1}+\frac{1}{\mathrm{q}_2}=\frac{1}{\mathrm{q}}\le 1$, $\frac{1}{\mathrm{p}_1}+\frac{1}{\mathrm{p}_2}=\frac{1}{\mathrm{p}}\le 1$, *then* ${}_\rho\widetilde{\|\varphi\phi\|}_{\mathrm{q},\mathrm{p}} \le {}_\rho\widetilde{\|\varphi\|}_{\mathrm{q}_1,\mathrm{p}_1}\,{}_\rho\widetilde{\|\phi\|}_{\mathrm{q}_2,\mathrm{p}_2}$.

For $\vartheta \in \mathcal{G}_{\mathrm{q},\mathrm{p}}$ there are $C_1, C_2 > 0$ with

$$\tfrac{1}{2} \le \tfrac{\rho}{\varsigma} \le 2 \implies C_1^{-1} \le \vartheta(\rho)/\vartheta(\varsigma) \le C_1, \tag{16}$$

$$(2^\iota \rho)^{\nu(\frac{1}{\mathrm{q}}+\frac{1}{\mathrm{p}})} \vartheta(2^{\iota+1}\rho) \le C_2\, 2^{\iota+1} \int_{2^\iota \rho}^{\rho} \vartheta(\varsigma) \varsigma^{\nu(\frac{1}{\mathrm{q}}+\frac{1}{\mathrm{p}})-1} d\varsigma, \qquad \iota \in \mathbb{Z}_{>0},\ \rho > 0. \tag{17}$$

Proposition 2 (Prop. 2.7(2), [69]). *If* $1 \le \mathrm{q} \le \mathrm{p}_1 \le \mathrm{p}_2 \le \infty$, $\vartheta \in \mathcal{G}_{\mathrm{q},\mathrm{p}_1}$, *then* $\|\varphi\|_{(L^{\mathrm{q}},L^{\mathrm{p}_2})^\vartheta(\mathbb{R}^\nu)} \lesssim \|\varphi\|_{(L^{\mathrm{q}},L^{\mathrm{p}_1})^\vartheta(\mathbb{R}^\nu)}$.

Proposition 3 (Section 2, [69]). *Let* $1 \le \mathrm{q}_1 \le \mathrm{p}_1 \le \infty$, $1 \le \mathrm{q}_2 \le \mathrm{p}_2 \le \infty$, $\vartheta \in \mathcal{G}_{\mathrm{q},\mathrm{p}}$, $\vartheta_1 \in \mathcal{G}_{\mathrm{q}_1,\mathrm{p}_1}$, $\vartheta_2 \in \mathcal{G}_{\mathrm{q}_2,\mathrm{p}_2}$, $\vartheta_1(\rho)\vartheta_2(\rho) \le \vartheta(\rho)$, $\frac{1}{\mathrm{q}_1}+\frac{1}{\mathrm{q}_2} = \frac{1}{\mathrm{q}} \le 1$, $\frac{1}{\mathrm{p}_1}+\frac{1}{\mathrm{p}_2} = \frac{1}{\mathrm{p}} \le 1$. *Then*

$$\|\varphi\phi\|_{(L^{\mathrm{q}},L^{\mathrm{p}})^\vartheta(\mathbb{R}^\nu)} \lesssim \|\varphi\|_{(L^{\mathrm{q}_1},L^{\mathrm{p}_1})^{\vartheta_1}(\mathbb{R}^\nu)} \|\phi\|_{(L^{\mathrm{q}_2},L^{\mathrm{p}_2})^{\vartheta_2}(\mathbb{R}^\nu)}. \tag{18}$$

Theorem 8 (Thm. 3.1, [69]). *Assume that* $1 < \mathrm{q} \le \mathrm{p} \le \infty$, $\vartheta \in \mathcal{G}_{\mathrm{q},\mathrm{p}}$ *with* $\int_\rho^\infty \vartheta^{\mathrm{q}}(\varsigma) \varsigma^{\frac{\nu \mathrm{q}}{\mathrm{p}}-1} d\varsigma \le C\vartheta^{\mathrm{q}}(\rho)\rho^{\frac{\nu \mathrm{q}}{\mathrm{p}}}$, $\forall \rho > 0$. *Then* $\|M\varphi\|_{(L^{\mathrm{q}},L^{\mathrm{p}})^\vartheta(\mathbb{R}^\nu)} \lesssim \|\varphi\|_{(L^{\mathrm{q}},L^{\mathrm{p}})^\vartheta(\mathbb{R}^\nu)}$, *M the Hardy–Littlewood maximal operator.*

For $0 \le \gamma < \nu$, $\beta \ge 0$, recall $T_{\gamma,\beta}\varphi(\xi) = \int_{\mathbb{R}^\nu} \varphi(\zeta)|\xi-\zeta|^{\gamma-\nu}(1+|\xi-\zeta|)^{-\beta} d\zeta$ and its fractional maximal counterpart

$$M_{\gamma,\beta}\varphi(\xi) = \sup_{\rho>0} |B(\xi,\rho)|^{\frac{\gamma}{\nu}-1}(1+\rho)^{-\beta} \int_{B(\xi,\rho)} |\varphi(\zeta)|\, d\zeta. \tag{19}$$

Lemma 4. *For* $0 < \gamma < \nu$, $\beta \ge 0$, φ *locally integrable,* $M_{\gamma,\beta}\varphi(\xi) \le C_\nu T_{\gamma,\beta}(|\varphi|)(\xi)$, $\xi \in \mathbb{R}^\nu$.

Proof. For $\zeta \in B(\xi,\rho)$, $|\xi-\zeta| < \rho$ gives $|\xi-\zeta|^{\gamma-\nu} \ge \rho^{\gamma-\nu}$ and $(1+|\xi-\zeta|)^{-\beta} \ge (1+\rho)^{-\beta}$, so

$$T_{\gamma,\beta}(|\varphi|)(\xi) \ge \rho^{\gamma-\nu}(1+\rho)^{-\beta} \int_{B(\xi,\rho)} |\varphi(\zeta)|\, d\zeta = C_\nu^{-1}|B(\xi,\rho)|^{\gamma/\nu-1}(1+\rho)^{-\beta} \int_{B(\xi,\rho)} |\varphi(\zeta)|\, d\zeta, \tag{20}$$

$C_\nu = |B(0,1)|^{1-\gamma/\nu}$; take $\sup_{\rho>0}$. □

2.4. Variable Exponent Fofana Spaces on $\mathbb{R}^\nu$

To make the role of the Bessel–Riesz decay parameter β genuinely non-trivial already at the level of unweighted boundedness (Section 3.1 below), we extend the variable exponent theory to the whole space $\mathbb{R}^\nu$. A variable exponent $\mathrm{p}(\cdot) : \mathbb{R}^\nu \to [1,\infty)$ is globally log-Hölder continuous, $\mathrm{p}(\cdot) \in LH(\mathbb{R}^\nu)$, if $\mathrm{p}(\cdot) \in LH_0(\mathbb{R}^\nu)$ (the local condition of Section 2.1, now on all of $\mathbb{R}^\nu$) and there are $\mathrm{p}_\infty \in [1,\infty)$, $C_\infty > 0$ such that

$$|\mathrm{p}(\xi) - \mathrm{p}_\infty| \le \frac{C_\infty}{\log(e+|\xi|)}, \qquad \xi \in \mathbb{R}^\nu. \tag{21}$$

This is the standard global log-Hölder class of variable Lebesgue space theory (Cruz-Uribe and Fiorenza [68], Ch. 4). We use the following two facts from [68], which extend Lemmas 1 and 2 from a bounded domain to $\mathbb{R}^\nu$, at the cost of requiring the exponent to be log-Hölder at infinity as well as locally.

Lemma 5 (cf. Cor. 4.30, [68]). *If* $\lambda(\cdot) \in LH(\mathbb{R}^\nu)$ *is bounded, there is* $C = C(\lambda)$ *such that for all* $\rho > 0$ *and* $\xi, \zeta \in \mathbb{R}^\nu$ *with* $|\xi-\zeta| \le \rho$,

$$C^{-1}\rho^{-\lambda(\zeta)} \le \rho^{-\lambda(\xi)} \le C\rho^{-\lambda(\zeta)}. \tag{22}$$

Lemma 6 (cf. Cor. 4.24, [68]). *Suppose* $\mathrm{p}(\cdot) \in \mathcal{P}(\mathbb{R}^\nu) \cap LH(\mathbb{R}^\nu)$ *with* $1 \le \mathrm{p}^- \le \mathrm{p}^+ < \infty$. *Then there exists a constant* $C = C(\mathrm{p}(\cdot))$, *depending only on* $\mathrm{p}(\cdot)$, *such that*

$$\|\chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)} \le C|B(\xi,\rho)|^{1/\mathrm{p}(\xi)}, \qquad \xi \in \mathbb{R}^\nu,\ \rho > 0. \tag{23}$$

Given $1 \le \mathrm{p}^+ \le \eta \le \infty$, let $\mathcal{G}_{\mathrm{p}^+,\eta}$ denote the class of Section 2.2 (with the roles of q, p there played by p^+, η). We extend Definition 2 to a variable inner exponent on $\mathbb{R}^\nu$ with a general (not necessarily pure-power) outer weight, exactly as (15) extends the classical Fofana space.

Definition 1. *Fix* $\eta \in [1,\infty)$, $\mathrm{p}(\cdot) \in \mathcal{P}(\mathbb{R}^\nu) \cap LH(\mathbb{R}^\nu)$ *with* $1 < \mathrm{p}^- \le \mathrm{p}^+ \le \eta < \infty$, *and* $\vartheta \in \mathcal{G}_{\mathrm{p}^+,\eta}$. *The variable exponent Fofana space* $\left(L^{\mathrm{p}(\cdot)}, L^\eta\right)^\vartheta(\mathbb{R}^\nu)$ *consists of* φ *with*

$$\|\varphi\|_{\left(L^{\mathrm{p}(\cdot)},L^\eta\right)^\vartheta(\mathbb{R}^\nu)} := \sup_{\rho>0} \frac{1}{\vartheta(\rho)} \left\{ \int_{\mathbb{R}^\nu} \left(\rho^{-\nu/\mathrm{p}(\xi)-\nu/\eta} \|\varphi\chi_{B(\xi,\rho)}\|_{L^{\mathrm{p}(\cdot)}(\mathbb{R}^\nu)} \right)^\eta d\xi \right\}^{1/\eta} < \infty. \tag{24}$$

At $\mathrm{p}(\cdot) \equiv \mathrm{q}$ constant, (24) reduces exactly to (15) with η in place of p; at $\vartheta(\rho) = \rho^{-\nu/\alpha}$ on a bounded Ω, it reduces to Definition 2. The following embedding lemma is the $\mathbb{R}^\nu$-analogue of Lemma 8, with the same proof.

Lemma 7. *With* $\vartheta \in \mathcal{G}_{\mathrm{p}^+,\eta}$ *as in Definition 1, for every* φ, $\xi_0 \in \mathbb{R}^\nu$, $\rho > 0$,

$$\|\varphi\chi_{B(\xi_0,\rho)}\|_{\mathrm{p}(\cdot)} \le C\vartheta(\rho)\rho^{\nu/\mathrm{p}(\xi_0)} \|\varphi\|_{\left(L^{\mathrm{p}(\cdot)},L^\eta\right)^\vartheta(\mathbb{R}^\nu)}. \tag{25}$$

Proof. Fix $\xi_0 \in \mathbb{R}^\nu$, $\rho > 0$. By (24), restricting the outer integral to $B(\xi_0, \rho/2)$ and expanding the η-th power,

$$\vartheta(\rho)^\eta \, \|\varphi\|^\eta_{\left(L^{\mathrm{p}(\cdot)},L^\eta\right)^\vartheta(\mathbb{R}^\nu)} \ge \int_{B(\xi_0,\rho/2)} \rho^{-\eta\nu/\mathrm{p}(\xi)-\nu} \, \|\varphi\chi_{B(\xi,\rho)}\|^\eta_{\mathrm{p}(\cdot)} \, d\xi, \tag{26}$$

using $\rho^{-\eta\nu/\eta} = \rho^{-\nu}$.

For $\xi \in B(\xi_0, \rho/2)$, we have $B(\xi_0, \rho/2) \subset B(\xi, \rho)$, so

$$\|\varphi\chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)} \ge \|\varphi\chi_{B(\xi_0,\rho/2)}\|_{\mathrm{p}(\cdot)};$$

by Lemma 5 (with radius ρ, applied to $1/\mathrm{p}(\cdot) \in LH(\mathbb{R}^\nu)$),

$$\rho^{-\eta\nu/\mathrm{p}(\xi)} \ge C^{-\eta}\rho^{-\eta\nu/\mathrm{p}(\xi_0)} \quad \text{on } B(\xi_0, \rho/2).$$

Hence, using $|B(\xi_0, \rho/2)| = c\rho^\nu$,

$$\begin{aligned} \vartheta(\rho)^\eta \, \|\varphi\|^\eta_{\left(L^{\mathrm{p}(\cdot)},L^\eta\right)^\vartheta(\mathbb{R}^\nu)} &\ge C^{-\eta}\rho^{-\eta\nu/\mathrm{p}(\xi_0)-\nu} \, \|\varphi\chi_{B(\xi_0,\rho/2)}\|^\eta_{\mathrm{p}(\cdot)} \cdot c\rho^\nu \\ &= C^{-\eta} c \, \rho^{-\eta\nu/\mathrm{p}(\xi_0)} \, \|\varphi\chi_{B(\xi_0,\rho/2)}\|^\eta_{\mathrm{p}(\cdot)}, \end{aligned} \tag{27}$$

the powers $\rho^{-\nu}$ and ρ^ν cancelling exactly.

Taking η-th roots and replacing $\rho/2$ by ρ gives (25). □

We also need the extension of Theorem 8 to a variable inner exponent, needed to close the estimates of Section 3.1.

Theorem 9. *Let* $\eta \in [1,\infty)$, $\mathrm{p}(\cdot) \in \mathcal{P}(\mathbb{R}^\nu) \cap LH(\mathbb{R}^\nu)$, $1 < \mathrm{p}^- \le \mathrm{p}^+ \le \eta < \infty$, $\vartheta \in \mathcal{G}_{\mathrm{p}^+,\eta}$ *satisfying*

$$\int_\rho^\infty \vartheta^{\mathrm{p}^+}(\varsigma)\varsigma^{\frac{\nu \mathrm{p}^+}{\eta}-1}d\varsigma \le C\vartheta^{\mathrm{p}^+}(\rho)\rho^{\frac{\nu \mathrm{p}^+}{\eta}}, \qquad \forall \rho > 0. \tag{28}$$

Then $\|M\varphi\|_{(L^{\mathrm{p}(\cdot)},L^\eta)^\vartheta(\mathbb{R}^\nu)} \lesssim \|\varphi\|_{(L^{\mathrm{p}(\cdot)},L^\eta)^\vartheta(\mathbb{R}^\nu)}$, *M the Hardy–Littlewood maximal operator.*

Proof. The argument is the same dyadic decomposition as in the proof of Theorem 8 [69], with the following two replacements: the classical Hardy–Littlewood–Wiener maximal theorem on $L^{\mathrm{p}^+}(\mathbb{R}^\nu)$ is replaced by Diening's variable exponent maximal theorem on $L^{\mathrm{p}(\cdot)}(\mathbb{R}^\nu)$, valid since $\mathrm{p}(\cdot) \in LH(\mathbb{R}^\nu)$, $1 < \mathrm{p}^- \le \mathrm{p}^+ < \infty$ (Thm. 4.4, [68]); and every comparison of the form $\rho^{\pm\nu/\mathrm{p}^+} \approx \rho^{\pm\nu/\mathrm{p}(\xi)}$ across a fixed dyadic scale is justified by Lemma 5 in place of a bare constant-exponent identity. Condition (28) is exactly the hypothesis of Theorem 8 with $\mathrm{q} = \mathrm{p}^+$, $\mathrm{p} = \eta$. □

2.5. Variable Exponent Fofana Spaces on a Bounded Domain

Throughout this subsection, $\Omega \subset \mathbb{R}^\nu$ is bounded, $\widetilde{B}(\xi,\rho) = B(\xi,\rho) \cap \Omega$.

Definition 2. *Fix* $\eta \in [1,\infty)$, $\mathrm{p}(\cdot) \in \mathcal{P}(\Omega) \cap LH_0(\Omega)$, *and* $\alpha(\cdot) : \Omega \to (0,\infty)$ *such that:*

(i) $1 < \mathrm{p}^- \le \mathrm{p}^+ < \infty$;

(ii) $\mathrm{p}(\xi) \le \alpha(\xi) \le \eta$ *for a.e.* $\xi \in \Omega$, *and* $\alpha(\cdot) \in LH_0(\Omega)$.

The variable exponent Fofana space $\left(L^{\mathrm{p}(\cdot)}, L^\eta\right)^{\alpha(\cdot)}(\Omega)$ *consists of* φ *with*

$$\|\varphi\|_{\left(L^{\mathrm{p}(\cdot)},L^\eta\right)^{\alpha(\cdot)}(\Omega)} := \sup_{\rho>0}\left\{\int_\Omega \left(|B(\xi,\rho)|^{\frac{1}{\alpha(\xi)}-\frac{1}{\mathrm{p}(\xi)}-\frac{1}{\eta}}\|\varphi\chi_{\widetilde{B}(\xi,\rho)}\|_{L^{\mathrm{p}(\cdot)}(\Omega)}\right)^\eta d\xi\right\}^{1/\eta} < \infty. \tag{29}$$

This is the bounded-domain, variable-exponent instance of (15): $\mathrm{p}(\cdot)$ plays the role of the inner exponent, η the fixed outer exponent, $\alpha(\cdot)$ the role of ϑ via $\vartheta(\rho) = \rho^{-\nu/\alpha}$.

For $\lambda(\cdot) : \Omega \to [0,\nu)$ with $0 < \lambda^- \le \lambda^+ < \nu$, $\lambda(\cdot) \in LH_0(\Omega)$, and $\mathrm{p}(\cdot)$ as above, set

$$\alpha_{\mathrm{p}}(\xi) := \frac{\nu\,\mathrm{p}(\xi)}{\nu - \lambda(\xi)}. \tag{30}$$

Since $\lambda(\cdot), \mathrm{p}(\cdot) \in LH_0(\Omega)$ give $\alpha_{\mathrm{p}}(\cdot) \in LH_0(\Omega)$, $\left(L^{\mathrm{p}(\cdot)}, L^\eta\right)^{\alpha_{\mathrm{p}}(\cdot)}(\Omega)$ is well defined once $\alpha_{\mathrm{p}}^+ \le \eta$.

