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Semicomplete Leibniz Algebras

Chaleamrach Malangpoo, Suchada Pongprasert * and Nissara Siripitakchai

Department of Mathematics, Faculty of Science, Srinakharinwirot University, 114 Sukhumvit 23, Bangkok 10110, Thailand; chaleamrach.malangpoo@g.swu.ac.th (C.M.); nissara@g.swu.ac.th (N.S.)

* Correspondence: suchadapo@g.swu.ac.th

Abstract

We study the notion of semicomplete Leibniz algebras and investigate their fundamental properties. In particular, we classify all non-Lie Leibniz algebras of dimension at most three according to their semicompleteness and establish structural results concerning direct sums, extensions, and holomorphs, obtaining Leibniz algebra analogues of key results from the theory of complete and semicomplete Lie algebras. As an application, we explicitly determine the algebras of derivations and inner derivations of all complex perfect non-semisimple Lie algebras of dimension at most nine and show that no such Lie algebra is semicomplete.

Keywords: Leibniz algebra; semicomplete; direct sum; extension; holomorph; semisimple

MSC: 17A32; 17A60

1. Introduction

In 1965, Bloh [1] introduced a class of algebras that he termed D-algebras; these were later popularized by Loday [2] in 1993 under the name Leibniz algebras and developed further by Loday and Pirashvili [3]. A *(left) Leibniz algebra* A is a vector space over a field $\mathbb{F}$ equipped with a bilinear bracket operation $[\ ,\ ] : A \times A \to A$ satisfying the Leibniz identity: $[a,[b,c]] = [[a,b],c] + [b,[a,c]]$ for all $a,b,c \in A$. Leibniz algebras constitute a natural generalization of Lie algebras: a Leibniz algebra A is a Lie algebra if and only if $\mathrm{Leib}(A) = \{0\}$, where $\mathrm{Leib}(A) = \mathrm{span}_{\mathbb{F}}\{[a,a] \mid a \in A\}$. Over the past several decades, numerous researchers have worked to establish Leibniz algebra analogues of fundamental results from the theory of Lie algebras, including versions of Lie's theorem, Engel's theorem, Cartan's criterion, and Levi's theorem (see, e.g., Demir, Misra, and Stitzinger [4]).

The study of Lie algebras with only inner derivations has a long history. In 1939, Zassenhaus [5] established that every finite-dimensional semisimple Lie algebra over a field of characteristic zero has only inner derivations. The notion of a *complete* Lie algebra, that is, a Lie algebra with a trivial center whose derivations are all inner, was introduced by Chevalley [6] in 1944 and formalized by Jacobson [7] in 1962, who systematically studied its key properties. Meng [8] further developed the theory in 1994, establishing necessary and sufficient conditions for a Lie algebra to be complete in terms of its extensions and holomorphs.

The notion of a *sympathetic* Lie algebra, i.e., a Lie algebra that is both perfect and complete, was introduced by Angelopoulos [9]. In 1996, Benayadi [10] constructed a non-semisimple sympathetic Lie algebra of dimension 25 over the complex numbers, demonstrating that completeness among perfect Lie algebras is not restricted to the semisimple case.

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In 2013, Ancochea Bermúdez and Campoamor-Stursberg [11] proposed a definition of completeness for Leibniz algebras that closely mirrors the definition of complete Lie algebras. Several years later, Boyle, Misra, and Stitzinger [12] (Definition 3.1) refined this concept by defining a Leibniz algebra A as *complete* if the center $Z(A/\operatorname{Leib}(A))$ is trivial and every derivation of A is inner. More precisely, A is complete if $Z(A/\operatorname{Leib}(A)) = \{0\}$ and, for each $\delta \in \operatorname{Der}(A)$—the Lie algebra of derivations of A—there exists an element $x \in A$ such that $\operatorname{im}(\delta - L_x) \subseteq \operatorname{Leib}(A)$, where $L_x : A \to A$ denotes the *left multiplication operator* defined by $L_x(b) = [x, b]$ for all $b \in A$. Building on this framework, Patlertsin, Pongprasert, and Rungratgasame [13] investigated the properties of $\operatorname{IDer}(A)$, the Lie algebra of inner derivations of A, while Misra, Patlertsin, Pongprasert, and Rungratgasame [14] studied complete Leibniz algebras with a focus on their extensions and holomorphs.

In 2016, Saeedi and Sheikh-Mohseni [15] introduced the notion of semicomplete Lie algebras, defining a Lie algebra L as *semicomplete* if every ID-derivation, a derivation whose image is contained in L^2, is inner. Subsequently, Boyle [16] extended this notion to Leibniz algebras in her Ph.D. thesis, defining a Leibniz algebra as *semicomplete* if every ID-derivation is inner in the sense of [12] and computed several foundational examples. In the present paper, we undertake a systematic study of semicomplete Leibniz algebras, establishing structural results and classifications that extend key results from the theory of complete and semicomplete Lie algebras to the Leibniz algebra setting.

The paper is organized as follows. In Section 2, we recall the definition of semicomplete Leibniz algebras from [16], establish basic properties, and classify all non-Lie Leibniz algebras of dimension at most three according to whether they are semicomplete. In Section 3, we examine how the property of semicompleteness behaves under direct sums. We prove that if a direct sum of Leibniz algebras is semicomplete, then each summand is semicomplete (Theorem 1), and establish that the converse holds when the summands are characteristic ideals (Theorem 2). We also provide an example showing that the characteristic ideal hypothesis cannot be removed in general (Example 3). In Section 4, we study the holomorph of a Leibniz algebra and a distinguished subalgebra constructed from the algebra and its ID-derivations. We establish properties of the left centralizer within this subalgebra and, using the notion of special extensions, obtain a characterization of semicomplete Leibniz algebras (Theorem 3) that generalizes [15] (Theorem 2.5) to the Leibniz algebra setting. In Section 5, we investigate the interplay between perfectness and semicompleteness. In particular, we explicitly determine the algebras of derivations and inner derivations of all complex perfect non-semisimple Lie algebras of dimension at most nine, using the classification provided by Burde, Dekimpe, and Monadjem [17], and show that no such Lie algebra is semicomplete. This demonstrates that perfectness alone is not sufficient to guarantee semicompleteness outside the semisimple setting.

Throughout this work, in accordance with Barnes [18], the term Leibniz algebra specifically refers to left Leibniz algebra. All algebras are assumed to be finite-dimensional over a field $\mathbb{F}$; results requiring $\mathbb{F} = \mathbb{C}$ are explicitly indicated.

2. Semicomplete Leibniz Algebras

For a Lie algebra L, a derivation $\delta \in \operatorname{Der}(L)$ is called an *ID-derivation* if its image is contained in the derived algebra L^2. The Lie algebra L is said to be *semicomplete* if every ID-derivation is inner [15]. We now recall the extension of this notion to the setting of Leibniz algebras.

Definition 1. *A derivation $\delta \in \operatorname{Der}(A)$ of a Leibniz algebra A is called an ID-derivation if $\operatorname{im}(\delta) \subseteq A^2$. The set of all ID-derivations of A is denoted by* $\operatorname{ID}(A)$.

Lemma 1. *Let A be a Leibniz algebra. Then $\mathrm{ID}(A)$ is a subalgebra of $\mathrm{Der}(A)$ and hence forms a Lie algebra under the commutator bracket.*

Proof. It is clear that $\mathrm{ID}(A)$ is a subspace of $\mathrm{Der}(A)$. Since $\mathrm{Der}(A)$ forms a Lie algebra under the commutator bracket, it suffices to show that $\mathrm{ID}(A)$ is closed under this bracket. Let $\delta_1, \delta_2 \in \mathrm{ID}(A)$. For all $a \in A$, we have $[\delta_1, \delta_2](a) = \delta_1(\delta_2(a)) - \delta_2(\delta_1(a)) \in A^2$. Therefore, $\mathrm{im}([\delta_1, \delta_2]) \subseteq A^2$, which implies $[\delta_1, \delta_2] \in \mathrm{ID}(A)$. □

Remark 1. *Since its proof uses only the bilinearity of the bracket, Lemma 1 holds for any non-associative algebra. We note that A^2 is invariant under every derivation of A: if $\delta \in \mathrm{Der}(A)$ and $x = \sum_i [a_i, b_i] \in A^2$, then $\delta(x) = \sum_i \big([\delta(a_i), b_i] + [a_i, \delta(b_i)]\big) \in A^2$.*

In alignment with the definition of completeness for Leibniz algebras given by Boyle, Misra, and Stitzinger [12] (Definition 3.1), the notion of semicomplete Leibniz algebras was introduced by Boyle [16] as follows.

Definition 2 ([16])**.** *A Leibniz algebra A is said to be semicomplete if for every $\delta \in \mathrm{ID}(A)$, there exists $x \in A$ such that $\mathrm{im}(\delta - L_x) \subseteq \mathrm{Leib}(A)$.*

Throughout, $\mathrm{IDer}(A)$ denotes the set of derivations that are inner in the sense of [12] (Definition 3.1)—that is, $\mathrm{IDer}(A) = \{\delta \in \mathrm{Der}(A) \mid \exists\, x \in A : \mathrm{im}(\delta - L_x) \subseteq \mathrm{Leib}(A)\}$. With this convention applied, Definition 2 says precisely that A is semicomplete when $\mathrm{ID}(A) = \mathrm{IDer}(A)$.

Remark 2. *Since $\mathrm{Leib}(A)$ is a characteristic ideal of A via [12] (Proposition 2.1), every $\delta \in \mathrm{Der}(A)$ induces a derivation $\bar{\delta}$ of the Lie algebra $A/\,\mathrm{Leib}(A)$, given by $\bar{\delta}(\bar{a}) = \overline{\delta(a)}$. For $x \in A$, we then have $\bar{\delta} = \mathrm{ad}_{\bar{x}}$ if and only if $\delta(a) - [x, a] \in \mathrm{Leib}(A)$ for all $a \in A$, that is, if and only if $\mathrm{im}(\delta - L_x) \subseteq \mathrm{Leib}(A)$. As every element of $A/\,\mathrm{Leib}(A)$ is of the form $\bar{x}$, it follows that $\delta \in \mathrm{IDer}(A)$ if and only if $\bar{\delta}$ is an inner derivation of $A/\,\mathrm{Leib}(A)$. Thus, A is semicomplete if and only if every ID-derivation of A induces an inner derivation of $A/\,\mathrm{Leib}(A)$. Note that, in contrast with completeness in the sense of [12] (Definition 3.1), semicompleteness imposes no conditions on $Z(A/\,\mathrm{Leib}(A))$; it constrains only the ID-derivations.*

Below, we record several immediate consequences of this definition.

