

Article

Rota–Baxter Operators and Systems on Hom- π -Algebras: Constructions, Derivation Chains, and Structural Equivalence

Xiuzhen Li, Ruoxi He, Yalan Zhang and Guodong Shi *

Faculty of Mathematics and Physics, Huai'an University, Huai'an 223003, China; 15329151612@163.com (X.L.); 13771038420@163.com (R.H.); zylyw@hau.edu.cn (Y.Z.)

* Correspondence: guodongs@hau.edu.cn

Abstract

(1) Background: Hom-type algebras, proposed by Yau, generalize classical algebras via twisting maps. Dendriform and tridendriform algebras, introduced by Loday and Vallette, decompose associative multiplications and play significant roles in algebraic K-theory and operad theory. Rota–Baxter operators, originating from analysis and probability, have become a vital bridge connecting multiple disciplines. The π -graded structure, a classical tool in algebra, decomposes algebraic objects into direct sums indexed by a monoid π . (2) Methods: We systematically investigate constructive approaches for Rota–Baxter operators on Hom- π -algebras, establish the stepwise structural derivation chain among Hom-tridendriform, Hom-dendriform, and Hom-associative π -algebras, and characterize the equivalence between generalized Rota–Baxter systems and dendriform structures in the Hom- π framework. (3) Results: We present several systematic constructive methods for Rota–Baxter Hom- π -algebras, including Yau twists, iterative endomorphism twisting, centroid element modifications, and monoid tensor lifting. We prove the structural equivalence between Rota–Baxter Hom- π -systems and Hom-dendriform π -algebras, and further extend the study to Hom-quadri- π -algebras with corresponding induction results. (4) Conclusions: The established structural equivalence and derivation relations provide a unified framework for these π -graded Hom-type algebraic structures, covering Rota–Baxter operators and various dendriform-derived algebras.

Keywords: tridendriform π -algebra; dendriform π -algebra; Rota–Baxter system; Rota–Baxter π -algebra

MSC: 16W99



Academic Editor: Xiaomin Tang

Received: 3 July 2026

Revised: 27 August 2026

Accepted: 2 September 2026

Published: 4 September 2026

Copyright: © 2026 by the authors.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

1. Introduction

Hom-type algebras, proposed by Yau, are important noncommutative algebras that generalize classical algebras via twisting maps [1]. Their core idea is to replace the identity map with a compatible linear endomorphism. The construction of twisted tensor products for Hom-type algebras has broadened their applications in mathematical physics, representation theory, and other fields [2]. Dendriform algebras and tridendriform algebras were introduced by Loday and Vallette [3,4]. By decomposing the associative multiplication, they reveal the internal rules of algebras and play a significant role in algebraic K-theory, operad theory, and combinatorics [5].

Originating from problems in analysis and probability theory, Rota–Baxter operators were formally defined and systematically developed by Rota and Baxter [6,7]. They have

become a vital bridge connecting multiple disciplines. The relations between Rota–Baxter operators and dendriform algebras have been thoroughly investigated [8], and relevant theories are widely applied in quantum field theory and the characterization of Hopf algebras. Recently, the interdisciplinary research on Rota–Baxter operators and Hom-type algebras has gradually emerged, giving rise to new structures such as Rota–Baxter Hom-algebras [9].

The π -graded structure is a classical tool in algebra research. It decomposes algebraic objects into direct sums of subspaces indexed by a monoid π while preserving the homogeneity of operations, and is widely used in algebraic geometry, topology and homological algebra [10]. Combining π -gradation with Rota–Baxter algebras, dendriform algebras and tridendriform algebras yields new structures including Rota–Baxter π -algebras, dendriform π -algebras and tridendriform π -algebras [11], which inherit the essential properties of the two kinds of fundamental algebraic structures.

This paper investigates the structural constructions and interrelations of these π -graded Hom-type algebras. Section 2 fixes unified notation conventions for six families of graded algebraic structures, streamlining the presentation of definitions and proofs throughout the paper.

Section 3 systematically develops multiple constructive approaches for Rota–Baxter Hom- π -algebras, ranging from Yau twists and iterative endomorphism twisting to centroid element modifications and monoid tensor lifting. We also extend the notion of Rota–Baxter pairs to the π -graded Hom framework, and illustrate all constructions with explicit examples on low-dimensional associative algebras.

Turning to dendriform structures, Section 4 formally defines Hom-tridendriform and Hom-dendriform π -algebras, verifies their Yau twist constructions, and establishes the stepwise derivation chain from tridendriform to dendriform and further to associative Hom- π -algebras. A core result is that all these dendriform structures are naturally induced by Rota–Baxter Hom- π -algebras.

Section 5 is devoted to the generalized framework of Rota–Baxter Hom- π -systems. We characterize decomposable systems via operator zero-product conditions, and prove the structural equivalence between these systems and Hom-dendriform π -algebras. The study is further extended to Hom-quadri- π -algebras, including their Yau twist construction.

2. Notations

Throughout the paper, we take $\mathbb{K}$ as our ground field and consider it to be of characteristic zero unless otherwise stated. In general, π is taken to be an arbitrary monoid unless otherwise stated, but note that some results remain valid if we take it to be an arbitrary group, and elements p, q, r will be assumed to be arbitrary elements of π . For a family of linear spaces $\{V_p\}_{p \in \pi}$, let id_p denote the identity map on V_p . We also have the following specified notation for the corresponding families of algebras used throughout the paper.

- (a) Hom- π -algebras $\mathcal{A}_H := (\{A_p\}_{p \in \pi}, \{m_{p,q}^A\}_{p,q \in \pi}, \{\alpha_p^A\}_{p \in \pi})$ and $\mathcal{B}_H := (\{B_p\}_{p \in \pi}, \{m_{p,q}^B\}_{p,q \in \pi}, \{\alpha_p^B\}_{p \in \pi})$, where “H” goes for “Hom”, and when Hom-type structures are not involved, then we just have the π -algebras [12] $\mathcal{A} := (\{A_p\}_{p \in \pi}, \{m_{p,q}^A\}_{p,q \in \pi})$ and $\mathcal{B} := (\{B_p\}_{p \in \pi}, \{m_{p,q}^B\}_{p,q \in \pi})$.
- (b) Rota–Baxter Hom- π -algebras ${}^R\mathcal{A}_H := (\{A_p\}_{p \in \pi}, \{m_{p,q}^A\}_{p,q \in \pi}, \{R_p^A\}_{p \in \pi}, \{\alpha_p^A\}_{p \in \pi})$ and ${}^R\mathcal{B}_H := (\{B_p\}_{p \in \pi}, \{m_{p,q}^B\}_{p,q \in \pi}, \{R_p^B\}_{p \in \pi}, \{\alpha_p^B\}_{p \in \pi})$, where “H” goes for “Hom” and “R” goes for the Rota–Baxter operator, and when Hom-type structures are not involved, then we just have the Rota–Baxter π -algebras [11] ${}^R\mathcal{A} := (\{A_p\}_{p \in \pi}, \{m_{p,q}^A\}_{p,q \in \pi}, \{R_p^A\}_{p \in \pi})$ and ${}^R\mathcal{B} := (\{B_p\}_{p \in \pi}, \{m_{p,q}^B\}_{p,q \in \pi}, \{R_p^B\}_{p \in \pi})$, where “R” goes for the Rota–Baxter operator.

- (c) Hom-tridendriform π -algebras ${}^{Td}\mathcal{A}_H := (\{A_p\}_{p \in \pi}, \{\prec_{p,q}^A\}_{p,q \in \pi}, \{\succ_{p,q}^A\}_{p,q \in \pi}, \{\bullet_{p,q}^A\}_{p,q \in \pi}, \{\alpha_p^A\}_{p \in \pi})$ and ${}^{Td}\mathcal{B}_H := (\{B_p\}_{p \in \pi}, \{\prec_{p,q}^B\}_{p,q \in \pi}, \{\succ_{p,q}^B\}_{p,q \in \pi}, \{\bullet_{p,q}^B\}_{p,q \in \pi}, \{\alpha_p^B\}_{p \in \pi})$, where “H” stands for “Hom” and “Td” stands for tridendriform. When Hom-type structures are not involved, we simply have the tridendriform π -algebras [11] ${}^{Td}\mathcal{A} := (\{A_p\}_{p \in \pi}, \{\prec_{p,q}^A\}_{p,q \in \pi}, \{\succ_{p,q}^A\}_{p,q \in \pi}, \{\bullet_{p,q}^A\}_{p,q \in \pi})$ and ${}^{Td}\mathcal{B} := (\{B_p\}_{p \in \pi}, \{\prec_{p,q}^B\}_{p,q \in \pi}, \{\succ_{p,q}^B\}_{p,q \in \pi}, \{\bullet_{p,q}^B\}_{p,q \in \pi})$.
- (d) Hom-dendriform π -algebras ${}^d\mathcal{A}_H := (\{A_p\}_{p \in \pi}, \{\prec_{p,q}^A\}_{p,q \in \pi}, \{\succ_{p,q}^A\}_{p,q \in \pi}, \{\alpha_p^A\}_{p \in \pi})$ and ${}^d\mathcal{B}_H := (\{B_p\}_{p \in \pi}, \{\prec_{p,q}^B\}_{p,q \in \pi}, \{\succ_{p,q}^B\}_{p,q \in \pi}, \{\alpha_p^B\}_{p \in \pi})$, where “H” goes for “Hom” and “d” goes for dendriform, and when Hom-type structures are not involved, then we just have the dendriform π -algebras [11] ${}^d\mathcal{A} := (\{A_p\}_{p \in \pi}, \{\prec_{p,q}^A\}_{p,q \in \pi}, \{\succ_{p,q}^A\}_{p,q \in \pi})$ and ${}^d\mathcal{B} := (\{B_p\}_{p \in \pi}, \{\prec_{p,q}^B\}_{p,q \in \pi}, \{\succ_{p,q}^B\}_{p,q \in \pi})$.
- (e) Rota–Baxter Hom- π -systems ${}^R\mathcal{A}_H^s := (\{A_p\}_{p \in \pi}, \{m_{p,q}^A\}_{p,q \in \pi}, \{R_p^A\}_{p \in \pi}, \{S_p^A\}_{p \in \pi}, \{\alpha_p^A\}_{p \in \pi})$ and ${}^R\mathcal{B}_H^s := (\{B_p\}_{p \in \pi}, \{m_{p,q}^B\}_{p,q \in \pi}, \{R_p^B\}_{p \in \pi}, \{S_p^B\}_{p \in \pi}, \{\alpha_p^B\}_{p \in \pi})$, where “H” goes for “Hom”, “s” goes for system, and “R” goes for the Rota–Baxter operator.
- (f) Hom-quadri- π -algebras ${}^q\mathcal{A}_H := (\{A_p\}_{p \in \pi}, \{\nwarrow_{p,q}^A\}_{p,q \in \pi}, \{\swarrow_{p,q}^A\}_{p,q \in \pi}, \{\nearrow_{p,q}^A\}_{p,q \in \pi}, \{\searrow_{p,q}^A\}_{p,q \in \pi}, \{\alpha_p^A\}_{p \in \pi})$ and ${}^q\mathcal{B}_H := (\{B_p\}_{p \in \pi}, \{\nwarrow_{p,q}^B\}_{p,q \in \pi}, \{\swarrow_{p,q}^B\}_{p,q \in \pi}, \{\nearrow_{p,q}^B\}_{p,q \in \pi}, \{\searrow_{p,q}^B\}_{p,q \in \pi}, \{\alpha_p^B\}_{p \in \pi})$, where “H” stands for “Hom” and “q” stands for quadri. When Hom-type structures are not involved, we simply have the quadri π -algebras ${}^q\mathcal{A} := (\{A_p\}_{p \in \pi}, \{\nwarrow_{p,q}^A\}_{p,q \in \pi}, \{\swarrow_{p,q}^A\}_{p,q \in \pi}, \{\nearrow_{p,q}^A\}_{p,q \in \pi}, \{\searrow_{p,q}^A\}_{p,q \in \pi})$ and ${}^q\mathcal{B} := (\{B_p\}_{p \in \pi}, \{\nwarrow_{p,q}^B\}_{p,q \in \pi}, \{\swarrow_{p,q}^B\}_{p,q \in \pi}, \{\nearrow_{p,q}^B\}_{p,q \in \pi}, \{\searrow_{p,q}^B\}_{p,q \in \pi})$.

3. Constructive Approaches to Rota–Baxter Hom- π Algebras

The π -graded algebraic structures studied in this paper originate from Turaev’s π -algebras, which were originally introduced in homotopy quantum field theory [12]. Hom- π -algebras are natural Hom-type generalizations of these classical graded structures.

Following the basic definitions and notation conventions in Section 2, we present several constructive approaches for Rota–Baxter Hom- π -algebras in this section. We establish three systematic constructive methods based on Yau twists, iterative endomorphism actions and centroid elements, and supplement them with explicit examples on low dimensional associative algebras.

Definition 1. A Hom- π -algebra $\mathcal{A}_H$ consists of a family of linear spaces $\{A_p\}_{p \in \pi}$ and two families of bilinear and linear operators $\{m_{p,q}^A : A_p \otimes A_q \longrightarrow A_{pq}, \alpha_p^A : A_p \longrightarrow A_p\}_{p,q \in \pi}$ such that for all $p, q, r \in \pi$, the equality $m_{p,q,r}(\alpha_p^A \otimes m_{q,r}^A) = m_{p,q,r}^A(m_{p,q}^A \otimes \alpha_r^A)$ holds.

A Hom- π -algebra homomorphism from $\mathcal{A}_H$ to $\mathcal{B}_H$ is a family of linear operators $\{f_p : A_p \longrightarrow B_p\}_{p \in \pi}$ such that for all $p, q, r \in \pi$, we have $f_{pq}m_{p,q}^A = m_{p,q}^B(f_p \otimes f_q)$, $f_p\alpha_p^A = \alpha_p^B f_p$.

A Hom- π -subalgebra of a Hom- π -algebra $\mathcal{A}_H$ is a Hom- π -algebra $\mathcal{B}_H$ that satisfies the following conditions:

- (a) B_p is a subspace of A_p for all $p \in \pi$;
- (b) $m_{p,q}^A(B_p \otimes B_q) \subseteq B_{pq}$ for all $p, q \in \pi$;
- (c) $\alpha_p^A(B_p) \subseteq B_p$ for all $p \in \pi$.

Remark 1. (a) A Hom- π -algebra $\mathcal{A}_H$ is called weakly non-degenerate if for any linear operators $f, g : A_p \rightarrow A_p$, the condition $m_{p,q}(f \otimes \text{id}_q) = m_{p,q}(g \otimes \text{id}_q)$ for all $q \in \pi$ implies $f = g$, and similarly, for any linear operators $g, h : A_q \rightarrow A_q$, the condition $m_{p,q}(\text{id}_p \otimes g) = m_{p,q}(\text{id}_p \otimes h)$ for all $p \in \pi$ implies $g = h$.