Lemma 8. *With* $\alpha_{\mathrm{p}}(\cdot)$ *as in* (30), *for every* φ, $\xi_0 \in \Omega$, $\rho > 0$,

$$\|\varphi\chi_{\widetilde{B}(\xi_0,\rho)}\|_{\mathrm{p}(\cdot)} \le C\rho^{\lambda(\xi_0)/\mathrm{p}(\xi_0)}\|\varphi\|_{\left(L^{\mathrm{p}(\cdot)},L^\eta\right)^{\alpha_{\mathrm{p}}(\cdot)}(\Omega)}. \tag{31}$$

Proof. Fix $\xi_0 \in \Omega$, $\rho > 0$; write

$$\Theta_{\mathrm{p}}(\xi) = \frac{1}{\alpha_{\mathrm{p}}(\xi)} - \frac{1}{\mathrm{p}(\xi)} - \frac{1}{\eta}.$$

By (29), restricting the outer integral to $\widetilde{B}(\xi_0,\rho/2)$,

$$\|\varphi\|^\eta_{\left(L^{\mathrm{p}(\cdot)},L^\eta\right)^{\alpha_{\mathrm{p}}(\cdot)}(\Omega)} \ge \int_{\widetilde{B}(\xi_0,\rho/2)}\left(|B(\xi,\rho)|^{\Theta_{\mathrm{p}}(\xi)}\,\|\varphi\chi_{\widetilde{B}(\xi,\rho)}\|_{\mathrm{p}(\cdot)}\right)^\eta d\xi. \tag{32}$$

For $\xi \in B(\xi_0, \rho/2)$, we have $B(\xi_0, \rho/2) \subset B(\xi, \rho)$, so

$$\|\varphi \chi_{\widetilde{B}(\xi,\rho)}\|_{\mathrm{p}(\cdot)} \geq \|\varphi \chi_{\widetilde{B}(\xi_0,\rho/2)}\|_{\mathrm{p}(\cdot)}.$$

Moreover $|B(\xi, \rho)| = c_\nu \rho^\nu$ is independent of ξ, and by Lemma 1 (with radius ρ, $\Theta_{\mathrm{p}}(\cdot) \in LH_0(\Omega)$),

$$(c_\nu \rho^\nu)^{\Theta_{\mathrm{p}}(\xi)} \geq C^{-1} (c_\nu \rho^\nu)^{\Theta_{\mathrm{p}}(\xi_0)} \quad \text{on } B(\xi_0, \rho/2).$$

Hence

$$\|\varphi\|^{\eta}_{\left(L^{\mathrm{p}(\cdot)}, L^{\eta}\right)^{\alpha_{\mathrm{p}}(\cdot)}(\Omega)} \geq C^{-\eta} (c_\nu \rho^\nu)^{\eta \Theta_{\mathrm{p}}(\xi_0)} \|\varphi \chi_{\widetilde{B}(\xi_0,\rho/2)}\|^{\eta}_{\mathrm{p}(\cdot)} \cdot c\rho^\nu. \tag{33}$$

Since

$$-\nu \Theta_{\mathrm{p}}(\xi_0) - \frac{\nu}{\eta} = \frac{\nu}{\mathrm{p}(\xi_0)} - \frac{\nu}{\alpha_{\mathrm{p}}(\xi_0)} = \frac{\lambda(\xi_0)}{\mathrm{p}(\xi_0)}$$

by (30), taking η-th roots and replacing $\rho/2$ by ρ gives (31). □

Remark 1. *Lemma 8 holds for every finite $\eta \geq 1$: $\left(L^{\mathrm{p}(\cdot)}, L^{\eta}\right)^{\alpha_{\mathrm{p}}(\cdot)}(\Omega)$ embeds into the variable exponent Morrey space $M^{\lambda(\cdot)}_{\mathrm{p}(\cdot)}(\Omega)$ of [19].*

We also recall the variable exponent Morrey space of [19], with modular

$$\varrho_{\mathrm{p}(\cdot),\lambda(\cdot)}(\varphi) = \sup_{\xi \in \Omega, \rho > 0} \rho^{-\lambda(\xi)} \int_{\widetilde{B}(\xi,\rho)} |\varphi(\zeta)|^{\mathrm{p}(\zeta)} d\zeta, \tag{34}$$

$$M^{\lambda(\cdot)}_{\mathrm{p}(\cdot)}(\Omega) = \left\{ \varphi : \|\varphi\|_{M^{\lambda(\cdot)}_{\mathrm{p}(\cdot)}(\Omega)} < \infty \right\}, \tag{35}$$

$$\|\varphi\|_{M^{\lambda(\cdot)}_{\mathrm{p}(\cdot)}(\Omega)} = \inf \left\{ \theta > 0 : \varrho_{\mathrm{p}(\cdot),\lambda(\cdot)}(\varphi/\theta) \leq 1 \right\}, \tag{36}$$

with the modular equivalence $\|\phi\|_{M^{\lambda(\cdot)}_{\mathrm{p}(\cdot)}(\Omega)} \leq C_1 \iff \varrho_{\mathrm{p}(\cdot),\lambda(\cdot)}(\phi) \leq C_2$ (Lemma 2.6, [20]).

Theorem 10 (Thms. 2, 3 & Cor. 2, [19]). *Let $\mathrm{p}(\cdot) \in \mathcal{P}(\Omega) \cap LH_0(\Omega)$, $1 < \mathrm{p}^- \leq \mathrm{p}^+ < \infty$, $\lambda(\cdot)$ measurable on Ω, such that:*

(i) $0 \leq \lambda^- \leq \lambda^+ < \nu$;

(ii) $\sup_{\xi \in \Omega} [\lambda(\xi) + \sigma \mathrm{p}(\xi)] < \nu$, *for a fixed* $\sigma \in [0, \nu)$.

Define $\mathrm{q}(\cdot)$ by $\frac{1}{\mathrm{q}(\xi)} = \frac{1}{\mathrm{p}(\xi)} - \frac{\sigma}{\nu - \lambda(\xi)}$. Then M_σ and I_σ are bounded from $M^{\lambda(\cdot)}_{\mathrm{p}(\cdot)}(\Omega)$ to $M^{\lambda(\cdot)}_{\mathrm{q}(\cdot)}(\Omega)$, where $M_\sigma \varphi(\xi) = \sup_{\zeta \in \Omega, \rho > 0} |B(\zeta, \rho)|^{\sigma/\nu - 1} \int_{\widetilde{B}(\zeta,\rho)} |\varphi| \, \chi_{B(\zeta,\rho)}(\xi)$.

2.6. BMO, Atoms, and Hardy Spaces on Variable Exponent Fofana Spaces

The classical John–Nirenberg inequality [43] states that, for $\varphi \in BMO(\mathbb{R}^\nu)$, $B \in \mathbb{B} := \{B(\xi_0, R) : \xi_0 \in \mathbb{R}^\nu, R > 0\}$, $\mathrm{t} > 0$,

$$|\{\xi \in B : |\varphi(\xi) - \varphi_B| \geq \mathrm{t}\}| \leq C_1 e^{-C_2 \mathrm{t}/\|\varphi\|_{BMO}} |B|, \qquad \varphi_B := \frac{1}{|B|} \int_B \varphi(\zeta) \, d\zeta, \tag{37}$$

where

$$\|\varphi\|_{BMO} := \sup_{B \in \mathbb{B}} \frac{1}{|B|} \int_B |\varphi(\zeta) - \varphi_B| \, d\zeta < \infty. \tag{38}$$

Rewriting (37) as $\|\chi_{\{\xi\in B:|\varphi-\varphi_B|\geq \mathfrak{t}\}}\|_{L^1} \leq C_1 e^{-C_2\mathfrak{t}/\|\varphi\|_{BMO}}\|\chi_B\|_{L^1}$ exhibits it as a norm inequality on $L^1(\mathbb{R}^\nu)$. Ho [44] replaced L^1 by the variable exponent Lebesgue norm $L^{\mathrm{p}(\cdot)}(\mathbb{R}^\nu)$, $\mathrm{p}(\cdot)\in LH(\mathbb{R}^\nu)$, $\mathrm{p}^+<\infty$:

$$\|\chi_{\{\xi\in B:|\varphi-\varphi_B|\geq \mathfrak{t}\}}\|_{L^{\mathrm{p}(\cdot)}(\mathbb{R}^\nu)} \leq C_1 e^{-C_2\mathfrak{t}/\|\varphi\|_{BMO}}\|\chi_B\|_{L^{\mathrm{p}(\cdot)}(\mathbb{R}^\nu)}. \tag{39}$$

Below we replace the norm $\|\cdot\|_{L^{\mathrm{p}(\cdot)}(\mathbb{R}^\nu)}$ in (39) by the norm of the variable exponent Fofana space $(L^{\mathrm{p}(\cdot)},L^\eta)^\vartheta(\mathbb{R}^\nu)$ of Definition 1. We collect the necessary tools in this subsection, following the atomic Hardy-space method of [44,71–73].

2.6.1. Convexification of Fofana Spaces

For a Banach function space $\mathcal{X}$ on $\mathbb{R}^\nu$ and $\mathfrak{r}\geq 1$, the $\mathfrak{r}$-*convexification* $\mathcal{X}^\mathfrak{r}$ is

$$\mathcal{X}^\mathfrak{r}=\{\varphi\in\mathcal{M}:|\varphi|^\mathfrak{r}\in\mathcal{X}\},\qquad \|\varphi\|_{\mathcal{X}^\mathfrak{r}}=\big\||\varphi|^\mathfrak{r}\big\|_{\mathcal{X}}^{1/\mathfrak{r}}, \tag{40}$$

$\mathcal{M}$ the set of measurable functions on $\mathbb{R}^\nu$; $\mathcal{X}^\mathfrak{r}$ is again a Banach function space with respect to $\|\cdot\|_{\mathcal{X}^\mathfrak{r}}$ [74]. For plain variable exponent Lebesgue spaces, $\big(L^{\mathrm{p}(\cdot)}(\mathbb{R}^\nu)\big)^\mathfrak{r}=L^{\mathfrak{r}\mathrm{p}(\cdot)}(\mathbb{R}^\nu)$, with $\|\varphi\|_{(L^{\mathrm{p}(\cdot)})^\mathfrak{r}}=\|\varphi\|_{L^{\mathfrak{r}\mathrm{p}(\cdot)}}$ [44].

Lemma 9. *Let $\eta\in[1,\infty)$, $\mathrm{p}(\cdot)\in\mathcal{P}(\mathbb{R}^\nu)\cap LH(\mathbb{R}^\nu)$, $1<\mathrm{p}^-\leq\mathrm{p}^+\leq\eta<\infty$, $\vartheta\in\mathcal{G}_{\mathrm{p}^+,\eta}$, and $\mathfrak{r}\geq 1$. Then*

$$\Big[(L^{\mathrm{p}(\cdot)},L^\eta)^\vartheta(\mathbb{R}^\nu)\Big]^\mathfrak{r}=(L^{\mathfrak{r}\mathrm{p}(\cdot)},L^{\mathfrak{r}\eta})^{\vartheta^{1/\mathfrak{r}}}(\mathbb{R}^\nu),\qquad \|\varphi\|_{[(L^{\mathrm{p}(\cdot)},L^\eta)^\vartheta]^\mathfrak{r}}=\|\varphi\|_{(L^{\mathfrak{r}\mathrm{p}(\cdot)},L^{\mathfrak{r}\eta})^{\vartheta^{1/\mathfrak{r}}}(\mathbb{R}^\nu)}. \tag{41}$$

Proof. Write $N(\xi,\rho):=\|\varphi\chi_{B(\xi,\rho)}\|_{L^{\mathfrak{r}\mathrm{p}(\cdot)}(\mathbb{R}^\nu)}$, so $\||\varphi|^\mathfrak{r}\chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)}=N(\xi,\rho)^\mathfrak{r}$ by the plain-exponent convexification recalled above.

By Definition 1 applied to $|\varphi|^\mathfrak{r}$,

$$\|\varphi\|_{[(L^{\mathrm{p}(\cdot)},L^\eta)^\vartheta]^\mathfrak{r}}^\mathfrak{r}=\sup_{\rho>0}\frac{1}{\vartheta(\rho)}\left\{\int_{\mathbb{R}^\nu}\Big(\rho^{-\nu/\mathrm{p}(\xi)-\nu/\eta}N(\xi,\rho)^\mathfrak{r}\Big)^\eta d\xi\right\}^{1/\eta}. \tag{42}$$

Raising to the power $1/\mathfrak{r}$ and regrouping exponents, using

$$\big(\rho^{-\nu/\mathrm{p}(\xi)-\nu/\eta}\big)^{\eta/\mathfrak{r}}=\big(\rho^{-\nu/(\mathfrak{r}\mathrm{p}(\xi))-\nu/(\mathfrak{r}\eta)}\big)^\eta,$$

we obtain

$$\|\varphi\|_{[(L^{\mathrm{p}(\cdot)},L^\eta)^\vartheta]^\mathfrak{r}}=\sup_{\rho>0}\frac{1}{\vartheta(\rho)^{1/\mathfrak{r}}}\left\{\int_{\mathbb{R}^\nu}\Big(\rho^{-\nu/(\mathfrak{r}\mathrm{p}(\xi))-\nu/(\mathfrak{r}\eta)}N(\xi,\rho)\Big)^{\mathfrak{r}\eta}d\xi\right\}^{1/(\mathfrak{r}\eta)}, \tag{43}$$

which is exactly $\|\varphi\|_{(L^{\mathfrak{r}\mathrm{p}(\cdot)},L^{\mathfrak{r}\eta})^{\vartheta^{1/\mathfrak{r}}}(\mathbb{R}^\nu)}$ by Definition 1 with exponents $\mathfrak{r}\,\mathrm{p}(\cdot),\mathfrak{r}\eta$ and weight $\vartheta^{1/\mathfrak{r}}\in\mathcal{G}_{(\mathfrak{r}\mathrm{p})^+,\mathfrak{r}\eta}$ (the class membership of $\vartheta^{1/\mathfrak{r}}$ follows from that of ϑ since $t\mapsto t^{1/\mathfrak{r}}$ is increasing). □

2.6.2. Norm of a Ball Indicator

Lemma 10. *Under the hypotheses of Definition 1, for every $B=B(\xi_0,R)\in\mathbb{B}$,*

$$\|\chi_B\|_{(L^{\mathrm{p}(\cdot)},L^\eta)^\vartheta(\mathbb{R}^\nu)}\approx\vartheta(R)^{-1}, \tag{44}$$

with implicit constants depending only on ν and the $LH(\mathbb{R}^\nu)$ constants of $\mathrm{p}(\cdot)$ and the $\mathcal{G}_{\mathrm{p}^+,\eta}$ constants of ϑ.

Proof. *Lower bound.* By Lemma 7 applied to $\varphi = \chi_B$ at $\xi_0, \rho = R$, and $\chi_B \chi_{B(\xi_0,R)} = \chi_B$, together with Lemma 6 ($\|\chi_B\|_{\mathrm{p}(\cdot)} \approx R^{\nu/\mathrm{p}(\xi_0)}$),

$$R^{\nu/\mathrm{p}(\xi_0)} \approx \|\chi_B\|_{\mathrm{p}(\cdot)} \le C\vartheta(R) R^{\nu/\mathrm{p}(\xi_0)} \|\chi_B\|_{(L^{\mathrm{p}(\cdot)}, L^\eta)^\vartheta(\mathbb{R}^\nu)}$$
$$\Longrightarrow \quad \|\chi_B\|_{(L^{\mathrm{p}(\cdot)}, L^\eta)^\vartheta(\mathbb{R}^\nu)} \ge \frac{c}{\vartheta(R)}. \tag{45}$$

Upper bound. Fix $\rho > 0$ in Definition 1, and split

$$\int_{\mathbb{R}^\nu} = \int_{|\xi - \xi_0| < 2R+\rho} + \int_{|\xi-\xi_0| \ge 2R+\rho};$$

the second integral vanishes since $B(\xi, \rho) \cap B = \emptyset$ there.

For $|\xi - \xi_0| < 2R + \rho$,

$$\|\chi_B \chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)} \le \|\chi_B\|_{\mathrm{p}(\cdot)} \approx R^{\nu/\mathrm{p}(\xi_0)}$$

by Lemma 6, since $\mathrm{p}(\cdot) \in LH(\mathbb{R}^\nu)$ gives $\mathrm{p}(\xi) \approx \mathrm{p}(\xi_0)$ on the ball of radius $\lesssim R + \rho$, once combined with Lemma 5.

Case $\rho \le R$. The integration region has measure $\lesssim R^\nu$, so

$$\int_{\mathbb{R}^\nu} \left(\rho^{-\nu/\mathrm{p}(\xi) - \nu/\eta} \|\chi_B \chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)}\right)^\eta d\xi \lesssim R^\nu \left(\rho^{-\nu/\mathrm{p}(\xi_0) - \nu/\eta} R^{\nu/\mathrm{p}(\xi_0)}\right)^\eta$$
$$= R^\nu \rho^{-\eta\nu/\mathrm{p}(\xi_0) - \nu} R^{\eta\nu/\mathrm{p}(\xi_0)}. \tag{46}$$

Directly bounding the supremum over $\rho \le R$ of $\rho^{-\nu} \cdot (R/\rho)^{\eta\nu/\mathrm{p}(\xi_0)}$ is handled by the almost-increasing property of $\rho \mapsto \vartheta(\rho)\rho^{\nu/\mathrm{p}^+}$ in $\mathcal{G}_{\mathrm{p}^+,\eta}$, which after raising to the η-th power and comparing gives, upon dividing by $\vartheta(\rho)^\eta$ and taking η-th roots, a bound $\lesssim \vartheta(R)^{-1}$ uniformly in $\rho \le R$.

Case $\rho > R$. The region has measure $\lesssim \rho^\nu$, and an entirely analogous computation using the almost-decreasing property of $\rho \mapsto \vartheta(\rho)\rho^{\nu/\eta}$ gives the same bound $\lesssim \vartheta(R)^{-1}$.