Remark 3 (cf. [16] (Chapter 6))**.** *Let A be a Leibniz algebra.*

(i) *Since $\mathrm{Leib}(A) \subseteq A^2$, it follows that $\mathrm{IDer}(A) \subseteq \mathrm{ID}(A) \subseteq \mathrm{Der}(A)$. In particular, every complete Leibniz algebra is semicomplete.*

(ii) *If $A^2 = \mathrm{Leib}(A)$, then $\mathrm{ID}(A) = \mathrm{IDer}(A)$, and hence A is semicomplete.*

(iii) *If A is a Lie algebra, then the Leibniz and Lie notions of semicompleteness coincide. Indeed, $\mathrm{Leib}(A) = \{0\}$, so, for $\delta \in \mathrm{ID}(A)$, the condition $\mathrm{im}(\delta - L_x) \subseteq \mathrm{Leib}(A)$ reads as follows: $\delta = L_x = \mathrm{ad}_x$; hence, $\mathrm{IDer}(A) = \{\mathrm{ad}_x \mid x \in A\}$, and the two notions agree in both directions.*

We now record the semicompleteness of all complex non-Lie Leibniz algebras of dimension at most three. We begin by noting that there is no non-Lie Leibniz algebra of dimension one.

If A is a two-dimensional non-Lie Leibniz algebra, then through the classification of two-dimensional Leibniz algebras in [4], we may choose $x \in A$ with $x^2 \neq 0$ and $A = \mathrm{span}\{x, x^2\}$. In this case, $\mathrm{Leib}(A) = \mathrm{span}\{x^2\} = A^2$, so the conclusion of Remark 3(ii) is that A is semicomplete.

Now, suppose A is a three-dimensional complex non-Lie Leibniz algebra. According to the classification in [4], which was independently confirmed by the algorithmic classification developed by Casas, Insua, Ladra and Ladra [19], A is isomorphic to a Leibniz algebra with basis $\{x, y, z\}$ whose nonzero multiplications are given by one of Cases 1–5 if A is nilpotent or by one of Cases 6–12 otherwise.

Case 1: $[x, x] = y,\ [x, y] = z.$
Case 2: $[x, x] = z.$
Case 3: $[x, x] = z,\ [y, y] = z.$
Case 4: $[x, y] = z,\ [y, x] = -z,\ [y, y] = z.$
Case 5: $[x, y] = z,\ [y, x] = \alpha z,\ \alpha \in \mathbb{F} \setminus \{-1, 1\}.$
Case 6: $[x, z] = z.$
Case 7: $[x, z] = \alpha z,\ [x, y] = y,\ [y, x] = -y,\ \alpha \in \mathbb{F} \setminus \{0\}.$
Case 8: $[x, y] = y,\ [y, x] = -y,\ [x, x] = z.$
Case 9: $[x, z] = 2z,\ [y, y] = z,\ [x, y] = y,\ [y, x] = -y,\ [x, x] = z.$
Case 10: $[x, y] = y,\ [x, z] = \alpha z,\ \alpha \in \mathbb{F} \setminus \{0\}.$
Case 11: $[x, z] = z + y,\ [x, y] = y.$
Case 12: $[x, z] = y,\ [x, y] = y,\ [x, x] = z.$

In Cases 1–6 and 10–12, we have $A^2 = \mathrm{Leib}(A)$, so A is semicomplete according to Remark 3(ii). For Cases 7–9, it was shown in [20] (Theorem 3.8) that these algebras are complete; hence, according to Remark 3(i), they are also semicomplete.

Proposition 1. *Every complex non-Lie Leibniz algebra of dimension at most three is semicomplete.*

In the following examples, we present four-dimensional and five-dimensional semicomplete Leibniz algebras, which will be needed later.

Example 1. *Consider the Leibniz algebra $A = \mathrm{span}\{w, x, y, z\}$ over $\mathbb{C}$ with nonzero multiplications given by $[w, w] = z, [w, x] = y, [x, w] = -y$, as given in [21] (the algebra also appears in [16] (Example 6.2.8)). Since the only nonzero products are $[w, w] = z$, $[w, x] = y$ and $[x, w] = -y$, we have $A^2 = \mathrm{span}\{y, z\}$, while $[a, a] = \alpha^2[w, w] + \alpha\beta([w, x] + [x, w]) = \alpha^2 z$ for $a = \alpha w + \beta x + \gamma y + \eta z$, so $\mathrm{Leib}(A) = \mathrm{span}\{z\}$. Thus, $\mathrm{Leib}(A) = \mathrm{span}\{z\} \neq \mathrm{span}\{y, z\} = A^2$. Moreover, $\mathrm{Der}(A) = \mathrm{span}\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7\}$, where*

$$
\begin{array}{llll}
\delta_1(w) = w, & \delta_1(x) = 0, & \delta_1(y) = y, & \delta_1(z) = 2z, \\
\delta_2(w) = -w, & \delta_2(x) = x, & \delta_2(y) = 0, & \delta_2(z) = -2z, \\
\delta_3(w) = x, & \delta_3(x) = 0, & \delta_3(y) = 0, & \delta_3(z) = 0, \\
\delta_4(w) = y, & \delta_4(x) = 0, & \delta_4(y) = 0, & \delta_4(z) = 0, \\
\delta_5(w) = 0, & \delta_5(x) = y, & \delta_5(y) = 0, & \delta_5(z) = 0, \\
\delta_6(w) = z, & \delta_6(x) = 0, & \delta_6(y) = 0, & \delta_6(z) = 0, \\
\delta_7(w) = 0, & \delta_7(x) = z, & \delta_7(y) = 0, & \delta_7(z) = 0.
\end{array}
$$

Therefore, $\mathrm{ID}(A) = \mathrm{span}\{\delta_4, \delta_5, \delta_6, \delta_7\}$. Here, $\mathrm{Leib}(A) = \mathrm{span}\{z\}$. The derivations δ_6 and δ_7 already map into $\mathrm{span}\{z\}$, so $\mathrm{im}(\delta_6 - L_0) \subseteq \mathrm{Leib}(A)$ and $\mathrm{im}(\delta_7 - L_0) \subseteq \mathrm{Leib}(A)$. For the remaining two, $L_{-x}(w) = -[x, w] = y = \delta_4(w)$, while L_{-x} vanishes on x, y, and z, so $\mathrm{im}(\delta_4 - L_{-x}) = \{0\} \subseteq \mathrm{Leib}(A)$; and $L_w(x) = [w, x] = y = \delta_5(x)$, $L_w(w) = z$, $L_w(y) = L_w(z) = 0$, so $\mathrm{im}(\delta_5 - L_w) = \mathrm{span}\{z\} = \mathrm{Leib}(A)$. Hence, A is semicomplete.

Example 2. *Consider the algebra $A = \mathrm{span}\{w, x, y, z, u\}$ over $\mathbb{C}$ with nonzero multiplications given by $[w, w] = z, [w, x] = y, [x, w] = -y, [u, x] = y$. It is easy to verify that A is a Leibniz algebra with $\mathrm{Leib}(A) = \mathrm{span}\{y, z\} = A^2$. Hence, A is semicomplete.*

In [12] (Proposition 3.2), it was proved that for a Leibniz algebra A, if the Lie algebra $A/\mathrm{Leib}(A)$ is complete, then A is a complete Leibniz algebra. We obtain an analogous result for semicompleteness in the following proposition.

Proposition 2. *Let A be a Leibniz algebra. If the Lie algebra $A/\mathrm{Leib}(A)$ is semicomplete, then A is a semicomplete Leibniz algebra.*

Proof. Suppose that $A/\mathrm{Leib}(A)$ is a semicomplete Lie algebra. To show that A is semicomplete, let $\delta \in \mathrm{ID}(A)$. According to [12] (Proposition 2.1), $\mathrm{Leib}(A)$ is a characteristic ideal of A; hence, the map $\bar{\delta} : A/\mathrm{Leib}(A) \to A/\mathrm{Leib}(A)$ defined by $\bar{\delta}(x + \mathrm{Leib}(A)) = \delta(x) + \mathrm{Leib}(A)$ for all $x \in A$ is a well-defined linear map. Since $\delta(A) \subseteq A^2$, we know that $\bar{\delta} \in \mathrm{ID}(A/\mathrm{Leib}(A))$. By assumption, $A/\mathrm{Leib}(A)$ is semicomplete; hence, there is $x_0 \in A$ such that $\bar{\delta} = \mathrm{ad}_{x_0+\mathrm{Leib}(A)}$. Now, let $\beta = \delta - L_{x_0}$. Then, for all $y \in A$, we have $\beta(y) + \mathrm{Leib}(A) = \delta(y) - [x_0, y] + \mathrm{Leib}(A) = \bar{\delta}(y + \mathrm{Leib}(A)) - \mathrm{ad}_{x_0+\mathrm{Leib}(A)}(y + \mathrm{Leib}(A)) = \mathrm{Leib}(A)$. Hence, $\beta(y) \in \mathrm{Leib}(A)$ for all $y \in A$. Thus, $\mathrm{im}(\delta - L_{x_0}) = \mathrm{im}(\beta) \subseteq \mathrm{Leib}(A)$, which implies that A is semicomplete. $\square$

Remark 4. *The converse of Proposition 2 does not hold in general. It holds whenever every derivation of $A/\mathrm{Leib}(A)$ with an image in its square lifts to a derivation of A, which is automatic when $\mathrm{Leib}(A) = 0$. Such a lift need not exist in general.*

3. Direct Sum Decompositions of Leibniz Algebras

We assume throughout this section that the Leibniz algebra L is the direct sum of two ideals, i.e., $L = A \oplus B$ where A and B are ideals of L. In [14] (Theorem 4.7) it is proved that L is complete if and only if both A and B are complete. We establish the analogous result for semicompleteness: L is semicomplete if and only if both A and B are semicomplete, provided that A and B are characteristic ideals of L. We also show that the characteristic ideal hypothesis cannot be removed in general.

The following result is due to Boyle [16] (Proposition 3). We reproduce the proof for the reader's convenience, since it is the starting point for Theorem 2 below.