(b) A Hom- π -algebra $\mathcal{A}_H$ is called strongly non-degenerate if for any linear operators $f, g : A_p \rightarrow A_p$, the condition $m_{p,q}(f \otimes \alpha_q^A) = m_{p,q}(g \otimes \alpha_q^A)$ for all $q \in \pi$ implies $f = g$, and similarly, for any linear operators $g, h : A_q \rightarrow A_q$, the condition $m_{p,q}(\alpha_p^A \otimes g) = m_{p,q}(\alpha_p^A \otimes h)$ for all $p \in \pi$ implies $g = h$.

(c) If each α_p^A is surjective, then the strongly non-degeneracy of A_H implies the weakly non-degeneracy of its underlying Hom- π -algebra A_H .

(d) Please note that non-degeneracy is a very important feature for an algebra in its applications in quantum mathematics, and for this reason, later on we will see some results that involve this feature.

Hom- π -algebras arise naturally from π -algebras under the Yau twist.

Lemma 1. If $\{\alpha_p^A\}_{p \in \pi}$ is a π -algebra endomorphism on $\mathcal{A}$, then $(\{A_p\}_{p \in \pi}, \{m_{p,q}^{A,*} = \alpha_{pq}^A m_{p,q}^A\}_{p,q \in \pi}, \{\alpha_p^A\}_{p \in \pi})$ is a Hom- π -algebra.

If $\{f_p : A_p \rightarrow B_p\}_{p \in \pi}$ is a π -algebra homomorphism from $\mathcal{A}$ to $\mathcal{B}$ and satisfies $\alpha_p^B f_p = f_p \alpha_p^A$ for all $p \in \pi$, then $\{f_p\}_{p \in \pi}$ is a Hom- π -algebra homomorphism from $(\{A_p\}_{p \in \pi}, \{m_{p,q}^{A,*}\}_{p,q \in \pi}, \{\alpha_p^A\}_{p \in \pi})$ to $(\{B_p\}_{p \in \pi}, \{m_{p,q}^{B,*}\}_{p,q \in \pi}, \{\alpha_p^B\}_{p \in \pi})$.

Proof. For any $p, q, r \in \pi$, we have

$$\begin{aligned} m_{p,q}^{A,*}(\alpha_p \otimes m_{q,r}^{A,*}) &= \alpha_{pq}^A m_{p,q}^A (\alpha_p^A \otimes \alpha_{qr}^A m_{q,r}^A) \\ &= (\alpha_{pqr}^A)^2 m_{p,q}^A (\text{id}_p \otimes m_{q,r}) \\ &= (\alpha_{pqr}^A)^2 m_{p,q}^A (m_{p,q}^A \otimes \text{id}_r) \\ &= \alpha_{pqr}^A m_{p,q}^A (\alpha_{pq}^A m_{p,q}^A \otimes \alpha_r^A) = m_{p,q}^{A,*} (m_{p,q}^{A,*} \otimes \alpha_r^A), \end{aligned}$$

and

$$f_{pq} m_{p,q}^{A,*} = f_{pq} \alpha_{pq}^A m_{p,q}^A = \alpha_{pq}^B f_{pq} m_{p,q}^A = \alpha_{pq}^B m_{p,q}^B (f_p \otimes f_q) = m_{p,q}^{B,*} (f_p \otimes f_q).$$

□

Definition 2. A Rota–Baxter operator of weight λ on a Hom- π -algebra $\mathcal{A}_H$ is a family of linear operators $\{R_p^A : A_p \rightarrow A_p\}_{p \in \pi}$ such that for all $p, q \in \pi$, we have

$$m_{p,q}^A (R_p^A \otimes R_q^A) = R_{pq}^A m_{p,q}^A (R_p^A \otimes \text{id}_q + \text{id}_p \otimes R_q^A + \lambda \text{id}_p \otimes \text{id}_q).$$

In this case, ${}^R\mathcal{A}_H$ is called a Rota–Baxter Hom- π -algebra of weight λ . (Unless otherwise specified, the weight of all Rota–Baxter Hom- π -algebras below is taken to be λ).

A Rota–Baxter Hom- π -algebra homomorphism from ${}^R\mathcal{A}_H$ to ${}^R\mathcal{B}_H$ is a Hom- π -algebra homomorphism $\{f_p : A_p \rightarrow B_p\}_{p \in \pi}$ such that $f_p R_p^A = R_p^B f_p$ for all $p \in \pi$.

Remark 2. Since the definition of Rota–Baxter operators on $\mathcal{A}_H$ is independent of the Hom-operator, it is easy to verify that many properties of ${}^R\mathcal{A}$ given in Reference [11] also hold for ${}^R\mathcal{A}_H$: If $\{R_p^A\}_{p \in \pi}$ is a Rota–Baxter operator of weight λ_1 on a Hom- π -algebra, then

- $\{-\lambda_1 \text{id}_p - R_p^A\}_{p \in \pi}$ is a Rota–Baxter operator of weight λ_1 , $\{\lambda_2 R_p^A\}_{p \in \pi}$ is a Rota–Baxter operator of weight $\lambda_1 \lambda_2$.
- Let $\{R_p : A_p \rightarrow A_p\}_{p \in \pi}$ be a family of linear operators on a Hom- π -algebra, and let $\lambda \in \mathbb{K}$ to not be a zero divisor on $\mathcal{A}_H$. Then $\{R_p\}_{p \in \pi}$ is a Rota–Baxter operator of weight λ on A_H if and only if for any $p, q \in \pi$, $a \in A_p$, $b \in A_q$, there exists $d \in A_{pq}$ such that $R_p(a)R_q(b) = R_{pq}(d)$, $(-\lambda \text{id}_p - R_p)(a)(-\lambda \text{id}_q - R_q)(b) = (-\lambda \text{id}_{pq} - R_{pq})(d)$.
- Let $\{R_p : A_p \rightarrow A_p\}_{p \in \pi}$ be a family of left $\mathcal{A}_H$ -linear operators on a unital Hom- π -algebra $\mathcal{A}_H$. Then $\{R_p\}_{p \in \pi}$ is a Rota–Baxter operator of weight λ on $\mathcal{A}_H$ if and only if $\{R_p\}_{p \in \pi}$ is π -quasi idempotent with respect to λ .

- (d) Let $\{R_p : A_p \rightarrow A_p\}_{p \in \pi}$ be a π -idempotent Rota–Baxter operator on a unital Hom- π -algebra A_H . Then $(1 + \lambda)R_{pq}m_{p,q}(id_p \otimes R_p) = (1 + \lambda)R_{pq}m_{p,q}(R_p \otimes id_p)$, $(1 + \lambda)(m_{p,q}(R_p \otimes R_q) - \lambda R_{pq}m_{p,q}) = 0$.

We begin with the most fundamental construction, which builds a two-way correspondence between Rota–Baxter π -algebras and Rota–Baxter Hom- π -algebras via the Yau twist.

- Theorem 1.** (a) Let $\{\alpha_p^A : A_p \rightarrow A_p\}_{p \in \pi}$ be an endomorphism of the Rota–Baxter π -algebra ${}^R\mathcal{A}$ such that $\alpha_p^A R_p^A = R_p^A \alpha_p^A$ for all $p \in \pi$. Then $(\{A_p\}_{p \in \pi}, \{m_{p,q}^{A,*} = \alpha_{pq}^A m_{p,q}^A\}_{p,q \in \pi}, \{R_p^A\}_{p \in \pi}, \{\alpha_p^A\}_{p \in \pi})$ is a Rota–Baxter Hom- π -algebra.
- (b) Let ${}^R\mathcal{A}_H$ be a Rota–Baxter Hom- π -algebra such that each $\alpha_p^A : A_p \rightarrow A_p$ is bijective, and we have $\alpha_p^A R_p^A = R_p^A \alpha_p^A$ and $\alpha_{pq}^A m_{p,q}^A = m_{p,q}^A(\alpha_p^A \otimes \alpha_q^A)$ for all $p, q \in \pi$. Then $(\{A_p\}_{p \in \pi}, \{m_{p,q}^{A,\#} = (\alpha_{pq}^A)^{-1} m_{p,q}^A\}_{p,q \in \pi}, \{R_p^A\}_{p \in \pi})$ is a Rota–Baxter π -algebra.

Proof. (a) By Lemma 1, $(\{A_p\}_{p \in \pi}, \{m_{p,q}^{A,*}\}_{p,q \in \pi}, \{\alpha_p^A\}_{p \in \pi})$ is a Hom- π -algebra. Moreover, we have

$$\begin{aligned} m_{p,q}^{A,*}(R_p^A \otimes R_q^A) &= \alpha_{pq}^A m_{p,q}^A(R_p^A \otimes R_q^A) \\ &= R_{pq}^A \alpha_{pq}^A m_{p,q}^A(R_p^A \otimes id_q + id_p \otimes R_q^A + \lambda id_p \otimes id_q) \\ &= R_{pq}^A m_{p,q}^{A,*}(R_p^A \otimes id_q + id_p \otimes R_q^A + \lambda id_p \otimes id_q). \end{aligned}$$

(b) For any $p, q, r \in \pi$, we have

$$\begin{aligned} m_{pq,r}^{A,\#}(m_{p,q}^{A,\#} \otimes id_r) &= (\alpha_{pqr}^A)^{-1} m_{pqr}^A(\alpha_{pq}^{-1} m_{p,q}^A \otimes id_r) \\ &= (\alpha_{pqr}^A)^{-1} m_{pqr}^A(\alpha_{pq}^{-1} \otimes \alpha_r^{-1})(m_{p,q}^A \otimes \alpha_r^A) \\ &= ((\alpha_{pqr}^A)^{-1})^2 m_{pqr}^A(m_{p,q}^A \otimes \alpha_r^A) \\ &= ((\alpha_{pqr}^A)^{-1})^2 m_{pqr}^A(\alpha_p^A \otimes m_{q,r}^A) \\ &= (\alpha_{pqr}^A)^{-1} m_{pqr}^A((\alpha_p^A)^{-1} \otimes (\alpha_{qr}^A)^{-1})(\alpha_p^A \otimes m_{q,r}^A) \\ &= (\alpha_{pqr}^A)^{-1} m_{p,q}^A(id_p \otimes (\alpha_{qr}^A)^{-1} m_{q,r}^A) = m_{p,q}^{A,\#}(id_p \otimes m_{q,r}^{A,\#}). \end{aligned}$$

hence $(\{A_p\}_{p \in \pi}, \{m_{p,q}^{A,\#}\}_{p,q \in \pi})$ is a π -algebra. By $(\alpha_p^A)^{-1} R_p^A = (\alpha_p^A)^{-1} R_p^A \alpha_p^A (\alpha_p^A)^{-1} = (\alpha_p^A)^{-1} \alpha_p^A R_p^A \alpha_p^{-1} = R_p^A (\alpha_p^A)^{-1}$, we have

$$\begin{aligned} m_{p,q}^{A,\#}(R_p^A \otimes R_q^A) &= \alpha_{pq}^{-1} m_{p,q}^A(R_p^A \otimes R_q^A) \\ &= \alpha_{pq}^{-1} R_{pq}^A m_{p,q}^A(R_p^A \otimes id_q + id_p \otimes R_q^A + \lambda id_p \otimes id_q) \\ &= R_{pq}^A m_{p,q}^{A,\#}(R_p \otimes id_q + id_p \otimes R_q^A + \lambda id_p \otimes id_q). \end{aligned}$$

hence $(\{A_p\}_{p \in \pi}, \{m_{p,q}^{A,\#}\}_{p,q \in \pi}, \{R_p^A\}_{p \in \pi})$ is a Rota–Baxter π -algebra. Please note that in this case we have $\alpha_{pq}^A m_{p,q}^{A,\#} = \alpha_{pq}^A (\alpha_{pq}^A)^{-1} m_{p,q}^A = m_{p,q}^A = (\alpha_{pq}^A)^{-1} \alpha_{pq}^A m_{p,q}^A = (\alpha_{pq}^A)^{-1} m_{p,q}^A (\alpha_p^A \otimes \alpha_q^A) = m_{p,q}^{A,\#}(\alpha_p^A \otimes \alpha_q^A)$, i.e., $\{\alpha_p^A\}_{p \in \pi}$ is a π -algebra endomorphism on $(\{A_p\}_{p \in \pi}, \{m_{p,q}^{A,\#}\}_{p,q \in \pi})$, so its Yau twist is exactly ${}^R\mathcal{A}_H$. $\square$

Inspired by the twisting idea above, we further derive an iterative construction method that produces an infinite sequence of Rota–Baxter Hom- π -algebras from a single initial structure.

Theorem 2. Let ${}^R\mathcal{A}_H$ be a Rota–Baxter Hom- π -algebra such that $\alpha_p^A R_p^A = R_p^A \alpha_p^A$ and $\alpha_{pq}^A m_{p,q}^A = m_{p,q}^A (\alpha_p^A \otimes \alpha_q^A)$ for all $p \in \pi$. Then for any positive integer n , $(\{A_p\}_{p \in \pi}, \{m_{p,q}^{(n)} = (\alpha_{pq}^A)^n m_{p,q}\}_{p,q \in \pi}, \{R_p^A\}_{p \in \pi}, \{(\alpha_p^A)^{n+1}\}_{p \in \pi})$ is a Rota–Baxter Hom- π -algebra.

Proof. We proceed by induction on n . For any $n = 1$, we have

$$\begin{aligned} m_{pq,r}^{(1)}(m_{p,q}^{(1)} \otimes (\alpha_r^A)^2) &= \alpha_{pqr}^A m_{pq,r}^A (\alpha_{pq}^A m_{p,q}^A \otimes (\alpha_r^A)^2) \\ &= (\alpha_{pqr}^A)^2 m_{pq,r}^A (m_{p,q}^A \otimes \alpha_r^A) = (\alpha_{pqr}^A)^2 m_{p,q,r}^A (\alpha_p^A \otimes m_{q,r}^A) \\ &= \alpha_{pqr}^A m_{p,q,r}^A ((\alpha_p^A)^2 \otimes \alpha_{qr}^A m_{q,r}^A) = m_{p,q,r}^{(1)}((\alpha_p^A)^2 \otimes m_{q,r}^{(1)}), \end{aligned}$$

$$\begin{aligned} m_{p,q}^{(1)}(R_p^A \otimes R_q^A) &= \alpha_{pq}^A m_{p,q}^A (R_p^A \otimes R_q^A) \\ &= \alpha_{pq}^A R_{pq}^A m_{p,q}^A (R_p^A \otimes \text{id}_q + \text{id}_p \otimes R_q^A + \lambda \text{id}_p \otimes \text{id}_q) \\ &= R_{pq}^A m_{p,q}^{(1)}(R_p^A \otimes \text{id}_q + \text{id}_p \otimes R_q^A + \lambda \text{id}_p \otimes \text{id}_q), \end{aligned}$$

thus $(\{A_p\}_{p \in \pi}, \{m_{p,q}^{(1)}\}_{p,q \in \pi}, \{R_p^A\}_{p \in \pi}, \{(\alpha_p^A)^2\}_{p \in \pi})$ is a Rota–Baxter Hom- π -algebra.