Taking $\sup_{\rho>0}$ gives (44). □

2.6.3. Boundedness of M with Explicit Constant, and Its Associate-Space Consequence

Theorem 11. *Let* $\mathrm{p}(\cdot) \in \mathcal{P}(\mathbb{R}^\nu) \cap LH(\mathbb{R}^\nu)$, $\mathrm{p}^- > 1$. *There is* $\mathcal{K} > 0$, *depending only on* ν *and the* $LH(\mathbb{R}^\nu)$ *constants of* $\mathrm{p}(\cdot)$, *such that*

$$\|M\varphi\|_{L^{\mathrm{p}(\cdot)}(\mathbb{R}^\nu)} \le \mathcal{K}(\mathrm{p}^-)' \|\varphi\|_{L^{\mathrm{p}(\cdot)}(\mathbb{R}^\nu)}, \qquad \varphi \in L^{\mathrm{p}(\cdot)}(\mathbb{R}^\nu). \tag{47}$$

Proof. This is Diening's theorem (Thm. 4.4, [68]), restated below with new assumptions; see also (Thm. 2.3, [44]). □

Corollary 1. *Let* $\mathrm{p}(\cdot) \in \mathcal{P}(\mathbb{R}^\nu) \cap LH(\mathbb{R}^\nu)$, $\mathrm{p}^+ < \infty$, $\mathfrak{r} \ge 1$. *There is* $\mathcal{K} > 0$, *independent of* $\mathfrak{r}$, *such that*

$$\|M\varphi\|_{L^{(\mathfrak{r}\mathrm{p}(\cdot))'}(\mathbb{R}^\nu)} \le \mathcal{K}\,\mathfrak{r}\,\mathrm{p}^+ \|\varphi\|_{L^{(\mathfrak{r}\mathrm{p}(\cdot))'}(\mathbb{R}^\nu)}, \qquad \varphi \in L^{(\mathfrak{r}\mathrm{p}(\cdot))'}(\mathbb{R}^\nu). \tag{48}$$

Proof. Write $\mathrm{q}(\cdot) := (\mathfrak{r}\,\mathrm{p}(\cdot))'$; since $x \mapsto x' = x/(x-1)$ is decreasing, $\mathrm{q}^- = (\mathfrak{r}\mathrm{p}^+)' = \frac{\mathfrak{r}\mathrm{p}^+}{\mathfrak{r}\mathrm{p}^+ - 1} > 1$ and, since $x'' = x$, $(\mathrm{q}^-)' = \mathfrak{r}\mathrm{p}^+$. Also $\mathrm{q}(\cdot) \in LH(\mathbb{R}^\nu)$ with constants independent of $\mathfrak{r}$, since conjugation is a fixed algebraic operation on $1/(\mathfrak{r}\mathrm{p}(\cdot)) \in LH(\mathbb{R}^\nu)$. Applying Theorem 11 to $\mathrm{q}(\cdot)$ gives (48) with $\mathcal{K}$ the constant from (47). □

2.6.4. Atoms and the Hardy Space $H^1(\mathbb{R}^\nu)$

Definition 3. *Let $\mathcal{X}$ be a Banach function space on $\mathbb{R}^\nu$. A function Ψ is a $(1,\mathcal{X})$-*atom *if there is $B(\xi_0,R)\in\mathbb{B}$ such that:*

(i) $\operatorname{supp}\Psi\subset 3B=B(\xi_0,3R)$;

(ii) $\displaystyle\int_{\mathbb{R}^\nu}\Psi(\xi)\,d\xi=0$;

(iii) $\|\Psi\|_{\mathcal{X}}\le\|\chi_B\|_{\mathcal{X}}|B|^{-1}$.

We call B the ball associated with Ψ, and write $\mathcal{A}_{1,\mathcal{X}}$ for the set of $(1,\mathcal{X})$-atoms.

Definition 4. *Let $\Phi\in\mathcal{S}(\mathbb{R}^\nu)$, $\int_{\mathbb{R}^\nu}\Phi(\xi)\,d\xi\neq 0$. The Hardy space $H^1(\mathbb{R}^\nu)$ consists of $\varphi\in\mathcal{S}'(\mathbb{R}^\nu)$ with*

$$\|\varphi\|_{H^1}=\|M_\Phi\varphi\|_{L^1(\mathbb{R}^\nu)}<\infty,\qquad M_\Phi\varphi(\xi):=\sup_{t>0}|(\varphi*\Phi_t)(\xi)|,\ \Phi_t(\xi):=t^{-\nu}\Phi(\xi/t).\tag{49}$$

Lemma 11. *Let $\mathfrak{r}\ge 1$, $\mathfrak{p}(\cdot)\in LH(\mathbb{R}^\nu)$, $\mathfrak{p}^+<\infty$. For every $\big(1,(L^{\mathfrak{r}\mathfrak{p}(\cdot)}(\mathbb{R}^\nu))'\big)$-atom Ψ,*

$$\|\Psi\|_{H^1}\le C\mathfrak{r}\mathfrak{p}^+\tag{50}$$

for some $C>0$ independent of Ψ and $\mathfrak{r}$.

Proof. Suppose $\operatorname{supp}\Psi\subset B=B(\xi_0,3h)$, $h>0$; without loss of generality $\xi_0=0$. Let $\Phi\in\mathcal{S}(\mathbb{R}^\nu)$, $\operatorname{supp}\Phi\subset B(0,1)$, $\int\Phi=1$, which is consistent with Definition 4 (a compactly supported smooth bump normalized to have unit integral is a valid choice of test function there). The cancellation exploited below comes entirely from the vanishing moment of the atom, $\int\Psi=0$ (Definition 3(ii)), via the Taylor expansion of $\Phi_t(\xi-\zeta)-\Phi_t(\xi)$; it does not require Φ itself to have zero mean. Since $\mathfrak{p}^+<\infty$, $M_\Phi\varphi\le CM\varphi$ for some $C>0$, so, writing $\mathfrak{q}(\cdot)=(\mathfrak{r}\mathfrak{p}(\cdot))'$, Corollary 1 gives

$$\|M_\Phi\varphi\|_{L^{\mathfrak{q}(\cdot)}(\mathbb{R}^\nu)}\le C\mathfrak{r}\mathfrak{p}^+\|\varphi\|_{L^{\mathfrak{q}(\cdot)}(\mathbb{R}^\nu)},\qquad\varphi\in L^{\mathfrak{q}(\cdot)}(\mathbb{R}^\nu).\tag{51}$$

Split

$$\|\Psi\|_{H^1}=\|M_\Phi\Psi\|_{L^1}\le 2\big(\|\chi_{2B}M_\Phi\Psi\|_{L^1}+\|(1-\chi_{2B})M_\Phi\Psi\|_{L^1}\big)=:2(I+II).$$

Estimate of I. By (10) (applied to $\mathfrak{q}(\cdot)$ and $\mathfrak{r}\mathfrak{p}(\cdot)$, mutually conjugate), (51), and Lemma 6,

$$\begin{aligned}I&\le 4\|M_\Phi\Psi\|_{L^{\mathfrak{q}(\cdot)}}\|\chi_{2B}\|_{L^{\mathfrak{q}(\cdot)}}\\&\le C\mathfrak{r}\mathfrak{p}^+\|\Psi\|_{L^{\mathfrak{q}(\cdot)}}\|\chi_{2B}\|_{L^{\mathfrak{q}(\cdot)}}\\&\le C\mathfrak{r}\mathfrak{p}^+\|\Psi\|_{L^{\mathfrak{q}(\cdot)}}\|\chi_B\|_{L^{\mathfrak{q}(\cdot)}},\end{aligned}\tag{52}$$

using $\|\chi_{2B}\|_{L^{\mathfrak{q}(\cdot)}}\lesssim\|\chi_B\|_{L^{\mathfrak{q}(\cdot)}}$ (Lemma 6 at comparable radii). By Definition 3(iii) with $\mathcal{X}=L^{\mathfrak{q}(\cdot)}=(L^{\mathfrak{r}\mathfrak{p}(\cdot)})'$,

$$\begin{aligned}I&\le C\mathfrak{r}\mathfrak{p}^+\|\chi_B\|_{L^{\mathfrak{q}(\cdot)}}|B|^{-1}\|\chi_B\|_{L^{\mathfrak{q}(\cdot)}}\\&=C\mathfrak{r}\mathfrak{p}^+\frac{\|\chi_B\|_{L^{\mathfrak{q}(\cdot)}}^2}{|B|}\\&\le C\mathfrak{r}\mathfrak{p}^+,\end{aligned}\tag{53}$$

the last step by the plain-exponent analogue of Lemma 10 at $\eta = \infty$, namely $\|\chi_B\|^2_{L^{\mathfrak{q}(\cdot)}} \lesssim |B|$ (Prop. 2.2, [44]), with constants independent of $\mathfrak{r}$ since the $LH(\mathbb{R}^\nu)$ constants of $\mathfrak{q}(\cdot)$ are independent of $\mathfrak{r}$ (Corollary 1).

Estimate of *II*. For $\xi \notin 2B$, the vanishing moment (Definition 3(ii)) and Taylor's formula for Φ_t give, for every $N > 0$,

$$\begin{aligned} |(\Psi * \Phi_t)(\xi)| &= \left| \int_{3B} \Psi(\zeta) \big(\Phi_t(\xi - \zeta) - \Phi_t(\xi)\big)\, d\zeta \right| \\ &\le t^{-\nu} \int_{3B} |\Psi(\zeta)| \frac{C_N |\zeta/t|}{(1 + |\xi/t|)^N}\, d\zeta \\ &\le \frac{C_N t^{-(1+\nu)}}{(1 + t^{-1}|\xi|)^N} \int_{3B} |\Psi(\zeta)||\zeta|\, d\zeta. \end{aligned} \tag{54}$$

Since B is centered at 0, $|\zeta| \lesssim |B|^{1/\nu}$ on $3B$; by Hölder's inequality (10) and Definition 3(iii),

$$\begin{aligned} \sup_{t>0} |(\Psi * \Phi_t)(\xi)| &\le C_N \frac{t^{-(1+\nu)} |B|^{1/\nu}}{(1 + t^{-1}|\xi|)^N} \|\Psi\|_{L^{\mathfrak{q}(\cdot)}} \|\chi_B\|_{\mathfrak{r}\mathfrak{p}(\cdot)} \\ &\le C_N \frac{|B|^{1/\nu}}{|\xi|^{1+\nu}} \end{aligned} \tag{55}$$

after optimizing $t \sim |\xi|$ and taking $N > 1 + \nu$, using

$$\|\Psi\|_{L^{\mathfrak{q}(\cdot)}} \|\chi_B\|_{\mathfrak{r}\mathfrak{p}(\cdot)} \le \|\chi_B\|_{L^{\mathfrak{q}(\cdot)}} |B|^{-1} \|\chi_B\|_{\mathfrak{r}\mathfrak{p}(\cdot)} \lesssim 1$$

(Hölder duality pairing bound, plain-exponent case).

Writing $h = 2^{\mathfrak{a}}$, $\mathfrak{a} \in \mathbb{Z}$, and integrating in ξ over dyadic shells $|\xi| \sim 2^j$, $j \ge \mathfrak{a}$,

$$II \le C 2^{\mathfrak{a}} \sum_{j=\mathfrak{a}}^{\infty} \frac{2^{j\nu}}{2^{j(1+\nu)}} \le C \tag{56}$$

for some $C > 0$ independent of $\Psi, \mathfrak{r}$. Combining (53) and (56) gives (50). □

3. Main Results

In this section we prove the following results: boundedness of $T_{\gamma,\beta}$ and $M_{\gamma,\beta}$ on variable exponent Fofana spaces on $\mathbb{R}^\nu$, with a norm equivalence between them, and with β entering non-trivially throughout (Section 3.1); boundedness of M_σ and $T_{\sigma,\beta}$ on variable exponent Fofana spaces on a bounded domain (Section 3.2); the Stein–Weiss inequality for $T_{\gamma,\beta}$ in variable exponent Fofana spaces, together with further corollaries, remarks, and an explicit example (Section 3.3); and, as a companion space-theoretic result, John–Nirenberg inequalities for $BMO(\mathbb{R}^\nu)$ in terms of the norm of variable exponent Fofana spaces on $\mathbb{R}^\nu$ (Section 3.4).

3.1. Norm Inequalities for the Bessel–Riesz Operator on Variable Exponent Fofana Spaces

We now establish the norm inequalities of this subsection with a genuinely variable inner exponent $\mathfrak{p}(\cdot)$ on $\mathbb{R}^\nu$, using Section 2.4; unlike Section 3.2 below, where $T_{\sigma,\beta}$ is bounded via the trivial pointwise domination $T_{\sigma,\beta}\varphi \le I_\sigma \varphi$, here the decay parameter β enters non-trivially, exactly as it did in the constant-exponent setting, and we track this explicitly.

Theorem 12. *Fix $\eta \in [1, \infty)$. Let $\mathfrak{p}(\cdot) \in \mathcal{P}(\mathbb{R}^\nu) \cap LH(\mathbb{R}^\nu)$, and suppose the following:*

(i) $1 < \mathfrak{p}^- \le \mathfrak{p}^+ \le \eta < \infty$;

(ii) $0 < \gamma < \dfrac{\nu}{\mathrm{p}^+}$, $\beta \geq 0$;

(iii) $\lambda \in \mathbb{R}$ *with* $-\dfrac{\nu}{\mathrm{p}^+} \leq \lambda < \beta - \gamma$; *set* $\mu := \lambda - \beta < -\gamma$;

(iv) $\vartheta \in \mathcal{G}_{\mathrm{p}^+,\eta}$ *satisfies* (28) *and* $\vartheta(\rho) \lesssim \rho^{\lambda}$, $\forall \rho > 0$.

Put

$$t := 1 + \frac{\gamma}{\mu} \in (0,1), \qquad \frac{1}{\bar{\mathrm{q}}(\xi)} = \frac{t}{\mathrm{p}(\xi)} \ (\xi \in \mathbb{R}^{\nu}), \qquad \frac{1}{\bar{\eta}} = \frac{t}{\eta}, \qquad \Theta(\rho) = \big(\vartheta(\rho)\big)^{t}. \tag{57}$$

Then $\|T_{\gamma,\beta}\varphi\|_{(L^{\bar{\mathrm{q}}(\cdot)},L^{\bar{\eta}})^{\Theta}(\mathbb{R}^{\nu})} \lesssim \|\varphi\|_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})}$ *for all* $\varphi \in (L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})$.

Remark 2. *Since* $\mu < -\gamma < 0$, $\gamma/\mu \in (-1,0)$, *so* $t \in (0,1)$; *hence* $\bar{\mathrm{q}}(\xi) = \mathrm{p}(\xi)/t > \mathrm{p}(\xi)$ *and* $\bar{\eta} = \eta/t > \eta$, *and, since* $\mathrm{p}(\xi) \leq \mathrm{p}^+ \leq \eta$ *for a.e.* ξ, *dividing by the same* t *gives* $\bar{\mathrm{q}}(\xi) \leq \bar{\eta}$ *automatically, so* $(L^{\bar{\mathrm{q}}(\cdot)},L^{\bar{\eta}})^{\Theta}(\mathbb{R}^{\nu})$ *is well defined by Definition 1. Also* $\bar{\mathrm{q}}(\cdot) \in LH(\mathbb{R}^{\nu})$, *being a constant multiple of* $\mathrm{p}(\cdot)$.

Proof of Theorem 12. Let $\varphi \in (L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu}) \setminus \{0\}$, $\rho > 0$, $\xi \in \mathbb{R}^{\nu}$, $\rho_{\iota} := 2^{\iota}\rho$.

For $\rho \leq 1$, write $T_{\gamma,\beta}\varphi(\xi) = A(\xi) + A'(\xi) + B(\xi)$,

$$A(\xi) = \int_{B(\xi,\rho)} \frac{\varphi(\zeta)\,d\zeta}{|\xi-\zeta|^{\nu-\gamma}(1+|\xi-\zeta|)^{\beta}}, \tag{58}$$

$$A'(\xi) = \int_{B(\xi,1)\setminus B(\xi,\rho)} \frac{\varphi(\zeta)\,d\zeta}{|\xi-\zeta|^{\nu-\gamma}(1+|\xi-\zeta|)^{\beta}}, \tag{59}$$

$$B(\xi) = \int_{\mathbb{R}^{\nu}\setminus B(\xi,1)} \frac{\varphi(\zeta)\,d\zeta}{|\xi-\zeta|^{\nu-\gamma}(1+|\xi-\zeta|)^{\beta}}. \tag{60}$$

On $B(\xi,\rho)$, $|\xi-\zeta| < \rho \leq 1$ gives $(1+|\xi-\zeta|)^{-\beta} \leq 1$, so

$$|A(\xi)| \leq \int_{B(\xi,\rho)} \frac{|\varphi(\zeta)|}{|\xi-\zeta|^{\nu-\gamma}} d\zeta \leq C\rho^{\gamma} M\varphi(\xi). \tag{61}$$

For $|\xi-\zeta| \geq \rho$, on each shell $\rho_{\iota} \leq |\xi-\zeta| < \rho_{\iota+1}$, $(1+|\xi-\zeta|)^{-\beta} \leq \rho_{\iota}^{-\beta} \wedge 1$; by Hölder's inequality (10) at the fixed point ξ, Lemma 6 applied to $\mathrm{p}'(\cdot)$, and Lemma 7,

$$\begin{aligned}
\int_{\rho_{\iota}\leq|\xi-\zeta|<\rho_{\iota+1}} |\varphi(\zeta)|\,d\zeta &\leq \|\varphi\chi_{B(\xi,\rho_{\iota+1})}\|_{\mathrm{p}(\cdot)} \|\chi_{B(\xi,\rho_{\iota+1})}\|_{\mathrm{p}'(\cdot)} \\
&\leq C\rho_{\iota+1}^{\nu/\mathrm{p}'(\xi)} \|\varphi\chi_{B(\xi,\rho_{\iota+1})}\|_{\mathrm{p}(\cdot)} \\
&\leq C\vartheta(\rho_{\iota+1})\rho_{\iota+1}^{\nu/\mathrm{p}'(\xi)+\nu/\mathrm{p}(\xi)} \|\varphi\|_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})} \\
&= C\vartheta(\rho_{\iota+1})\rho_{\iota+1}^{\nu} \|\varphi\|_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})},
\end{aligned} \tag{62}$$

using $1/\mathrm{p}(\xi) + 1/\mathrm{p}'(\xi) = 1$. Hence, using $\vartheta(\varsigma) \lesssim \varsigma^{\lambda}$,

$$\begin{aligned}
|A'(\xi) + B(\xi)| &\leq C\|\varphi\|_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})} \sum_{\iota=0}^{\infty} \rho_{\iota}^{-\nu+\gamma}\big(\rho_{\iota}^{-\beta} \wedge 1\big)\rho_{\iota+1}^{\nu+\lambda} \\
&\leq C\|\varphi\|_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})} \sum_{\iota=0}^{\infty} \rho_{\iota}^{\gamma+\lambda}\big(\rho_{\iota}^{-\beta} \wedge 1\big).
\end{aligned} \tag{63}$$

With $\iota_0 = \max\{\iota \geq 0 : \rho_{\iota} \leq 1\}$, split

$$\Sigma_1 = \sum_{\iota=0}^{\iota_0} \rho_{\iota}^{\gamma+\lambda}, \qquad \Sigma_2 = \sum_{\iota>\iota_0} \rho_{\iota}^{\gamma+\lambda-\beta} :$$

since $\mu \le \lambda$ ($\beta \ge 0$), $\rho^{\gamma+\lambda} \le \rho^{\gamma+\mu}$ for $\rho \le 1$, so $\Sigma_1 \lesssim \rho^{\gamma+\mu}$; since $\gamma + \mu < 0$, $\Sigma_2 \lesssim \rho_{\iota_0}^{\gamma+\mu} \lesssim 1 \lesssim \rho^{\gamma+\mu}$ ($\rho \le 1$). Hence,

$$|A'(\xi) + B(\xi)| \le C\rho^{\gamma+\mu}\|\varphi\|_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})}. \tag{64}$$

For $\rho > 1$, only shells $\rho_\iota \ge 1$ occur outside $B(\xi,\rho)$, giving $\sum_\iota \rho_\iota^{\gamma+\lambda}(\rho_\iota^{-\beta} \wedge 1) \lesssim \rho^{\gamma+\mu}$ directly; (61)–(64) hold for every $\rho > 0$.

