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Solution and Stability of a Multi-Quadratic Functional Equation

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Abstract

In this paper, we characterize the general solution to a system of functional equations with respect to the multi-quadratic mappings by proving that any such mapping can be uniquely represented via a $2n$-additive pairwise symmetric mapping. And we prove the stability results with respect to multi-quadratic mappings in two different algebraic settings.

Keywords: multi-quadratic mapping; functional equation; general solution; stability

MSC: 39B52; 39B72

1. Introduction

The study of functional equations and their stability was initiated by S. M. Ulam [1] in 1940 regarding the stability of group homomorphisms. Ulam posed a fundamental question: under what conditions does there exist an exact homomorphism near an approximate homomorphism? In 1941, D. H. Hyers [2] gave a brilliant affirmative answer to Ulam's problem for the case of additive mappings in Banach spaces, thereby establishing the foundation of what is now known as the Hyers–Ulam stability. In 1978, T. M. Rassias [3] generalized the Hyers–Ulam stability. In 1994, P. Găvruta [4] generalized the Hyers–Ulam–Rassias stability. In 2003, L. Cădariu and V. Radu [5] presented a fixed point method for the stability theorems of functional equations.

A classical quadratic functional equation is given by

$$g(x+y)+g(x-y)=2g(x)+2g(y). \tag{1}$$

Every solution of the quadratic functional equation is called a quadratic mapping. This equation originates from the well-known parallelogram law in inner product spaces. In 1983, F. Skof [6] investigated the stability of quadratic mappings defined on a subset of $\mathbb{R}$. In 1984, P. W. Cholewa [7] investigated the stability of quadratic mappings defined on an abelian group. In 1992, S. Czerwik [8] generalized the stability of quadratic mappings proposed by P. W. Cholewa.

As the study of single-variable functional equations reached maturity, scholarly interest naturally expanded toward exploring the structural properties of mappings in multi-variable spaces. In 2005, Park and Bae [9] introduced and investigated a system of coordinate-wise functional equations:

$$f(x+y,z)+f(x-y,z)=2f(x,z)+2f(y,z),$$
$$f(x,y+z)+f(x,y-z)=2f(x,y)+2f(x,z)$$

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and the single bi-quadratic functional equation:

$$f(x+y,z+w)+f(x+y,z-w)+f(x-y,z+w)+f(x-y,z-w)$$
$$=4[f(x,z)+f(x,w)+f(y,z)+f(y,w)],$$

deriving its general solution and establishing its generalized Hyers–Ulam stability.

In 2003, C. Park [10] investigated the stability of a single functional equation for multi-quadratic mappings. In 2011, Ciepliński [11] investigated the stability of a system of functional equations for multi-quadratic mappings.

From now on, X denotes an abelian group divisible by 2, and Y denotes an abelian group divisible by 4.

In 2013, Zhao et al. [12] introduced a new single functional equation

$$\sum_{s_1,\cdots,s_n\in\{-1,1\}} f(x_{11}+s_1x_{12},\cdots,x_{n1}+s_nx_{n2}) = 2^n \sum_{i_1,\cdots,i_n\in\{1,2\}} f(x_{1i_1},\cdots,x_{ni_n}), \quad (2)$$

and investigated its solution and stability. They obtained the following theorem:

Theorem 1. *Suppose that $\varphi : X^{2n} \to [0,\infty)$ is a function such that*

$$\sum_{j=0}^{\infty} \frac{1}{4^{jn}} \varphi(2^j x_1, 2^j y_1, \cdots, 2^j x_n, 2^j y_n) < \infty$$

for all $x_1, y_1, \cdots, x_n, y_n \in X$. If a mapping $f : X^n \to Y$ with $f(x_1,\cdots,x_n) = 0$ whenever at least one $x_i = 0$ satisfies the inequality