Assume that for $n = k$, $(\{A_p\}_{p \in \pi}, \{m_{p,q}^{(k)}\}_{p,q \in \pi}, \{R_p^A\}_{p \in \pi}, \{(\alpha_p^A)^{k+1}\}_{p \in \pi})$ is a Rota–Baxter Hom- π -algebra, then for $n = k + 1$, we have

$$\begin{aligned} m_{pq,r}^{(k+1)}(m_{p,q}^{(k+1)} \otimes (\alpha_r^A)^{k+2}) &= \alpha_{pqr}^A m_{pq,r}^{(k)} (\alpha_{pq}^A m_{p,q}^{(k)} \otimes (\alpha_r^A)^{k+2}) \\ &= (\alpha_{pqr}^A)^2 m_{pq,r}^{(k)} (m_{p,q}^{(k)} \otimes (\alpha_r^A)^{k+1}) = (\alpha_{pqr}^A)^2 m_{p,q,r}^{(k)} ((\alpha_p^A)^{k+1} \otimes m_{q,r}^{(k)}) \\ &= \alpha_{pqr}^A m_{p,q,r}^{(k)} ((\alpha_p^A)^{k+2} \otimes \alpha_{qr}^A m_{q,r}^{(k)}) = m_{p,q,r}^{(k+1)}((\alpha_p^A)^{k+2} \otimes m_{q,r}^{(k+1)}), \end{aligned}$$

$$\begin{aligned} m_{p,q}^{(k+1)}(R_p^A \otimes R_q^A) &= \alpha_{pq}^A m_{p,q}^{(k)} (R_p^A \otimes R_q^A) \\ &= R_{pq}^A \alpha_{pq}^A m_{p,q}^{(k)} (R_p^A \otimes \text{id}_q + \text{id}_p \otimes R_q^A + \lambda \text{id}_p \otimes \text{id}_q) \\ &= R_{pq}^A m_{p,q}^{(k+1)} (R_p^A \otimes \text{id}_q + \text{id}_p \otimes R_q^A + \lambda \text{id}_p \otimes \text{id}_q), \end{aligned}$$

thus $(\{A_p\}_{p \in \pi}, \{m_{p,q}^{(k+1)}\}_{p,q \in \pi}, \{R_p^A\}_{p \in \pi}, \{(\alpha_p^A)^{k+2}\}_{p \in \pi})$ is a Rota–Baxter Hom- π -algebra. By the induction principle, the proof is complete. $\square$

Definition 3. Let $\mathcal{A}$ be a π -algebra. A centroid element of $\mathcal{A}$ is a family of linear operators $\{\alpha_p^A : A_p \rightarrow A_p\}_{p \in \pi}$ such that $\alpha_{pq}^A m_{p,q}^A = m_{p,q}^A (\alpha_p^A \otimes \text{id}_q) = m_{p,q}^A (\text{id}_p \otimes \alpha_q^A)$ and $\alpha_p^A R_p^A = R_p^A \alpha_p^A$ hold for all $p, q \in \pi$. The notion of centroids for ungraded algebras was first introduced in [2]; we extend this notion to the graded π -setting here.

Rota–Baxter π -algebras can also be used to construct Rota–Baxter Hom- π -algebras via their centroid elements. Beyond endomorphism-induced Yau twists, a broader class of twisted structures can be obtained using centroid elements of the Rota–Baxter π -algebra. We present two generalized constructions in the following theorem.

Theorem 3. Let ${}^R\mathcal{A}$ be a Rota–Baxter π -algebra, and let $\{\alpha_p^A : A_p \rightarrow A_p\}_{p \in \pi}$ be a centroid element on ${}^R\mathcal{A}$ as defined above. Then $(\{A_p\}_{p \in \pi}, \{m_{p,q}^{A,*} = m_{p,q}(\alpha_p^A \otimes \text{id}_q)\}_{p,q \in \pi}, \{R_p^A\}_{p \in \pi}, \{\alpha_p^A\}_{p \in \pi})$ and $(\{A_p\}_{p \in \pi}, \{m_{p,q}^{A,\#} = m_{p,q}(\alpha_p^A \otimes \alpha_q^A)\}_{p,q \in \pi}, \{R_p^A\}_{p \in \pi}, \{\alpha_p^A\}_{p \in \pi})$ are both Rota–Baxter Hom- π -algebras.

Proof. By

$$\begin{aligned}
 m_{pq,r}^{A,*}(m_{p,q}^{A,*} \otimes \alpha_r^A) &= m_{pq,r}^A(\alpha_{pq}^A \otimes \text{id}_r)(m_{p,q}^A(\text{id}_p \otimes \alpha_q^A) \otimes \alpha_r^A) \\
 &= m_{pq,r}^A(\alpha_{pq}^A m_{p,q}^A \otimes \text{id}_r)(\text{id}_p \otimes \alpha_q^A \otimes \alpha_r^A) \\
 &= m_{pq,r}^A(m_{p,q}^A \otimes \text{id}_r)(\alpha_p^A \otimes \alpha_q^A \otimes \alpha_r^A) \\
 &= m_{p,q,r}^A(\text{id}_p \otimes m_{q,r}^A)(\alpha_p^A \otimes \alpha_q^A \otimes \alpha_r^A) \\
 &= m_{p,q,r}^A(\text{id}_p \otimes \alpha_{qr}^A m_{q,r}^A)(\alpha_p^A \otimes \alpha_q^A \otimes \text{id}_r) \\
 &= m_{p,q,r}^A(\text{id}_p \otimes \alpha_{qr}^A)(\alpha_p \otimes m_{q,r}^A(\alpha_q^A \otimes \text{id}_r)) \\
 &= m_{p,q,r}^{A,*}(\alpha_p^A \otimes m_{q,r}^A(\alpha_q^A \otimes \text{id}_r)) = m_{p,q,r}^{A,*}(\alpha_p^A \otimes m_{q,r}^{A,*}),
 \end{aligned}$$

$$\begin{aligned}
 m_{p,q}^{A,*}(R_p^A \otimes R_q^A) &= m_{p,q}^A(\alpha_p^A \otimes \text{id}_q)(R_p^A \otimes R_q^A) \\
 &= m_{p,q}^A(R_p^A \otimes R_q^A)(\alpha_p^A \otimes \text{id}_q) \\
 &= R_{pq}^A m_{p,q}^A(R_p^A \otimes \text{id}_q + \text{id}_p \otimes R_q^A + \lambda \text{id}_p \otimes \text{id}_q)(\alpha_p^A \otimes \text{id}_q) \\
 &= R_{pq}^A m_{p,q}^A(\alpha_p^A \otimes \text{id}_q)(R_p^A \otimes \text{id}_q + \text{id}_p \otimes R_q^A + \lambda \text{id}_p \otimes \text{id}_q) \\
 &= R_{pq}^A m_{p,q}^{A,*}(R_p^A \otimes \text{id}_q + \text{id}_p \otimes R_q^A + \lambda \text{id}_p \otimes \text{id}_q),
 \end{aligned}$$

$(\{A_p\}_{p \in \pi}, \{m_{p,q}^{A,*}\}_{p,q \in \pi}, \{R_p^A\}_{p \in \pi}, \{\alpha_p^A\}_{p \in \pi})$ is a Rota–Baxter Hom- π -algebra. Moreover,

$$\begin{aligned}
 m_{pq,r}^{A,\#}(m_{p,q}^{A,\#} \otimes \alpha_r^A) &= m_{pq,r}^A(\alpha_{pq}^A \otimes \alpha_r^A)(m_{p,q}^A \otimes \text{id}_r)(\alpha_p^A \otimes \alpha_q^A \otimes \alpha_r^A) \\
 &= m_{pq,r}^A(m_{p,q}^A \otimes \text{id}_r)((\alpha_p^A)^2 \otimes \alpha_q^A \otimes \alpha_r^2) \\
 &= m_{p,q,r}^A(\text{id}_p \otimes m_{q,r}^A)((\alpha_p^A)^2 \otimes \alpha_q^A \otimes \alpha_r^2) \\
 &= m_{p,q,r}^A(\alpha_p^A \otimes \alpha_{qr}^A)(\text{id}_p \otimes m_{q,r}^A)(\alpha_p^A \otimes \alpha_q^A \otimes (\alpha_r^A)^2) = m_{p,q,r}^{A,\#}(\alpha_p^A \otimes m_{q,r}^{A,\#}),
 \end{aligned}$$

$$\begin{aligned}
 m_{p,q}^{A,\#}(R_p^A \otimes R_q^A) &= m_{p,q}^A(\alpha_p^A \otimes \alpha_q^A)(R_p^A \otimes R_q^A) \\
 &= m_{p,q}^A(R_p^A \otimes R_q^A)(\alpha_p^A \otimes \alpha_q^A) \\
 &= R_{pq}^A m_{p,q}^A(R_p^A \otimes \text{id}_q + \text{id}_p \otimes R_q^A + \lambda \text{id}_p \otimes \text{id}_q)(\alpha_p^A \otimes \alpha_q^A) \\
 &= R_{pq}^A m_{p,q}^A(\alpha_p^A \otimes \alpha_q^A)(R_p^A \otimes \text{id}_q + \text{id}_p \otimes R_q^A + \lambda \text{id}_p \otimes \text{id}_q) \\
 &= R_{pq}^A m_{p,q}^{A,\#}(R_p^A \otimes \text{id}_q + \text{id}_p \otimes R_q^A + \lambda \text{id}_p \otimes \text{id}_q),
 \end{aligned}$$

therefore $(\{A_p\}_{p \in \pi}, \{m_{p,q}^{A,\#}\}_{p,q \in \pi}, \{R_p^A\}_{p \in \pi}, \{\alpha_p^A\}_{p \in \pi})$ is a Rota–Baxter Hom- π -algebra. $\square$

All the above constructions are performed on existing π -algebras. For an ordinary Rota–Baxter Hom-algebra without gradation, one can naturally obtain a Rota–Baxter Hom- π -algebra by means of the monoid tensor lifting.

For any $p \in \pi$, and (A, m, α, R) a Rota–Baxter Hom-algebra in its usual meaning (note that in this case the gradation is not involved), consider the one-dimensional linear space $\mathbb{K}p := \{kp \mid k \in \mathbb{K}\}$ that naturally gives rise to a family of linear spaces $\{\mathbb{K}p\}_{p \in \pi}$ which is a requirement for the graded case. Based on $\{\mathbb{K}p\}_{p \in \pi}$, we define the following families:

- (a) a family of linear spaces $\{A \otimes \mathbb{K}p\}_{p \in \pi}$;
- (b) a family of bilinear operators $\{m_{p,q}^\otimes: (A \otimes \mathbb{K}p) \otimes (A \otimes \mathbb{K}q) \rightarrow A \otimes \mathbb{K}(pq), \quad a \otimes k_1p \otimes b \otimes k_2q \mapsto ab \otimes k_1k_2pq\}_{p,q \in \pi}$;
- (c) a family of linear operators $\{R_p: A \otimes \mathbb{K}p \rightarrow A \otimes \mathbb{K}p, \quad a \otimes kp \mapsto R(a) \otimes kp\}_{p \in \pi}$;
- (d) a family of linear operators $\{\alpha_p: A \otimes \mathbb{K}p \rightarrow A \otimes \mathbb{K}p, \quad a \otimes kp \mapsto \alpha(a) \otimes kp\}_{p \in \pi}$.

Then, as a result, it will be straightforward to verify the following result, and hence we will omit its proof here.

Lemma 2. *Following the above notation, the tuple*

$$(\{A \otimes \mathbb{K}p\}_{p \in \pi}, \{m_{p,q}^{\otimes}\}_{p,q \in \pi}, \{R_p\}_{p \in \pi}, \{\alpha_p\}_{p \in \pi})$$

is a Rota–Baxter Hom- π -algebra.

To demonstrate the above constructive methods properly, we provide comprehensively detailed examples following those in [11], concerning the complete classification of Rota–Baxter operators, compatible endomorphisms, and centroid elements for low-dimensional cases.

Example 1. Let $A_2 = \mathbb{K}u_1 \oplus \mathbb{K}u_2$ be a 2-dimensional associative algebra with multiplication $u_1^2 = u_1$, $u_1u_2 = u_2u_1 = u_2$, $u_2^2 = u_2$. There exist several Rota–Baxter operators of weight λ on A_2 , listed as follows [11]:

- (a) $R_a^2(u_1) = -\lambda u_1$, $R_a^2(u_2) = 0$;
- (b) $R_b^2(u_1) = 0$, $R_b^2(u_2) = -\lambda u_2$;
- (c) $R_c^2(u_1) = -\lambda u_1$, $R_c^2(u_2) = -\lambda u_2$;
- (d) $R_d^2(u_1) = -\lambda u_2$, $R_d^2(u_2) = -\lambda u_2$;
- (e) $R_e^2(u_1) = -\lambda u_1$, $R_e^2(u_2) = -\lambda u_1$;
- (f) $R_f^2(u_1) = \lambda u_2$, $R_f^2(u_2) = 0$;
- (g) $R_g^2(u_1) = 0$, $R_g^2(u_2) = \lambda u_1 - \lambda u_2$;
- (h) $R_h^2(u_1) = -2\lambda u_1 + \lambda u_2$, $R_h^2(u_2) = -\lambda u_1$;
- (i) $R_i^2(u_1) = -\lambda u_1 - \lambda u_2$, $R_i^2(u_2) = -\lambda u_2$;
- (j) $R_j^2(u_1) = -\lambda u_1 + \lambda u_2$, $R_j^2(u_2) = 0$;
- (k) $R_k^2(u_1) = \lambda u_1 - \lambda u_2$, $R_k^2(u_2) = \lambda u_1 - \lambda u_2$.