Combining (61) and (64),

$$|T_{\gamma,\beta}\varphi(\xi)| \le C\big(\rho^{\gamma}M\varphi(\xi) + \rho^{\gamma+\mu}\|\varphi\|_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})}\big), \qquad \forall \rho > 0.$$

Taking $\rho = \big(\|\varphi\|_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})}/M\varphi(\xi)\big)^{1/\mu}$,

$$|T_{\gamma,\beta}\varphi(\xi)| \le C\big(M\varphi(\xi)\big)^{t}\|\varphi\|^{-\gamma/\mu}_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})}, \qquad \xi \in \mathbb{R}^{\nu}, \tag{65}$$

recalling $t = 1 + \gamma/\mu$.

Since t is a constant (not depending on ξ), for any measurable $g \ge 0$ and $\xi_0 \in \mathbb{R}^{\nu}, \rho > 0$,

$$\|g^{t}\chi_{B(\xi_0,\rho)}\|_{\bar{\mathrm{q}}(\cdot)} = \|g\chi_{B(\xi_0,\rho)}\|^{t}_{t\bar{\mathrm{q}}(\cdot)} = \|g\chi_{B(\xi_0,\rho)}\|^{t}_{\mathrm{p}(\cdot)},$$

using $t\bar{\mathrm{q}}(\xi) = \mathrm{p}(\xi)$. Applying this with $g = M\varphi$ to (65),

$$\|(T_{\gamma,\beta}\varphi)\chi_{B(\xi_0,\rho)}\|_{\bar{\mathrm{q}}(\cdot)} \le C\|\varphi\|^{-\gamma/\mu}_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})}\|(M\varphi)\chi_{B(\xi_0,\rho)}\|^{t}_{\mathrm{p}(\cdot)}, \qquad \xi_0 \in \mathbb{R}^{\nu},\ \rho > 0. \tag{66}$$

Multiplying (66) by $\rho^{-\nu/\bar{\mathrm{q}}(\xi_0)-\nu/\bar{\eta}}/\Theta(\rho)$ and using $\nu/\bar{\mathrm{q}}(\xi_0) + \nu/\bar{\eta} = t(\nu/\mathrm{p}(\xi_0) + \nu/\eta)$, $\Theta(\rho) = \vartheta(\rho)^{t}$,

$$\frac{\rho^{-\nu/\bar{\mathrm{q}}(\xi_0)-\nu/\bar{\eta}}}{\Theta(\rho)}\|(T_{\gamma,\beta}\varphi)\chi_{B(\xi_0,\rho)}\|_{\bar{\mathrm{q}}(\cdot)} \le C\|\varphi\|^{-\gamma/\mu}_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})} \times \left(\frac{\rho^{-\nu/\mathrm{p}(\xi_0)-\nu/\eta}}{\vartheta(\rho)}\|(M\varphi)\chi_{B(\xi_0,\rho)}\|_{\mathrm{p}(\cdot)}\right)^{t}. \tag{67}$$

Raising to the power $\bar{\eta}$, integrating $d\xi_0$ over $\mathbb{R}^{\nu}$, and using $t\bar{\eta} = \eta$,

$$\int_{\mathbb{R}^{\nu}}\left(\frac{\rho^{-\nu/\bar{\mathrm{q}}(\xi_0)-\nu/\bar{\eta}}}{\Theta(\rho)}\|(T_{\gamma,\beta}\varphi)\chi_{B(\xi_0,\rho)}\|_{\bar{\mathrm{q}}(\cdot)}\right)^{\bar{\eta}} d\xi_0 \le C\|\varphi\|^{-\gamma\bar{\eta}/\mu}_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})}\int_{\mathbb{R}^{\nu}}\left(\frac{\rho^{-\nu/\mathrm{p}(\xi_0)-\nu/\eta}}{\vartheta(\rho)}\|(M\varphi)\chi_{B(\xi_0,\rho)}\|_{\mathrm{p}(\cdot)}\right)^{\eta} d\xi_0. \tag{68}$$

Taking the $\bar{\eta}$-th root of the left side of the last display, and the η-th root of the integral on the right raised to the power $\eta/\bar{\eta} = t$, then $\sup_{\rho>0}$,

$$\|T_{\gamma,\beta}\varphi\|_{(L^{\bar{\mathrm{q}}(\cdot)},L^{\bar{\eta}})^{\Theta}(\mathbb{R}^{\nu})} \le C\|\varphi\|^{-\gamma/\mu}_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})}\|M\varphi\|^{t}_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})}. \tag{69}$$

By Theorem 9, $\|M\varphi\|_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})} \lesssim \|\varphi\|_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})}$, so

$$\|T_{\gamma,\beta}\varphi\|_{(L^{\bar{\mathrm{q}}(\cdot)},L^{\bar{\eta}})^{\Theta}(\mathbb{R}^{\nu})} \lesssim \|\varphi\|^{-\gamma/\mu}_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})}\|\varphi\|^{t}_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})} = \|\varphi\|_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})}, \quad \square \tag{70}$$

since $t - \gamma/\mu = 1 + \gamma/\mu - \gamma/\mu = 1$.

Corollary 2. *Under the hypotheses of Theorem 12,*

$$\|M_{\gamma,\beta}\varphi\|_{(L^{\bar{\mathrm{q}}(\cdot)},L^{\bar{\eta}})^{\Theta}(\mathbb{R}^\nu)} \lesssim \|\varphi\|_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^\nu)}, \qquad \varphi \in (L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^\nu).$$

Proof. By Lemma 4, $M_{\gamma,\beta}\varphi \le C_\nu T_{\gamma,\beta}(|\varphi|)$; apply Theorem 12 to $|\varphi|$. □

Remark 3. *At $\beta = 0$, $\mu = \lambda$ and $T_{\gamma,0} = I_\gamma$: Theorem 12 gives the mapping properties of the fractional integral on variable exponent Fofana spaces with admissible range $\lambda < -\gamma$; for $\beta > 0$ the range $\lambda < \beta - \gamma$ is strictly larger, exactly as in the constant-exponent case, with β entering identically (through $\mu = \lambda - \beta$) whether $\mathrm{p}(\cdot)$ is constant or variable. At $\mathrm{p}(\cdot) \equiv \mathrm{q}$ constant, $\bar{\mathrm{q}}(\cdot) \equiv \mathrm{q}/t =: \bar{\mathrm{q}}$ and $\bar{\eta} = \eta/t =: \bar{\mathrm{p}}$ are both constants satisfying $\frac{1}{\bar{\mathrm{q}}} = \frac{1}{\mathrm{q}}(1 + \gamma/\mu)$, $\frac{1}{\bar{\mathrm{p}}} = \frac{1}{\eta}(1 + \gamma/\mu)$, recovering the classical constant-exponent statement with η in place of the classical outer exponent* p.

Theorem 13. *Fix $\eta \in [1,\infty)$. Let $\mathrm{p}(\cdot) \in \mathcal{P}(\mathbb{R}^\nu) \cap LH(\mathbb{R}^\nu)$, $1 \le \mathrm{p}^- \le \mathrm{p}^+ \le \eta < \infty$, $0 < \gamma < \nu$, $\beta \ge 0$, and let $\vartheta \in \mathcal{G}_{\mathrm{p}^+,\eta}$ satisfy*

$$\int_\rho^\infty \vartheta(\varsigma)\varsigma^{\nu\left(\frac{1}{\mathrm{p}^+}+\frac{1}{\eta}\right)-1}d\varsigma \le C\vartheta(\rho)\rho^{\nu\left(\frac{1}{\mathrm{p}^+}+\frac{1}{\eta}\right)}, \qquad \forall \rho > 0. \tag{71}$$

Then, for every non-negative $\varphi \in L^{\mathrm{p}(\cdot)}_{\mathrm{loc}}(\mathbb{R}^\nu)$, $\|T_{\gamma,\beta}\varphi\|_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^\nu)} \approx \|M_{\gamma,\beta}\varphi\|_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^\nu)}$.

Proof. By Lemma 4, $\|M_{\gamma,\beta}\varphi\|_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^\nu)} \lesssim \|T_{\gamma,\beta}\varphi\|_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^\nu)}$. Fix $\rho > 0$, $\zeta \in \mathbb{R}^\nu$. By (Lemma 4.2, [75]), with weight $(1+\varsigma)^{-\beta}$ and $\mathrm{p}(\cdot)$ in place of a fixed exponent,

$$\begin{aligned}\|(T_{\gamma,\beta}\varphi)\chi_{B(\zeta,\rho)}\|_{\mathrm{p}(\cdot)} &\approx \|(M_{\gamma,\beta}\varphi)\chi_{B(\zeta,\rho)}\|_{\mathrm{p}(\cdot)} \\ &\quad + |B(\zeta,\rho)|^{1/\mathrm{p}(\zeta)}(1+\rho)^{-\beta}\int_{\mathbb{R}^\nu\setminus B(\zeta,\rho)}\frac{\varphi(\xi)}{|\xi-\zeta|^{\nu-\gamma}}d\xi.\end{aligned} \tag{72}$$

Decomposing dyadically over $B(\zeta,\rho_{\iota+1}) \setminus B(\zeta,\rho_\iota)$, $\iota \ge 0$, with $(1+\rho_\iota)^{-\beta} \le (1+\rho)^{-\beta}$, Lemma 6, and (Thm. 5.2, [75]) at ζ,

$$|B(\zeta,\rho)|^{1/\mathrm{p}(\zeta)}(1+\rho)^{-\beta}\int_{\mathbb{R}^\nu\setminus B(\zeta,\rho)}\frac{\varphi(\xi)}{|\xi-\zeta|^{\nu-\gamma}}d\xi \lesssim \sum_{\iota=0}^{\infty}(2^\iota)^{-\nu/\mathrm{p}^+}\|(M_{\gamma,\beta}\varphi)\chi_{B(\zeta,\rho_{\iota+1})}\|_{\mathrm{p}(\cdot)}. \tag{73}$$

Raising (72) to power η, dividing by $\vartheta(\rho)^\eta\rho^{\eta(\nu/\mathrm{p}(\zeta)+\nu/\eta)}$, integrating $d\zeta$, and applying (16)–(17) (p^+,η for q,p) with (71), as in Theorem 9,

$$\begin{aligned}&\left\{\int_{\mathbb{R}^\nu}\left(\frac{\rho^{-\nu/\mathrm{p}(\zeta)-\nu/\eta}}{\vartheta(\rho)}\|(T_{\gamma,\beta}\varphi)\chi_{B(\zeta,\rho)}\|_{\mathrm{p}(\cdot)}\right)^\eta d\zeta\right\}^{1/\eta} \\ &\qquad \lesssim \|M_{\gamma,\beta}\varphi\|_{(L^{\mathrm{p}(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^\nu)};\end{aligned} \tag{74}$$

$\sup_{\rho>0}$ concludes. □

3.2. Boundedness of M_σ and $T_{\sigma,\beta}$ on Variable Exponent Fofana Spaces

Theorem 14. *Let $\Omega \subset \mathbb{R}^\nu$ be bounded, $\mathrm{p}(\cdot),\mathrm{q}(\cdot) \in \mathcal{P}(\Omega) \cap LH_0(\Omega)$, and suppose the following:*
(i) $1 < \mathrm{p}^- \le \mathrm{p}^+ < \infty$, $1 < \mathrm{q}^- \le \mathrm{q}^+ < \infty$, $\mathrm{p}(\xi) \le \mathrm{q}(\xi)$;

(ii) $\lambda(\cdot):\Omega\to[0,\nu)$, $0<\lambda^-\le\lambda^+<\nu$, $\lambda(\cdot)\in LH_0(\Omega)$, $\sup_{\xi\in\Omega}[\lambda(\xi)+\sigma\,\mathrm{p}(\xi)]<\nu$ *for fixed* $\sigma\in[0,\nu)$;
(iii) $\mathrm{q}(\cdot)$ *is defined by* $\frac{1}{\mathrm{q}(\xi)}=\frac{1}{\mathrm{p}(\xi)}-\frac{\sigma}{\nu-\lambda(\xi)}$;
(iv) $\eta\in[1,\infty)$ *is fixed, and* $\alpha_{\mathrm{p}}(\cdot),\alpha_{\mathrm{q}}(\cdot)$ *are given by* (30).

Then M_σ *is bounded from* $\left(L^{\mathrm{p}(\cdot)},L^\eta\right)^{\alpha_{\mathrm{p}}(\cdot)}(\Omega)$ *to* $\left(L^{\mathrm{q}(\cdot)},L^\eta\right)^{\alpha_{\mathrm{q}}(\cdot)}(\Omega)$; *the same holds for* I_σ, *and, since* $T_{\sigma,\beta}\varphi\le I_\sigma\varphi$ *pointwise for* $\varphi\ge0$, *for* $T_{\sigma,\beta}$, $\beta\ge0$.

Proof. The hypotheses (i)–(iii) match Theorem 10, so $M_\sigma, I_\sigma: M^{\lambda(\cdot)}_{\mathrm{p}(\cdot)}(\Omega)\to M^{\lambda(\cdot)}_{\mathrm{q}(\cdot)}(\Omega)$ are bounded by Theorem 10. Since Lemma 8 controls the Fofana norm by the Morrey norm only one way, boundedness on $\left(L^{\mathrm{p}(\cdot)},L^\eta\right)^{\alpha_{\mathrm{p}}(\cdot)}(\Omega)$ does not follow automatically and is reproved directly at the level of (29), via the same dyadic scheme as Theorem 10.

Fix $\zeta\in\Omega$, $\rho>0$, $\phi=I_\sigma\varphi$ (the argument for M_σ is identical and simpler, with no tail sum). For $\xi\in\widetilde{B}(\zeta,\rho)$,

$$\phi(\xi)=\int_{B(\xi,\rho)}\frac{\varphi(\zeta')}{|\xi-\zeta'|^{\nu-\sigma}}d\zeta'+\int_{\Omega\setminus B(\xi,\rho)}\frac{\varphi(\zeta')}{|\xi-\zeta'|^{\nu-\sigma}}d\zeta'=:\phi_1(\xi)+\phi_2(\xi).\tag{75}$$

By (10) and Lemma 2,

$$\begin{aligned}|\phi_1(\xi)|&\lesssim\rho^{\sigma-\nu}\|\varphi\chi_{\widetilde{B}(\xi,\rho)}\|_{\mathrm{p}(\cdot)}\|\chi_{B(\xi,\rho)}\|_{\mathrm{p}'(\cdot)}\\&\lesssim\rho^{\sigma-\nu+\nu/\mathrm{p}'(\xi)}\|\varphi\chi_{\widetilde{B}(\xi,\rho)}\|_{\mathrm{p}(\cdot)}.\end{aligned}\tag{76}$$

For ϕ_2, decompose $\Omega\setminus B(\xi,\rho)$ into dyadic shells $B(\xi,\rho_{\iota+1})\setminus B(\xi,\rho_\iota)$, $\iota\ge0$, insert $|B(\xi,\rho_{\iota+1})|^{\pm(1/\alpha_{\mathrm{p}}(\xi)-1/\mathrm{p}(\xi)-1/\eta)}$, and apply (11) as in (Thm. 2, [19]), combined with the dyadic summation of Theorem 12 ($\gamma\to\sigma$, $\vartheta\to$ the Fofana weight); this gives, for every $\xi\in\Omega$, $\rho>0$,

$$|B(\xi,\rho)|^{\frac{1}{\alpha_{\mathrm{q}}(\xi)}-\frac{1}{\mathrm{q}(\xi)}-\frac{1}{\eta}}\|\phi\chi_{\widetilde{B}(\xi,\rho)}\|_{\mathrm{q}(\cdot)}\lesssim\|\varphi\|_{\left(L^{\mathrm{p}(\cdot)},L^\eta\right)^{\alpha_{\mathrm{p}}(\cdot)}(\Omega)},\tag{77}$$

whence, taking the $L^\eta(d\xi)$-norm and $\sup_{\rho>0}$, $\|\phi\|_{\left(L^{\mathrm{q}(\cdot)},L^\eta\right)^{\alpha_{\mathrm{q}}(\cdot)}(\Omega)}\lesssim\|\varphi\|_{\left(L^{\mathrm{p}(\cdot)},L^\eta\right)^{\alpha_{\mathrm{p}}(\cdot)}(\Omega)}$. The statement for $T_{\sigma,\beta}$ follows from that for I_σ by monotonicity of (29) under pointwise domination. □

3.3. A Weighted Stein–Weiss Inequality in the Variable Exponent Fofana Space Setting

Here $\Omega\subset\mathbb{R}^\nu$ is bounded, $\xi_0\in\Omega$, and $T_{\gamma,\beta}\varphi(\xi)=\int_\Omega\varphi(\zeta)|\xi-\zeta|^{\gamma-\nu}(1+|\xi-\zeta|)^{-\beta}d\zeta$, $0<\gamma<\nu$, $\beta\ge0$.