$$\left\| \sum_{s_1,\cdots,s_n\in\{-1,1\}} f(x_{11}+s_1x_{12},\cdots,x_{n1}+s_nx_{n2}) - 2^n \sum_{i_1,\cdots,i_n\in\{1,2\}} f(x_{1i_1},\cdots,x_{ni_n}) \right\|_Y \quad (3)$$
$$\leq \varphi(x_{11},x_{12},\cdots,x_{n1},x_{n2})$$

for all $x_{11}, x_{12}, \cdots, x_{n1}, x_{n2} \in X$, then there exists a unique multi-quadratic mapping $Q : X^n \to Y$, which satisfies the inequality

$$\|f(x_1,\cdots,x_n) - Q(x_1,\cdots,x_n)\|_Y \leq \tilde{\Phi}(x_1,\cdots,x_n)$$

where

$$\tilde{\Phi}(x_1,\cdots,x_n) := \sum_{j=0}^{\infty} \frac{1}{4^{(j+1)n}} \varphi(2^j x_1, 2^j x_1, \cdots, 2^j x_n, 2^j x_n)$$

for all $x_1,\cdots,x_n \in X$. The mapping Q is given by

$$Q(x_1,\cdots,x_n) := \lim_{k\to\infty} \frac{1}{4^{nk}} f(2^k x_1,\cdots,2^k x_n)$$

for all $x_1,\cdots,x_n \in X$.

In 2020, A. Bodaghi et al. [13] investigated generalized multi-quadratic mappings and their stability. In 2021, A. Bodaghi [14] introduced a generalized multi-quadratic functional equation and investigated the stability of it. In 2022, C. Park et al. [15] investigated some stability and hyperstability in random normed spaces for multi-quadratic mappings. In 2023, EL-Fassi [16] investigated a generalized multi-quadratic type functional equation in Lipschitz spaces. In 2026, Bae and Park [17] investigatedthe stability of multi-variable additive-quadratic functional equations with complex cross-variable interactions.

In this paper, we extend the results of Park and Bae [9] by defining a multi-quadratic functional equation for n variables and proving its general solution. Unlike the approach utilized by Ciepliński [11], which defines a multi-quadratic mapping through a system of n independent equations acting quadratically on each variable separately, we introduce a single, unified n-variable functional equation. This single-equation approach provides a more compact algebraic structure and serves as a direct generalization of the bi-quadratic case. We characterize the general solution of the multi-quadratic functional equation by proving that any such mapping can be uniquely represented via a $2n$-additive pairwise symmetric mapping. This rigorous generalization provides deep insights into multi-linear and multi-quadratic forms within functional analysis. Building upon these foundations, Ciepliński [11] formally defined the concept of multi-quadratic mappings in n-variables and proved their generalized Hyers–Ulam stability in both Banach and complete non-Archimedean spaces. However, Ciepliński's approach primarily relied on defining the mapping through a coordinate-wise system of equations (i.e., being quadratic in each variable independently) rather than presenting a single unified functional equation.

While classic Hyers–Ulam stability results are typically established over real Banach spaces, extending these formulations to Banach modules over C^*-algebras represents a substantial conceptual and analytical advancement. From an algebraic perspective, substituting scalar multipliers in $\mathbb{C}$ with unitary elements $u \in \mathcal{U}(\mathcal{A})$ of a C^*-algebra $\mathcal{A}$ allows one to control the operator-theoretic behaviors of the mapping. More precisely, establishing the functional equation over Banach $\mathcal{A}$-modules guarantees that the approximating mapping inherits $\mathcal{A}$-quadraticity, i.e., $f(ax_1, \cdots, a_n x_n) = a_1^2 \cdots a_n^2 f(x_1, \cdots, x_n)$ for all $a_1, \cdots, a_n \in \mathcal{A}$. Consequently, this module-theoretic generalization not only unifies and extends previous stability results in standard Banach spaces (e.g., Zhao et al. [12]), but also provides essential analytical tools for investigating automatic continuity, C^*-algebra isomorphisms, and derivations in non-commutative geometry and quantum operator algebras.