Some non-trivial algebra endomorphisms (excluding the identity and zero endomorphism) on A_2 and their commuting Rota–Baxter operators are listed as follows:

- (a) $\alpha_1^2: \alpha_1^2(u_1) = u_1$, $\alpha_1^2(u_2) = 0$, commutes with Rota–Baxter operators of type (a), (b), (c);
- (b) $\alpha_2^2: \alpha_2^2(u_1) = u_1$, $\alpha_2^2(u_2) = u_1$, commutes with Rota–Baxter operators of type (c), (e), (g);
- (c) $\alpha_3^2: \alpha_3^2(u_1) = u_1$, $\alpha_3^2(u_2) = u_1 - u_2$, commutes with Rota–Baxter operators of type (c);
- (d) $\alpha_4^2: \alpha_4^2(u_1) = u_2$, $\alpha_4^2(u_2) = 0$, commutes with Rota–Baxter operators of type (c), (f), (i);
- (e) $\alpha_5^2: \alpha_5^2(u_1) = u_2$, $\alpha_5^2(u_2) = u_2$, commutes with Rota–Baxter operators of type (c), (d), (j);
- (f) $\alpha_6^2: \alpha_6^2(u_1) = u_1 - u_2$, $\alpha_6^2(u_2) = 0$, commutes with Rota–Baxter operators of type (c), (d), (j);
- (g) $\alpha_7^2: \alpha_7^2(u_1) = u_1 - u_2$, $\alpha_7^2(u_2) = u_1 - u_2$, commutes with Rota–Baxter operators of type (c), (h), (k).

Let (α, R) be any commuting pair. By Theorem 1, one obtains a 2-dimensional Rota–Baxter Hom-algebra $\text{Yau}(A_2, \alpha, R) := (A_2, \alpha m, \alpha, R)$.

For $k_2 \neq 0$, the following centroid element on A_2 commutes with the three types of Rota–Baxter operators (c), (d), and (j): $\beta_1^2(u_1) = k_1 u_1 + k_2 u_2$, $\beta_1^2(u_2) = (k_1 + k_2)u_2$.

Let (β, R) be any commuting pair. By Theorem 3, one obtains a 2-dimensional Rota–Baxter Hom-algebra $\text{GYau}(A_2, \beta, R) := (A_2, m(\beta \otimes \text{id}), \beta, R)$.

Example 2. Let $A_3 = \mathbb{K}u_1 \oplus \mathbb{K}u_2 \oplus \mathbb{K}u_3$ be a 3-dimensional commutative associative algebra with multiplication $u_1^2 = u_1$, $u_1u_2 = u_2u_1 = u_2$, $u_1u_3 = u_3u_1 = u_3$, $u_2^2 = u_2$, $u_2u_3 = u_3u_2 = u_3$, $u_3^2 = 0$. There exist several Rota–Baxter operators of weight λ on A_3 , listed as follows [11]:

- (a) $R_a^3(u_1) = -\lambda u_1$, $R_a^3(u_2) = 0$, $R_a^3(u_3) = 0$;

- (b) $R_b^3(u_1) = 0, R_b^3(u_2) = -\lambda u_2, R_b^3(u_3) = 0;$
- (c) $R_c^3(u_1) = 0, R_c^3(u_2) = 0, R_c^3(u_3) = -\lambda u_3;$
- (d) $R_d^3(u_1) = -\lambda u_1, R_d^3(u_2) = -\lambda u_2, R_d^3(u_3) = 0;$
- (e) $R_e^3(u_1) = 0, R_e^3(u_2) = -\lambda u_2, R_e^3(u_3) = -\lambda u_3;$
- (f) $R_f^3(u_1) = -\lambda u_1, R_f^3(u_2) = 0, R_f^3(u_3) = -\lambda u_3;$
- (g) $R_g^3(u_1) = -\lambda u_1, R_g^3(u_2) = -\lambda u_2, R_g^3(u_3) = -\lambda u_3;$
- (h) $R_h^3(u_1) = -\lambda u_2, R_h^3(u_2) = -\lambda u_2, R_h^3(u_3) = 0;$
- (i) $R_i^3(u_1) = -\lambda u_2, R_i^3(u_2) = -\lambda u_2, R_i^3(u_3) = -\lambda u_3;$
- (j) $R_j^3(u_1) = \lambda u_2, R_j^3(u_2) = 0, R_j^3(u_3) = 0;$
- (k) $R_k^3(u_1) = \lambda u_2, R_k^3(u_2) = 0, R_k^3(u_3) = -\lambda u_3;$
- (l) $R_l^3(u_1) = 0, R_l^3(u_2) = \lambda u_1 - \lambda u_2, R_l^3(u_3) = 0;$
- (m) $R_m^3(u_1) = 0, R_m^3(u_2) = \lambda u_1 - \lambda u_2, R_m^3(u_3) = -\lambda u_3;$
- (n) $R_n^3(u_1) = -2\lambda u_1 + \lambda u_2, R_n^3(u_2) = -\lambda u_1, R_n^3(u_3) = 0;$
- (o) $R_o^3(u_1) = -2\lambda u_1 + \lambda u_2, R_o^3(u_2) = -\lambda u_1, R_o^3(u_3) = -\lambda u_3;$
- (p) $R_p^3(u_1) = -\lambda u_1 - \lambda u_2, R_p^3(u_2) = -\lambda u_2, R_p^3(u_3) = 0;$
- (q) $R_q^3(u_1) = -\lambda u_1 - \lambda u_2, R_q^3(u_2) = -\lambda u_2, R_q^3(u_3) = -\lambda u_3;$
- (r) $R_r^3(u_1) = -\lambda u_1 + \lambda u_2, R_r^3(u_2) = 0, R_r^3(u_3) = 0;$
- (s) $R_s^3(u_1) = -\lambda u_1 + \lambda u_2, R_s^3(u_2) = 0, R_s^3(u_3) = -\lambda u_3;$
- (t) $R_t^3(u_1) = -\lambda u_1, R_t^3(u_2) = -\lambda u_1, R_t^3(u_3) = 0;$
- (u) $R_u^3(u_1) = -\lambda u_1, R_u^3(u_2) = -\lambda u_1, R_u^3(u_3) = -\lambda u_3;$
- (v) $R_v^3(u_1) = \lambda u_1 - \lambda u_2, R_v^3(u_2) = \lambda u_1 - \lambda u_2, R_v^3(u_3) = 0;$
- (w) $R_w^3(u_1) = \lambda u_1 - \lambda u_2, R_w^3(u_2) = \lambda u_1 - \lambda u_2, R_w^3(u_3) = -\lambda u_3.$

Some non-trivial algebra endomorphisms (excluding the identity and zero endomorphism) on A_3 and their commuting Rota–Baxter operators are listed as follows:

- (a) $\alpha_1^3: \alpha_1^3(u_1) = u_1, \alpha_1^3(u_2) = 0, \alpha_1^3(u_3) = 0$, commutes with types (a), (b), (c), (d), (e), (f), (g);
- (b) $\alpha_2^3(k): \alpha_2^3(k)(u_1) = u_1, \alpha_2^3(k)(u_2) = u_1, \alpha_2^3(k)(u_3) = ku_3$ ($k \in \mathbb{K}$), commutes with types (c), (d), (g), (l), (m), (t), (u);
- (c) $\alpha_3^3(k): \alpha_3^3(k)(u_1) = u_1, \alpha_3^3(k)(u_2) = u_2, \alpha_3^3(k)(u_3) = ku_3$ ($k \in \mathbb{K}, k \neq 1$), commutes with all types (a)–(w);
- (d) $\alpha_4^3: \alpha_4^3(u_1) = u_1, \alpha_4^3(u_2) = u_1 - u_2, \alpha_4^3(u_3) = 0$, commutes with types (c), (d), (g);
- (e) $\alpha_5^3: \alpha_5^3(u_1) = u_1 - u_2, \alpha_5^3(u_2) = 0, \alpha_5^3(u_3) = 0$, commutes with types (c), (d), (g), (h), (i), (r), (s);
- (f) $\alpha_6^3: \alpha_6^3(u_1) = u_1 - u_2, \alpha_6^3(u_2) = u_1 - u_2, \alpha_6^3(u_3) = 0$, commutes with types (c), (d), (g), (n), (o), (v), (w);
- (g) $\alpha_7^3: \alpha_7^3(u_1) = u_2, \alpha_7^3(u_2) = 0, \alpha_7^3(u_3) = 0$, commutes with types (c), (d), (g), (j), (k), (p), (q);
- (h) $\alpha_8^3(k): \alpha_8^3(k)(u_1) = u_2, \alpha_8^3(k)(u_2) = u_2, \alpha_8^3(k)(u_3) = ku_3$ ($k \in \mathbb{K}$), commutes with types (c), (d), (g), (h), (i), (r), (s).

Let (α, R) be any commuting pair. By Theorem 1, one obtains a 3-dimensional Rota–Baxter Hom–algebra $\text{Yau}(A_3, \alpha, R) := (A_3, \alpha m, \alpha, R)$.

Some centroid elements on A_3 commuting with Rota–Baxter operators are listed as follows:

- (a) $\beta_1^3(u_1) = k_1 u_1 + k_2 u_2, \beta_1^3(u_2) = (k_1 + k_2) u_2, \beta_1^3(u_3) = (k_1 + k_2) u_3$, which commutes with seven types of Rota–Baxter operators (c), (d), (g), (h), (i), (r), (s);
- (b) $\beta_2^3(u_1) = k_1 u_1 + k_2 u_2 + k_3 u_3, \beta_2^3(u_2) = (k_1 + k_2) u_2 + k_3 u_3, \beta_2^3(u_3) = (k_1 + k_2) u_3$, which commutes with three types of Rota–Baxter operators (g), (i), (r).

Let (β, R) be any commuting pair. By Theorem 3, one obtains a 3-dimensional Rota–Baxter Hom–algebra $\text{GYau}(A_3, \beta, R) := (A_3, m(\beta \otimes \text{id}), \beta, R)$.

Combining Lemma 2 with the classification results in Example 1 and Example 2, we can derive a large number of explicit Rota–Baxter Hom- π -algebras. A typical construction is presented as follows.

Example 3. Take A_2 , R_c^2 and α_3^2 from Example 1, and let $\pi = \mathbb{Z}_2 = \{e, g\}$ with $g^2 = e$. By Theorem 1 and Lemma 2, the resulting Rota–Baxter Hom- π -algebra is given by:

- (a) Graded linear spaces: for each $p \in \pi$, $A_p = A_2 \otimes \mathbb{K}p = \text{span}_{\mathbb{K}}\{u_1 \otimes p, u_2 \otimes p\}$
- (b) Graded Hom-multiplication: for all $p, q \in \pi$, $(u_1 \otimes p)(u_1 \otimes q) = u_1 \otimes pq$, $(u_1 \otimes p)(u_2 \otimes q) = (u_2 \otimes p)(u_1 \otimes q) = (u_2 \otimes p)(u_2 \otimes q) = (u_1 - u_2) \otimes pq$.
- (c) Graded Rota–Baxter operators: for each $p \in \pi$, $R_p(u_1 \otimes p) = -\lambda u_1 \otimes p$, $R_p(u_2 \otimes p) = -\lambda u_2 \otimes p$.
- (d) Graded Hom-operators: for each $p \in \pi$, $\alpha_p(u_1 \otimes p) = u_1 \otimes p$, $\alpha_p(u_2 \otimes p) = (u_1 - u_2) \otimes p$.

The notion of Rota–Baxter pairs is introduced in [11] to facilitate another approach to construct Rota–Baxter π -algebras. This notion can also be regarded as a generalization of the usual Rota–Baxter operators. Later on, we will study another generalization of Rota–Baxter algebras, which is called the Rota–Baxter system.

Definition 4. A Rota–Baxter pair of weight λ on a Hom-associative algebra (A, m, α) consists of two Rota–Baxter operators R, S of weight λ satisfying

$$\begin{aligned} m(R \otimes S) &= Sm(R \otimes \text{id} + \text{id} \otimes S + \lambda \text{id} \otimes \text{id}), \\ m(S \otimes R) &= Sm(S \otimes \text{id} + \text{id} \otimes R + \lambda \text{id} \otimes \text{id}). \end{aligned}$$

Lemma 3. Suppose (R, S) is a Rota–Baxter pair on a Hom-associative algebra (A, m, α) , and take the two-element monoid $\pi = \{e, g\}$ with e being the unit element and $g^2 = g$. By Lemma 2, we obtain a Hom-associative algebra $(\{A \otimes \mathbb{K}p\}_{p \in \pi}, \{m_{p,q}^{\otimes}\}_{p,q \in \pi}, \{\alpha_p\}_{p \in \pi})$. Define two linear mappings $R_e: A \otimes \mathbb{K}e \rightarrow A \otimes \mathbb{K}e$, $a \otimes ke \mapsto R(a) \otimes ke$ and $R_g: A \otimes \mathbb{K}g \rightarrow A \otimes \mathbb{K}g$, $a \otimes kg \mapsto S(a) \otimes kg$, then $(\{A \otimes \mathbb{K}p\}_{p \in \pi}, \{m_{p,q}^{\otimes}\}_{p,q \in \pi}, \{R_p\}_{p \in \pi}, \{\alpha_p\}_{p \in \pi})$ forms a Rota–Baxter Hom- π -algebra.

The proof of Lemma 3 is essentially the same as the case without Hom-operators in [11].

Remark 3. (i) Let A_2 and A_3 be as in Example 1 and Example 2, respectively. Consider the Rota–Baxter pairs (R, S) on A_2/A_3 listed in Example 2.12 and Example 2.13 of [11]. Please note that since the defining formulas of a Rota–Baxter pair do not involve Hom-operators for most Rota–Baxter pairs (R, S) , if both R and S commute with the algebra endomorphism α_i on A_2/A_3 , then (R, S) remains a Rota–Baxter pair for the Hom-associative algebras $\text{Yau}(A_2, \alpha_i)/\text{Yau}(A_3, \alpha_i)$ induced by the Yau twist. For example:

- (a) On $\text{Yau}(A_2, \alpha_1^2)$, (R_a^2, R_c^2) and (R_b^2, R_c^2) are Rota–Baxter pairs;
- (b) On $\text{Yau}(A_2, \alpha_5^2)$, (R_c^2, R_d^2) and (R_d^2, R_j^2) are Rota–Baxter pairs;
- (c) On $\text{Yau}(A_3, \alpha_1^3)$, (R_a^3, R_d^3) and (R_a^3, R_g^3) are Rota–Baxter pairs.

Other cases follow similarly, and we omit the full list here.

(ii) If a Rota–Baxter pair (R, S) commutes with the centroid element β_i on A_2/A_3 , then (R, S) remains a Rota–Baxter pair on the Hom-associative algebras $\text{GYau}(A_2, \beta_i)/\text{GYau}(A_3, \beta_i)$ induced by the generalized Yau twist via centroid elements. For example:

- (a) On $\text{GYau}(A_2, \beta_1^2)$, (R_c^2, R_d^2) , (R_c^2, R_j^2) and (R_d^2, R_j^2) are Rota–Baxter pairs;

- (b) On $\text{GYau}(A_3, \beta_1^3)$, (R_c^3, R_g^3) , (R_c^3, R_i^3) , (R_c^3, R_r^3) , (R_c^3, R_s^3) , (R_d^3, R_g^3) , (R_d^3, R_h^3) , (R_d^3, R_i^3) , (R_d^3, R_r^3) , (R_g^3, R_i^3) , (R_g^3, R_r^3) , (R_g^3, R_s^3) , (R_h^3, R_i^3) , (R_h^3, R_r^3) , (R_i^3, R_r^3) and (R_i^3, R_s^3) are all Rota–Baxter pairs;
- (c) On $\text{GYau}(A_3, \beta_2^3)$, (R_g^3, R_i^3) , (R_g^3, R_r^3) and (R_i^3, R_r^3) are all Rota–Baxter pairs.