Theorem 15. *Let* $\xi_0\in\mathbb{R}^\nu$, $\mathrm{p}(\cdot),\mathrm{q}(\cdot)\in\mathcal{P}(\mathbb{R}^\nu)\cap LH(\mathbb{R}^\nu)$, $\lambda(\cdot):\mathbb{R}^\nu\to[0,\nu]$, *and suppose the following:*

(i) $1<\mathrm{p}^-\le\mathrm{p}^+<\infty$, $1<\mathrm{q}^-\le\mathrm{q}^+<\infty$, $\mathrm{p}(\xi)\le\mathrm{q}(\xi)$;
(ii) $0<\lambda^-\le\lambda^+<\nu$, $\lambda(\cdot)\in LH(\mathbb{R}^\nu)$;
(iii) $\beta>0$, *and constants* κ,τ *satisfy*

$$0\le\tau-\kappa\le\gamma,\tag{78}$$

$$\lambda^+-\frac{\nu}{\mathrm{q}^+}<\kappa\le\tau<\frac{\nu}{(\mathrm{p}')^+},\tag{79}$$

$$0<\lambda^-\le\lambda(\xi)<\nu-(\gamma-\tau+\kappa)\mathrm{p}(\xi),\tag{80}$$

$$\frac{\gamma+\kappa-\tau}{\nu-\lambda(\xi)}=\frac{1}{\mathrm{p}(\xi)}-\frac{1}{\mathrm{q}(\xi)};\tag{81}$$

(iv) $\eta \in [1,\infty)$ *satisfies*

$$\eta > \frac{\nu\, \mathrm{q}^+}{\nu - \lambda^+}; \tag{82}$$

(v) $\alpha_{\mathrm{p}}(\cdot), \alpha_{\mathrm{q}}(\cdot)$ *are given by* (30).

Then there is $C > 0$ *such that*

$$\left\| |\xi - \xi_0|^{\kappa} T_{\gamma,\beta}\varphi \right\|_{\left(L^{\mathrm{q}(\cdot)}, L^{\eta}\right)^{\alpha_{\mathrm{q}}(\cdot)}(\mathbb{R}^{\nu})} \le C \left\| |\xi - \xi_0|^{\tau} \varphi \right\|_{\left(L^{\mathrm{p}(\cdot)}, L^{\eta}\right)^{\alpha_{\mathrm{p}}(\cdot)}(\mathbb{R}^{\nu})} \tag{83}$$

for every φ.

Proof. By homogeneity,

$$\left\| |\cdot - \xi_0|^{\tau} \varphi \right\|_{\left(L^{\mathrm{p}(\cdot)}, L^{\eta}\right)^{\alpha_{\mathrm{p}}(\cdot)}(\mathbb{R}^{\nu})} = 1; \tag{84}$$

by positivity of $T_{\gamma,\beta}$, $\varphi \ge 0$. By Lemma 8 and (84), for every $\rho > 0$,

$$\left\| \varphi \chi_{B(\xi_0,\rho)} (\cdot - \xi_0)^{\tau} \right\|_{\mathrm{p}(\cdot)} \lesssim \rho^{\lambda(\xi_0)/\mathrm{p}(\xi_0)}. \tag{85}$$

For $\xi \in \mathbb{R}^{\nu}$, write $r := |\xi - \xi_0|$ and

$$\begin{aligned} T_{\gamma,\beta}\varphi(\xi) = {} & \int_{B(\xi_0,\frac{r}{2})} \frac{\varphi(\zeta)\, d\zeta}{|\xi - \zeta|^{\nu-\gamma}(1 + |\xi - \zeta|)^{\beta}} + \int_{B(\xi_0,2r)\setminus B(\xi_0,\frac{r}{2})} \frac{\varphi(\zeta)\, d\zeta}{|\xi - \zeta|^{\nu-\gamma}(1 + |\xi - \zeta|)^{\beta}} \\ & + \int_{\mathbb{R}^{\nu}\setminus B(\xi_0,2r)} \frac{\varphi(\zeta)\, d\zeta}{|\xi - \zeta|^{\nu-\gamma}(1 + |\xi - \zeta|)^{\beta}} = Z_1(\xi) + Z_2(\xi) + Z_3(\xi). \end{aligned} \tag{86}$$

Estimate of Z_1. For $\zeta \in B(\xi_0, \frac{r}{2})$,

$$\frac{r}{2} \le |\xi - \zeta| \le \frac{3}{2} r, \tag{87}$$

so, since $(1 + \varsigma)^{-\beta}$ is decreasing,

$$\left(1 + \frac{3}{2} r\right)^{-\beta} \le (1 + |\xi - \zeta|)^{-\beta} \le \left(1 + \frac{r}{2}\right)^{-\beta} \le 2^{\beta}(1 + r)^{-\beta}. \tag{88}$$

Hence, with $c_{\beta} := 2^{\beta}$,

$$Z_1(\xi) \le c_{\beta}(1 + r)^{-\beta} \int_{B(\xi_0,\frac{r}{2})} \frac{\varphi(\zeta)}{|\xi - \zeta|^{\nu-\gamma}}\, d\zeta \le c_{\beta}(1 + r)^{-\beta} \left(\frac{r}{2}\right)^{\gamma-\nu} \int_{B(\xi_0,\frac{r}{2})} \varphi(\zeta)\, d\zeta. \tag{89}$$

Decompose $B(\xi_0, \frac{r}{2}) = \bigcup_{\iota=1}^{\infty} A_{\iota}$, $A_{\iota} := B(\xi_0, 2^{-\iota} r) \setminus B(\xi_0, 2^{-\iota-1} r)$. On A_{ι}, $|\zeta - \xi_0| \ge 2^{-\iota-1} r$, so

$$\int_{A_{\iota}} \varphi(\zeta)\, d\zeta \le (2^{-\iota-1} r)^{-\tau} \int_{A_{\iota}} |\zeta - \xi_0|^{\tau} \varphi(\zeta)\, d\zeta. \tag{90}$$

By Hölder (10), Lemma 2, and (85) at $\rho = 2^{-\iota} r$,

$$\begin{aligned} \int_{A_{\iota}} |\zeta - \xi_0|^{\tau} \varphi(\zeta)\, d\zeta &\le \left\| (\cdot - \xi_0)^{\tau} \varphi \chi_{B(\xi_0, 2^{-\iota} r)} \right\|_{\mathrm{p}(\cdot)} \left\| \chi_{A_{\iota}} \right\|_{\mathrm{p}'(\cdot)} \\ &\lesssim (2^{-\iota} r)^{\lambda(\xi_0)/\mathrm{p}(\xi_0)} (2^{-\iota} r)^{\nu/\mathrm{p}'(\xi_0)}. \end{aligned} \tag{91}$$

Therefore

$$\int_{A_{\iota}} \varphi(\zeta)\, d\zeta \lesssim (2^{-\iota} r)^{-\tau + \nu/\mathrm{p}'(\xi_0) + \lambda(\xi_0)/\mathrm{p}(\xi_0)}, \tag{92}$$

and, summing over $\iota \geq 1$ (the series converges since $-\tau + \nu/\mathrm{p}'(\xi_0) \geq -\tau + \nu/(\mathrm{p}')^+ > 0$ by (79)),

$$\begin{aligned} Z_1(\xi) &\lesssim c_\beta (1+r)^{-\beta} r^{\gamma-\nu} \sum_{\iota=1}^{\infty} (2^{-\iota} r)^{-\tau+\nu/\mathrm{p}'(\xi_0)+\lambda(\xi_0)/\mathrm{p}(\xi_0)} \\ &\lesssim (1+r)^{-\beta} r^{\gamma-\tau-\frac{\nu-\lambda(\xi_0)}{\mathrm{p}(\xi_0)}}. \end{aligned} \tag{93}$$

By Lemma 1 and (81),

$$Z_1(\xi) \lesssim (1+r)^{-\beta} r^{-\kappa-\frac{\nu-\lambda(\xi)}{\mathrm{q}(\xi)}} \leq r^{-\kappa-\frac{\nu-\lambda(\xi)}{\mathrm{q}(\xi)}} = |\xi-\xi_0|^{-\kappa-\frac{\nu-\lambda(\xi)}{\mathrm{q}(\xi)}}. \tag{94}$$

Set $g(\xi) := |\xi-\xi_0|^{\lambda(\xi)-\nu}$; then $\left[|\xi-\xi_0|^{\kappa} Z_1(\xi)\right]^{\mathrm{q}(\xi)} \lesssim g(\xi)$. Fix $\zeta \in \mathbb{R}^\nu$, $\rho > 0$.
Case $|\zeta-\xi_0| \leq 2\rho$: $B(\zeta,\rho) \subset B(\xi_0, 3\rho)$, so by Lemma 3 and Lemma 1,

$$\int_{B(\zeta,\rho)} g(\xi)\, d\xi \leq \int_{B(\xi_0,3\rho)} g(\xi)\, d\xi \lesssim \rho^{\lambda(\xi_0)} \lesssim \rho^{\lambda(\zeta)}. \tag{95}$$

Case $|\zeta-\xi_0| > 2\rho$: for $\xi \in B(\zeta,\rho)$, $\frac{|\zeta-\xi_0|}{2} < |\xi-\xi_0| < \frac{3}{2}|\zeta-\xi_0|$, so $|\xi-\xi_0| \approx |\zeta-\xi_0|$ and, by Lemma 1, $\lambda(\xi) \approx \lambda(\zeta)$ on $B(\zeta,\rho)$, so

$$\int_{B(\zeta,\rho)} g(\xi)\, d\xi \lesssim |B(\zeta,\rho)|\, |\zeta-\xi_0|^{\lambda(\zeta)-\nu} = c_\nu \rho^\nu |\zeta-\xi_0|^{\lambda(\zeta)-\nu}. \tag{96}$$

By Lemma 2 and the identity

$$-\nu\Big(\frac{1}{\alpha_{\mathrm{q}}(\zeta)} - \frac{1}{\mathrm{q}(\zeta)} - \frac{1}{\eta}\Big) + \frac{\lambda(\zeta)}{\mathrm{q}(\zeta)} = -\frac{\nu}{\eta}, \tag{97}$$

define

$$H_\rho(\zeta) := |B(\zeta,\rho)|^{\frac{1}{\alpha_{\mathrm{q}}(\zeta)} - \frac{1}{\mathrm{q}(\zeta)} - \frac{1}{\eta}} \left\| (\cdot-\xi_0)^{\kappa} Z_1 \chi_{B(\zeta,\rho)} \right\|_{\mathrm{q}(\cdot)}. \tag{98}$$

By (95) and (97), for $|\zeta-\xi_0| \leq 2\rho$,

$$H_\rho(\zeta) \lesssim \left(c_\nu \rho^\nu\right)^{\frac{1}{\alpha_{\mathrm{q}}(\zeta)} - \frac{1}{\mathrm{q}(\zeta)} - \frac{1}{\eta}} \rho^{\lambda(\zeta)/\mathrm{q}(\zeta)} \lesssim \rho^{-\nu/\eta}. \tag{99}$$

By (96) and (97), for $|\zeta-\xi_0| > 2\rho$, since $\rho < |\zeta-\xi_0|/2$ and $\nu - \lambda(\zeta) \geq \nu - \lambda^+ > 0$ by (80),

$$H_\rho(\zeta) \lesssim \left(\frac{\rho}{|\zeta-\xi_0|}\right)^{\frac{\nu-\lambda(\zeta)}{\mathrm{q}(\zeta)}} \rho^{-\nu/\eta} \leq \rho^{-\nu/\eta}. \tag{100}$$

Let $E_\rho := \{\zeta : |\zeta-\xi_0| \leq 2\rho\}$, $|E_\rho| = c_\nu (2\rho)^\nu$. By (99),

$$\int_{E_\rho} H_\rho(\zeta)^\eta\, d\zeta \lesssim \rho^{-\nu} |E_\rho| = c_\nu 2^\nu. \tag{101}$$

By (100), $\nu - \lambda(\zeta) \geq \nu - \lambda^+$, and hypothesis (82), $\eta(\nu-\lambda^+)/\mathrm{q}^+ > \nu$, so

$$\begin{aligned} \int_{\mathbb{R}^\nu \setminus E_\rho} H_\rho(\zeta)^\eta\, d\zeta &\leq \rho^{-\nu} \int_{|\zeta-\xi_0|>2\rho} \left(\frac{\rho}{|\zeta-\xi_0|}\right)^{\eta(\nu-\lambda^+)/\mathrm{q}^+} d\zeta \\ &= \rho^{\eta(\nu-\lambda^+)/\mathrm{q}^+ - \nu}\, c_\nu \nu \int_{2\rho}^{\infty} s^{\nu-1-\eta(\nu-\lambda^+)/\mathrm{q}^+}\, ds \\ &= \rho^{\eta(\nu-\lambda^+)/\mathrm{q}^+ - \nu}\, c_\nu \nu \cdot \frac{(2\rho)^{\nu-\eta(\nu-\lambda^+)/\mathrm{q}^+}}{\eta(\nu-\lambda^+)/\mathrm{q}^+ - \nu} = O(1). \end{aligned} \tag{102}$$

Adding (101) and (102), taking η-th roots and $\sup_{\rho>0}$,

$$\left\| |\cdot - \xi_0|^{\kappa} Z_1 \right\|_{\left(L^{\mathfrak{q}(\cdot)}, L^{\eta}\right)^{\alpha \mathfrak{q}(\cdot)}(\mathbb{R}^{\nu})} \lesssim 1. \tag{103}$$

Estimate of Z_3. For $|\zeta - \xi_0| \geq 2r$,

$$|\zeta - \xi_0| \leq |\zeta - \xi| + r \leq |\zeta - \xi| + \frac{|\zeta - \xi_0|}{2} \implies \frac{|\zeta - \xi_0|}{2} \leq |\zeta - \xi|, \tag{104}$$

so, since $(1+\varsigma)^{-\beta}$ is decreasing,

$$|\xi - \zeta|^{\gamma - \nu}(1 + |\xi - \zeta|)^{-\beta} \leq \left(\frac{|\zeta - \xi_0|}{2}\right)^{\gamma - \nu} \left(1 + \frac{|\zeta - \xi_0|}{2}\right)^{-\beta}. \tag{105}$$

By (79) and (81),

$$\varkappa := \gamma - \tau - \frac{\nu - \lambda(\xi_0)}{\mathrm{p}(\xi_0)} < 0. \tag{106}$$

Decompose $\mathbb{R}^{\nu} \setminus B(\xi_0, 2r) = \bigcup_{\iota=1}^{\infty} D_{\iota}$, $D_{\iota} := B(\xi_0, 2^{\iota+1} r) \setminus B(\xi_0, 2^{\iota} r)$. As for Z_1, inserting the weight $|\zeta - \xi_0|^{-\tau} |\zeta - \xi_0|^{\tau}$, Hölder (10), Lemma 2, and (85) at $\rho = 2^{\iota+1} r$, and using (105) with $(1+\varsigma)^{-\beta} \leq \min(1, \varsigma^{-\beta})$,

$$Z_3(\xi) \lesssim \sum_{\iota=1}^{\infty} (2^{\iota} r)^{\varkappa} \min\left(1, (2^{\iota} r)^{-\beta}\right). \tag{107}$$

Let $\iota_0 := \max\{\iota \geq 1 : 2^{\iota} r \leq 1\}$, $\iota_0 := 0$ if $r > 1/2$. Split

$$\Sigma_1 := \sum_{\iota=1}^{\iota_0} (2^{\iota} r)^{\varkappa}, \qquad \Sigma_2 := \sum_{\iota > \iota_0} (2^{\iota} r)^{\varkappa - \beta}. \tag{108}$$

Case $r \leq 1/2$: since $\varkappa < 0$,

$$\Sigma_1 = r^{\varkappa} \sum_{\iota=1}^{\iota_0} 2^{\iota \varkappa} \leq r^{\varkappa} \frac{2^{\varkappa}}{1 - 2^{\varkappa}} \lesssim r^{\varkappa}, \tag{109}$$

and, since $\varkappa - \beta < 0$, $2^{\iota_0} r \in (1/2, 1]$,

$$\Sigma_2 = (2^{\iota_0} r)^{\varkappa - \beta} \sum_{\iota > \iota_0} 2^{(\iota - \iota_0)(\varkappa - \beta)} \lesssim (2^{\iota_0} r)^{\varkappa - \beta} \lesssim 1. \tag{110}$$

Hence,

$$Z_3(\xi) \lesssim r^{\varkappa} + 1 \lesssim r^{\varkappa}, \qquad r \leq 1/2. \tag{111}$$

Case $r > 1/2$: $\iota_0 = 0$, $\Sigma_1 = 0$,

$$\Sigma_2 = \sum_{\iota=1}^{\infty} (2^{\iota} r)^{\varkappa - \beta} = r^{\varkappa - \beta} \frac{2^{\varkappa - \beta}}{1 - 2^{\varkappa - \beta}} \lesssim r^{\varkappa - \beta}, \tag{112}$$

so

$$Z_3(\xi) \lesssim r^{\varkappa - \beta}, \qquad r > 1/2. \tag{113}$$

By Lemma 1 and (81), (111) gives

$$Z_3(\xi) \lesssim |\xi - \xi_0|^{-\kappa - \frac{\nu - \lambda(\xi)}{\mathfrak{q}(\xi)}}, \qquad r \leq 1/2, \tag{114}$$

and, since $\varkappa - \beta = \gamma - \tau - \beta - \frac{\nu - \lambda(\xi_0)}{\mathrm{p}(\xi_0)}$, and $r^{-\beta} \le 2^{\beta}$ for $r > 1/2$, (113) gives

$$Z_3(\xi) \lesssim |\xi - \xi_0|^{-\kappa - \beta - \frac{\nu - \lambda(\xi)}{\mathrm{q}(\xi)}} \le 2^{\beta} |\xi - \xi_0|^{-\kappa - \frac{\nu - \lambda(\xi)}{\mathrm{q}(\xi)}}, \qquad r > 1/2. \tag{115}$$

Combining (114) and (115), for all $\xi \in \mathbb{R}^{\nu}$,

$$Z_3(\xi) \lesssim |\xi - \xi_0|^{-\kappa - \frac{\nu - \lambda(\xi)}{\mathrm{q}(\xi)}}, \tag{116}$$

which is of the same form as (94). Hence, $\left[|\xi - \xi_0|^{\kappa} Z_3(\xi)\right]^{\mathrm{q}(\xi)} \lesssim g(\xi)$, and the estimates (95)–(102) apply with Z_1 replaced by Z_3, giving

$$\left\| |\cdot - \xi_0|^{\kappa} Z_3 \right\|_{\left(L^{\mathrm{q}(\cdot)}, L^{\eta}\right)^{\alpha \mathrm{q}(\cdot)}(\mathbb{R}^{\nu})} \lesssim 1. \tag{117}$$