The remainder of this paper is organized as follows. In Section 2, we provide a new simple method for the characterization of the general solution of the multi-quadratic functional equation. Finally, Section 3 is devoted to proving the Hyers–Ulam stability in Banach modules.

2. Solution to a System of Functional Equations

We begin by defining a multi-quadratic mapping as a system of coordinate-wise functional equations.

Definition 1. *A mapping $f : X^n \to Y$ is called multi-quadraticif f satisfies the quadratic functional equation in each of its n arguments individually. That is, for every $i \in \{1, \cdots, n\}$ and all $x_1, \cdots, x_n, y_i \in X$,*

$$\begin{aligned} &f(x_1, \cdots, x_{i-1}, x_i + y_i, x_{i+1}, \cdots, x_n) + f(x_1, \cdots, x_{i-1}, x_i - y_i, x_{i+1}, \cdots, x_n) \\ &= 2f(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) + 2f(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n). \end{aligned} \tag{4}$$

In 2003, C. Park [10] investigated the stability of the single n-variable functional equation

$$\begin{aligned} &\sum_{i=1}^{n} \left[f(x_1, \cdots, x_{i-1}, x_i + y_i, x_{i+1}, \cdots, x_n) + f(x_1, \cdots, x_{i-1}, x_i - y_i, x_{i+1}, \cdots, x_n) \right] \\ &= 2nf(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) - 2\sum_{i=1}^{n} f(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n). \end{aligned}$$

In [12], Zhao et al. show the equivalence of (2) and (4) and that a multi-quadratic mapping can be represented by a $2n$-additive mapping. It is worth noting that they characterized

the general solution by reducing the equation to a multi-additive mapping. Their representation, however, does not separate the individual coordinates and therefore fails to reflect the natural symmetry of the mapping explicitly. To overcome this structural limitation, we employ a completely different approach based on a $2n$-additive symmetric mapping. By constructing an explicit representation formula, we show that every multi-quadratic mapping can be uniquely determined and structurally decomposed by a $2n$-additive symmetric mapping. This algebraic characterization not only streamlines the verification of structural equivalence but also establishes a more rigorous and systematic general solution template.

Lemma 1. *Let X, Y be vector spaces over $\mathbb{R}$ and let $M : X^{2n} \to Y$ be a $2n$-additive mapping which is symmetric in each pair of arguments corresponding to the same coordinate, i.e.,*

$$\begin{aligned} M(x_1, y_1, \cdots, x_{i-1}, y_{i-1}, u, v, x_{i+1}, y_{i+1}, \cdots, x_n, y_n) \\ = M(x_1, y_1, \cdots, x_{i-1}, y_{i-1}, v, u, x_{i+1}, y_{i+1}, \cdots, x_n, y_n) \end{aligned} \tag{5}$$

for all $x_1, y_1, \cdots, x_{i-1}, y_{i-1}, x_{i+1}, y_{i+1}, \cdots, x_n, y_n, u, v \in X$ and all $i \in \{1, \cdots, n\}$. Then

$$M(x_1, y_1, \cdots, x_n, y_n) = \frac{1}{4^n} \sum_{s_1, \cdots, s_n \in \{-1,1\}} s_1 \cdots s_n \, M_\Delta(x_1 + s_1 y_1, \cdots, x_n + s_n y_n) \tag{6}$$

for all $x_1, y_1, \cdots, x_n, y_n \in X$, where

$$M_\Delta(x_1, \cdots, x_n) := M(x_1, x_1, \cdots, x_n, x_n)$$

for all $x_1, \cdots, x_n \in X$.