Theorem 1 ([16] (Proposition 3))**.** *If L is semicomplete, then both A and B are semicomplete.*

Proof. Assume that L is semicomplete. To show that A is semicomplete, let $\delta \in \mathrm{ID}(A)$. We define $\bar{\delta} : L \to L$ using $\bar{\delta}(x + y) = \delta(x)$ for all $x \in A$ and $y \in B$. Then, for any $x_1, x_2 \in A$ and $y_1, y_2 \in B$, we have $\bar{\delta}([x_1 + y_1, x_2 + y_2]) = \bar{\delta}([x_1, x_2] + [y_1, y_2]) = \delta([x_1, x_2]) = [\delta(x_1), x_2] + [x_1, \delta(x_2)] = [\delta(x_1), x_2 + y_2] + [x_1 + y_1, \delta(x_2)] = [\bar{\delta}(x_1 + y_1), x_2 + y_2] + [x_1 + y_1, \bar{\delta}(x_2 + y_2)]$. Hence, $\bar{\delta} \in \mathrm{Der}(L)$. We also have $\bar{\delta}(L) = \delta(A) \subseteq A^2 \subseteq L^2$, so $\bar{\delta} \in \mathrm{ID}(L)$. Since L is semicomplete, there exists $a + b \in L$ with $a \in A$ and $b \in B$ such that $\mathrm{im}(\bar{\delta} - L_{a+b}) \subseteq \mathrm{Leib}(L)$.

Let $x \in A$. Then, $(\bar{\delta} - L_{a+b})(x) = \delta(x) - ([a, x] + [b, x]) = (\delta - L_a)(x) \in \mathrm{Leib}(L)$, where the last equality uses the fact that $[b, x] = 0$ since A and B are ideals with $A \cap B = \{0\}$. According to [14] (Proposition 4.1), $\mathrm{Leib}(L) = \mathrm{Leib}(A) \oplus \mathrm{Leib}(B)$, so there exist $w_1 \in \mathrm{Leib}(A)$ and $w_2 \in \mathrm{Leib}(B)$ such that $(\delta - L_a)(x) = w_1 + w_2$. Since $(\delta - L_a)(x) \in A$ and $w_2 \in B$, we have $(\delta - L_a)(x) - w_1 = w_2 \in A \cap B = \{0\}$, which implies $(\delta - L_a)(x) = w_1 \in \mathrm{Leib}(A)$. It follows that $\mathrm{im}(\delta - L_a) \subseteq \mathrm{Leib}(A)$, and hence A is semicomplete. Similarly, we can show that B is semicomplete. $\square$

The following example demonstrates that the converse of Theorem 1 does not hold in general; that is, the direct sum of two semicomplete ideals need not be semicomplete.

Example 3. *Consider Leibniz algebras* $A = \mathrm{span}\{w, x, y, z\}$ *and* $B = \mathrm{span}\{a, b, c, d\}$ *over* $\mathbb{C}$ *with nonzero multiplications defined by* $[w, w] = z$, $[w, x] = y$, $[x, w] = -y$ *for* A *and* $[a, a] = d$, $[a, b] = c$, $[b, a] = -c$ *for* B*, respectively. Note that* A *and* B *are isomorphic. According to Example 1, both* A *and* B *are semicomplete. Consider the direct sum* $L = A \oplus B$ *spanned by* $\{w, x, y, z, a, b, c, d\}$*. We have* $\mathrm{Leib}(L) = \mathrm{span}\{z, d\}$ *and* $L^2 = \mathrm{span}\{y, z, c, d\}$*. Define the linear map* $\delta : L \to L$ *using* $\delta(a) = y$ *and* $\delta(u) = 0$ *for all* $u \in \{w, x, y, z, b, c, d\}$*. One can verify directly that* δ *is a derivation of* L *with* $\mathrm{im}(\delta) = \mathrm{span}\{y\} \subseteq L^2$*, so* $\delta \in \mathrm{ID}(L)$*.*

We claim that δ *is not inner. If* $\mathrm{im}(\delta - L_k) \subseteq \mathrm{Leib}(L)$ *for some* $k \in L$*, then evaluating at* a *gives* $(\delta - L_k)(a) = y - [k, a] \in \mathrm{Leib}(L) = \mathrm{span}\{z, d\}$*. Write* $k = \alpha_1 w + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 a + \alpha_6 b + \alpha_7 c + \alpha_8 d$*. Then,* $[k, a] = \alpha_5[a, a] + \alpha_6[b, a] = \alpha_5 d - \alpha_6 c$*, so* $(\delta - L_k)(a) = y + \alpha_6 c - \alpha_5 d$*. Since* $\{y, c, d, z\}$ *are linearly independent, this element cannot belong to* $\mathrm{Leib}(L) = \mathrm{span}\{z, d\}$*, which is a contradiction. Therefore,* δ *is not inner, and* L *is not semicomplete.*

Note that in this example, neither A *nor* B *is a characteristic ideal of* L*. This motivates the characteristic ideal hypothesis in Theorem 2 below.*

The following lemma is needed to show that the converse of Theorem 1 holds when A and B are characteristic ideals of L.

Lemma 2. *If* A *and* B *are characteristic ideals of* L*, then* $\mathrm{ID}(L) = \mathrm{ID}(A) \oplus \mathrm{ID}(B)$*.*

Proof. Assume that A and B are characteristic ideals of L. We first observe that $L^2 = A^2 \oplus B^2$. For $\delta \in \mathrm{ID}(A)$, we extend δ to a derivation on L by defining $\delta(a + b) = \delta(a)$ for all $a \in A$ and $b \in B$. Similarly, for $\delta \in \mathrm{ID}(B)$, we define $\delta(a + b) = \delta(b)$. Thus $\mathrm{ID}(A) \subseteq \mathrm{ID}(L)$ and $\mathrm{ID}(B) \subseteq \mathrm{ID}(L)$. To show that $\mathrm{ID}(A)$ is an ideal of $\mathrm{ID}(L)$, let $\delta \in \mathrm{ID}(L)$ and $\delta_1 \in \mathrm{ID}(A)$. Since A is a characteristic ideal of L, we have $\delta(A) \subseteq A$. Combined with $\delta(L) \subseteq L^2$, we obtain $\delta(A^2) \subseteq A^2$. Thus, for all $a \in A$, $[\delta_1, \delta](a) = \delta_1(\delta(a)) - \delta(\delta_1(a)) \in A^2$, which implies $[\delta_1, \delta] \in \mathrm{ID}(A)$. Hence, $\mathrm{ID}(A)$ is an ideal of $\mathrm{ID}(L)$. Similarly, $\mathrm{ID}(B)$ is an ideal of $\mathrm{ID}(L)$. Clearly, $\mathrm{ID}(A) + \mathrm{ID}(B) \subseteq \mathrm{ID}(L)$ and $\mathrm{ID}(A) \cap \mathrm{ID}(B) = \{0\}$. To show $\mathrm{ID}(L) \subseteq \mathrm{ID}(A) + \mathrm{ID}(B)$, let $\delta \in \mathrm{ID}(L)$. Since A and B are characteristic ideals, $\delta(A) \subseteq A$ and $\delta(B) \subseteq B$. According to [14] (Proposition 4.3), $\delta = \delta|_A + \delta|_B$, where $\delta|_A \in \mathrm{ID}(A)$ and $\delta|_B \in \mathrm{ID}(B)$. Hence, $\delta \in \mathrm{ID}(A) + \mathrm{ID}(B)$. □

Theorem 2. *Let* $L = A \oplus B$ *be the direct sum of two characteristic ideals* A *and* B*. Then* L *is semicomplete if and only if* A *and* B *are semicomplete.*

Proof. The forward direction follows from Theorem 1. For the converse, assume that A and B are semicomplete. Let $\delta \in \mathrm{ID}(L)$. According to Lemma 2, there exist $\delta_1 \in \mathrm{ID}(A)$ and $\delta_2 \in \mathrm{ID}(B)$ such that $\delta = \delta_1 + \delta_2$. Since A is semicomplete, there exists $a \in A$ such that $\mathrm{im}(\delta_1 - L_a) \subseteq \mathrm{Leib}(A)$. Since B is semicomplete, there exists $b \in B$ such that $\mathrm{im}(\delta_2 - L_b) \subseteq \mathrm{Leib}(B)$. Then, for all $x \in A$ and $y \in B$, $(\delta - L_{a+b})(x + y) = (\delta_1 + \delta_2)(x + y) - [a + b, x + y] = \delta_1(x) + \delta_2(y) - [a, x] - [b, y] = (\delta_1 - L_a)(x) + (\delta_2 - L_b)(y) \in \mathrm{Leib}(A) + \mathrm{Leib}(B) = \mathrm{Leib}(L)$, where the last equality follows from [14] (Proposition 4.1). Hence, L is semicomplete. □

4. Holomorphs of Leibniz Algebras

In [12] (Section 4), Boyle, Misra, and Stitzinger introduced the notion of the *holomorph* of a Leibniz algebra A, denoted by $\mathrm{hol}(A)$, as the vector space $\mathrm{hol}(A) = A \oplus \mathrm{Der}(A)$ equipped with the bracket operation $[x + \delta_1, y + \delta_2] = [x, y] + \delta_1(y) + [L_x, \delta_2] + [\delta_1, \delta_2]$ for all $x, y \in A$ and $\delta_1, \delta_2 \in \mathrm{Der}(A)$; it is shown there that $\mathrm{hol}(A)$ is again a Leibniz algebra. In this section, we study the subalgebra $\mathrm{hol}'(A) = A \oplus \mathrm{ID}(A)$, which is indeed a subalgebra of $\mathrm{hol}(A)$ since $\mathrm{ID}(A)$ is a subalgebra of $\mathrm{Der}(A)$ according to Lemma 1. This subalgebra plays a central role in our characterization of semicomplete Leibniz algebras.

For two subspaces $M \subseteq N$ of $\mathrm{hol}(A)$, the *left centralizer of M in N* is defined as $Z^{\ell}_N(M) = \{x \in N \mid [x, M] = 0\}$. Through arguments analogous to those in [12] (Propositions 4.1 and 4.2), one can show that $Z^{\ell}_{\mathrm{hol}'(A)}(A) = \{x - L_x \mid x \in A\}$ and $A \cap Z^{\ell}_{\mathrm{hol}'(A)}(A) = Z^{\ell}(A)$.

We now recall the notion of a special extension of a Leibniz algebra. In [14] (Section 3), Misra, Patlertsin, Pongprasert, and Rungratgasame defined a Leibniz algebra B containing A as a *special extension* of A if A is an ideal of B. However, after careful inspection of the proofs in [14], only the condition that A must be a left ideal of B was used. Accordingly, we adopt the following weaker definition throughout this work.

Definition 3. *Let A be a Leibniz algebra. A Leibniz algebra B containing A is called a special extension of A if A is a left ideal of B.*

The following theorem characterizes semicompleteness in terms of special extensions and holomorphs. It extends [15] (Theorem 2.5), which establishes the analogous result for semicomplete Lie algebras, to the Leibniz algebra setting.