Estimate of Z_2. Let $\mathcal{A} := B(\xi_0, 2r) \setminus B(\xi_0, \frac{r}{2})$. On $\mathcal{A}$, the quantity $|\xi - \zeta|$ ranges over the full interval $(0, 3r]$ as ζ ranges over $\mathcal{A}$ (indeed $\xi \in \overline{\mathcal{A}}$), so no single comparison of $(1 + |\xi - \zeta|)^{-\beta}$ to a power of r is possible; the decay factor must instead be retained shell by shell. Cover $\mathcal{A} \subset \bigcup_{\iota=0}^{\infty} D_{\iota}$, $D_{\iota} := \{\zeta \in \mathcal{A} : 3 \cdot 2^{-\iota-1} r \le |\xi - \zeta| < 3 \cdot 2^{-\iota} r\} \subset B(\xi, 3 \cdot 2^{-\iota} r)$. On D_{ι}, since $3 \cdot 2^{-\iota-1} r \le |\xi - \zeta| < 3 \cdot 2^{-\iota} r$ (ratio of endpoints equal to 2), monotonicity of $(1 + \varsigma)^{-\beta}$ gives

$$(1 + 3 \cdot 2^{-\iota} r)^{-\beta} \le (1 + |\xi - \zeta|)^{-\beta} \le (1 + 3 \cdot 2^{-\iota-1} r)^{-\beta} \le 2^{\beta} (1 + 3 \cdot 2^{-\iota} r)^{-\beta}, \tag{118}$$

so

$$(1 + |\xi - \zeta|)^{-\beta} \approx (1 + 2^{-\iota} r)^{-\beta}, \qquad \zeta \in D_{\iota}, \tag{119}$$

with constants depending only on β. Using (119) and $|\xi - \zeta| \approx 2^{-\iota} r$ on D_{ι},

$$Z_2(\xi) \le \sum_{\iota=0}^{\infty} \int_{D_{\iota}} \frac{\varphi(\zeta)}{|\xi - \zeta|^{\nu - \gamma} (1 + |\xi - \zeta|)^{\beta}} \, d\zeta \lesssim \sum_{\iota=0}^{\infty} (2^{-\iota} r)^{\gamma - \nu} (1 + 2^{-\iota} r)^{-\beta} \int_{D_{\iota}} \varphi(\zeta) \, d\zeta. \tag{120}$$

Subcase $\tau > \kappa$. By (78), $\sigma := \gamma - \tau + \kappa \ge 0$. On $D_{\iota} \subset \mathcal{A}$, $|\zeta - \xi_0| \ge r/2$, so $|\zeta - \xi_0|^{-\tau} \le 2^{\tau} r^{-\tau}$; inserting $|\zeta - \xi_0|^{-\tau} |\zeta - \xi_0|^{\tau}$ into (120),

$$\begin{aligned} Z_2(\xi) &\lesssim r^{-\tau} \sum_{\iota=0}^{\infty} (2^{-\iota} r)^{\gamma - \nu} (1 + 2^{-\iota} r)^{-\beta} \int_{D_{\iota}} |\zeta - \xi_0|^{\tau} \varphi(\zeta) \, d\zeta \\ &= r^{-\kappa} \sum_{\iota=0}^{\infty} 2^{-\iota(\tau - \kappa)} (2^{-\iota} r)^{\sigma - \nu} (1 + 2^{-\iota} r)^{-\beta} \int_{D_{\iota}} |\zeta - \xi_0|^{\tau} \varphi(\zeta) \, d\zeta. \end{aligned} \tag{121}$$

Since $D_{\iota} \subset B(\xi, 3 \cdot 2^{-\iota} r)$, each term is bounded by $2^{-\iota(\tau - \kappa)}$ times

$$(3 \cdot 2^{-\iota} r)^{\sigma - \nu} \left(1 + \frac{3 \cdot 2^{-\iota} r}{3}\right)^{-\beta} \int_{B(\xi, 3 \cdot 2^{-\iota} r)} |\zeta - \xi_0|^{\tau} \varphi(\zeta) \, d\zeta \lesssim M_{\sigma,\beta}\big((\cdot - \xi_0)^{\tau} \varphi\big)(\xi), \tag{122}$$

where the decay-weighted fractional maximal operator is defined by

$$M_{\sigma,\beta} f(\xi) := \sup_{s > 0} s^{\sigma - \nu} (1 + s)^{-\beta} \int_{B(\xi, s)} f(\zeta) \, d\zeta. \tag{123}$$

Since $\tau - \kappa > 0$, $\sum_{\iota=0}^{\infty} 2^{-\iota(\tau - \kappa)} < \infty$, and summing,

$$Z_2(\xi) \lesssim |\xi - \xi_0|^{-\kappa} M_{\sigma,\beta}\big((\cdot - \xi_0)^{\tau} \varphi\big)(\xi). \tag{124}$$

Because $(1+s)^{-\beta} \le 1$ for every $s > 0$, comparing (123) with the plain fractional maximal operator $M_\sigma f(\xi) = \sup_{s>0} s^{\sigma-\nu} \int_{B(\xi,s)} f(\zeta)\, d\zeta$ gives the pointwise domination

$$M_{\sigma,\beta} f(\xi) \le M_\sigma f(\xi), \qquad f \ge 0, \tag{125}$$

so that, by Theorem 14 ((79)–(81) verifying its hypotheses) applied to M_σ, (125), and (84),

$$\begin{aligned} \big\| |\cdot - \xi_0|^\kappa Z_2 \big\|_{\left(L^{\mathfrak{q}(\cdot)}, L^\eta\right)^{\alpha \mathfrak{q}(\cdot)}(\mathbb{R}^\nu)} &\lesssim \big\| M_{\sigma,\beta}\big((\cdot - \xi_0)^\tau \varphi\big) \big\|_{\left(L^{\mathfrak{q}(\cdot)}, L^\eta\right)^{\alpha \mathfrak{q}(\cdot)}(\mathbb{R}^\nu)} \\ &\le \big\| M_\sigma\big((\cdot - \xi_0)^\tau \varphi\big) \big\|_{\left(L^{\mathfrak{q}(\cdot)}, L^\eta\right)^{\alpha \mathfrak{q}(\cdot)}(\mathbb{R}^\nu)} \\ &\lesssim \big\| (\cdot - \xi_0)^\tau \varphi \big\|_{\left(L^{\mathfrak{p}(\cdot)}, L^\eta\right)^{\alpha \mathfrak{p}(\cdot)}(\mathbb{R}^\nu)} = 1. \end{aligned} \tag{126}$$

Subcase $\tau = \kappa$. Here $\sigma = \gamma$, and $\sum_\iota 2^{-\iota(\tau-\kappa)} = \sum_\iota 1 = \infty$, so the maximal-function route above is unavailable; instead, (120) is summed directly. Since $|\xi - \xi_0|^{-\kappa} |\zeta - \xi_0|^\tau = |\xi - \xi_0|^{-\tau} |\zeta - \xi_0|^\tau \approx 1$ on $\mathcal{A}$ (as $|\zeta - \xi_0| \approx |\xi - \xi_0|$ there), and keeping the decay factor,

$$\begin{aligned} Z_2(\xi) &\lesssim |\xi - \xi_0|^{-\kappa} \int_{\mathcal{A}} \frac{|\zeta - \xi_0|^\tau \varphi(\zeta)}{|\xi - \zeta|^{\nu-\gamma} (1 + |\xi - \zeta|)^\beta}\, d\zeta \\ &\le |\xi - \xi_0|^{-\kappa} \int_{\mathbb{R}^\nu} \frac{|\zeta - \xi_0|^\tau \varphi(\zeta)}{|\xi - \zeta|^{\nu-\gamma} (1 + |\xi - \zeta|)^\beta}\, d\zeta = |\xi - \xi_0|^{-\kappa} T_{\gamma,\beta}\big((\cdot - \xi_0)^\tau \varphi\big)(\xi), \end{aligned} \tag{127}$$

where $T_{\gamma,\beta}$ appears here with its full kernel, retaining the decay parameter β throughout. By Theorem 14 applied directly to $T_{\gamma,\beta}$ (with $\sigma = \gamma$; (79)–(81) verify its hypotheses) and (84),

$$\begin{aligned} \big\| |\cdot - \xi_0|^\kappa Z_2 \big\|_{\left(L^{\mathfrak{q}(\cdot)}, L^\eta\right)^{\alpha \mathfrak{q}(\cdot)}(\mathbb{R}^\nu)} &\lesssim \big\| T_{\gamma,\beta}\big((\cdot - \xi_0)^\tau \varphi\big) \big\|_{\left(L^{\mathfrak{q}(\cdot)}, L^\eta\right)^{\alpha \mathfrak{q}(\cdot)}(\mathbb{R}^\nu)} \\ &\lesssim \big\| (\cdot - \xi_0)^\tau \varphi \big\|_{\left(L^{\mathfrak{p}(\cdot)}, L^\eta\right)^{\alpha \mathfrak{p}(\cdot)}(\mathbb{R}^\nu)} = 1. \end{aligned} \tag{128}$$

Combining (126) and (128), in either subcase,

$$\big\| |\cdot - \xi_0|^\kappa Z_2 \big\|_{\left(L^{\mathfrak{q}(\cdot)}, L^\eta\right)^{\alpha \mathfrak{q}(\cdot)}(\mathbb{R}^\nu)} \lesssim 1. \tag{129}$$

Conclusion. By (103), (117), (129), and the triangle inequality in $\left(L^{\mathfrak{q}(\cdot)}, L^\eta\right)^{\alpha \mathfrak{q}(\cdot)}(\mathbb{R}^\nu)$,

$$\big\| |\cdot - \xi_0|^\kappa T_{\gamma,\beta} \varphi \big\|_{\left(L^{\mathfrak{q}(\cdot)}, L^\eta\right)^{\alpha \mathfrak{q}(\cdot)}(\mathbb{R}^\nu)} \le \sum_{i=1}^{3} \big\| |\cdot - \xi_0|^\kappa Z_i \big\|_{\left(L^{\mathfrak{q}(\cdot)}, L^\eta\right)^{\alpha \mathfrak{q}(\cdot)}(\mathbb{R}^\nu)} \lesssim 1, \tag{130}$$

which, undoing (84), is (83). □

Remark 4. *Hypotheses (78)–(81) and (82) involve only $\mathfrak{p}(\cdot), \mathfrak{q}(\cdot), \lambda(\cdot), \kappa, \tau, \eta$ and place no restriction on β beyond $\beta > 0$; this last requirement, however, is not vacuous on $\mathbb{R}^\nu$. Unlike the estimate on a bounded domain, where β enters only through the far-field term, here β is tracked explicitly in the treatment of Z_1, Z_2, and Z_3 alike: in Z_1 and Z_3 the factor $(1 + |\xi - \zeta|)^{-\beta}$ is shown to be comparable to $(1 + |\xi - \xi_0|)^{-\beta}$ and $(1 + |\zeta - \xi_0|/2)^{-\beta}$ respectively, and it is exactly the strict positivity of β that makes the dyadic tail sums Σ_2 in (107) converge over the infinitely many shells of $\mathbb{R}^\nu \setminus B(\xi_0, 2r)$; in Z_2, the same factor is retained shell by shell in (119) and survives into the decay-weighted maximal operator $M_{\sigma,\beta}$ of (123), or, in the borderline case $\tau = \kappa$, directly into the bound by $T_{\gamma,\beta}$ itself in (127). Consequently $\beta = 0$ is not an admissible value in Theorem 15: at $\beta = 0$ the sum defining Σ_2 in the estimate of Z_3 no longer converges, since $\mathbb{R}^\nu \setminus B(\xi_0, 2r)$ has infinite measure.*

Corollary 3. *At $\eta = \infty$, $\left(L^{\mathrm{p}(\cdot)}, L^{\infty}\right)^{\alpha_{\mathrm{p}}(\cdot)}(\mathbb{R}^{\nu}) = M^{\lambda(\cdot)}_{\mathrm{p}(\cdot)}(\mathbb{R}^{\nu})$, and (83) yields the Stein–Weiss inequality for the Bessel–Riesz operator $T_{\gamma,\beta}$, $\beta > 0$, in variable-exponent Morrey spaces on all of $\mathbb{R}^{\nu}$. This complements, rather than reduces to, Theorem 5 of [20]: the latter is recovered only formally in the limit $\beta \to 0^{+}$, since $\beta = 0$ itself is excluded from the hypotheses of Theorem 15 by the convergence requirement noted in the Remark above.*

Remark 5. *Theorem 15 generalizes Theorem 1 in three independent directions: it replaces the constant exponents p, q by variable exponents $\mathrm{p}(\cdot), \mathrm{q}(\cdot)$; it replaces the Riesz kernel $|\xi - \zeta|^{\gamma-\nu}$ by the Bessel–Riesz kernel with decay parameter $\beta > 0$; and it replaces a bounded domain Ω by all of $\mathbb{R}^{\nu}$, which forces $\beta > 0$ to be used non-trivially throughout the proof rather than merely in a far-field correction. Restricting Theorem 15 to $\nu = 1$, functions supported on $\mathbb{R}_{+} \subset \mathbb{R}$, $\kappa = 0$, $\eta = \infty$, $\lambda(\cdot) \equiv 0$ recovers the unweighted boundedness statement of Theorem, once the exponent identity (81) is matched to the formula for $\mathrm{q}(\cdot)$ given there.*

Corollary 4 (Unweighted case)**.** *Suppose, in addition to the hypotheses of Theorem 15, that $\kappa = \tau = 0$ and $\lambda^{+} < \nu/\mathrm{q}^{+}$ (so that (79) holds with $\kappa = \tau = 0$). Then*

$$\|T_{\gamma,\beta}\varphi\|_{\left(L^{\mathrm{q}(\cdot)}, L^{\eta}\right)^{\alpha_{\mathrm{q}}(\cdot)}(\mathbb{R}^{\nu})} \leq C\|\varphi\|_{\left(L^{\mathrm{p}(\cdot)}, L^{\eta}\right)^{\alpha_{\mathrm{p}}(\cdot)}(\mathbb{R}^{\nu})}, \tag{131}$$

where now $\frac{\gamma}{\nu-\lambda(\xi)} = \frac{1}{\mathrm{p}(\xi)} - \frac{1}{\mathrm{q}(\xi)}$ by (81). Since $\tau = \kappa$, this bound arises from the borderline subcase of the estimate of Z_2 in the proof of Theorem 15, where Z_2 is controlled directly by $T_{\gamma,\beta}$ itself via (127), and Corollary 4 is precisely Theorem 14 applied to $T_{\gamma,\beta}$ at $\sigma = \gamma$ with its decay parameter $\beta > 0$ intact.

Corollary 5 (Constant exponents)**.** *If $\mathrm{p}(\cdot) \equiv \mathrm{p}_0$, $\mathrm{q}(\cdot) \equiv \mathrm{q}_0$, $\lambda(\cdot) \equiv \lambda_0$ are constant on $\mathbb{R}^{\nu}$ and $\eta = \infty$, the hypotheses of Theorem 15 reduce to (78)–(79) with fixed λ_0, $\frac{1}{\mathrm{p}_0} - \frac{1}{\mathrm{q}_0} = \frac{\gamma+\kappa-\tau}{\nu-\lambda_0}$, and (83) becomes the classical (constant-exponent) global Morrey–Stein–Weiss inequality for the Bessel–Riesz operator,*

$$\left\||\xi - \xi_0|^{\kappa} T_{\gamma,\beta}\varphi\right\|_{M^{\lambda_0}_{\mathrm{q}_0}(\mathbb{R}^{\nu})} \leq C\left\||\xi - \xi_0|^{\tau}\varphi\right\|_{M^{\lambda_0}_{\mathrm{p}_0}(\mathbb{R}^{\nu})}, \qquad \beta > 0. \tag{132}$$

Example 1. *Let $\nu = 2$, $\Omega = \mathbb{R}^2$, $\xi_0 = 0$, $\gamma = \frac{1}{2}$, $\beta = 1$. Take constant exponents $\mathrm{p}(\xi) \equiv \mathrm{p}_0 = \frac{3}{2}$, and $\lambda(\xi) \equiv \lambda_0 = \frac{1}{2}$, and weights $\kappa = 0$, $\tau = \frac{1}{4}$. Then:*

(i) $\beta = 1 > 0$, *as required by hypothesis (iii);*

(ii) $\tau - \kappa = \frac{1}{4} \in [0, \gamma] = [0, \frac{1}{2}]$, *so (78) holds;*

(iii) q_0 *is determined by (81):* $\dfrac{\gamma + \kappa - \tau}{\nu - \lambda_0} = \dfrac{\frac{1}{2} - \frac{1}{4}}{2 - \frac{1}{2}} = \dfrac{1}{6} = \dfrac{1}{\mathrm{p}_0} - \dfrac{1}{\mathrm{q}_0} = \dfrac{2}{3} - \dfrac{1}{\mathrm{q}_0}$, *giving* $\mathrm{q}_0 = 2$;

(iv) $\lambda^{+} - \dfrac{\nu}{\mathrm{q}^{+}} = \frac{1}{2} - 1 = -\frac{1}{2} < 0 = \kappa \leq \tau = \frac{1}{4} < \dfrac{\nu}{(\mathrm{p}')^{+}} = \dfrac{2}{3}$, *so (79) holds;*

(v) $\nu - (\gamma - \tau + \kappa)\mathrm{p}_0 = 2 - \left(\frac{1}{2} - \frac{1}{4}\right) \cdot \frac{3}{2} = 2 - \frac{3}{8} = \frac{13}{8}$, *and* $0 < \lambda_0 = \frac{1}{2} < \frac{13}{8}$, *so (80) holds.*

Thus, by Theorem 15 (with $\eta = \infty$, so that $(L^{\mathrm{p}_0}, L^{\infty})^{\alpha_{\mathrm{p}_0}}(\mathbb{R}^2) = M^{\lambda_0}_{\mathrm{p}_0}(\mathbb{R}^2)$ by Corollary 3), there is $C > 0$ such that for every φ,

$$\left\||\xi|^{0} T_{\frac{1}{2},1}\varphi\right\|_{M^{1/2}_{2}(\mathbb{R}^2)} = \|T_{\frac{1}{2},1}\varphi\|_{M^{1/2}_{2}(\mathbb{R}^2)} \leq C\left\||\xi|^{1/4}\varphi\right\|_{M^{1/2}_{3/2}(\mathbb{R}^2)}. \tag{133}$$

Unlike in the bounded-domain setting, hypotheses (i)–(v) here do not *also validate the corresponding inequality for the pure Riesz potential $I_{1/2}$ in place of $T_{1/2,1}$: taking $\beta = 0$ would violate hypothesis (iii), since the tail sum in the estimate of Z_3 diverges once the exterior region $\mathbb{R}^2 \setminus B(0, 2|\xi|)$ is unbounded. The pointwise domination $T_{1/2,1}\varphi \leq I_{1/2}\varphi$ for $\varphi \geq 0$ still holds,*

but it no longer transfers boundedness from $T_{1/2,1}$ to $I_{1/2}$ or vice versa on $\mathbb{R}^2$, since $I_{1/2} = T_{1/2,0}$ is simply outside the range of Theorem 15.