Proof. For $i \in \{1, \cdots, n\}$, write $c_i(0) := x_i$ and $c_i(1) := y_i$. Since M is additive in each of its $2n$ arguments, it is in particular $\mathbb{Z}$-homogeneous in each argument, so that $M(\cdots, -u, \cdots) = -M(\cdots, u, \cdots)$. Expanding each of the $2n$ arguments of

$$M(x_1 + s_1 y_1, x_1 + s_1 y_1, \cdots, x_n + s_n y_n, x_n + s_n y_n)$$

by additivity and pulling out the signs $s_i \in \{-1, 1\}$, we have

$$\begin{aligned} &M(x_1 + s_1 y_1, x_1 + s_1 y_1, \cdots, x_n + s_n y_n, x_n + s_n y_n) \\ &\quad = \sum_{(a_1, \cdots, a_n), (b_1, \cdots, b_n) \in \{0,1\}^n} \Big(\prod_{i=1}^{n} s_i^{a_i + b_i} \Big) M\big(c_1(a_1), c_1(b_1), \cdots, c_n(a_n), c_n(b_n)\big), \end{aligned}$$

for all $x_1, y_1, \cdots, x_n, y_n \in X$. Note that

$$\begin{aligned} \sum_{s_1, \cdots, s_n \in \{-1,1\}} \prod_{i=1}^{n} s_i^{1 + a_i + b_i} &= \sum_{s_1 \in \{-1,1\}} \cdots \sum_{s_n \in \{-1,1\}} \prod_{i=1}^{n} s_i^{1 + a_i + b_i} \\ &= \Big(\sum_{s_1 \in \{-1,1\}} s_1^{1 + a_1 + b_1} \Big) \cdots \Big(\sum_{s_n \in \{-1,1\}} s_n^{1 + a_n + b_n} \Big) = \prod_{i=1}^{n} \Big(\sum_{s_i \in \{-1,1\}} s_i^{1 + a_i + b_i} \Big) \\ &= \begin{cases} 2^n, & a_i \neq b_i \quad \text{for all } i, \\ 0, & a_i = b_i \quad \text{for some } i. \end{cases} \end{aligned}$$

By the pairwise symmetry (5), we obtain

$$\begin{aligned}
&\sum_{s_1,\cdots,s_n\in\{-1,1\}} s_1\cdots s_n M_\Delta(x_1+s_1y_1,\cdots,x_n+s_ny_n)\\
&=\sum_{s_1,\cdots,s_n\in\{-1,1\}} s_1\cdots s_n M(x_1+s_1y_1,x_1+s_1y_1,\cdots,x_n+s_ny_n,x_n+s_ny_n)\\
&=\sum_{s_1,\cdots,s_n\in\{-1,1\}} s_1\cdots s_n \sum_{(a_1,\cdots,a_n),(b_1,\cdots,b_n)\in\{0,1\}^n}\left(\prod_{i=1}^n s_i^{a_i+b_i}\right)M(c_1(a_1),c_1(b_1),\cdots,c_n(a_n),c_n(b_n))\\
&=\sum_{(a_1,\cdots,a_n),(b_1,\cdots,b_n)\in\{0,1\}^n}\ \sum_{s_1,\cdots,s_n\in\{-1,1\}}\left(\prod_{i=1}^n s_i^{1+a_i+b_i}\right)M(c_1(a_1),c_1(b_1),\cdots,c_n(a_n),c_n(b_n))\\
&=2^n\sum_{(a_1,\cdots,a_n),(b_1,\cdots,b_n\in\{0,1\}^n} M(c_1(a_1),c_1(b_1),\cdots,c_n(a_n),c_n(b_n))\\
&=2^n\sum_{(a_1,\cdots,a_n),(b_1,\cdots,b_n)\in E_n} M(c_1(a_1),c_1(b_1),\cdots,c_n(a_n),c_n(b_n))\\
&=4^nM(x_1,y_1,\cdots,x_n,y_n),
\end{aligned}$$

where $E_n:=\left\{(a_1,\cdots,a_n),\,(b_1,\cdots,b_n)\in\{0,1\}^n\,\middle|\,a_1\neq b_1,\cdots,a_n\neq b_n\right\}$. Hence, we see that M satisfies Equation (6). □

Lemma 2. *Let $g:X\to Y$ be a mapping satisfying Equation (1). Put*

$$P(x,y):=g(x+y)-g(x-y)$$

for all $x,y\in X$. Then, the following is true

(i) $g(0)=0$, $g(-x)=g(x)$ *and* $g(2x)=4g(x)$ *for all* $x\in X$;
(ii) $P(x,y)=P(y,x)$ *for all* $x,y\in X$;
(iii) $P(u+v,y)+P(u-v,y)=2P(u,y)$ *for all* $u,v,y\in X$;
(iv) P *is additive in each of its two variables;*
(v) $P(x,x)=4g(x)$ *for all* $x\in X$.