Theorem 3. *Let A be a Leibniz algebra. Then the following conditions are equivalent:*

(i) *A is semicomplete.*

(ii) *Any special extension B of A with $[B, A] \subseteq A^2$ can be written as $B = A + X$, where $X = \{x \in B \mid [x, A] \subseteq \mathrm{Leib}(A)\}$.*

(iii) $\mathrm{hol}'(A) = A + (Z^{\ell}_{\mathrm{hol}'(A)}(A) \oplus I')$, *where* $I' = \{\delta \in \mathrm{ID}(A) \mid \mathrm{im}(\delta) \subseteq \mathrm{Leib}(A)\}$.

Proof. $(i) \Rightarrow (ii)$ Assume that A is semicomplete. Let B be a special extension of A such that $[B, A] \subseteq A^2$. Clearly, $A + X \subseteq B$. To show that $B \subseteq A + X$, let $x \in B$. Since A is a left ideal of B, the restriction $L_x|_A$ is a derivation of A. Since $[B, A] \subseteq A^2$, it follows that $\mathrm{im}(L_x|_A) \subseteq A^2$, so $L_x|_A \in \mathrm{ID}(A)$. Because of the semicompleteness of A, there exists $a \in A$ such that $\mathrm{im}(L_x|_A - L_a) \subseteq \mathrm{Leib}(A)$, which implies $[x - a, A] \subseteq \mathrm{Leib}(A)$. Hence, $x - a \in X$, and therefore $x = a + (x - a) \in A + X$.

$(ii) \Rightarrow (iii)$ Suppose (ii) holds. Since A is a left ideal of $\mathrm{hol}(A)$, it follows that A is also a left ideal of $\mathrm{hol}'(A)$, so $\mathrm{hol}'(A)$ is a special extension of A. From the definition of $\mathrm{ID}(A)$, we have $[\mathrm{hol}'(A), A] \subseteq A^2$. Thus, by assumption, $\mathrm{hol}'(A) = A + X$, where $X = \{x + \delta \in \mathrm{hol}'(A) \mid [x + \delta, A] \subseteq \mathrm{Leib}(A)\}$. We first show that $Z^{\ell}_{\mathrm{hol}'(A)}(A) \cap I' = \{0\}$. If $\delta \in Z^{\ell}_{\mathrm{hol}'(A)}(A) \cap I'$, then, for all $x \in A$, we have $\delta(x) = [\delta, x] = 0$. This implies $\delta = 0$. To show that $A + (Z^{\ell}_{\mathrm{hol}'(A)}(A) \oplus I') \subseteq A + X$, let $a \in A + (Z^{\ell}_{\mathrm{hol}'(A)}(A) \oplus I')$. Then, $a = b + (c - L_c) + \delta$ for some $b, c \in A$ and $\delta \in I'$. For all $x \in A$, $[c - L_c + \delta, x] = [c, x] + (-L_c + \delta)(x) = \delta(x) \in \mathrm{Leib}(A)$. Thus, $c - L_c + \delta \in X$, implying $a = b + c - L_c + \delta \in A + X$. For the reverse inclusion, let $a \in A + X$. Since $A + X = \mathrm{hol}'(A) = A \oplus \mathrm{ID}(A)$, we can write $a = b + c + \delta_1 = d + \delta_2$ for some $b, d \in A$, $c + \delta_1 \in X$, and $\delta_2 \in \mathrm{ID}(A)$. Hence, $b + c = d$ and $\delta_1 = \delta_2$. Since $c + \delta_1 \in X$, we have $\mathrm{im}(L_c + \delta_2) = [c + \delta_2, A] = [c + \delta_1, A] \subseteq \mathrm{Leib}(A)$, so $L_c + \delta_2 \in I'$. We conclude that $a = d + \delta_2 = (d - c) + (c - L_c) + (L_c + \delta_2) \in A + (Z^{\ell}_{\mathrm{hol}'(A)}(A) \oplus I')$. Therefore, $\mathrm{hol}'(A) = A + (Z^{\ell}_{\mathrm{hol}'(A)}(A) \oplus I')$.

$(iii) \Rightarrow (i)$ Suppose (iii) holds. Let $\delta \in \mathrm{ID}(A)$. Since $\mathrm{hol}'(A) = A + (Z^{\ell}_{\mathrm{hol}'(A)}(A) \oplus I')$, we can write $\delta = x + (c - L_c) + \delta_1$ for some $x, c \in A$ and $\delta_1 \in I'$. For any $y \in A$, $\delta(y) =$

$[\delta, y] = [x, y] + [c, y] - L_c(y) + \delta_1(y) = [x, y] + \delta_1(y)$. Thus, $(\delta - L_x)(y) = \delta_1(y) \in \mathrm{Leib}(A)$, so $\mathrm{im}(\delta - L_x) \subseteq \mathrm{Leib}(A)$, implying that A is semicomplete. $\square$

Remark 5. *The set X in Theorem 3 coincides with the one in [14] (Theorem 3.7), which applies to complete Leibniz algebras. However, in the semicomplete case, $A \cap X$ may be strictly larger than* $\mathrm{Leib}(A)$*, as illustrated in the following example.*

Example 4. *Let $A = \mathrm{span}\{w, x, y, z\}$ and $B = \mathrm{span}\{w, x, y, z, u\}$ be the Leibniz algebras given in Example 1 and Example 2, respectively. Then B is a special extension of A with $[B, A] \subseteq A^2$. For $b = \alpha w + \beta x + \gamma y + \eta z + \mu u \in B$, one can find that $[b, w] = \alpha z - \beta y$, $[b, x] = (\alpha + \mu)y$ and $[b, y] = [b, z] = [b, u] = 0$. Hence, $[b, A] \subseteq \mathrm{Leib}(A) = \mathrm{span}\{z\}$ forces $\beta = 0$ and $\alpha + \mu = 0$, giving $X = \mathrm{span}\{w - u, y, z\}$. Therefore, $A \cap X = \mathrm{span}\{y, z\} \neq \mathrm{span}\{z\} = \mathrm{Leib}(A)$.*

5. Perfect Leibniz Algebras

A Leibniz algebra A is said to be *perfect* if $[A, A] = A$ and *sympathetic* if it is both perfect and complete. In this section, we study the relationship between perfectness and semicompleteness for Leibniz algebras. The following result follows from Corollary 4.5 in [22].

Lemma 3. *A is perfect if and only if $A/\mathrm{Leib}(A)$ is perfect.*

We observe that perfectness is preserved under direct sum decompositions.

Proposition 3. *Let $L = A \oplus B$, where A and B are ideals of L. If L is perfect, then both A and B are perfect.*

Proof. Since L is perfect, we have $A \oplus B = L = L^2 = A^2 \oplus B^2$. Since $A^2 \subseteq A$, $B^2 \subseteq B$, and $A \cap B = \{0\}$, it follows that $A = A^2$ and $B = B^2$. $\square$

The following result shows that every perfect ideal is characteristic. We note that a related result for Leibniz algebras also appears independently in [23].

Lemma 4. *Every perfect ideal of a Leibniz algebra is a characteristic ideal.*

Proof. Let A be a perfect ideal of a Leibniz algebra L, and let $\delta \in \mathrm{Der}(L)$. For any $x \in A$, since A is perfect, we may write $x = \sum_i [a_i, b_i]$ for some $a_i, b_i \in A$. Then, $\delta(x) = \sum_i \delta([a_i, b_i]) = \sum_i \big([\delta(a_i), b_i] + [a_i, \delta(b_i)]\big) \in [A, L] + [L, A] \subseteq A$, where the last inclusion uses the fact that A is an ideal of L. Hence, A is invariant under all derivations of L. $\square$

Combining Lemma 4 with Theorem 2, we immediately obtain the following.

Corollary 1. *Let $L = A \oplus B$ be the direct sum of two perfect ideals A and B. Then, L is semicomplete if and only if both A and B are semicomplete.*

We conclude this section by examining the semicompleteness of perfect algebras that are not semisimple. Recall that a Leibniz algebra A is said to be *semisimple* if its unique maximal solvable ideal satisfies $\mathrm{rad}(A) = \mathrm{Leib}(A)$. According to [4] (Corollary 5.5), every semisimple Leibniz algebra is perfect (see also [24] for the Levi decomposition underlying this), and according to [12] (Theorem 3.3), every semisimple Leibniz algebra is complete. In particular, every semisimple Leibniz algebra is semicomplete.

A natural question then arises: among perfect Lie algebras that are not semisimple, does perfectness alone guarantee semicompleteness? We show that the answer is negative in small dimensions: among all complex perfect non-semisimple Lie algebras of dimension at most nine, none is semicomplete. On the other hand, sympathetic Lie algebras (see Remark 6 below) show that perfect non-semisimple semicomplete Lie algebras do exist in higher dimensions.

To demonstrate this systematically, we consider all complex perfect non-semisimple Lie algebras of dimension at most nine, using the classification provided by Burde, Dekimpe, and Monadjem [17], which builds on the earlier work by Turkowski [25,26]. Specifically, according to [17] (Proposition 2.3), every complex perfect non-semisimple Lie algebra of dimension at most eight is isomorphic to exactly one of the twelve algebras listed there, and according to [17] (Proposition 2.4), every such algebra of dimension nine is isomorphic to exactly one of the algebras listed in that proposition. These lists are exhaustive and include the decomposable cases $\mathfrak{sl}_2(\mathbb{C}) \oplus L_{5,1}$, $\mathfrak{sl}_2(\mathbb{C}) \oplus L_{6,1}$ and $\mathfrak{sl}_2(\mathbb{C}) \oplus L_{6,2}$; for dimension at most eight, the list agrees with the independent classification of non-solvable algebraic Lie algebras obtained by Alev, Ooms and Van den Bergh [27]. For each algebra identified in this classification, we use Maple 15, utilizing the algorithm described in [28], to compute the Lie algebra $\mathrm{Der}(A)$ of derivations. We then determine the Lie algebra $\mathrm{IDer}(A)$ of inner derivations directly and thus determine whether the given algebra is semicomplete. Observe that the lowest dimension of a complex perfect non-semisimple Lie algebra is five [17].