3.4. John–Nirenberg Inequalities in Variable Exponent Fofana Spaces

Throughout this subsection, $\mathrm{p}(\cdot) \in \mathcal{P}(\mathbb{R}^\nu) \cap LH(\mathbb{R}^\nu)$, $\mathrm{p}^+ < \infty$, $\eta \in [1, \infty)$, $\vartheta \in \mathcal{G}_{\mathrm{p}^+,\eta}$, and $(L^{\mathrm{p}(\cdot)}, L^\eta)^\vartheta(\mathbb{R}^\nu)$ is as in Definition 1. We first record the Fofana-space analogue of the BMO estimate of Lemma 3.1 of [44], at the level of the associate space $\big(L^{\mathfrak{r}\mathrm{p}(\cdot)}(\mathbb{R}^\nu)\big)'$.

Lemma 12. *Let $\mathrm{p}(\cdot) \in \mathcal{P}(\mathbb{R}^\nu) \cap LH(\mathbb{R}^\nu)$, with $1 \le \mathrm{p}^- \le \mathrm{p}^+ < \infty$. Let $\mathfrak{r} \ge 1$. There is $C > 0$, independent of $\mathfrak{r}$ and φ, such that*

$$\frac{\big\|(\varphi - \varphi_B)\chi_B\big\|_{L^{\mathfrak{r}\mathrm{p}(\cdot)}(\mathbb{R}^\nu)}}{\|\chi_B\|^{1/\mathfrak{r}}_{L^{\mathrm{p}(\cdot)}(\mathbb{R}^\nu)}} \le C\mathfrak{r}\,\mathrm{p}^+\|\varphi\|_{BMO}, \qquad \varphi \in BMO(\mathbb{R}^\nu),\ B \in \mathbb{B}. \tag{134}$$

Proof. Fix $\varphi \in BMO(\mathbb{R}^\nu)$, $B \in \mathbb{B}$, and $k \ge 1$. By the layer-cake formula and (5),

$$\int_B |\varphi(\zeta) - \varphi_B|^k\, d\zeta = k\int_0^\infty t^{k-1}\big|\{\xi \in B : |\varphi(\xi) - \varphi_B| > t\}\big|\, dt \le kC_1|B|\int_0^\infty t^{k-1}e^{-C_2 t/\|\varphi\|_{BMO}}\, dt. \tag{135}$$

Substituting $s = C_2 t/\|\varphi\|_{BMO}$, $dt = (\|\varphi\|_{BMO}/C_2)\, ds$,

$$\int_0^\infty t^{k-1}e^{-C_2 t/\|\varphi\|_{BMO}}\, dt = \Big(\frac{\|\varphi\|_{BMO}}{C_2}\Big)^k\int_0^\infty s^{k-1}e^{-s}\, ds = \Big(\frac{\|\varphi\|_{BMO}}{C_2}\Big)^k\Gamma(k), \tag{136}$$

so, using $k\Gamma(k) = \Gamma(k+1)$,

$$\int_B |\varphi - \varphi_B|^k\, d\zeta \le C_1\Gamma(k+1)\Big(\frac{\|\varphi\|_{BMO}}{C_2}\Big)^k|B|. \tag{137}$$

By Stirling's formula there is an absolute constant $C_*' \ge 1$ with $\Gamma(k+1)^{1/k} \le C_*' k$ for every $k \ge 1$; also $C_1^{1/k} \le \max(1, C_1) =: C_1''$. Raising (137) to the power $1/k$ and setting $C_* := C_1''C_*'/C_2$,

$$\left(\int_B |\varphi - \varphi_B|^k\, d\zeta\right)^{1/k} \le C_* k\|\varphi\|_{BMO}\,|B|^{1/k}, \qquad k \ge 1. \tag{138}$$

Since $\mathrm{p}(\cdot) \in LH(\mathbb{R}^\nu)$, there is $\xi_B \in \overline{B}$ and $C_1' = C_1'(\nu, \mathrm{p}^+) \ge 1$, independent of B, such that, writing $p_B := \mathrm{p}(\xi_B) \in [\mathrm{p}^-, \mathrm{p}^+]$,

$$(C_1')^{-1}|B|^{1/p_B} \le \|\chi_B\|_{\mathrm{p}(\cdot)} \le C_1'|B|^{1/p_B}. \tag{139}$$

The same log-Hölder bound on the oscillation of $\mathrm{p}(\cdot)$ over B that yields (139) also gives, for every $h \ge 0$ supported on B and every $\mathfrak{r} \ge 1$,

$$\varrho_{\mathfrak{r}\mathrm{p}(\cdot)}(h) = \int_B h(\zeta)^{\mathfrak{r}\mathrm{p}(\zeta)}\, d\zeta \le C_2'\int_B h(\zeta)^{\mathfrak{r}p_B}\, d\zeta = C_2'\,\varrho_{\mathfrak{r}p_B}(h), \tag{140}$$

with $C_2' = C_2'(\nu, \mathrm{p}^+) \ge 1$ independent of $\mathfrak{r}$ and B, since the exponent $\mathfrak{r}\mathrm{p}(\cdot)$ has the same log-Hölder modulus of continuity as $\mathrm{p}(\cdot)$ up to the fixed multiplicative factor $\mathfrak{r}$, which cancels in the ratio $\mathfrak{r}\mathrm{p}(\zeta)/(\mathfrak{r}p_B) = \mathrm{p}(\zeta)/p_B$ controlling the exponent oscillation.

Set $f := (\varphi - \varphi_B)\chi_B$, $\rho := \|\chi_B\|_{\mathrm{p}(\cdot)}$, and

$$\lambda := C_3\mathfrak{r}\,\mathrm{p}^+\|\varphi\|_{BMO}\,\rho^{1/\mathfrak{r}}, \tag{141}$$

with $C_3 = C_3(\nu, \mathrm{p}^+) \geq 1$ to be fixed. By (140) at $h = f/\lambda$,

$$\varrho_{\mathfrak{r}\mathrm{p}(\cdot)}(f/\lambda) \leq C_2' \varrho_{\mathfrak{r}p_B}(f/\lambda) = C_2' \lambda^{-\mathfrak{r}p_B} \int_B |\varphi - \varphi_B|^{\mathfrak{r}p_B}\, d\zeta. \tag{142}$$

By (138) at $k = \mathfrak{r}p_B \geq \mathfrak{r} \geq 1$,

$$\int_B |\varphi - \varphi_B|^{\mathfrak{r}p_B}\, d\zeta \leq \left(C_* \mathfrak{r} p_B \|\varphi\|_{BMO}\right)^{\mathfrak{r}p_B} |B|, \tag{143}$$

and by (139),

$$|B| \leq (C_1')^{p_B} \rho^{p_B}. \tag{144}$$

Substituting (144) into (143) and then into (142),

$$\varrho_{\mathfrak{r}\mathrm{p}(\cdot)}(f/\lambda) \leq C_2' (C_1')^{p_B} \left(C_* \mathfrak{r} p_B \|\varphi\|_{BMO}\right)^{\mathfrak{r}p_B} \lambda^{-\mathfrak{r}p_B} \rho^{p_B}. \tag{145}$$

Since $\lambda^{\mathfrak{r}p_B} = \left(C_3 \mathfrak{r}\, \mathrm{p}^+ \|\varphi\|_{BMO}\right)^{\mathfrak{r}p_B} \rho^{p_B}$, the factor ρ^{p_B} in (145) cancels against $\lambda^{-\mathfrak{r}p_B}\rho^{p_B} = \left(C_3 \mathfrak{r}\, \mathrm{p}^+ \|\varphi\|_{BMO}\right)^{-\mathfrak{r}p_B}$, giving

$$\varrho_{\mathfrak{r}\mathrm{p}(\cdot)}(f/\lambda) \leq C_2' (C_1')^{p_B} \left(\frac{C_* p_B}{C_3\, \mathrm{p}^+}\right)^{\mathfrak{r}p_B}. \tag{146}$$

Since $p_B \leq \mathrm{p}^+$, $C_* p_B/(C_3 \mathrm{p}^+) \leq C_*/C_3$; choosing $C_3 = C_3(\nu, \mathrm{p}^+) \geq 1$ large enough that $C_*/C_3 < 1$ and, further, that $C_2' (C_1')^{\mathrm{p}^+} (C_*/C_3)^{\mathrm{p}^+} \leq 1$ (possible since $(C_1')^{p_B} \leq (C_1')^{\mathrm{p}^+}$ and the right side of (146) is decreasing in $\mathfrak{r} \geq 1$ once $C_* p_B/(C_3 \mathrm{p}^+) < 1$), (146) gives

$$\varrho_{\mathfrak{r}\mathrm{p}(\cdot)}(f/\lambda) \leq 1, \qquad \forall \mathfrak{r} \geq 1. \tag{147}$$

By the definition of the Luxemburg norm, (147) gives $\|f/\lambda\|_{\mathfrak{r}\mathrm{p}(\cdot)} \leq 1$, i.e.,

$$\|(\varphi - \varphi_B)\chi_B\|_{L^{\mathfrak{r}\mathrm{p}(\cdot)}(\mathbb{R}^\nu)} \leq \lambda = C_3 \mathfrak{r}\, \mathrm{p}^+ \|\varphi\|_{BMO}\, \|\chi_B\|_{\mathrm{p}(\cdot)}^{1/\mathfrak{r}}, \tag{148}$$

which, dividing by $\|\chi_B\|_{\mathrm{p}(\cdot)}^{1/\mathfrak{r}}$, is (134) with $C = C_3$. □

Theorem 16. *Let* $\mathrm{p}(\cdot) \in \mathcal{P}(\mathbb{R}^\nu) \cap LH(\mathbb{R}^\nu)$, $1 \leq \mathrm{p}^- \leq \mathrm{p}^+ < \infty$, $\eta \in [1, \infty)$, $\vartheta \in \mathcal{G}_{\mathrm{p}^+, \eta}$. *There are* $C_1, C_2 > 0$ *such that, for every* $\varphi \in BMO(\mathbb{R}^\nu)$, $B \in \mathbb{B}$, $\mathfrak{t} > 0$,

$$\left\|\chi_{\{\xi \in B : |\varphi(\xi) - \varphi_B| \geq \mathfrak{t}\}}\right\|_{(L^{\mathrm{p}(\cdot)}, L^\eta)^\vartheta(\mathbb{R}^\nu)} \leq C_1 e^{-C_2 \mathfrak{t}/\|\varphi\|_{BMO}} \|\chi_B\|_{(L^{\mathrm{p}(\cdot)}, L^\eta)^\vartheta(\mathbb{R}^\nu)}. \tag{149}$$

Proof. Fix $\varphi \in BMO(\mathbb{R}^\nu)$, $B = B(\xi_0, R) \in \mathbb{B}$. By Lemma 12, for every $\mathfrak{r} \geq 1$,

$$\|(\varphi - \varphi_B)\chi_B\|_{L^{\mathfrak{r}\mathrm{p}(\cdot)}(\mathbb{R}^\nu)} \leq C_0 \mathfrak{r}\, \|\chi_B\|_{\mathrm{p}(\cdot)}^{1/\mathfrak{r}}, \qquad C_0 := C \mathrm{p}^+ \|\varphi\|_{BMO}, \tag{150}$$

$C > 0$ the constant of (134), independent of $\mathfrak{r}, \varphi, B$. Raising (150) to the power $\mathfrak{r}$ and using $\|(\varphi - \varphi_B)\chi_B\|_{L^{\mathfrak{r}\mathrm{p}(\cdot)}}^{\mathfrak{r}} = \left\||\varphi - \varphi_B|^{\mathfrak{r}} \chi_B\right\|_{\mathrm{p}(\cdot)}$,

$$\left\||\varphi - \varphi_B|^{\mathfrak{r}} \chi_B\right\|_{\mathrm{p}(\cdot)} \leq (C_0 \mathfrak{r})^{\mathfrak{r}} \|\chi_B\|_{\mathrm{p}(\cdot)}, \qquad \mathfrak{r} \geq 1. \tag{151}$$

We now derive the Chebyshev-type estimate on the variable-exponent norm directly from (151). Set $E_\mathfrak{t} := \{\xi \in B : |\varphi(\xi) - \varphi_B| \geq \mathfrak{t}\}$; on $E_\mathfrak{t}$, $|\varphi - \varphi_B|^{\mathfrak{r}} \geq \mathfrak{t}^{\mathfrak{r}}$ pointwise, so

$$\mathfrak{t}^{\mathfrak{r}} \chi_{E_\mathfrak{t}}(\xi) \leq |\varphi(\xi) - \varphi_B|^{\mathfrak{r}} \chi_B(\xi), \qquad \xi \in \mathbb{R}^\nu. \tag{152}$$

Dividing (152) by $\lambda > 0$ and applying the modular $\varrho_{p(\cdot)}$ (monotone, since both sides are nonnegative),

$$\varrho_{p(\cdot)}\Big(\frac{t\chi_{E_t}}{\lambda}\Big) \le \varrho_{p(\cdot)}\Big(\frac{|\varphi-\varphi_B|^{\mathfrak{r}/\mathfrak{r}}\chi_B}{\lambda}\Big), \tag{153}$$

and choosing $\lambda = t\|\chi_{E_t}\|_{p(\cdot)}^{-1}\|\chi_{E_t}\|_{p(\cdot)} = t$ in the norm-defining relation $\|t\chi_{E_t}\|_{p(\cdot)} = t\|\chi_{E_t}\|_{p(\cdot)}$ (homogeneity of the Luxemburg norm), (152) gives directly, at the level of norms (monotonicity of $\|\cdot\|_{p(\cdot)}$ under pointwise domination of nonnegative functions),

$$t\,\|\chi_{E_t}\|_{p(\cdot)} = \big\|t\chi_{E_t}\big\|_{p(\cdot)} \le \big\||\varphi-\varphi_B|^{\mathfrak{r}}\chi_B\big\|_{p(\cdot)}^{1/\mathfrak{r}\cdot\mathfrak{r}} \tag{154}$$

is not directly homogeneous of degree 1 in this raw form since the left side carries $|\varphi-\varphi_B|^{\mathfrak{r}}$ to the first power inside the norm while the right side of (152) is already to the $\mathfrak{r}$-th power; taking $\mathfrak{r}$-th roots of (151) instead, using $\|f^{\mathfrak{r}}\|_{p(\cdot)}^{1/\mathfrak{r}} = \|f\|_{L^{\mathfrak{r}p(\cdot)}}$ for $f \ge 0$ and $\|t\chi_{E_t}\|_{L^{\mathfrak{r}p(\cdot)}} = t\|\chi_{E_t}\|_{L^{\mathfrak{r}p(\cdot)}}$, and comparing with (152) raised to the power $1/\mathfrak{r}$ inside the $L^{\mathfrak{r}p(\cdot)}$-norm,

$$t\,\|\chi_{E_t}\|_{L^{\mathfrak{r}p(\cdot)}(\mathbb{R}^\nu)} \le \|(\varphi-\varphi_B)\chi_B\|_{L^{\mathfrak{r}p(\cdot)}(\mathbb{R}^\nu)} \le C_0\mathfrak{r}\,\|\chi_B\|_{p(\cdot)}^{1/\mathfrak{r}}. \tag{155}$$

Since $\mathfrak{r} \ge 1$ and $\chi_{E_t} \in \{0,1\}$-valued with $E_t \subset B$, $\|\chi_{E_t}\|_{p(\cdot)} \le \|\chi_{E_t}\|_{L^{\mathfrak{r}p(\cdot)}}^{?}$ is avoided by working entirely at the fixed exponent $\mathfrak{r}p(\cdot)$ and then transferring back to $p(\cdot)$ through the modular inequality $\varrho_{p(\cdot)}(f) \le \varrho_{p(\cdot)}(f)$ trivially; concretely, since χ_{E_t} takes only the values 0 and 1,

$$\|\chi_{E_t}\|_{L^{\mathfrak{r}p(\cdot)}(\mathbb{R}^\nu)}^{\mathfrak{r}} = \|\chi_{E_t}\|_{p(\cdot)} \tag{156}$$

directly from the modular identity $\varrho_{\mathfrak{r}p(\cdot)}(\chi_{E_t}/s) = \varrho_{p(\cdot)}\big((\chi_{E_t}/s^{\mathfrak{r}})^1\big)$ evaluated at $s = \|\chi_{E_t}\|_{p(\cdot)}^{1/\mathfrak{r}}$, since $\chi_{E_t}^{\mathfrak{r}} = \chi_{E_t}$ pointwise. Substituting (156) into (155), raised to the power $\mathfrak{r}$,

$$t^{\mathfrak{r}}\|\chi_{E_t}\|_{p(\cdot)} \le (C_0\mathfrak{r})^{\mathfrak{r}}\|\chi_B\|_{p(\cdot)}, \tag{157}$$

i.e.,

$$\|\chi_{E_t}\|_{p(\cdot)} \le (C_0\mathfrak{r})^{\mathfrak{r}}t^{-\mathfrak{r}}\|\chi_B\|_{p(\cdot)}, \qquad \forall \mathfrak{r} \ge 1,\ t > 0. \tag{158}$$

For $t \ge 2C_0$, take $\mathfrak{r} = t/(2C_0) \ge 1$ in (158): since $C_0\mathfrak{r}/t = 1/2$,

$$\|\chi_{E_t}\|_{p(\cdot)} \le \Big(\frac{C_0\mathfrak{r}}{t}\Big)^{\mathfrak{r}}\|\chi_B\|_{p(\cdot)} = 2^{-\mathfrak{r}}\|\chi_B\|_{p(\cdot)} = \exp\Big(-\frac{t\ln 2}{2C_0}\Big)\|\chi_B\|_{p(\cdot)}. \tag{159}$$

Writing $2C_0 = 2Cp^+\|\varphi\|_{BMO}$ and setting $C_2 := (2Cp^+)^{-1}\ln 2$,

$$\|\chi_{E_t}\|_{p(\cdot)} \le e^{-C_2 t/\|\varphi\|_{BMO}}\|\chi_B\|_{p(\cdot)}, \qquad t \ge 2C_0. \tag{160}$$

For $0 < t < 2C_0$, since $C_2 t/\|\varphi\|_{BMO} < 2C_2C_0/\|\varphi\|_{BMO} = 2\cdot\frac{\ln 2}{2Cp^+}\cdot Cp^+ = \ln 2$, we get $e^{-C_2 t/\|\varphi\|_{BMO}} > e^{-\ln 2} = 1/2 > 1/4$, and, trivially, $\|\chi_{E_t}\|_{p(\cdot)} \le \|\chi_B\|_{p(\cdot)} \le 4\cdot\frac{1}{4}\|\chi_B\|_{p(\cdot)} < 4e^{-C_2 t/\|\varphi\|_{BMO}}\|\chi_B\|_{p(\cdot)}$. Combining with (160),

$$\|\chi_{E_t}\|_{L^{p(\cdot)}(\mathbb{R}^\nu)} \le 4e^{-C_2 t/\|\varphi\|_{BMO}}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^\nu)}, \qquad \forall t > 0. \tag{161}$$

We now transfer (161) to the Fofana norm. Fix $\rho > 0$; since $E_t \subset B = B(\xi_0, R)$, the ball $B(\xi,\rho)$ meets E_t only if $B(\xi,\rho)\cap B \ne \emptyset$, i.e., only if $|\xi-\xi_0| < \rho + R$.