Proof. (i) Setting $x=y=0$ in (1) gives $g(0)=0$. Setting $x=0$ in (1) gives $g(-y)=g(y)$ for all $y\in X$. Setting $y=x$ in (1) gives $g(2x)=4g(x)$ for all $x\in X$.

(ii) By the evenness of g in (i),

$$P(y,x)=g(y+x)-g(y-x)=g(x+y)-g(x-y)=P(x,y)$$

for all $x,y\in X$.

(iii) Applying (1) to the pair $(u+y,v)$ and to the pair $(u-y,v)$ we obtain

$$g(u+y+v)+g(u+y-v)=2g(u+y)+2g(v)$$

and

$$g(u-y+v)+g(u-y-v)=2g(u-y)+2g(v)$$

for all $u,v,y\in X$. Subtracting the second identity from the first, the terms $2g(v)$ cancel and, after regrouping the four terms on the left-hand side as

$$\left[g(u+v+y)-g(u+v-y)\right]+\left[g(u-v+y)-g(u-v-y)\right],$$

we get $P(u+v,y)+P(u-v,y)=2g(u+y)-2g(u-y)=2P(u,y)$ for all $u,v,y\in X$.

(iv) By (ii) it suffices to treat the first variable. By (i), $P(0,y)=g(y)-g(-y)=0$ for all $y\in X$. Putting $v=u$ in (iii), we have $P(2u,y)=2P(u,y)$ for all $u,y\in X$. Now let $r,t\in X$

and put $u = \frac{1}{2}(r+t)$, $v = \frac{1}{2}(r-t)$, which is legitimate since X is an abelian group divisible by 2. By (iii), we have

$$P(r,y) + P(t,y) = P(u+v,y) + P(u-v,y) = 2P(u,y) = 2P\left(\tfrac{r+t}{2}, y\right) = P(r+t,y)$$

for all $r, t, y \in X$.

(v) By (i), $P(x,x) = g(2x) - g(0) = 4g(x)$ for all $x \in X$. □

Lemma 3. *Let $f : X^n \to Y$ satisfy the system (4) of equations, fix $i \in \{1, \cdots, n\}$ and fix $a_j, b_j \in X$ for all $j \in \{k \in \{1, \cdots, n\} | \, k \neq i\}$. Then, the mapping $G_i : X \to Y$ defined by*

$$G_i(x) := 4^{1-n} \sum_{s_j \in \{-1,1\}, j \neq i} \left(\prod_{j \neq i} s_j \right) f(a_1 + s_1 b_1, \cdots, a_{i-1} + s_{i-1} b_{i-1}, \, x, \, a_{i+1} + s_{i+1} b_{i+1}, \cdots, a_n + s_n b_n)$$

satisfies Equation (1).

Proof. For each fixed choice of signs $\sigma_i := (s_j)_{j \neq i}$, define a mapping $g_{i,\sigma_i} : X \to Y$ by

$$g_{i,\sigma_i}(x) := f(a_1 + s_1 b_1, \cdots, a_{i-1} + s_{i-1} b_{i-1}, x, a_{i+1} + s_{i+1} b_{i+1}, \cdots, a_n + s_n b_n)$$

for all $x \in X$. By (4), the mapping g_{i,σ_i} satisfies (1). Note that

$$G_i(x) = 4^{1-n} \sum_{s_j \in \{-1,1\}, j \neq i} \left(\prod_{j \neq i} s_j \right) g_{i,\sigma_i}(x)$$

for all $x \in X$. Since G_i is a sum of $\pm 4^{1-n} g_{i,\sigma_i}$'s and Y is an abelian group divisible by 4, G_i also satisfies (1). □

We provide the existence and uniqueness of the general solution of (4) in the next theorem.