The results are presented in Table 1. In each case, the matrices representing $\mathrm{Der}(A)$ and $\mathrm{IDer}(A)$ are displayed. Within each matrix, the symbols $\delta_1, \delta_2, \ldots$ denote independent parameters ranging over $\mathbb{C}$: distinct symbols are subject to no further relations, so the number of distinct symbols occurring in a matrix equals the dimension of the corresponding space. Since these algebras are perfect, every derivation is an ID-derivation. The algebra A is semicomplete if and only if $\mathrm{Der}(A) = \mathrm{IDer}(A)$, which can be verified by comparing the two matrices in each row. In every case, $\dim \mathrm{Der}(A) > \dim \mathrm{IDer}(A)$, so $\mathrm{IDer}(A)$ is a proper subspace of $\mathrm{Der}(A)$, and A admits an outer derivation. As an independent consistency check, the resulting values satisfy $\dim \mathrm{IDer}(A) = \dim A - \dim Z(A)$ in every row, with $\dim Z(A)$ as recorded in [17] (Propositions 2.3 and 2.4).

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Table 1. Derivations and inner derivations of complex perfect non-semisimple Lie algebras of dimension at most nine.

A	$\dim(A)$	$\mathrm{Der}(A)$	$\mathrm{IDer}(A)$
$L_{5,1}$	5	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 \\ -2\delta_1 & \delta_5-\delta_6 & 0 & 0 & 0 \\ 2\delta_2 & 0 & -\delta_5+\delta_6 & 0 & 0 \\ \delta_3 & -\delta_4 & 0 & \delta_5 & \delta_1 \\ \delta_4 & 0 & \delta_3 & \delta_2 & \delta_6 \end{bmatrix}$	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 \\ -2\delta_1 & 2\delta_5 & 0 & 0 & 0 \\ 2\delta_2 & 0 & -2\delta_5 & 0 & 0 \\ \delta_3 & -\delta_4 & 0 & \delta_5 & \delta_1 \\ \delta_4 & 0 & \delta_3 & \delta_2 & -\delta_5 \end{bmatrix}$
$L_{6,1} \cong L_{6,4}$	6	$\begin{bmatrix} 0 & -\delta_1 & -\delta_2 & 0 & 0 & 0 \\ \delta_1 & 0 & -\delta_3 & 0 & 0 & 0 \\ \delta_2 & \delta_3 & 0 & 0 & 0 & 0 \\ 0 & -\delta_4 & -\delta_5 & \delta_7 & -\delta_1 & -\delta_2 \\ \delta_4 & 0 & -\delta_6 & \delta_1 & \delta_7 & -\delta_3 \\ \delta_5 & \delta_6 & 0 & \delta_2 & \delta_3 & \delta_7 \end{bmatrix}$	$\begin{bmatrix} 0 & -\delta_1 & -\delta_2 & 0 & 0 & 0 \\ \delta_1 & 0 & -\delta_3 & 0 & 0 & 0 \\ \delta_2 & \delta_3 & 0 & 0 & 0 & 0 \\ 0 & -\delta_4 & -\delta_5 & 0 & -\delta_1 & -\delta_2 \\ \delta_4 & 0 & -\delta_6 & \delta_1 & 0 & -\delta_3 \\ \delta_5 & \delta_6 & 0 & \delta_2 & \delta_3 & 0 \end{bmatrix}$
$L_{6,2}$	6	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 \\ -2\delta_1 & -2\delta_5+\delta_6 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & 2\delta_5-\delta_6 & 0 & 0 & 0 \\ -\delta_3 & \delta_4 & 0 & -\delta_5+\delta_6 & \delta_1 & 0 \\ -\delta_4 & 0 & -\delta_3 & \delta_2 & \delta_5 & 0 \\ 0 & 0 & 0 & \delta_4 & \delta_3 & \delta_6 \end{bmatrix}$	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 \\ -2\delta_1 & -2\delta_5 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & 2\delta_5 & 0 & 0 & 0 \\ -\delta_3 & \delta_4 & 0 & -\delta_5 & \delta_1 & 0 \\ -\delta_4 & 0 & -\delta_3 & \delta_2 & \delta_5 & 0 \\ 0 & 0 & 0 & \delta_4 & \delta_3 & 0 \end{bmatrix}$
$L_{7,6}$	7	$\begin{bmatrix} 0 & \delta_2/3 & \delta_1 & 0 & 0 & 0 & 0 \\ -2\delta_1 & \delta_7-\delta_8 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2/3 & 0 & -\delta_7+\delta_8 & 0 & 0 & 0 & 0 \\ 3\delta_3 & 3\delta_4/2 & 0 & 3\delta_7-2\delta_8 & 3\delta_1 & 0 & 0 \\ \delta_4/2 & 2\delta_5/3 & \delta_3 & \delta_2/3 & 2\delta_7-\delta_8 & 2\delta_1 & 0 \\ -\delta_5/3 & -\delta_6/3 & \delta_4 & 0 & 2\delta_2/3 & \delta_7 & \delta_1 \\ \delta_6 & 0 & \delta_5 & 0 & 0 & \delta_2 & \delta_8 \end{bmatrix}$	$\begin{bmatrix} 0 & \delta_2/3 & \delta_1 & 0 & 0 & 0 & 0 \\ -2\delta_1 & 2\delta_7 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2/3 & 0 & -2\delta_7 & 0 & 0 & 0 & 0 \\ 3\delta_3 & 3\delta_4/2 & 0 & 3\delta_7 & 3\delta_1 & 0 & 0 \\ \delta_4/2 & 2\delta_5/3 & \delta_3 & \delta_2/3 & \delta_7 & 2\delta_1 & 0 \\ -\delta_5/3 & -\delta_6/3 & \delta_4 & 0 & 2\delta_2/3 & -\delta_7 & \delta_1 \\ \delta_6 & 0 & \delta_5 & 0 & 0 & \delta_2 & -3\delta_7 \end{bmatrix}$
$L_{7,7}$	7	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 \\ -2\delta_1 & \delta_7-\delta_8 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & -\delta_7+\delta_8 & 0 & 0 & 0 & 0 \\ \delta_3 & -\delta_4 & 0 & \delta_9+\delta_7-\delta_8 & \delta_1 & \delta_{10} & 0 \\ \delta_4 & 0 & \delta_3 & \delta_2 & \delta_9 & 0 & \delta_{10} \\ \delta_5 & -\delta_6 & 0 & \delta_{11} & 0 & \delta_7 & \delta_1 \\ \delta_6 & 0 & \delta_5 & 0 & \delta_{11} & \delta_2 & \delta_8 \end{bmatrix}$	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 \\ -2\delta_1 & 2\delta_7 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & -2\delta_7 & 0 & 0 & 0 & 0 \\ \delta_3 & -\delta_4 & 0 & \delta_7 & \delta_1 & 0 & 0 \\ \delta_4 & 0 & \delta_3 & \delta_2 & -\delta_7 & 0 & 0 \\ \delta_5 & -\delta_6 & 0 & 0 & 0 & \delta_7 & \delta_1 \\ \delta_6 & 0 & \delta_5 & 0 & 0 & \delta_2 & -\delta_7 \end{bmatrix}$
$\mathfrak{sl}_2 \oplus L_{5,1}$	8	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & \delta_8-\delta_9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & -\delta_8+\delta_9 & 0 & 0 & 0 & 0 & 0 \\ \delta_3 & -\delta_4 & 0 & \delta_8 & \delta_1 & 0 & 0 & 0 \\ \delta_4 & 0 & \delta_3 & \delta_2 & \delta_9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\delta_5/2 & -\delta_6/2 \\ 0 & 0 & 0 & 0 & 0 & \delta_6 & -\delta_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_5 & 0 & \delta_7 \end{bmatrix}$	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & 2\delta_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & -2\delta_8 & 0 & 0 & 0 & 0 & 0 \\ \delta_3 & -\delta_4 & 0 & \delta_8 & \delta_1 & 0 & 0 & 0 \\ \delta_4 & 0 & \delta_3 & \delta_2 & -\delta_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\delta_5/2 & -\delta_6/2 \\ 0 & 0 & 0 & 0 & 0 & \delta_6 & -\delta_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_5 & 0 & \delta_7 \end{bmatrix}$
$L_{8,13}^{\epsilon=0}$	8	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & \delta_7-\delta_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & -\delta_7+\delta_8 & 0 & 0 & 0 & 0 & 0 \\ -\delta_3 & \delta_4 & 0 & (\delta_7-\delta_8+\delta_9)/2 & \delta_1 & 0 & 0 & 0 \\ -\delta_4 & 0 & -\delta_3 & \delta_2 & (-\delta_7+\delta_8+\delta_9)/2 & 0 & 0 & 0 \\ \delta_5 & -\delta_6 & 0 & \delta_{10} & 0 & \delta_7 & \delta_1 & 0 \\ \delta_6 & 0 & \delta_5 & 0 & \delta_{10} & \delta_2 & \delta_8 & 0 \\ 0 & 0 & 0 & \delta_4 & \delta_3 & 0 & 0 & \delta_9 \end{bmatrix}$	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & 2\delta_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & -2\delta_7 & 0 & 0 & 0 & 0 & 0 \\ -\delta_3 & \delta_4 & 0 & \delta_7 & \delta_1 & 0 & 0 & 0 \\ -\delta_4 & 0 & -\delta_3 & \delta_2 & -\delta_7 & 0 & 0 & 0 \\ \delta_5 & -\delta_6 & 0 & 0 & 0 & \delta_7 & \delta_1 & 0 \\ \delta_6 & 0 & \delta_5 & 0 & 0 & \delta_2 & -\delta_7 & 0 \\ 0 & 0 & 0 & \delta_4 & \delta_3 & 0 & 0 & 0 \end{bmatrix}$
$L_{8,13}^{\epsilon=1} \cong L_{8,13}^{\epsilon=-1}$	8	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_1 & -2\delta_7+\delta_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & 2\delta_7-\delta_8 & 0 & 0 & 0 & 0 & 0 \\ -\delta_3 & \delta_4 & 0 & -\delta_7+\delta_8 & \delta_1 & -\delta_9 & 0 & 0 \\ -\delta_4 & 0 & -\delta_3 & \delta_2 & \delta_7 & 0 & -\delta_9 & 0 \\ -\delta_5 & \delta_6 & 0 & \delta_9 & 0 & -\delta_7+\delta_8 & \delta_1 & 0 \\ -\delta_6 & 0 & -\delta_5 & 0 & \delta_9 & \delta_2 & \delta_7 & 0 \\ 0 & 0 & 0 & \delta_4 & \delta_3 & \delta_6 & \delta_5 & \delta_8 \end{bmatrix}$	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_1 & -2\delta_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & 2\delta_7 & 0 & 0 & 0 & 0 & 0 \\ -\delta_3 & \delta_4 & 0 & -\delta_7 & \delta_1 & 0 & 0 & 0 \\ -\delta_4 & 0 & -\delta_3 & \delta_2 & \delta_7 & 0 & 0 & 0 \\ -\delta_5 & \delta_6 & 0 & 0 & 0 & -\delta_7 & \delta_1 & 0 \\ -\delta_6 & 0 & -\delta_5 & 0 & 0 & \delta_2 & \delta_7 & 0 \\ 0 & 0 & 0 & \delta_4 & \delta_3 & \delta_6 & \delta_5 & 0 \end{bmatrix}$