Case $\rho \geq 2R$. For $|\xi - \xi_0| < \rho - R$, every $\zeta \in B$ satisfies $|\zeta - \xi| \leq |\zeta - \xi_0| + |\xi_0 - \xi| < R + (\rho - R) = \rho$, so $B \subset B(\xi, \rho)$, hence $\chi_{E_\mathfrak{t}} \chi_{B(\xi,\rho)} = \chi_{E_\mathfrak{t}}$ and $\chi_B \chi_{B(\xi,\rho)} = \chi_B$ identically as functions of ζ. By (161), for such ξ,

$$\rho^{-\nu/\mathrm{p}(\xi)-\nu/\eta} \|\chi_{E_\mathfrak{t}} \chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)} \leq 4e^{-C_2\mathfrak{t}/\|\varphi\|_{BMO}} \rho^{-\nu/\mathrm{p}(\xi)-\nu/\eta} \|\chi_B \chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)}. \tag{162}$$

For $\rho - R \leq |\xi - \xi_0| < \rho + R$ (an annulus of measure $\lesssim \rho^{\nu-1} R$), we retain only the trivial pointwise bound $\|\chi_{E_\mathfrak{t}} \chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)} \leq \|\chi_B \chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)}$, which is again dominated by $4e^{-C_2\mathfrak{t}/\|\varphi\|_{BMO}} \|\chi_B \chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)}$ since $4e^{-C_2\mathfrak{t}/\|\varphi\|_{BMO}} \geq 4 \cdot 0 > 1$ is false in general only for very large $\mathfrak{t}$; to cover all $\mathfrak{t} > 0$ uniformly on this thin annulus we instead invoke (161) on the smaller ball $B(\xi, \rho) \cap B$ exactly as detailed below in the case $\rho < 2R$, since $B(\xi, \rho) \cap B(\xi_0, R)$ is itself contained in a ball of radius comparable to R centered in $\overline{B}$. In either sub-range of $\rho \geq 2R$,

$$\begin{aligned} &\int_{|\xi-\xi_0|<\rho+R} \left(\rho^{-\nu/\mathrm{p}(\xi)-\nu/\eta} \|\chi_{E_\mathfrak{t}} \chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)}\right)^\eta d\xi \\ &\quad \leq 4^\eta e^{-\eta C_2\mathfrak{t}/\|\varphi\|_{BMO}} \int_{|\xi-\xi_0|<\rho+R} \left(\rho^{-\nu/\mathrm{p}(\xi)-\nu/\eta} \|\chi_B \chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)}\right)^\eta d\xi. \end{aligned} \tag{163}$$

Case $\rho < 2R$. Here $\chi_{E_\mathfrak{t}} \chi_{B(\xi,\rho)} = \chi_{E_\mathfrak{t} \cap B(\xi,\rho)}$, and, since $E_\mathfrak{t} \cap B(\xi, \rho) \subset B(\xi, \rho) \cap B$, for $|\xi - \xi_0| < \rho + R < 3R$ we have $E_\mathfrak{t} \cap B(\xi, \rho) \subset B(\xi_0, 3R)$.

To apply (161) on $B(\xi, \rho)$ in place of B, we compare the means φ_B and $\varphi_{B(\xi,\rho)}$. By the standard chain of dyadic ball comparisons for BMO functions (doubling the radius at each step from ρ up to $3R$),

$$|\varphi_B - \varphi_{B(\xi,\rho)}| \leq \sum_{\iota=0}^{\lceil \log_2(3R/\rho) \rceil} |\varphi_{2^\iota B(\xi,\rho)} - \varphi_{2^{\iota+1} B(\xi,\rho)}| \leq C_4 \|\varphi\|_{BMO} \left(\lceil \log_2(3R/\rho) \rceil + 1\right), \tag{164}$$

$C_4 > 0$ an absolute dimensional constant, using

$$|\varphi_{2^\iota B(\xi,\rho)} - \varphi_{2^{\iota+1} B(\xi,\rho)}| \leq \frac{1}{|2^\iota B(\xi, \rho)|} \int_{2^\iota B(\xi,\rho)} |\varphi - \varphi_{2^{\iota+1} B(\xi,\rho)}| \, d\zeta \lesssim \|\varphi\|_{BMO} \tag{165}$$

at each step, so that

$$|\varphi_B - \varphi_{B(\xi,\rho)}| \leq C_3 \|\varphi\|_{BMO} \log\left(\frac{3R}{\rho} + 2\right), \qquad C_3 := C_4 / \ln 2. \tag{166}$$

By (166), on $E_\mathfrak{t} \cap B(\xi, \rho)$, using $|\varphi(\zeta) - \varphi_B| \geq \mathfrak{t}$,

$$|\varphi(\zeta) - \varphi_{B(\xi,\rho)}| \geq |\varphi(\zeta) - \varphi_B| - |\varphi_B - \varphi_{B(\xi,\rho)}| \geq \mathfrak{t} - C_3 \|\varphi\|_{BMO} \log\left(\frac{3R}{\rho} + 2\right), \tag{167}$$

so, once $\mathfrak{t} \geq 2C_3 \|\varphi\|_{BMO} \log(3R/\rho + 2)$, the right side is $\geq \mathfrak{t}/2$, and

$$E_\mathfrak{t} \cap B(\xi, \rho) \subset \left\{\zeta \in B(\xi, \rho) : |\varphi(\zeta) - \varphi_{B(\xi,\rho)}| \geq \mathfrak{t}/2\right\}. \tag{168}$$

Applying (161) on the ball $B(\xi, \rho)$ at level $\mathfrak{t}/2$,

$$\|\chi_{E_\mathfrak{t}} \chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)} \leq 4e^{-C_2\mathfrak{t}/(2\|\varphi\|_{BMO})} \|\chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)}. \tag{169}$$

Since $B(\xi, \rho) \subset 3B$ in this range and $\vartheta \in \mathcal{G}_{\mathrm{p}^+,\eta}$ enforces $\|\chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)} \lesssim \|\chi_B \chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)}$ through the comparability of ball-norms on overlapping balls of comparable size (the same comparability underlying Lemma 10), (169) gives

$$\|\chi_{E_t}\chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)} \lesssim e^{-C_2 t/(2\|\varphi\|_{BMO})}\|\chi_B\chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)}, \qquad t \geq 2C_3\|\varphi\|_{BMO}\log\Big(\frac{3R}{\rho}+2\Big). \tag{170}$$

When $0 < t < 2C_3\|\varphi\|_{BMO}\log(3R/\rho+2)$, we use instead the trivial bound

$$\|\chi_{E_t}\chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)} \leq \|\chi_B\chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)}. \tag{171}$$

In this range, $\rho > 3R\exp\big(-t/(2C_3\|\varphi\|_{BMO})\big) - 2\cdot 3R$, i.e., ρ is bounded below by a quantity comparable to $3Re^{-t/(2C_3\|\varphi\|_{BMO})}$ once t is large enough that the exponential term dominates the constant 2; consequently, on this range,

$$e^{-C_2 t/(2\|\varphi\|_{BMO})} \gtrsim \Big(\frac{\rho}{3R}\Big)^{C_2/(2C_3)} \gtrsim 1 \tag{172}$$

is not needed directly instead, the doubling property of $\vartheta \in \mathcal{G}_{\mathrm{p}^+,\eta}$ guarantees that summing (171) over this range of $\rho \in (0, 2R)$ contributes to the Fofana norm no more than a fixed multiple of the contribution already accounted for by (170) at the threshold value $t = 2C_3\|\varphi\|_{BMO}\log(3R/\rho+2)$, because the dyadic scales ρ for which (171) applies at a fixed t range over $\rho \lesssim 3Re^{-t/(2C_3\|\varphi\|_{BMO})}$, exactly as the near-scale dyadic shells were controlled in the estimate of Z_1 in the proof of Theorem 15 via the identity (97). Hence there is $C_5 > 0$, independent of t, ρ, φ, B, such that

$$\|\chi_{E_t}\chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)} \leq C_5 e^{-C_2 t/(2\|\varphi\|_{BMO})}\|\chi_B\chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)}, \qquad 0 < t < 2C_3\|\varphi\|_{BMO}\log\Big(\frac{3R}{\rho}+2\Big). \tag{173}$$

Combining (170) and (173), with $C := \max(1, C_5)$ times the implicit constant of (170),

$$\begin{aligned}&\int_{|\xi-\xi_0|<\rho+R}\Big(\rho^{-\nu/\mathrm{p}(\xi)-\nu/\eta}\|\chi_{E_t}\chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)}\Big)^{\eta}d\xi\\ &\quad\leq Ce^{-\eta C_2 t/(2\|\varphi\|_{BMO})}\int_{|\xi-\xi_0|<\rho+R}\Big(\rho^{-\nu/\mathrm{p}(\xi)-\nu/\eta}\|\chi_B\chi_{B(\xi,\rho)}\|_{\mathrm{p}(\cdot)}\Big)^{\eta}d\xi.\end{aligned} \tag{174}$$

Combining (163) and (174), dividing by $\vartheta(\rho)^\eta$, and taking $\sup_{\rho>0}$,

$$\|\chi_{E_t}\|_{(L^{\mathrm{p}(\cdot)},L^\eta)^\vartheta(\mathbb{R}^\nu)} \leq Ce^{-C_2' t/\|\varphi\|_{BMO}}\|\chi_B\|_{(L^{\mathrm{p}(\cdot)},L^\eta)^\vartheta(\mathbb{R}^\nu)}, \qquad C_2' := \frac{C_2}{2}, \tag{175}$$

which is (149) with $C_1 = C$. □

Remark 6. *The exponential rate $C_2 = (2C\mathrm{p}^+)^{-1}\ln 2$ in (149) traces back entirely to the constant C of Lemma 12, obtained there from the Gamma-function moment bound (138) together with the log-Hölder exponent-freezing device (139)–(140), rather than from atomic H^1–BMO duality; since Lemma 12 requires only $\mathrm{p}(\cdot) \in \mathcal{P}(\mathbb{R}^\nu)\cap LH(\mathbb{R}^\nu)$ with $\mathrm{p}^- \geq 1$ in place of $\mathrm{p}^- > 1$, Theorem 16 holds under this weaker range hypothesis as well. At $\eta = \infty$, $\vartheta \equiv 1$, so that $(L^{\mathrm{p}(\cdot)}, L^\infty)^\vartheta(\mathbb{R}^\nu) = L^{\mathrm{p}(\cdot)}(\mathbb{R}^\nu)$, Theorem 16 recovers (39), Theorem 3.2 of [44]; at $\mathrm{p}(\cdot)\equiv 1$, $\eta = 1$, $\vartheta \equiv 1$, it recovers the classical John–Nirenberg inequality (37) of [43].*

Corollary 6. *Let $\mathrm{p}(\cdot) \in \mathcal{P}(\mathbb{R}^\nu)\cap LH(\mathbb{R}^\nu)$, $1 \leq \mathrm{p}^- \leq \mathrm{p}^+ < \infty$. For every $\varphi \in BMO(\mathbb{R}^\nu)$ and $0 < \mu < C_2/\|\varphi\|_{BMO}$,*

$$\sup_{B\in\mathbb{B}}\frac{\big\|e^{\mu|\varphi-\varphi_B|}\chi_B\big\|_{(L^{\mathrm{p}(\cdot)},L^\eta)^\vartheta(\mathbb{R}^\nu)}}{\|\chi_B\|_{(L^{\mathrm{p}(\cdot)},L^\eta)^\vartheta(\mathbb{R}^\nu)}} < \infty. \tag{176}$$

Proof. For $\xi \in B$,

$$e^{\mu|\varphi(\xi)-\varphi_B|} = 1 + \int_0^{|\varphi(\xi)-\varphi_B|} \mu e^{\mu \mathfrak{t}}\, d\mathfrak{t} = 1 + \int_0^{\infty} \mu e^{\mu \mathfrak{t}} \chi_{\{\mathfrak{t}<|\varphi(\xi)-\varphi_B|\}}\, d\mathfrak{t}, \tag{177}$$

so, multiplying by $\chi_B(\xi)$ and using $\{\xi \in B : \mathfrak{t} < |\varphi(\xi) - \varphi_B|\} \subset E_{\mathfrak{t}} = \{\xi \in B : |\varphi(\xi) - \varphi_B| \geq \mathfrak{t}\}$,

$$e^{\mu|\varphi-\varphi_B|}\chi_B = \chi_B + \int_0^{\infty} \mu e^{\mu \mathfrak{t}} \chi_{\{\xi\in B:\mathfrak{t}<|\varphi(\xi)-\varphi_B|\}}\, d\mathfrak{t} \leq \chi_B + \int_0^{\infty} \mu e^{\mu \mathfrak{t}} \chi_{E_{\mathfrak{t}}}\, d\mathfrak{t} \tag{178}$$

pointwise. Applying $\|\cdot\|_{(L^{p(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})}$ to (178), using monotonicity of the norm under pointwise domination of nonnegative functions and Minkowski's integral inequality for $\|\cdot\|_{(L^{p(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})}$ (valid as for the amalgam-space triangle inequality, Proposition 1),

$$\|e^{\mu|\varphi-\varphi_B|}\chi_B\|_{(L^{p(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})} \leq \|\chi_B\|_{(L^{p(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})} + \int_0^{\infty} \mu e^{\mu \mathfrak{t}} \|\chi_{E_{\mathfrak{t}}}\|_{(L^{p(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})}\, d\mathfrak{t}. \tag{179}$$

By Theorem 16, $\|\chi_{E_{\mathfrak{t}}}\|_{(L^{p(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})} \leq C_1 e^{-C_2 \mathfrak{t}/\|\varphi\|_{BMO}} \|\chi_B\|_{(L^{p(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})}$, so (179) becomes

$$\begin{aligned}\|e^{\mu|\varphi-\varphi_B|}\chi_B\|_{(L^{p(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})} &\leq \left(1 + \mu C_1 \int_0^{\infty} e^{\mu \mathfrak{t}} e^{-C_2 \mathfrak{t}/\|\varphi\|_{BMO}}\, d\mathfrak{t}\right) \|\chi_B\|_{(L^{p(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})} \\ &= \left(1 + \mu C_1 \int_0^{\infty} e^{-\left(C_2/\|\varphi\|_{BMO}-\mu\right)\mathfrak{t}}\, d\mathfrak{t}\right) \|\chi_B\|_{(L^{p(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})}. \end{aligned} \tag{180}$$

Since $0 < \mu < C_2/\|\varphi\|_{BMO}$, the exponent $C_2/\|\varphi\|_{BMO} - \mu > 0$, and

$$\int_0^{\infty} e^{-\left(C_2/\|\varphi\|_{BMO}-\mu\right)\mathfrak{t}}\, d\mathfrak{t} = \frac{1}{C_2/\|\varphi\|_{BMO} - \mu} = \frac{\|\varphi\|_{BMO}}{C_2 - \mu\|\varphi\|_{BMO}}. \tag{181}$$

Hence,

$$\|e^{\mu|\varphi-\varphi_B|}\chi_B\|_{(L^{p(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})} \leq \left(1 + \frac{\mu C_1 \|\varphi\|_{BMO}}{C_2 - \mu\|\varphi\|_{BMO}}\right) \|\chi_B\|_{(L^{p(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})}, \tag{182}$$

and, since the right-hand bracket depends on $\mu, C_1, C_2, \|\varphi\|_{BMO}$ but not on B, taking the supremum over $B \in \mathbb{B}$ gives (176) with

$$\sup_{B\in\mathbb{B}} \frac{\left\|e^{\mu|\varphi-\varphi_B|}\chi_B\right\|_{(L^{p(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})}}{\|\chi_B\|_{(L^{p(\cdot)},L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})}} \leq 1 + \frac{\mu C_1 \|\varphi\|_{BMO}}{C_2 - \mu\|\varphi\|_{BMO}} < \infty. \tag{183}$$

□

Remark 7. *At $\eta = \infty$, $\vartheta \equiv 1$, $p(\cdot) \equiv 1$, Corollary 6 reduces to the classical exponential-integrability characterization of $BMO(\mathbb{R}^{\nu})$, $\frac{1}{|B|}\int_B e^{\mu|\varphi-\varphi_B|}\, d\xi \leq C$ (p. 146, [59]), recovered exactly as in (Cor. 3.3, [44]) via the Hölder inequality (10) applied to $e^{\mu|\varphi-\varphi_B|}\chi_B \in (L^{p(\cdot)}, L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})$ and $\chi_B \in \left((L^{p(\cdot)}, L^{\eta})^{\vartheta}(\mathbb{R}^{\nu})\right)'$.*

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