Combining Lemma 3 with the Rota–Baxter pairs given in Remark 3, we can construct various concrete Rota–Baxter Hom- π -algebras. A representative example is provided below.

Example 4. Consider α_5^2 and the Rota–Baxter pair (R_c^2, R_d^2) on A_2 from Example 1, and let $\pi = \{e, g\}$ as in Lemma 3. Then by following Lemma 3 and letting $R_e := R_c^2$ and $R_g := R_d^2$, the resulting structure is as follows:

- (a) Graded linear spaces: $A_e = \text{span}_{\mathbb{K}}\{u_1 \otimes e, u_2 \otimes e\}$, $A_g = \text{span}_{\mathbb{K}}\{u_1 \otimes g, u_2 \otimes g\}$.
- (b) Graded Hom-multiplication: for all $p, q \in \pi$, $(u_1 \otimes p)(u_1 \otimes q) = u_2 \otimes pq$, $(u_1 \otimes p)(u_2 \otimes q) = u_2 \otimes pq$, $(u_2 \otimes p)(u_1 \otimes q) = u_2 \otimes pq$, $(u_2 \otimes p)(u_2 \otimes q) = u_2 \otimes pq$.
- (c) Graded Hom-operators: for each $p \in \pi$, $\alpha_p(u_1 \otimes p) = u_2 \otimes p$, $\alpha_p(u_2 \otimes p) = u_2 \otimes p$.
- (d) Component-wise Rota–Baxter operators: $R_e(u_1 \otimes e) = -\lambda u_1 \otimes e$, $R_e(u_2 \otimes e) = -\lambda u_2 \otimes e$, $R_g(u_1 \otimes g) = -\lambda u_2 \otimes g$, $R_g(u_2 \otimes g) = -\lambda u_2 \otimes g$.

4. Hom-(Tri)dendriform π -Algebras: Derivation Chains and Rota–Baxter Inductions

Based on the structures and notions introduced in Section 3, this section defines Hom-tridendriform and Hom-dendriform π -algebras, establishes their Yau twist and derivation chain, and proves that Rota–Baxter Hom- π -algebras naturally induce such dendriform structures.

We start with formal definitions of these dendriform structures, followed by their Yau twist and stepwise derivation results.

Definition 5. A Hom-tridendriform π -algebra ${}^{\text{Td}}\mathcal{A}_H$ consists of a family of linear vector spaces $\{A_p\}_{p \in \pi}$ together with three families of bilinear operators $\{\prec_{p,q}^A, \succ_{p,q}^A, \bullet_{p,q}^A: A_p \otimes A_q \rightarrow A_{pq}\}_{p,q \in \pi}$ and a family of linear operators $\{\alpha_p^A: A_p \rightarrow A_p\}_{p \in \pi}$ such that for all $p, q, r \in \pi$, the following identities hold:

$$\prec_{pq,r}^A (\prec_{p,q}^A \otimes \alpha_r^A) = \prec_{p,q,r}^A (\alpha_p^A \otimes (\prec_{q,r}^A + \succ_{q,r}^A + \bullet_{q,r}^A)), \quad (1)$$

$$\prec_{pq,r}^A (\succ_{p,q}^A \otimes \alpha_r^A) = \succ_{p,q,r}^A (\alpha_p^A \otimes \prec_{q,r}^A), \quad (2)$$

$$\succ_{pq,r}^A (\alpha_p^A \otimes \succ_{q,r}^A) = \succ_{p,q,r}^A ((\prec_{p,q}^A + \succ_{p,q}^A + \bullet_{p,q}^A) \otimes \alpha_r^A), \quad (3)$$

$$\bullet_{pq,r}^A (\prec_{p,q}^A \otimes \alpha_r^A) = \bullet_{p,q,r}^A (\alpha_p^A \otimes \succ_{q,r}^A), \quad (4)$$

$$\bullet_{pq,r}^A (\succ_{p,q}^A \otimes \alpha_r^A) = \succ_{p,q,r}^A (\alpha_p^A \otimes \bullet_{q,r}^A), \quad (5)$$

$$\prec_{pq,r}^A (\bullet_{p,q}^A \otimes \alpha_r^A) = \bullet_{p,q,r}^A (\alpha_p^A \otimes \prec_{q,r}^A), \quad (6)$$

$$\bullet_{pq,r}^A (\bullet_{p,q}^A \otimes \alpha_r^A) = \bullet_{p,q,r}^A (\alpha_p^A \otimes \bullet_{q,r}^A). \quad (7)$$

A Hom-tridendriform π -algebra homomorphism from ${}^{\text{Td}}\mathcal{A}_H$ to ${}^{\text{Td}}\mathcal{B}_H$ is a family of linear operators $\{f_p: A_p \rightarrow B_p\}_{p \in \pi}$ such that for all $p, q \in \pi$, we have $f_{pq} \prec_{p,q}^A = \prec_{p,q}^B (f_p \otimes f_q)$, $f_{pq} \succ_{p,q}^A = \succ_{p,q}^B (f_p \otimes f_q)$, $f_{pq} \bullet_{p,q}^A = \bullet_{p,q}^B (f_p \otimes f_q)$, $\alpha_p^B f_p = f_p \alpha_p^A$.

Remark 4. Tridendriform π -algebras are special cases of Hom-tridendriform π -algebras with $\alpha_p = \text{id}_p$ for all $p \in \pi$.

Theorem 4. For an endomorphism $\{\alpha_p^A\}_{p \in \pi}$ of the tridendriform π -algebra ${}^{Td}\mathcal{A}$, the quintuple $(\{A_p\}_{p \in \pi}, \{\prec_{p,q}^{A,*} = \alpha_{pq}^A \prec_{p,q}^A\}_{p,q \in \pi}, \{\succ_{p,q}^{A,*} = \alpha_{pq}^A \succ_{p,q}^A\}_{p,q \in \pi}, \{\bullet_{p,q}^{A,*} = \alpha_{pq}^A \bullet_{p,q}^A\}_{p,q \in \pi}, \{\alpha_p^A\}_{p \in \pi})$ is a Hom-tridendriform π -algebra, called the Yau twist of the former.

If $\{f_p\}_{p \in \pi}$ is a tridendriform π -algebra homomorphisms from ${}^{Td}\mathcal{A}$ to ${}^{Td}\mathcal{B}$ that satisfies $\alpha_p^B f_p = f_p \alpha_p^A$ for all $p \in \pi$. Then $\{f_p\}_{p \in \pi}$ will be a Hom-tridendriform π -algebra homomorphism between the two Yau twists ${}^{Td}\mathcal{A}_H$ and ${}^{Td}\mathcal{B}_H$.

Proof. For any $p, q, r \in \pi$, we have

$$\begin{aligned} \prec_{pq,r}^{A,*} (\prec_{p,q}^{A,*} \otimes \alpha_r^A) &= \alpha_{pqr}^A \prec_{pq,r}^A (\alpha_{pq}^A \prec_{p,q}^A \otimes \alpha_r^A) \\ &= \alpha_{pqr}^2 \prec_{pq,r}^A (\prec_{p,q}^A \otimes \text{id}_r) \\ &= \alpha_{pqr}^2 \prec_{p,q,r}^A (\text{id}_p \otimes (\prec_{q,r}^A + \succ_{q,r}^A + \bullet_{q,r}^A)) \\ &= \alpha_{pqr}^A \prec_{p,q,r}^A (\alpha_p^A \otimes \alpha_{qr}^A (\prec_{q,r}^A + \succ_{q,r}^A + \bullet_{q,r}^A)) \\ &= \prec_{p,q,r}^{A,*} (\alpha_p^A \otimes (\prec_{q,r}^{A,*} + \succ_{q,r}^{A,*} + \bullet_{q,r}^{A,*})), \end{aligned}$$

the other defining identities of the Hom-tridendriform π -algebra can be verified similarly. Moreover,

$$f_{pq} \prec_{p,q}^{A,*} = f_{pq} \alpha_{pq}^A \prec_{p,q}^A = \alpha_{pq}^B f_{pq} \prec_{p,q}^A = \alpha_{pq}^B \prec_{p,q}^B (f_p \otimes f_q) = \prec_{p,q}^{B,*} (f_p \otimes f_q),$$

the other defining identities of the Hom-tridendriform π -algebra homomorphism can be verified similarly. $\square$

Definition 6. A Hom-dendriform π -algebra ${}^d\mathcal{A}_H$ consists of a family of linear spaces $\{A_p\}_{p \in \pi}$ together with two families of bilinear operators $\{\prec_{p,q}^A, \succ_{p,q}^A : A_p \otimes A_q \rightarrow A_{pq}\}_{p,q \in \pi}$ and a family of linear operators $\{\alpha_p^A : A_p \rightarrow A_p\}_{p \in \pi}$ such that for all $p, q, r \in \pi$, the following identities hold:

$$\begin{cases} \prec_{pq,r}^A (\prec_{p,q}^A \otimes \alpha_r^A) = \prec_{p,q,r}^A (\alpha_p^A \otimes (\prec_{q,r}^A + \succ_{q,r}^A)), \\ \prec_{pq,r}^A (\succ_{p,q}^A \otimes \alpha_r^A) = \succ_{p,q,r}^A (\alpha_p^A \otimes \prec_{q,r}^A), \\ \succ_{p,q,r}^A (\alpha_p^A \otimes \succ_{q,r}^A) = \succ_{p,q,r}^A ((\succ_{p,q}^A + \prec_{p,q}^A) \otimes \alpha_r^A). \end{cases}$$

Remark 5. A Hom-dendriform π -algebra is a special case of a Hom-tridendriform π -algebra with $\bullet_{p,q}^A = 0$ for all $p, q \in \pi$. Meanwhile, a Hom-tridendriform π -algebra induces a Hom-dendriform π -algebra.

Theorem 5. For a Hom-tridendriform π -algebra ${}^{Td}\mathcal{A}_H$, $(\{A_p\}_{p \in \pi}, \{\prec_{p,q}^{A'} = \prec_{p,q}^A + \bullet_{p,q}^A\}_{p,q \in \pi}, \{\succ_{p,q}^{A'} = \succ_{p,q}^A\}_{p,q \in \pi}, \{\alpha_p^A\}_{p \in \pi})$ is a Hom-dendriform π -algebra.

Proof. For any $p, q, r \in \pi$, we have

$$\begin{aligned} \prec_{pq,r}^{A'} (\prec_{p,q}^{A'} \otimes \alpha_r^A) &= (\prec_{pq,r}^A + \bullet_{pq,r}^A)((\prec_{p,q}^A + \bullet_{p,q}^A) \otimes \alpha_r^A), \\ &= \prec_{pq,r}^A (\prec_{p,q}^A \otimes \alpha_r^A) + \prec_{pq,r}^A (\bullet_{p,q}^A \otimes \alpha_r^A) \\ &\quad + \bullet_{pq,r}^A (\prec_{p,q}^A \otimes \alpha_r^A) + \bullet_{pq,r}^A (\bullet_{p,q}^A \otimes \alpha_r^A), \\ &= \prec_{p,q,r}^A (\alpha_p^A \otimes (\prec_{q,r}^A + \succ_{q,r}^A + \bullet_{q,r}^A)) + \bullet_{p,q,r}^A (\alpha_p^A \otimes \prec_{q,r}^A) \\ &\quad + \bullet_{p,q,r}^A (\alpha_p^A \otimes \succ_{q,r}^A) + \bullet_{p,q,r}^A (\alpha_p^A \otimes \bullet_{q,r}^A) \\ &= (\prec_{p,q,r}^A + \bullet_{p,q,r}^A)(\alpha_p^A \otimes (\prec_{q,r}^A + \succ_{q,r}^A + \bullet_{q,r}^A)) \\ &= \prec_{p,q,r}^{A'} (\alpha_p^A \otimes (\prec_{q,r}^{A'} + \succ_{q,r}^{A'})), \end{aligned}$$

the other defining identities of the Hom-dendriform π -algebra can be verified similarly. $\square$

Every Hom-dendriform π -algebra naturally gives rise to a Hom- π -algebra by summing its left and right dendriform products, which is demonstrated in the following theorem.

Theorem 6. For a Hom-dendriform π -algebra ${}^dA_H$, $(\{A_p\}_{p \in \pi}, \{*_p^A = \prec_{p,q}^A + \succ_{p,q}^A\}_{p,q \in \pi}, \{\alpha_p^A\}_{p \in \pi})$ will be a Hom- π -algebra.

Proof. For any $p, q, r \in \pi$, we have

$$\begin{aligned} *_p^A (*_{p,q}^A \otimes \alpha_r^A) &= \prec_{p,q,r}^A (\prec_{p,q}^A \otimes \alpha_r^A) + \succ_{p,q,r}^A (\prec_{p,q}^A \otimes \alpha_r^A) + \prec_{p,q,r}^A (\succ_{p,q}^A \otimes \alpha_r^A) \\ &\quad + \succ_{p,q,r}^A (\succ_{p,q}^A \otimes \alpha_r^A) \\ &= \prec_{p,q,r}^A (\alpha_p^A \otimes \prec_{q,r}^A) + \succ_{p,q,r}^A (\alpha_p^A \otimes \prec_{q,r}^A) + \prec_{p,q,r}^A (\alpha_p^A \otimes \succ_{q,r}^A) \\ &\quad + \succ_{p,q,r}^A (\alpha_p^A \otimes \succ_{q,r}^A) = *_p^A (\alpha_p^A \otimes *_q^A). \end{aligned}$$

$\square$

Combining the results obtained in Theorem 4 concerning the Yau twist with the results obtained in Theorems 5 and 6 will yield the following Corollary.

Corollary 1. If $\{\alpha_p^A\}_{p \in \pi}$ is a tridendriform π -algebra endomorphism on T^dA , then we have:

- A Hom-dendriform π -algebra $(\{A_p\}_{p \in \pi}, \{\prec_{p,q}^{A''} = \alpha_{pq}^A (\prec_{p,q}^A + \bullet_{p,q}^A)\}_{p,q \in \pi}, \{\succ_{p,q}^{A''} = \alpha_{pq}^A \succ_{p,q}^A\}_{p,q \in \pi}, \{\alpha_p^A\}_{p \in \pi})$;
- A Hom- π -algebra $(\{A_p\}_{p \in \pi}, \{*_p^A = \alpha_{pq}^A (\prec_{p,q}^A + \succ_{p,q}^A + \bullet_{p,q}^A)\}_{p,q \in \pi}, \{\alpha_p^A\}_{p \in \pi})$.