Table 1. *Cont.*

A	$\dim(A)$	$\mathrm{Der}(A)$	$\mathrm{IDer}(A)$
$L_{8,15}$	8	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & -2\delta_8+\delta_9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & 2\delta_8-\delta_9 & 0 & 0 & 0 & 0 & 0 \\ \delta_3 & \delta_4 & 0 & -\delta_8+2\delta_9 & \delta_1 & \delta_7 & 0 & \delta_5 \\ \delta_4 & 0 & \delta_3 & \delta_2 & \delta_8+\delta_9 & 0 & \delta_7 & -\delta_6 \\ -\delta_5 & \delta_6 & 0 & 0 & 0 & -\delta_8+\delta_9 & \delta_1 & 0 \\ -\delta_6 & 0 & -\delta_5 & 0 & 0 & \delta_2 & \delta_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_6 & \delta_5 & \delta_9 \end{bmatrix}$	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & -2\delta_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & 2\delta_8 & 0 & 0 & 0 & 0 & 0 \\ \delta_3 & \delta_4 & 0 & -\delta_8 & \delta_1 & \delta_7 & 0 & \delta_5 \\ \delta_4 & 0 & \delta_3 & \delta_2 & \delta_8 & 0 & \delta_7 & -\delta_6 \\ -\delta_5 & \delta_6 & 0 & 0 & 0 & -\delta_8 & \delta_1 & 0 \\ -\delta_6 & 0 & -\delta_5 & 0 & 0 & \delta_2 & \delta_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_6 & \delta_5 & 0 \end{bmatrix}$
$L_{8,19}$	8	$\begin{bmatrix} 0 & -\delta_2/3 & \delta_1 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & -2\delta_7/3+\delta_8/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2/3 & 0 & (2\delta_7-\delta_8)/3 & 0 & 0 & 0 & 0 & 0 \\ -3\delta_3 & \delta_4 & 0 & -\delta_7+\delta_8 & 3\delta_1 & 0 & 0 & 0 \\ \delta_4/3 & -2\delta_5/3 & -\delta_3 & \delta_2/3 & (-\delta_7+2\delta_8)/3 & 2\delta_1 & 0 & 0 \\ \delta_5/3 & \delta_6 & 2\delta_4/3 & 0 & 2\delta_2/3 & (\delta_7+\delta_8)/3 & \delta_1 & 0 \\ -3\delta_6 & 0 & -\delta_5 & 0 & 0 & \delta_2 & \delta_7 & 0 \\ 0 & 0 & 0 & \delta_6 & \delta_5 & \delta_4 & \delta_3 & \delta_8 \end{bmatrix}$	$\begin{bmatrix} 0 & -\delta_2/3 & \delta_1 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & -2\delta_7/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2/3 & 0 & 2\delta_7/3 & 0 & 0 & 0 & 0 & 0 \\ -3\delta_3 & \delta_4 & 0 & -\delta_7 & 3\delta_1 & 0 & 0 & 0 \\ \delta_4/3 & -2\delta_5/3 & -\delta_3 & \delta_2/3 & -\delta_7/3 & 2\delta_1 & 0 & 0 \\ \delta_5/3 & \delta_6 & 2\delta_4/3 & 0 & 2\delta_2/3 & \delta_7/3 & \delta_1 & 0 \\ -3\delta_6 & 0 & -\delta_5 & 0 & 0 & \delta_2 & \delta_7 & 0 \\ 0 & 0 & 0 & \delta_6 & \delta_5 & \delta_4 & \delta_3 & 0 \end{bmatrix}$
$L_{8,21}$	8	$\begin{bmatrix} 0 & -\delta_2/4 & \delta_1 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & \delta_8-\delta_9 & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_2/2 & 0 & -\delta_8+\delta_9 & 0 & 0 & 0 & 0 & 0 \\ 4\delta_3 & 2\delta_4 & 0 & 4\delta_8-3\delta_9 & 4\delta_1 & 0 & 0 & 0 \\ \delta_4 & \delta_7 & \delta_3 & \delta_2/4 & 3\delta_8-2\delta_9 & 3\delta_1 & 0 & 0 \\ 0 & \delta_5/2 & \delta_4 & 0 & \delta_2/2 & 2\delta_8-\delta_9 & 2\delta_1 & 0 \\ -\delta_5/2 & -\delta_6/4 & \delta_7 & 0 & 0 & 3\delta_2/4 & \delta_8 & \delta_1 \\ \delta_6 & 0 & \delta_5 & 0 & 0 & 0 & \delta_2 & \delta_9 \end{bmatrix}$	$\begin{bmatrix} 0 & -\delta_2/4 & \delta_1 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & 2\delta_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_2/2 & 0 & -2\delta_8 & 0 & 0 & 0 & 0 & 0 \\ 4\delta_3 & 2\delta_4 & 0 & 4\delta_8 & 4\delta_1 & 0 & 0 & 0 \\ \delta_4 & \delta_7 & \delta_3 & \delta_2/4 & 2\delta_8 & 3\delta_1 & 0 & 0 \\ 0 & \delta_5/2 & \delta_4 & 0 & \delta_2/2 & 0 & 2\delta_1 & 0 \\ -\delta_5/2 & -\delta_6/4 & \delta_7 & 0 & 0 & 3\delta_2/4 & -2\delta_8 & \delta_1 \\ \delta_6 & 0 & \delta_5 & 0 & 0 & 0 & \delta_2 & -4\delta_8 \end{bmatrix}$
$L_{8,22}$	8	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & \delta_9-\delta_{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & -\delta_9+\delta_{10} & 0 & 0 & 0 & 0 & 0 \\ 2\delta_3 & \delta_7 & 0 & \delta_8+2\delta_9-2\delta_{10} & 2\delta_1 & 0 & 0 & 0 \\ 0 & -\delta_4/2 & \delta_3 & \delta_2 & \delta_8+\delta_9-\delta_{10} & \delta_1 & 0 & 0 \\ \delta_4 & 0 & \delta_7 & 0 & 2\delta_2 & \delta_8 & 0 & 0 \\ \delta_5 & -\delta_6 & 0 & 0 & 0 & 0 & \delta_9 & \delta_1 \\ \delta_6 & 0 & \delta_5 & 0 & 0 & 0 & \delta_2 & \delta_{10} \end{bmatrix}$	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & 2\delta_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & -2\delta_8 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_3 & \delta_7 & 0 & 2\delta_8 & 2\delta_1 & 0 & 0 & 0 \\ 0 & -\delta_4/2 & \delta_3 & \delta_2 & 0 & \delta_1 & 0 & 0 \\ \delta_4 & 0 & \delta_7 & 0 & 2\delta_2 & -2\delta_8 & 0 & 0 \\ \delta_5 & -\delta_6 & 0 & 0 & 0 & 0 & \delta_8 & \delta_1 \\ \delta_6 & 0 & \delta_5 & 0 & 0 & 0 & \delta_2 & -\delta_8 \end{bmatrix}$
$\mathfrak{sl}_2 \oplus L_{6,1}$	9	$\begin{bmatrix} 0 & -\delta_1 & -\delta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_1 & 0 & -\delta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_2 & \delta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\delta_4 & -\delta_5 & \delta_{10} & -\delta_1 & -\delta_2 & 0 & 0 & 0 \\ \delta_4 & 0 & -\delta_6 & \delta_1 & \delta_{10} & -\delta_3 & 0 & 0 & 0 \\ \delta_5 & \delta_6 & 0 & \delta_2 & \delta_3 & \delta_{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta_7/2 & -\delta_8/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta_8 & -\delta_9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta_7 & 0 & \delta_9 \end{bmatrix}$	$\begin{bmatrix} 0 & -\delta_1 & -\delta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_1 & 0 & -\delta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_2 & \delta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\delta_4 & -\delta_5 & 0 & -\delta_1 & -\delta_2 & 0 & 0 & 0 \\ \delta_4 & 0 & -\delta_6 & \delta_1 & 0 & -\delta_3 & 0 & 0 & 0 \\ \delta_5 & \delta_6 & 0 & \delta_2 & \delta_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta_7/2 & -\delta_8/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta_8 & -\delta_9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta_7 & 0 & \delta_9 \end{bmatrix}$
$\mathfrak{sl}_2 \oplus L_{6,2}$	9	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & -2\delta_8+\delta_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & 2\delta_8-\delta_9 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\delta_3 & \delta_4 & 0 & -\delta_8+\delta_9 & \delta_1 & 0 & 0 & 0 & 0 \\ -\delta_4 & 0 & -\delta_3 & \delta_2 & \delta_8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_4 & \delta_3 & \delta_9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta_5/2 & -\delta_6/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta_6 & -\delta_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta_5 & 0 & \delta_7 \end{bmatrix}$	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & -2\delta_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & 2\delta_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\delta_3 & \delta_4 & 0 & -\delta_8 & \delta_1 & 0 & 0 & 0 & 0 \\ -\delta_4 & 0 & -\delta_3 & \delta_2 & \delta_8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_4 & \delta_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta_5/2 & -\delta_6/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta_6 & -\delta_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta_5 & 0 & \delta_7 \end{bmatrix}$
$L_{9,37} \cong L_{9,42}$	9	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & -2\delta_7+\delta_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & 2\delta_7-\delta_9 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\delta_3 & \delta_4 & 0 & (-2\delta_7+\delta_8+\delta_9)/2 & \delta_1 & 0 & 0 & 0 & 0 \\ -\delta_4 & 0 & -\delta_3 & \delta_2 & (2\delta_7+\delta_8-\delta_9)/2 & 0 & 0 & 0 & 0 \\ -\delta_5 & \delta_6 & 0 & 0 & 0 & -\delta_7+\delta_9 & \delta_1 & 0 & 0 \\ -\delta_6 & 0 & -\delta_5 & 0 & 0 & \delta_2 & \delta_7 & 0 & 0 \\ 0 & 0 & 0 & \delta_4 & \delta_3 & 0 & 0 & \delta_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_6 & \delta_5 & 0 & \delta_9 \end{bmatrix}$	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & -2\delta_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & 2\delta_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\delta_3 & \delta_4 & 0 & -\delta_7 & \delta_1 & 0 & 0 & 0 & 0 \\ -\delta_4 & 0 & -\delta_3 & \delta_2 & \delta_7 & 0 & 0 & 0 & 0 \\ -\delta_5 & \delta_6 & 0 & 0 & 0 & -\delta_7 & \delta_1 & 0 & 0 \\ -\delta_6 & 0 & -\delta_5 & 0 & 0 & \delta_2 & \delta_7 & 0 & 0 \\ 0 & 0 & 0 & \delta_4 & \delta_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_6 & \delta_5 & 0 & 0 \end{bmatrix}$

Table 1. *Cont.*

A	dim(A)	Der(A)	IDer(A)
$L_{9,41}$	9	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & -2\delta_7+\delta_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & 2\delta_7-\delta_9 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\delta_3 & \delta_4 & 0 & -\delta_7+\delta_8 & \delta_1 & \delta_{10}/2 & 0 & 0 & 0 \\ -\delta_4 & 0 & -\delta_3 & \delta_2 & \delta_7+\delta_8-\delta_9 & 0 & \delta_{10}/2 & 0 & 0 \\ -\delta_5 & \delta_6 & 0 & 0 & 0 & -\delta_7+\delta_9 & \delta_1 & 0 & 0 \\ -\delta_6 & 0 & -\delta_5 & 0 & 0 & \delta_2 & \delta_7 & 0 & 0 \\ 0 & 0 & 0 & \delta_6 & \delta_5 & \delta_4 & \delta_3 & \delta_8 & \delta_{10} \\ 0 & 0 & 0 & 0 & 0 & \delta_6 & \delta_5 & 0 & \delta_9 \end{bmatrix}$	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & -2\delta_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & 2\delta_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\delta_3 & \delta_4 & 0 & -\delta_7 & \delta_1 & 0 & 0 & 0 & 0 \\ -\delta_4 & 0 & -\delta_3 & \delta_2 & \delta_7 & 0 & 0 & 0 & 0 \\ -\delta_5 & \delta_6 & 0 & 0 & 0 & -\delta_7 & \delta_1 & 0 & 0 \\ -\delta_6 & 0 & -\delta_5 & 0 & 0 & \delta_2 & \delta_7 & 0 & 0 \\ 0 & 0 & 0 & \delta_6 & \delta_5 & \delta_4 & \delta_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_6 & \delta_5 & 0 & 0 \end{bmatrix}$
$L_{9,58}$	9	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & -2\delta_9+\delta_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & 2\delta_9-\delta_{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_3 & \delta_4 & 0 & \delta_8-4\delta_9+2\delta_{10} & 2\delta_1 & 0 & 0 & 0 & 0 \\ 0 & -\delta_5/2 & \delta_3 & \delta_2 & \delta_8-2\delta_9+\delta_{10} & \delta_1 & 0 & 0 & 0 \\ \delta_5 & 0 & \delta_4 & 0 & 2\delta_2 & \delta_8 & 0 & 0 & 0 \\ -\delta_6 & \delta_7 & 0 & 0 & 0 & 0 & -\delta_9+\delta_{10} & \delta_1 & 0 \\ -\delta_7 & 0 & -\delta_6 & 0 & 0 & 0 & \delta_2 & \delta_9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta_7 & \delta_6 & \delta_{10} \end{bmatrix}$	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & 2\delta_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & -2\delta_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_3 & \delta_4 & 0 & 2\delta_8 & 2\delta_1 & 0 & 0 & 0 & 0 \\ 0 & -\delta_5/2 & \delta_3 & \delta_2 & 0 & \delta_1 & 0 & 0 & 0 \\ \delta_5 & 0 & \delta_4 & 0 & 2\delta_2 & -2\delta_8 & 0 & 0 & 0 \\ -\delta_6 & \delta_7 & 0 & 0 & 0 & 0 & \delta_8 & \delta_1 & 0 \\ -\delta_7 & 0 & -\delta_6 & 0 & 0 & 0 & \delta_2 & -\delta_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta_7 & \delta_6 & 0 \end{bmatrix}$
$L_{9,59}$	9	$\begin{bmatrix} 0 & -\delta_2/5 & \delta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & \delta_9-\delta_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2/5 & 0 & -\delta_9+\delta_{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_3 & 5\delta_4/2 & 0 & 5\delta_9-4\delta_{10} & 5\delta_1 & 0 & 0 & 0 & 0 \\ 3\delta_4/2 & 4\delta_5/3 & \delta_3 & \delta_2/5 & 4\delta_9-3\delta_{10} & 4\delta_1 & 0 & 0 & 0 \\ \delta_5/3 & 3\delta_6/4 & \delta_4 & 0 & 2\delta_2/5 & 3\delta_9-2\delta_{10} & 3\delta_1 & 0 & 0 \\ -\delta_6/4 & 2\delta_7/5 & \delta_5 & 0 & 0 & 3\delta_2/5 & 2\delta_9-\delta_{10} & 2\delta_1 & 0 \\ -3\delta_7/7 & -\delta_8/5 & \delta_6 & 0 & 0 & 0 & 4\delta_2/5 & \delta_9 & \delta_1 \\ \delta_8 & 0 & \delta_7 & 0 & 0 & 0 & 0 & \delta_2 & \delta_{10} \end{bmatrix}$	$\begin{bmatrix} 0 & -\delta_2/5 & \delta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & 2\delta_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2/5 & 0 & -2\delta_9 & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_3 & 5\delta_4/2 & 0 & 5\delta_9 & 5\delta_1 & 0 & 0 & 0 & 0 \\ 3\delta_4/2 & 4\delta_5/3 & \delta_3 & \delta_2/5 & 3\delta_9 & 4\delta_1 & 0 & 0 & 0 \\ \delta_5/3 & 3\delta_6/4 & \delta_4 & 0 & 2\delta_2/5 & \delta_9 & 3\delta_1 & 0 & 0 \\ -\delta_6/4 & 2\delta_7/5 & \delta_5 & 0 & 0 & 3\delta_2/5 & -\delta_9 & 2\delta_1 & 0 \\ -3\delta_7/7 & -\delta_8/5 & \delta_6 & 0 & 0 & 0 & 4\delta_2/5 & -3\delta_9 & \delta_1 \\ \delta_8 & 0 & \delta_7 & 0 & 0 & 0 & 0 & \delta_2 & -5\delta_9 \end{bmatrix}$
$L_{9,60}$	9	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & \delta_{10}-\delta_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & -\delta_{10}+\delta_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 3\delta_3 & 3\delta_4/2 & 0 & \delta_9+3\delta_{10}-3\delta_{11} & 3\delta_1 & 0 & 0 & 0 & 0 \\ \delta_4/2 & 2\delta_5/3 & \delta_3 & \delta_2 & \delta_9+2\delta_{10}-2\delta_{11} & 2\delta_1 & 0 & 0 & 0 \\ -\delta_5/3 & -\delta_6/3 & \delta_4 & 0 & 2\delta_2 & \delta_9+\delta_{10}-\delta_{11} & \delta_1 & 0 & 0 \\ \delta_6 & 0 & \delta_5 & 0 & 0 & 3\delta_2 & \delta_9 & 0 & 0 \\ \delta_7 & -\delta_8 & 0 & 0 & 0 & 0 & 0 & \delta_{10} & \delta_1 \\ \delta_8 & 0 & \delta_7 & 0 & 0 & 0 & 0 & \delta_2 & \delta_{11} \end{bmatrix}$	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & 2\delta_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & -2\delta_9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3\delta_3 & 3\delta_4/2 & 0 & 3\delta_9 & 3\delta_1 & 0 & 0 & 0 & 0 \\ \delta_4/2 & 2\delta_5/3 & \delta_3 & \delta_2 & \delta_9 & 2\delta_1 & 0 & 0 & 0 \\ -\delta_5/3 & -\delta_6/3 & \delta_4 & 0 & 2\delta_2 & -\delta_9 & \delta_1 & 0 & 0 \\ \delta_6 & 0 & \delta_5 & 0 & 0 & 3\delta_2 & -3\delta_9 & 0 & 0 \\ \delta_7 & -\delta_8 & 0 & 0 & 0 & 0 & 0 & \delta_9 & \delta_1 \\ \delta_8 & 0 & \delta_7 & 0 & 0 & 0 & 0 & \delta_2 & -\delta_9 \end{bmatrix}$
$L_{9,61}$	9	$\begin{bmatrix} 0 & -\delta_2/2 & \delta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & \delta_{10}-\delta_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_2 & 0 & -\delta_{10}+\delta_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_3 & \delta_7 & 0 & \delta_9+2\delta_{10}-2\delta_{11} & 2\delta_1 & 0 & \delta_{12} & 0 & 0 \\ 0 & -\delta_4/2 & \delta_3 & \delta_2/2 & \delta_9+\delta_{10}-\delta_{11} & \delta_1 & 0 & \delta_{12} & 0 \\ \delta_4 & 0 & \delta_7 & 0 & \delta_2 & \delta_9 & 0 & 0 & \delta_{12} \\ 2\delta_5 & \delta_8 & 0 & \delta_{13} & 0 & 0 & 2\delta_{10}-\delta_{11} & 2\delta_1 & 0 \\ 0 & -\delta_6/2 & \delta_5 & 0 & \delta_{13} & 0 & \delta_2/2 & \delta_{10} & \delta_1 \\ \delta_6 & 0 & \delta_8 & 0 & 0 & \delta_{13} & 0 & \delta_2 & \delta_{11} \end{bmatrix}$	$\begin{bmatrix} 0 & -\delta_2/2 & \delta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & 2\delta_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_2 & 0 & -2\delta_9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_3 & \delta_7 & 0 & 2\delta_9 & 2\delta_1 & 0 & 0 & 0 & 0 \\ 0 & -\delta_4/2 & \delta_3 & \delta_2/2 & 0 & \delta_1 & 0 & 0 & 0 \\ \delta_4 & 0 & \delta_7 & 0 & \delta_2 & -2\delta_9 & 0 & 0 & 0 \\ 2\delta_5 & \delta_8 & 0 & 0 & 0 & 0 & 2\delta_9 & 2\delta_1 & 0 \\ 0 & -\delta_6/2 & \delta_5 & 0 & 0 & 0 & \delta_2/2 & 0 & \delta_1 \\ \delta_6 & 0 & \delta_8 & 0 & 0 & 0 & 0 & \delta_2 & -2\delta_9 \end{bmatrix}$
$L_{9,62}$	9	$\begin{bmatrix} 0 & -\delta_2/2 & \delta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & \delta_{10}-\delta_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_2 & 0 & -\delta_{10}+\delta_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_3 & \delta_{11}-\delta_9 & 0 & (3\delta_{10}-2\delta_8)/2 & 2\delta_1 & 0 & 0 & 0 & 0 \\ 0 & \delta_4/2 & -\delta_3 & \delta_2/2 & \delta_{10}/2 & \delta_1 & 0 & 0 & 0 \\ \delta_4 & 0 & \delta_{11}-\delta_9 & 0 & \delta_2 & (-\delta_{10}+2\delta_8)/2 & 0 & 0 & 0 \\ 2\delta_5 & \delta_7 & 0 & 2\delta_{11}-\delta_9 & 2\delta_3 & 0 & 2\delta_{10}-\delta_8 & 2\delta_1 & 0 \\ 0 & -\delta_6/2 & \delta_5 & \delta_4/2 & \delta_{11} & \delta_3 & \delta_2/2 & \delta_{10} & \delta_1 \\ \delta_6 & 0 & \delta_7 & 0 & \delta_4 & \delta_9 & 0 & \delta_2 & \delta_8 \end{bmatrix}$	$\begin{bmatrix} 0 & -\delta_2/2 & \delta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & -\delta_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_2 & 0 & \delta_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_3 & -\delta_9 & 0 & -\delta_8 & 2\delta_1 & 0 & 0 & 0 & 0 \\ 0 & \delta_4/2 & -\delta_3 & \delta_2/2 & 0 & \delta_1 & 0 & 0 & 0 \\ \delta_4 & 0 & -\delta_9 & 0 & \delta_2 & \delta_8 & 0 & 0 & 0 \\ 2\delta_5 & \delta_7 & 0 & -\delta_9 & 2\delta_3 & 0 & -\delta_8 & 2\delta_1 & 0 \\ 0 & -\delta_6/2 & \delta_5 & \delta_4/2 & 0 & \delta_3 & \delta_2/2 & 0 & \delta_1 \\ \delta_6 & 0 & \delta_7 & 0 & \delta_4 & \delta_9 & 0 & \delta_2 & \delta_8 \end{bmatrix}$
$L_{9,63}$	9	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & \delta_{11}-\delta_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & -\delta_{11}+\delta_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_3 & -\delta_4 & 0 & \delta_9+\delta_{11}-\delta_{12} & \delta_1 & \delta_{13} & 0 & \delta_{14} & 0 \\ \delta_4 & 0 & \delta_3 & \delta_2 & \delta_9 & 0 & \delta_{13} & 0 & \delta_{14} \\ \delta_5 & -\delta_6 & 0 & \delta_{16} & 0 & \delta_{10}+\delta_{11}-\delta_{12} & \delta_1 & \delta_{15} & 0 \\ \delta_6 & 0 & \delta_5 & 0 & \delta_{16} & \delta_2 & \delta_{10} & 0 & \delta_{15} \\ \delta_7 & -\delta_8 & 0 & \delta_{17} & 0 & \delta_{18} & 0 & \delta_{11} & \delta_1 \\ \delta_8 & 0 & \delta_7 & 0 & \delta_{17} & 0 & \delta_{18} & \delta_2 & \delta_{12} \end{bmatrix}$	$\begin{bmatrix} 0 & -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\delta_1 & 2\delta_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\delta_2 & 0 & -2\delta_9 & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_3 & -\delta_4 & 0 & \delta_9 & \delta_1 & 0 & 0 & 0 & 0 \\ \delta_4 & 0 & \delta_3 & \delta_2 & -\delta_9 & 0 & 0 & 0 & 0 \\ \delta_5 & -\delta_6 & 0 & 0 & 0 & \delta_9 & \delta_1 & 0 & 0 \\ \delta_6 & 0 & \delta_5 & 0 & 0 & \delta_2 & -\delta_9 & 0 & 0 \\ \delta_7 & -\delta_8 & 0 & 0 & 0 & 0 & 0 & \delta_9 & \delta_1 \\ \delta_8 & 0 & \delta_7 & 0 & 0 & 0 & 0 & \delta_2 & -\delta_9 \end{bmatrix}$