An explicit example is provided below to demonstrate the full twist and derivation chain described above.

Example 5. Let π and A_p be as in Example 4. Then we have

(a) Tridendriform π -algebra. The graded operations $\prec_{p,q}, \succ_{p,q}, \bullet_{p,q} : A_p \otimes A_q \rightarrow A_{pq}$ are defined as follows.

On the component $A_e \otimes A_e$: $(u_1 \otimes e) \prec_{e,e} (u_1 \otimes e) = -\lambda u_1 \otimes e$, $(u_1 \otimes e) \prec_{e,e} (u_2 \otimes e) = -\lambda u_2 \otimes e$, $(u_2 \otimes e) \prec_{e,e} (u_1 \otimes e) = -\lambda u_2 \otimes e$, $(u_2 \otimes e) \prec_{e,e} (u_2 \otimes e) = -\lambda u_2 \otimes e$, $(u_1 \otimes e) \succ_{e,e} (u_1 \otimes e) = -\lambda u_1 \otimes e$, $(u_1 \otimes e) \succ_{e,e} (u_2 \otimes e) = -\lambda u_2 \otimes e$, $(u_2 \otimes e) \succ_{e,e} (u_1 \otimes e) = -\lambda u_2 \otimes e$, $(u_2 \otimes e) \succ_{e,e} (u_2 \otimes e) = -\lambda u_2 \otimes e$, $(u_1 \otimes e) \bullet_{e,e} (u_1 \otimes e) = \lambda u_1 \otimes e$, $(u_1 \otimes e) \bullet_{e,e} (u_2 \otimes e) = \lambda u_2 \otimes e$, $(u_2 \otimes e) \bullet_{e,e} (u_1 \otimes e) = \lambda u_2 \otimes e$, $(u_2 \otimes e) \bullet_{e,e} (u_2 \otimes e) = \lambda u_2 \otimes e$.

For all $p, q \in \pi$ with $q = g$: $(u_i \otimes p) \prec_{p,q} (u_j \otimes q) = -\lambda u_2 \otimes g$, $i, j = 1, 2$.

For all $p, q \in \pi$ with $p = g$: $(u_i \otimes p) \succ_{p,q} (u_j \otimes q) = -\lambda u_2 \otimes g$, $i, j = 1, 2$.

For all $p, q \in \pi$ with $pq = g$: $(u_i \otimes p) \bullet_{p,q} (u_j \otimes q) = \lambda u_i u_j \otimes g$, $i, j = 1, 2$.

(b) Tridendriform π -algebra endomorphisms. Two non-trivial families of endomorphisms preserving the tridendriform structure are listed as follows.

Case 1: For all $p \in \pi$, $\alpha_p(u_1 \otimes p) = u_2 \otimes p$, $\alpha_p(u_2 \otimes p) = u_2 \otimes p$.

Case 2: For all $p \in \pi$, $\alpha_p(u_1 \otimes p) = (u_1 - u_2) \otimes p$, $\alpha_p(u_2 \otimes p) = 0$.

(c) Hom-tridendriform π -algebras via Yau twist.

Case 1: For all $p, q \in \pi$ and $i, j = 1, 2$, $(u_i \otimes p) \prec_{p,q}^* (u_j \otimes q) = -\lambda u_2 \otimes pq$, $(u_i \otimes p) \succ_{p,q}^* (u_j \otimes q) = -\lambda u_2 \otimes pq$, $(u_i \otimes p) \bullet_{p,q}^* (u_j \otimes q) = \lambda u_2 \otimes pq$.

(i) Induced Hom-dendriform π -algebra: $(u_i \otimes p) \prec_{p,q}' (u_j \otimes q) = 0$, $(u_i \otimes p) \succ_{p,q}' (u_j \otimes q) = -\lambda u_2 \otimes pq$, $i, j = 1, 2$.

(ii) Induced Hom- π -associative algebra: $(u_i \otimes p) *_p^A (u_j \otimes q) = -\lambda u_2 \otimes pq$, $i, j = 1, 2$, $p, q \in \pi$.

Case 2: On the component $A_e \otimes A_e$: $(u_1 \otimes e) \prec_{e,e}^* (u_1 \otimes e) = -\lambda(u_1 - u_2) \otimes e$, $(u_1 \otimes e) \prec_{e,e}^* (u_2 \otimes e) = 0$, $(u_2 \otimes e) \prec_{e,e}^* (u_1 \otimes e) = 0$, $(u_2 \otimes e) \prec_{e,e}^* (u_2 \otimes e) = 0$, $(u_1 \otimes e) \succ_{e,e}^* (u_1 \otimes e) = -\lambda(u_1 - u_2) \otimes e$, $(u_1 \otimes e) \succ_{e,e}^* (u_2 \otimes e) = 0$, $(u_2 \otimes e) \succ_{e,e}^* (u_1 \otimes e) = 0$, $(u_2 \otimes e) \succ_{e,e}^* (u_2 \otimes e) = 0$, $(u_1 \otimes e) \bullet_{e,e}^* (u_1 \otimes e) = \lambda(u_1 - u_2) \otimes e$, $(u_1 \otimes e) \bullet_{e,e}^* (u_2 \otimes e) = 0$, $(u_2 \otimes e) \bullet_{e,e}^* (u_1 \otimes e) = 0$, $(u_2 \otimes e) \bullet_{e,e}^* (u_2 \otimes e) = 0$.

For all $p, q \in \pi$ with $q = g$: $(u_i \otimes p) \prec_{p,q}^* (u_j \otimes q) = 0$, $i, j = 1, 2$. For all $p, q \in \pi$ with $p = g$: $(u_i \otimes p) \succ_{p,q}^* (u_j \otimes q) = 0$, $i, j = 1, 2$. For all $p, q \in \pi$ with $pq = g$: $(u_1 \otimes p) \bullet_{p,q}^* (u_1 \otimes q) = \lambda(u_1 - u_2) \otimes g$, and all other basis combinations yield 0.

(i) Induced Hom-dendriform π -algebra: For all $p, q \in \pi$ with $q = e$: $(u_i \otimes p) \prec'_{p,q} (u_j \otimes q) = 0$, $i, j = 1, 2$; For all $p, q \in \pi$ with $q = g$: $(u_1 \otimes p) \prec'_{p,q} (u_1 \otimes q) = \lambda(u_1 - u_2) \otimes g$, and all other basis combinations yield 0. For all $p, q \in \pi$: $(u_i \otimes p) \succ'_{p,q} (u_j \otimes q) = \succ_{p,q}^* (u_i \otimes p, u_j \otimes q)$, $i, j = 1, 2$.

(ii) Induced Hom- π -associative algebra: For all $p, q \in \pi$: $(u_1 \otimes p) *_p (u_1 \otimes q) = -\lambda(u_1 - u_2) \otimes pq$, and all other basis combinations yield 0.

As the paper might look bulky with many newly structured structures, the need for a mediator result seems unavoidable, and hence we propose the following result that might play such a role between the Rota–Baxter operators on Hom- π -algebras and the many types of newly defined structures from Section 4.

Theorem 7. If ${}^R A_H$ is a Rota–Baxter Hom- π -algebra and $\alpha_p^A R_p^A = R_p^A \alpha_p^A$ for all $p \in \pi$, then we obtain

- a Hom- π -subalgebra $(\{R_p^A(A_p)\}_{p \in \pi}, \{m_{p,q}^A\}_{p,q \in \pi}, \{\alpha_p^A\}_{p \in \pi})$;
- a Hom-tridendriform π -algebra $(\{A_p\}_{p \in \pi}, \{\prec_{p,q}^A = m_{p,q}^A(\text{id}_p \otimes R_q^A)\}_{p,q \in \pi}, \{\succ_{p,q}^A = m_{p,q}^A(R_p^A \otimes \text{id}_q)\}_{p,q \in \pi}, \{\bullet_{p,q}^A = m_{p,q}^A(\lambda \text{id}_p \otimes \text{id}_q)\}_{p,q \in \pi}, \{\alpha_p^A\}_{p \in \pi})$;
- a Hom-dendriform π -algebra $(\{A_p\}_{p \in \pi}, \{\prec_{p,q}^A = m_{p,q}^A(\text{id}_p \otimes R_q^A + \lambda \text{id}_p \otimes \text{id}_q)\}_{p,q \in \pi}, \{\succ_{p,q}^A = m_{p,q}^A(R_p^A \otimes \text{id}_q)\}_{p,q \in \pi}, \{\alpha_p^A\}_{p \in \pi})$;
- a Hom- π -algebra $(\{A_p\}_{p \in \pi}, \{*_p^A = m_{p,q}^A(R_p^A \otimes \text{id}_q + \text{id}_p \otimes R_q^A + \lambda \text{id}_p \otimes \text{id}_q)\}_{p,q \in \pi}, \{\alpha_p^A\}_{p \in \pi})$.

Proof. (a) For all $p, q \in \pi$, $R_p^A(a) \in R_p^A(A_p)$, $R_q^A(b) \in R_q^A(A_q)$, we have

$$R_p^A(a)R_q^A(b) = R_{pq}^A(R_p^A(a)b + aR_q^A(b) + \lambda ab) \in R_{pq}^A(A_{pq}).$$

(b) For any $p, q, r \in \pi$, we have

$$\begin{aligned} \prec_{pq,r}^A (\prec_{p,q}^A \otimes \alpha_r^A) &= m_{pq,r}^A(m_{p,q}^A(\text{id}_p \otimes R_q^A) \otimes R_r^A \alpha_r^A) \\ &= m_{pq,r}^A(m_{p,q}^A \otimes \alpha_r^A)(\text{id}_p \otimes R_q^A \otimes R_r^A) \\ &= m_{p,q,r}^A(\alpha_p^A \otimes m_{q,r}^A)(\text{id}_p \otimes R_q^A \otimes R_r^A) \\ &= m_{p,q,r}^A(\alpha_p^A \otimes R_{qr} m_{q,r}^A(R_q^A \otimes \text{id}_r + \text{id}_q \otimes R_r^A + \lambda \text{id}_q \otimes \text{id}_r)) \\ &= \prec_{p,q,r}^A (\alpha_p^A \otimes (\prec_{q,r}^A + \succ_{q,r}^A + \bullet_{q,r}^A)). \end{aligned}$$

$$\begin{aligned} \prec_{pq,r}^A (\succ_{p,q}^A \otimes \alpha_r^A) &= m_{pq,r}^A(m_{p,q}^A(R_p^A \otimes \text{id}_q) \otimes R_r^A \alpha_r^A) \\ &= m_{pq,r}^A(m_{p,q}^A \otimes \alpha_r^A)(R_p^A \otimes \text{id}_q \otimes R_r^A) \\ &= m_{p,q,r}^A(\alpha_p^A \otimes m_{q,r}^A)(R_p^A \otimes \text{id}_q \otimes R_r^A) = \succ_{p,q,r}^A (\alpha_p^A \otimes \prec_{q,r}^A). \end{aligned}$$

Thus, Equations (1) and (2) are proved. Equation (3) can be proved similarly. Moreover,

$$\begin{aligned}\bullet_{pq,r}^A(\prec_{p,q}^A \otimes \alpha_r^A) &= m_{pq,r}^A(\lambda m_{p,q}^A(\text{id}_p \otimes R_q^A) \otimes \alpha_r^A) \\ &= m_{pq,r}^A(m_{p,q}^A \otimes \alpha_r^A)(\lambda \text{id}_p \otimes R_q^A \otimes \text{id}_r) \\ &= m_{p,q,r}^A(\alpha_p^A \otimes m_{q,r}^A)(\lambda \text{id}_p \otimes R_q^A \otimes \text{id}_r) \\ &= m_{p,q,r}^A(\lambda \alpha_p^A \otimes m_{q,r}^A(R_q^A \otimes \text{id}_r)) = \bullet_{p,q,r}^A(\alpha_p^A \otimes \succ_{q,r}^A).\end{aligned}$$

$$\begin{aligned}\bullet_{pq,r}^A(\succ_{p,q}^A \otimes \alpha_r^A) &= m_{pq,r}^A(\lambda m_{p,q}^A(R_p^A \otimes \text{id}_q) \otimes \alpha_r^A) \\ &= m_{pq,r}^A(m_{p,q}^A \otimes \alpha_r^A)(\lambda R_p^A \otimes \text{id}_q \otimes \text{id}_r) \\ &= m_{p,q,r}^A(\alpha_p^A \otimes m_{q,r}^A)(R_p^A \otimes \lambda \text{id}_q \otimes \text{id}_r) \\ &= m_{p,q,r}^A(R_p^A \alpha_p^A \otimes m_{q,r}^A(\lambda \text{id}_q \otimes \text{id}_r)) = \succ_{p,q,r}^A(\alpha_p^A \otimes \bullet_{q,r}^A).\end{aligned}$$

Thus, Equations (4) and (5) are proved. Equation (6) can be proved similarly. Now we prove Equation (7):

$$\begin{aligned}\bullet_{pq,r}^A(\bullet_{p,q}^A \otimes \alpha_r^A) &= m_{pq,r}^A(\lambda m_{p,q}^A(\lambda \text{id}_p \otimes \text{id}_q) \otimes \alpha_r^A) \\ &= m_{pq,r}^A(m_{p,q}^A \otimes \alpha_r^A)(\lambda^2 \text{id}_p \otimes \text{id}_q \otimes \text{id}_r) \\ &= m_{p,q,r}^A(\alpha_p^A \otimes m_{q,r}^A)(\lambda \text{id}_p \otimes \lambda \text{id}_q \otimes \text{id}_r) \\ &= m_{p,q,r}^A(\lambda \alpha_p^A \otimes m_{q,r}^A(\lambda \text{id}_q \otimes \text{id}_r)) = \bullet_{p,q,r}^A(\alpha_p^A \otimes \bullet_{q,r}^A).\end{aligned}$$

Combining the conclusion of (b) with Theorems 5 and 6, we obtain (c) and (d). $\square$

As a verification of the results obtained in Theorem 7, we have the following example.

Example 6. Take the Rota–Baxter Hom- π -algebra from Example 3. By Theorem 7, we obtain a non-trivial Hom-tridendriform π -algebra with the following explicit structure.

(a) Graded Hom-map: For each $p \in \pi$, $\alpha_p(u_1 \otimes p) = u_1 \otimes p$, $\alpha_p(u_2 \otimes p) = (u_1 - u_2) \otimes p$.