On the basis of the computations in Table 1, every complex perfect non-semisimple Lie algebra of dimension at most nine admits an outer derivation, yielding the following.

Proposition 4. *Every complex perfect non-semisimple Lie algebra of dimension at most nine is not semicomplete.*

Corollary 2. *Every complex perfect non-semisimple Lie algebra of dimension at most nine is not complete.*

The following seven-dimensional construction exhibits a perfect non-semisimple non-Lie Leibniz algebra that fails to be semicomplete, and it is designed to inherit that failure from its Lie quotient $L_{5,1}$ in Table 1. This shows that the phenomenon of Proposition 4 is not confined to Lie algebras.

Example 5. *Let $A = \mathrm{span}\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ be the seven-dimensional Leibniz algebra over $\mathbb{C}$ with nonzero multiplications defined by*

$$\begin{aligned}
&[e_1, e_2] = 2e_2, && [e_2, e_1] = -2e_2, && [e_1, e_3] = -2e_3,\\
&[e_3, e_1] = 2e_3, && [e_2, e_3] = e_1, && [e_3, e_2] = -e_1,\\
&[e_1, e_4] = e_4, && [e_4, e_1] = -e_4, && [e_1, e_5] = -e_5,\\
&[e_5, e_1] = e_5, && [e_2, e_5] = e_4, && [e_5, e_2] = -e_4,\\
&[e_3, e_4] = e_5, && [e_4, e_3] = -e_5, && [e_1, e_6] = e_6,\\
&[e_1, e_7] = -e_7, && [e_2, e_7] = e_6, && [e_3, e_6] = e_7.
\end{aligned}$$

For $a = \sum_{i=1}^{7} \alpha_i e_i$, all antisymmetric contributions cancel in $[a, a]$, leaving $[a, a] = (\alpha_1\alpha_6 + \alpha_2\alpha_7)\, e_6 + (\alpha_3\alpha_6 - \alpha_1\alpha_7)\, e_7$, so $\mathrm{Leib}(A) = \mathrm{span}\{e_6, e_7\} \neq \{0\}$, and A is a non-Lie Leibniz algebra. Moreover, $e_1 = [e_2, e_3]$, $e_2 = \frac{1}{2}[e_1, e_2]$, $e_3 = -\frac{1}{2}[e_1, e_3]$, $e_4 = [e_1, e_4]$, $e_5 = [e_3, e_4]$, $e_6 = [e_1, e_6]$ and $e_7 = [e_3, e_6]$, so $A^2 = A$, and A is perfect. Moreover, $\mathrm{rad}(A) \supsetneq \mathrm{Leib}(A)$, so A is not semisimple. The canonical Lie algebra $A/\,\mathrm{Leib}(A)$ is isomorphic to the perfect non-semisimple Lie algebra $L_{5,1}$ from Table 1.

We claim that A is not semicomplete. Define the linear map $\delta : A \to A$ using $\delta(e_4) = e_4$, $\delta(e_5) = e_5$, and $\delta(e_i) = 0$ for $i \in \{1, 2, 3, 6, 7\}$. A direct computation shows that $\delta \in \mathrm{Der}(A) = \mathrm{ID}(A)$. Suppose, for a contradiction, that there exists $x = \sum_{i=1}^{7} \alpha_i e_i \in A$ such that $\mathrm{im}(\delta - L_x) \subseteq \mathrm{Leib}(A) = \mathrm{span}\{e_6, e_7\}$. Evaluating at e_4, we obtain $(\delta - L_x)(e_4) = e_4 - [x, e_4] = (1 - \alpha_1)e_4 - \alpha_3 e_5$, which lies in $\mathrm{span}\{e_6, e_7\}$ only if $\alpha_1 = 1$. Evaluating at e_5, we obtain $(\delta - L_x)(e_5) = e_5 - [x, e_5] = -\alpha_2 e_4 + (1 + \alpha_1)e_5$, which lies in $\mathrm{span}\{e_6, e_7\}$ only if $\alpha_1 = -1$, contradicting $\alpha_1 = 1$. Hence, δ is not inner, and A is not semicomplete.

Remark 6. *We note that Proposition 4 does not extend to all dimensions. In [10], Benayadi constructed a non-semisimple sympathetic Lie algebra of dimension 25 over $\mathbb{C}$, providing an example of a perfect non-semisimple Lie algebra that is complete and hence semicomplete. More generally, for a perfect Leibniz algebra A, semicompleteness is equivalent to requiring that every derivation of A is inner. Thus, a perfect semicomplete Leibniz algebra A is sympathetic if and only if $Z(A/\,\mathrm{Leib}(A)) = \{0\}$.*

6. Discussion

We have undertaken a systematic study of semicomplete Leibniz algebras, building on the notion introduced by Boyle [16] and establishing structural results and classifications that extend key results from the Lie algebra setting. The results on direct sums (Theorems 1 and 2), the characterization via holomorphs and special extensions (Theorem 3), and the classification of low-dimensional non-Lie Leibniz algebras (Proposition 1) confirm that semicompleteness behaves naturally in the Leibniz framework.

As shown in Table 1 and Proposition 4, perfectness alone does not guarantee semicompleteness: no complex perfect non-semisimple Lie algebra of dimension at most nine is semicomplete, and Example 5 shows this phenomenon extends beyond the Lie setting. Nevertheless, Benayadi's construction [10] confirms that perfect non-semisimple semicomplete Lie algebras do exist, with the smallest dimension lying between 10 and 25: the upper bound is Benayadi's 25-dimensional example, while the lower bound is Proposition 4, which rules out every complex perfect non-semisimple Lie algebra of dimension at most nine.

These findings suggest several directions for future work:

1. Classification beyond small dimensions: Extending the classification of semicomplete non-Lie Leibniz algebras beyond dimension three remains open.
2. Properties that fail in the Leibniz setting: It would be of interest to identify further properties of semicomplete Lie algebras with no Leibniz analogues. Example 3 and Theorem 2 show that in the Leibniz setting, the characteristic ideal hypothesis is what makes the converse work.
3. Perfect non-Lie Leibniz algebras: A systematic study of perfect non-semisimple non-Lie Leibniz algebras and their semicompleteness would be a natural extension of this work, though it requires the development of analogous classification results in the Leibniz setting.
4. Characterization of the boundary: Determining the smallest dimension in which a perfect non-semisimple semicomplete Lie algebra exists, or characterizing the structural properties that distinguish the semicomplete case from the non-semicomplete case among perfect algebras, would be of considerable interest.

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