(b) Graded Hom-tridendriform operations: For all $p, q \in \pi$, the left product $\prec_{p,q}: A_p \otimes A_q \rightarrow A_{pq}$ is given by $(u_1 \otimes p) \prec_{p,q} (u_1 \otimes q) = -\lambda u_1 \otimes pq$, $(u_1 \otimes p) \prec_{p,q} (u_2 \otimes q) = (u_2 \otimes p) \prec_{p,q} (u_1 \otimes q) = (u_2 \otimes p) \prec_{p,q} (u_2 \otimes q) = -\lambda(u_1 - u_2) \otimes pq$;

The right product $\succ_{p,q}: A_p \otimes A_q \rightarrow A_{pq}$ is given by $(u_1 \otimes p) \succ_{p,q} (u_1 \otimes q) = -\lambda u_1 \otimes pq$, $(u_1 \otimes p) \succ_{p,q} (u_2 \otimes q) = (u_2 \otimes p) \succ_{p,q} (u_1 \otimes q) = (u_2 \otimes p) \succ_{p,q} (u_2 \otimes q) = -\lambda(u_1 - u_2) \otimes pq$;

The dot product $\bullet_{p,q}: A_p \otimes A_q \rightarrow A_{pq}$ is given by $(u_1 \otimes p) \bullet_{p,q} (u_1 \otimes q) = \lambda u_1 \otimes pq$, $(u_1 \otimes p) \bullet_{p,q} (u_2 \otimes q) = (u_2 \otimes p) \bullet_{p,q} (u_1 \otimes q) = (u_2 \otimes p) \bullet_{p,q} (u_2 \otimes q) = \lambda(u_1 - u_2) \otimes pq$.

Furthermore, by Theorem 7, we derive the associated Hom-dendriform π -algebra: The left dendriform product $\prec'_{p,q}$ satisfies $(u_i \otimes p) \prec'_{p,q} (u_j \otimes q) = 0$ for $i, j = 1, 2$;

The right dendriform product $\succ'_{p,q}$ satisfies $(u_1 \otimes p) \succ'_{p,q} (u_1 \otimes q) = -\lambda u_1 \otimes pq$, $(u_1 \otimes p) \succ'_{p,q} (u_2 \otimes q) = (u_2 \otimes p) \succ'_{p,q} (u_1 \otimes q) = (u_2 \otimes p) \succ'_{p,q} (u_2 \otimes q) = -\lambda(u_1 - u_2) \otimes pq$;

And the Hom- π -algebra with multiplication $*_{p,q}$ satisfies $(u_1 \otimes p) *_{p,q} (u_1 \otimes q) = -\lambda u_1 \otimes pq$, $(u_1 \otimes p) *_{p,q} (u_2 \otimes q) = (u_2 \otimes p) *_{p,q} (u_1 \otimes q) = (u_2 \otimes p) *_{p,q} (u_2 \otimes q) = -\lambda(u_1 - u_2) \otimes pq$.

5. Structural Equivalence Between Rota–Baxter Hom- π -Systems and Hom-Dendriform π -Algebras

Rota–Baxter systems, introduced by Brzeziński [13] as a natural generalization of classical Rota–Baxter operators, admit an essential structural equivalence with dendriform algebras and provide a unified framework for covariant bialgebras, operad splitting and

set-theoretic Yang–Baxter solutions. Motivated by this, we extend the framework to the π -graded Hom setting in this section, defining Rota–Baxter Hom- π -systems and establishing their structural equivalence with Hom-dendriform π -algebras.

Definition 7. A Rota–Baxter Hom- π -system ${}^R\mathcal{A}_H^s$ consists of a Hom- π -algebra $\mathcal{A}_H$ and two families of linear operators $\{R_p^A, S_p^A : A_p \longrightarrow A_p\}_{p \in \pi}$ such that for all $p, q \in \pi$, we have

$$m_{p,q}^A(R_p^A \otimes R_q^A) = R_{pq}^A m_{p,q}^A(R_p^A \otimes \text{id}_q + \text{id}_p \otimes S_q^A), \quad (8)$$

$$m_{p,q}^A(S_p^A \otimes S_q^A) = S_{pq}^A m_{p,q}^A(R_p^A \otimes \text{id}_q + \text{id}_p \otimes S_q^A). \quad (9)$$

Every Rota–Baxter Hom- π -algebra naturally induces two such systems.

Proposition 1. If ${}^R\mathcal{A}_H$ is a Rota–Baxter Hom- π -algebra, then $(\{A_p\}_{p \in \pi}, \{m_{p,q}^A\}_{p,q \in \pi}, \{R_p^A\}_{p \in \pi}, \{R_p^A + \lambda \text{id}_p\}_{p \in \pi}, \{\alpha_p^A\}_{p \in \pi})$ and $(\{A_p\}_{p \in \pi}, \{m_{p,q}^A\}_{p,q \in \pi}, \{R_p^A + \lambda \text{id}_p\}_{p \in \pi}, \{R_p^A\}_{p \in \pi}, \{\alpha_p^A\}_{p \in \pi})$ are both Rota–Baxter Hom- π -systems.

Proof. For $(\{A_p^A\}_{p \in \pi}, \{m_{p,q}^A\}_{p,q \in \pi}, \{R_p^A\}_{p \in \pi}, \{R_p^A + \lambda \text{id}_p\}_{p \in \pi}, \{\alpha_p^A\}_{p \in \pi})$, Equation (8) is precisely the definition of a Rota–Baxter π -algebra. We now verify Equation (9):

$$\begin{aligned} m_{p,q}^A((R_p^A \otimes \lambda \text{id}_p) \otimes (R_q^A + \lambda \text{id}_q)) &= m_{p,q}^A(R_p^A \otimes R_q^A) + m_{p,q}^A(R_p^A \otimes \lambda \text{id}_q + \lambda \text{id}_p \otimes R_q^A \\ &\quad + \lambda \text{id}_p \otimes \lambda \text{id}_q) \\ &= R_{pq}^A m_{p,q}^A(R_p^A \otimes \text{id}_q + \text{id}_p \otimes R_q^A + \text{id}_p \otimes \lambda \text{id}_q) \\ &\quad + \lambda m_{p,q}^A(R_p^A \otimes \text{id}_q + \text{id}_p \otimes R_q^A + \text{id}_p \otimes \lambda \text{id}_q) \\ &= (R_{pq}^A + \lambda \text{id}_{pq}) m_{p,q}^A(R_p^A \otimes \text{id}_q + \text{id}_p \otimes R_q^A \\ &\quad + \text{id}_p \otimes \lambda \text{id}_q) \\ &= (R_{pq}^A + \lambda \text{id}_{pq}) m_{p,q}^A(R_p^A \otimes \text{id}_q + \text{id}_p \otimes (R_q^A + \lambda \text{id}_q)). \end{aligned}$$

Thus $(\{A_p^A\}_{p \in \pi}, \{m_{p,q}^A\}_{p,q \in \pi}, \{R_p^A\}_{p \in \pi}, \{R_p^A + \lambda \text{id}_p\}_{p \in \pi}, \{\alpha_p^A\}_{p \in \pi})$ forms a Rota–Baxter Hom- π -system. Similarly, one can prove that $(\{A_p\}_{p \in \pi}, \{m_{p,q}^A\}_{p,q \in \pi}, \{R_p^A + \lambda \text{id}_p\}_{p \in \pi}, \{R_p^A\}_{p \in \pi}, \{\alpha_p^A\}_{p \in \pi})$ is also a Rota–Baxter Hom- π -system. $\square$

Following the Rota–Baxter Hom- π -algebras studied in Example 3 and Example 4, we will establish an explicit construction of the Rota–Baxter Hom- π -systems in the following example.

Example 7. (a) Induced from Example 3.

Case 1. Pair (R_p, S_p) with $S_p = R_p + \lambda \text{id}_p$: $R_p(u_i \otimes p) = -\lambda u_i \otimes p$, $S_p(u_i \otimes p) = 0$ for $i = 1, 2$.

Case 2. Pair (R'_p, S'_p) with $R'_p = R_p + \lambda \text{id}_p$, $S'_p = R_p$: $R'_p(u_i \otimes p) = 0$, $S'_p(u_i \otimes p) = -\lambda u_i \otimes p$ for $i = 1, 2$.

(b) Induced from Example 4.

Case 1. Pair (R_p, S_p) with $S_p = R_p + \lambda \text{id}_p$: On A_e : $R_e(u_i \otimes e) = -\lambda u_i \otimes e$, $S_e(u_i \otimes e) = 0$ for $i = 1, 2$; On A_g : $R_g(u_i \otimes g) = -\lambda u_2 \otimes g$ for $i = 1, 2$, $S_g(u_1 \otimes g) = \lambda(u_1 - u_2) \otimes g$, $S_g(u_2 \otimes g) = 0$.

Case 2. Pair (R'_p, S'_p) with $R'_p = R_p + \lambda \text{id}_p$, $S'_p = R_p$: On A_e : $R'_e(u_i \otimes e) = 0$, $S'_e(u_i \otimes e) = -\lambda u_i \otimes e$ for $i = 1, 2$; On A_g : $R'_g(u_1 \otimes g) = \lambda(u_1 - u_2) \otimes g$, $R'_g(u_2 \otimes g) = 0$, $S'_g(u_i \otimes g) = -\lambda u_2 \otimes g$ for $i = 1, 2$.

A characterization of decomposable Rota–Baxter Hom- π -systems is given next.

Theorem 8. Let $\mathcal{A}_H$ be a Hom- π -algebra, and $\{R_p, S_p : A_p \longrightarrow A_p\}_{p \in \pi}$ be two families of linear operators such that for all $p, q \in \pi$, we have $R_{pq} = m_{p,q}(\text{id}_p \otimes R_q)$, $S_{pq} = m_{p,q}(S_p \otimes \text{id}_q)$, $R_p \alpha_p = \alpha_p R_p$, $S_p \alpha_p = \alpha_p S_p$. Then

- (a) ${}^R\mathcal{A}_H^s$ is a Rota–Baxter Hom- π -system if and only if for all $p, q \in \pi$, we have $m_{p,q}(\text{id}_p \otimes R_q S_q) = 0 = m_{p,q}(S_p R_p \otimes \text{id}_q)$.
- (b) If $\mathcal{A}_H$ is weakly a non-degenerate Hom- π -algebra, then ${}^R\mathcal{A}_H^s$ is a Rota–Baxter Hom- π -system if and only if for all $p \in \pi$, we have $R_p S_p = 0 = S_p R_p$.

Proof. (a) (proof of necessity) From $R_p(a)R_q(b) = R_{pq}(R_p(a)b + aS_q(b)) = R_{pq}(R_p(a)b) + R_{pq}(aS_q(b)) = R_p(a)R_q(b) + aR_q S_q(b)$, we obtain $aR_q S_q(b) = 0$. Similarly, $S_p R_p(a)b = 0$ can be proved.

(proof of sufficiency) $R_{pq}(R_p(a)b + aS_q(b)) = R_{pq}(R_p(a)b) + R_{pq}(aS_q(b)) = R_p(a)R_q(b) + aR_q S_q(b) = R_p(a)R_q(b)$, and Equation (9) can be proved similarly.

(b) The proof of (b) follows directly from (a) combined with the definition of a weakly non-degenerate Hom- π -algebra. $\square$

We illustrate Theorem 8 with a decomposable Rota–Baxter Hom- π -system constructed from the data in Example 1 and Example 3.

Example 8. Let π and A_p be as in Example 3, and let the underlying Hom- π -algebra be given by the Yau twist via α_5^2 from Example 1. For all $p, q \in \pi$, the graded multiplication satisfies $m_{p,q}(u_i \otimes p, u_j \otimes q) = \alpha_5^2(u_i u_j) \otimes pq$, and the graded Hom-operator satisfies $\alpha_p(u_i \otimes p) = \alpha_5^2(u_i) \otimes p$, where $\alpha_5^2(u_1) = u_2$, $\alpha_5^2(u_2) = u_2$.

Define component-wise linear operators for each $p \in \pi$: $R_p(u_i \otimes p) = -\lambda u_2 \otimes p$ for $i = 1, 2$, $S_p(u_1 \otimes p) = \lambda(u_1 - u_2) \otimes p$, $S_p(u_2 \otimes p) = 0$.

For all basis elements $u_i \otimes p, u_j \otimes q$, direct computations give $R_{pq}m_{p,q}(u_i \otimes p, u_j \otimes q) = -\lambda u_2 \otimes pq = m_{p,q}(\text{id}_p \otimes R_q)(u_i \otimes p, u_j \otimes q)$, $S_{pq}m_{p,q}(u_i \otimes p, u_j \otimes q) = 0 = m_{p,q}(S_p \otimes \text{id}_q)(u_i \otimes p, u_j \otimes q)$, $\alpha_p R_p(u_i \otimes p) = -\lambda u_2 \otimes p = R_p \alpha_p(u_i \otimes p)$, $\alpha_p S_p(u_i \otimes p) = 0 = S_p \alpha_p(u_i \otimes p)$, $R_p S_p(u_i \otimes p) = 0 = S_p R_p(u_i \otimes p)$.

By Theorem 8 (a), the above structure forms a Rota–Baxter Hom- π -system.

Now we can state the main result of this section as a direct generalization of Proposition 2.5 from [13], that presents a structural intercommunication between Rota–Baxter Hom- π -systems and Hom-dendriform π -algebras, where for $\{\alpha_p^A\}$ the set of identity operators, we exactly will obtain the result obtained in [13].

Theorem 9. Let $\mathcal{A}_H$ be a Hom- π -algebra, and let $\{R_p, S_p : A_p \longrightarrow A_p\}_{p \in \pi}$ be two families of linear operators such that for all $p, q \in \pi$, we have $R_p \alpha_p = \alpha_p R_p$, $S_p \alpha_p = \alpha_p S_p$. Define two families of linear operators $\{\prec_{p,q} = m_{p,q}(\text{id}_p \otimes S_q)\}_{p,q \in \pi}$ and $\{\succ_{p,q} = m_{p,q}(R_p \otimes \text{id}_q)\}_{p,q \in \pi}$.

- (a) If ${}^R\mathcal{A}_H^s$ is a Rota–Baxter Hom- π -system, then ${}^d\mathcal{A}_H$ is a Hom-dendriform π -algebra.
- (b) If ${}^d\mathcal{A}_H$ is a Hom-dendriform π -algebra, and $\mathcal{A}_H$ is a strongly non-degenerate Hom- π -algebra, then ${}^R\mathcal{A}_H^s$ is a Rota–Baxter Hom- π -system.

Proof. (a) For any $p, q, r \in \pi$, we have

$$\begin{aligned}\succ_{p,q,r} (\alpha_p \otimes \succ_{q,r}) &= m_{p,q,r} (R_p \otimes \text{id}_r) (\alpha_p \otimes m_{q,r} (R_q \otimes \text{id}_r)) \\ &= m_{p,q,r} (\alpha_p \otimes m_{q,r}) (R_p \otimes R_q \otimes \text{id}_r) \\ &= m_{p,q,r} (m_{p,q} \otimes \text{id}_r) (R_p \otimes R_q \otimes \text{id}_r) \\ &= m_{p,q,r} (m_{p,q} (R_p \otimes R_q) \otimes \alpha_r) \\ &= m_{p,q,r} (R_{pq} m_{p,q} (R_p \otimes \text{id}_q + \text{id}_p \otimes S_q)) \otimes \alpha_r \\ &= m_{p,q,r} (R_{pq} \otimes \text{id}_r) (m_{p,q} (R_p \otimes \text{id}_q + \text{id}_p \otimes S_q) \otimes \alpha_r) \\ &= \succ_{pq,r} ((\succ_{p,q} + \prec_{p,q}) \otimes \alpha_r),\end{aligned}$$

$$\begin{aligned}\prec_{p,q,r} (\succ_{p,q} \otimes \alpha_r) &= m_{p,q,r} (\text{id}_{pq} \otimes S_r) (m_{p,q} (R_p \otimes \text{id}_q) \otimes \alpha_r) \\ &= m_{p,q,r} (m_{p,q} \otimes \alpha_r) (R_p \otimes \text{id}_q \otimes S_r) \\ &= m_{p,q,r} (\alpha_p \otimes m_{q,r}) (R_p \otimes \text{id}_q \otimes S_r) \\ &= m_{p,q,r} (R_p \otimes \text{id}_{qr}) (\alpha_p \otimes m_{q,r} (\text{id}_q \otimes S_r)) = \succ_{p,q,r} (\alpha_p \otimes \prec_{q,r}).\end{aligned}$$

Similarly, one can prove $\prec_{p,q,r} (\prec_{p,q} \otimes \alpha_r) = \prec_{p,q,r} (\alpha_p \otimes (\prec_{q,r} + \succ_{q,r}))$.

(b) For any $p, q, r \in \pi$, we have

$$\begin{aligned}m_{p,q,r} (m_{p,q} (R_p \otimes R_q) \otimes \alpha_r) &= m_{p,q,r} (m_{p,q} \otimes \alpha_r) (R_p \otimes R_q \otimes \text{id}_r) \\ &= m_{p,q,r} (\alpha_p \otimes m_{q,r}) (R_p \otimes R_q \otimes \text{id}_r) \\ &= m_{p,q,r} (R_p \otimes \text{id}_{qr}) (\alpha_p \otimes m_{q,r} (R_q \otimes \text{id}_r)) \\ &= \succ_{p,q,r} (\alpha_p \otimes \succ_{q,r}) \\ &= \succ_{pq,r} ((\succ_{p,q} + \prec_{p,q}) \otimes \alpha_r) \\ &= m_{p,q,r} (R_{pq} (\succ_{p,q} + \prec_{p,q}) \otimes \alpha_r) \\ &= m_{p,q,r} (R_{pq} m_{p,q} (R_p \otimes \text{id}_q + \text{id}_p \otimes S_q) \otimes \alpha_r).\end{aligned}$$

Combined with non-degeneracy, Equation (8) is proved. Equation (9) can be proved similarly.

□

Furthermore, by Theorem 6, we have

Corollary 2. If ${}^R\mathcal{A}_H^s$ is a Rota–Baxter Hom- π -system, then we obtain a new Hom- π -algebra $(\{A_p\}_{p \in \pi}, \{m_{p,q}^* = m_{p,q} (R_p \otimes \text{id}_q + \text{id}_p \otimes S_q)\}_{p,q \in \pi}, \{\alpha_p\}_{p \in \pi})$.

Closely related structures to the notions studied in this paper are Hom-quadri- π -algebras as a generalization of the ordinary quadri-algebras introduced in [14]. So we will devote the final part of this paper to these structures and their connection with the Yau twist. But before presenting the definition of a Hom-quadri- π -algebra, we fix the following notation conventions:

$$\begin{aligned}\succ_{p,q}^A &= \nearrow_{p,q}^A + \searrow_{p,q}^A, & \prec_{p,q}^A &= \nwarrow_{p,q}^A + \swarrow_{p,q}^A, & \vee_{p,q}^A &= \swarrow_{p,q}^A + \searrow_{p,q}^A, \\ \wedge_{p,q}^A &= \nwarrow_{p,q}^A + \nearrow_{p,q}^A, & *_{p,q}^A &= \searrow_{p,q}^A + \swarrow_{p,q}^A + \nwarrow_{p,q}^A + \nearrow_{p,q}^A = \prec_{p,q}^A + \succ_{p,q}^A = \vee_{p,q}^A + \wedge_{p,q}^A.\end{aligned}$$

Definition 8. A Hom-quadri- π -algebra ${}^q\mathcal{A}_H$ consists of a family of linear spaces $\{A_p\}_{p \in \pi}$ and four families of bilinear operators $\{\nwarrow_{p,q}^A, \swarrow_{p,q}^A, \nearrow_{p,q}^A, \searrow_{p,q}^A: A_p \otimes A_q \longrightarrow A_{pq}\}_{p,q \in \pi}$ and a family of linear operators $\{\alpha_p^A: A_p \longrightarrow A_p\}_{p \in \pi}$ such that for all $p, q, r \in \pi$, we have:

$$\begin{aligned}
\leftharpoonup_{pq,r}^A (\leftharpoonup_{p,q}^A \otimes \alpha_r^A) &= \leftharpoonup_{p,q,r}^A (\alpha_p^A \otimes *_{q,r}^A), \\
\leftharpoonup_{pq,r}^A (\rightharpoonup_{p,q}^A \otimes \alpha_r^A) &= \rightharpoonup_{p,q,r}^A (\alpha_p^A \otimes \leftharpoonup_{q,r}^A), \\
\rightharpoonup_{pq,r}^A (\wedge_{p,q}^A \otimes \alpha_r^A) &= \rightharpoonup_{p,q,r}^A (\alpha_p^A \otimes \succ_{q,r}^A), \\
\leftharpoonup_{pq,r}^A (\swarrow_{p,q}^A \otimes \alpha_r^A) &= \swarrow_{p,q,r}^A (\alpha_p^A \otimes \wedge_{q,r}^A), \\
\leftharpoonup_{pq,r}^A (\searrow_{p,q}^A \otimes \alpha_r^A) &= \searrow_{p,q,r}^A (\alpha_p^A \otimes \leftharpoonup_{q,r}^A), \\
\rightharpoonup_{pq,r}^A (\vee_{p,q}^A \otimes \alpha_r^A) &= \searrow_{p,q,r}^A (\alpha_p^A \otimes \rightharpoonup_{q,r}^A), \\
\swarrow_{pq,r}^A (\leftharpoonup_{p,q}^A \otimes \alpha_r^A) &= \swarrow_{p,q,r}^A (\alpha_p^A \otimes \vee_{q,r}^A), \\
\swarrow_{pq,r}^A (\succ_{p,q}^A \otimes \alpha_r^A) &= \searrow_{p,q,r}^A (\alpha_p^A \otimes \swarrow_{q,r}^A), \\
\searrow_{pq,r}^A (*_{p,q}^A \otimes \alpha_r^A) &= \searrow_{p,q,r}^A (\alpha_p^A \otimes \searrow_{q,r}^A).
\end{aligned}$$

A homomorphism from ${}^q\mathcal{A}_H$ to ${}^q\mathcal{B}_H$ is a family of linear operators $\{f_p : A_p \longrightarrow B_p\}_{p \in \pi}$ such that for all $p, q \in \pi$, we have $f_{pq} \leftharpoonup_{p,q}^A = \leftharpoonup_{p,q}^B (f_p \otimes f_q)$, $f_{pq} \swarrow_{p,q}^A = \swarrow_{p,q}^B (f_p \otimes f_q)$, $f_{pq} \rightharpoonup_{p,q}^A = \rightharpoonup_{p,q}^B (f_p \otimes f_q)$, $f_{pq} \searrow_{p,q}^A = \searrow_{p,q}^B (f_p \otimes f_q)$, $f_p \alpha_p^A = \alpha_p^B f_p$.

The Yau twist construction carries over to this setting as well, with a proof entirely analogous to that of Theorem 4.

Theorem 10. If $\{\alpha_p\}_{p \in \pi}$ is an endomorphism of the quadri- π -algebra ${}^q\mathcal{A}$, then ${}^q\mathcal{A}$ is a Hom-quadri- π -algebra, called the Yau twist of the former.

If $\{f_p\}_{p \in \pi}$ is a π -quadri-algebra homomorphism from ${}^q\mathcal{A}$ to ${}^q\mathcal{B}$, then $\{f_p\}_{p \in \pi}$ is a Hom-quadri- π -algebra homomorphism between the two Yau twists ${}^q\mathcal{A}_H$ and ${}^q\mathcal{B}_H$.

Remark 6. Let ${}^q\mathcal{A}_H$ be a Hom-quadri π -algebra.

(a) Define two families of bilinear operations by $\leftharpoonup_{p,q}^A = \leftharpoonup_{p,q}^A + \swarrow_{p,q}^A$, $\succ_{p,q}^A = \rightharpoonup_{p,q}^A + \searrow_{p,q}^A$ for any $p, q \in \pi$. Directly from the definition, one checks that $(\{A_p\}_{p \in \pi}, \{\leftharpoonup_{p,q}^A\}_{p,q \in \pi}, \{\succ_{p,q}^A\}_{p,q \in \pi}, \{\alpha_p^A\}_{p \in \pi})$ is a Hom-dendriform π -algebra.

(b) Define the total multiplication by $*_{p,q}^A = \leftharpoonup_{p,q}^A + \succ_{p,q}^A = \leftharpoonup_{p,q}^A + \swarrow_{p,q}^A + \rightharpoonup_{p,q}^A + \searrow_{p,q}^A$ for any $p, q \in \pi$. By Theorem 6, $(\{A_p\}_{p \in \pi}, \{*_{p,q}^A\}_{p,q \in \pi}, \{\alpha_p^A\}_{p \in \pi})$ is a Hom-associative π -algebra.

6. Conclusions

This paper presents a systematic study of Rota–Baxter operators and multi-order dendriform algebras in the Hom- π -graded setting, with a focus on constructive methods, structural derivation relations, and equivalence characterizations.

Our main findings are summarized below. We established four constructive approaches for Rota–Baxter Hom- π -algebras, namely Yau twist, iterative endomorphism twist, centroid element twist, and monoid tensor lifting. We also built a stepwise derivation chain running from Hom-tridendriform π -algebras to Hom-dendriform π -algebras, and further to Hom- π -algebras, and confirmed that all these dendriform structures can be naturally induced by Rota–Baxter Hom- π -algebras. Finally, we provided a zero-product characterization of Rota–Baxter Hom- π -systems, proved their structural equivalence to Hom-dendriform π -algebras under the graded non-degeneracy condition, and extended this equivalence result to Hom-quadri π -algebras.

This work makes several meaningful improvements compared to its counterparts among other papers in this subject. First, it delivers a unified, self-contained structural framework for the Hom- π -graded context, integrating Rota–Baxter operators/systems with multiple families of dendriform algebras, rather than presenting scattered, isolated generalizations. Second, we verified the non-trivial compatibility constraints that graded indices imposed on multi-operation structures. The graded non-degeneracy condition

used in this work is more refined than its classical ungraded counterpart, and applies to a wider class of algebraic scenarios. Third, the cross-component construction method for Rota–Baxter systems introduced here enriches the construction toolkit for such structures, and has no direct analog in the standard ungraded single-algebra setting.

There are several natural directions for future work. One is to combine this graded framework with Hopf group-coalgebra theory, to explore potential applications in homotopy quantum field theory and group representation theory. Another is to investigate quantized (tri)dendriform π -algebras and their connections with multiplier Hopf structures. It would also be worthwhile to construct more non-trivial examples in noncommutative and non-unital settings, to further validate the generality of the framework.

Author Contributions: Conceptualization, X.L., R.H., Y.Z. and G.S.; Methodology, X.L.; Formal analysis, X.L., R.H., Y.Z. and G.S.; Writing—original draft, X.L.; Writing—review and editing, R.H., Y.Z. and G.S.; Supervision, Y.Z. and G.S.; Project administration, Y.Z. and G.S.; Funding acquisition, Y.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Natural Science Foundation of China (Grant No. 12505111). The APC was funded by Huai'an University.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

References

1. Yau, D. Hom-algebras and homology. *J. Lie Theory* **2009**, *19*, 409–421. [[CrossRef](#)]
2. Makhlouf, A.; Panaite, F. Twisting operators, twisted tensor products and smash products for Hom-associative algebras. *Glasg. Math. J.* **2016**, *58*, 513–538. [[CrossRef](#)]
3. Loday, J.-L. Dialgebras. In *Dialgebras and Related Operads*; Lecture Notes in Mathematics; Springer: Berlin, Germany, 2001; Volume 1763, pp. 7–66.
4. Loday, J.-L.; Ronco, M. Trialgebras and families of polytopes. In *Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology and Algebraic K-Theory*; Contemporary Mathematics; AMS: Providence, RI, USA, 2004; Volume 346, pp. 369–398.
5. Loday, J.-L.; Vallette, B. *Algebraic Operads*; Springer: Berlin, Germany, 2012.
6. Baxter, G. An analytic problem whose solution follows from a simple algebraic identity. *Pac. J. Math.* **1960**, *10*, 731–742. [[CrossRef](#)]
7. Rota, G.C. On the foundations of combinatorial theory I: Theory of Möbius functions. *Z. Wahrscheinlichkeitstheorie Verw. Geb.* **1964**, *2*, 340–368. [[CrossRef](#)]
8. Ebrahimi-Fard, K.; Li, G. Rota–Baxter algebras and dendriform algebras. *J. Pure Appl. Algebra* **2008**, *212*, 320–339. [[CrossRef](#)]
9. Makhlouf, A. Hom-dendriform algebras and Rota–Baxter Hom-algebras. In *Operads and Universal Algebra*; Nankai Series in Pure, Applied Mathematics and Theoretical Physics; World Scientific: Hackensack, NJ, USA, 2012; Volume 9, pp. 147–171.
10. Weibel, C.A. *An Introduction to Homological Algebra*; Cambridge University Press: Cambridge, UK, 1994.
11. Ma, T.; Li, J.; Chen, L.; Wang, S. Rota–Baxter operators on Turaev’s Hopf group (co)algebras I: Basic definitions and related algebraic structures. *J. Geom. Phys.* **2022**, *175*, 104469. [[CrossRef](#)]
12. Turaev, V. *Homotopy Quantum Field Theory*; EMS Tracts in Mathematics; European Mathematical Society: Zürich, Switzerland, 2010; Volume 10.
13. Brzeziński, T. Rota–Baxter systems, dendriform algebras and covariant bialgebras. *J. Algebra* **2016**, *460*, 1–25. [[CrossRef](#)]
14. Aguiar, M.; Loday, J.-L. Quadri-algebras. *J. Pure Appl. Algebra* **2004**, *191*, 205–221. [[CrossRef](#)]

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.