Theorem 2. *A mapping $f : X^n \to Y$ satisfies the system (4) of equations if and only if there exists a unique $2n$-additive mapping $M : X^{2n} \to Y$ satisfying (5) such that*

$$f(x_1, \cdots, x_n) = M_\Delta(x_1, \cdots, x_n) \tag{7}$$

for all $x_1, \cdots, x_n \in X$.

Proof. Suppose a mapping f satisfies the system (4) of equations. Define a mapping $M : X^{2n} \to Y$ by

$$M(x_1, y_1, \cdots, x_n, y_n) := \frac{1}{4^n} \sum_{s_1, \cdots, s_n \in \{-1,1\}} s_1 \cdots s_n f(x_1 + s_1 y_1, \cdots, x_n + s_n y_n) \tag{8}$$

for all $x_1, y_1, \cdots, x_n, y_n \in X$.

Let $i \in \{1, \cdots, n\}$ and let $a_j, b_j \in X$ for $j \in \{k \in \{1, \cdots, n\} | \, k \neq i\}$. With G_i as in Lemma 3, we have

$$\begin{aligned} &M(a_1, b_1, \cdots, a_{i-1}, b_{i-1}, x, y, a_{i+1}, b_{i+1}, \cdots, a_n, b_n) \\ &\qquad = \frac{1}{4} \sum_{s_i \in \{-1,1\}} s_i \, G_i(x + s_i y) = \frac{1}{4} \left[G_i(x+y) - G_i(x-y) \right] \end{aligned}$$

for all $x, y \in X$. By Lemma 3, G_i satisfies (1), so Lemma 2 (iv) shows that the left-hand side is additive in x and additive in y, and Lemma 2 (ii) shows that it is unchanged when x and y are interchanged. Since i and the values a_j, b_j $(j \neq i)$ are arbitrary, M is additive in each of its $2n$ arguments and satisfies the pairwise symmetry condition (5).

Setting $y_1 = x_1, \cdots, y_n = x_n$ in (8), every summand containing some $s_j = -1$ has $x_j + s_j x_j = 0$ in its j-th slot and hence vanishes, because $f(x_1, \cdots, x_n) = 0$ whenever at least one argument is 0 by Lemma 2 (i). Only the term with $s_1 = \cdots = s_n = 1$ survives, so that, by n applications of Lemma 2 (i),

$$M_\Delta(x_1, \cdots, x_n) = M(x_1, x_1, \cdots, x_n, x_n) = \frac{1}{4^n} f(2x_1, \cdots, 2x_n) = f(x_1, \cdots, x_n)$$

for all $x_1, \cdots, x_n \in X$, which is (7).

Suppose that $M, M' : X^{2n} \to Y$ are $2n$-additive mappings, both satisfying (5), with $M_\Delta(x_1, \cdots, x_n) = f(x_1, \cdots, x_n) = M'_\Delta(x_1, \cdots, x_n)$ for all $x_1, \cdots, x_n \in X$. Applying Lemma 1 to M and to M' separately, by Equation (6), we obtain that

$$\begin{aligned}
&M(x_1, y_1, \cdots, x_n, y_n) \\
&\quad = \frac{1}{4^n} \sum_{s_1, \cdots, s_n \in \{-1,1\}} s_1 \cdots s_n \, M_\Delta(x_1 + s_1 y_1, \cdots, x_n + s_n y_n) \\
&\quad = \frac{1}{4^n} \sum_{s_1, \cdots, s_n \in \{-1,1\}} s_1 \cdots s_n f(x_1 + s_1 y_1, \cdots, x_n + s_n y_n) \\
&\quad = \frac{1}{4^n} \sum_{s_1, \cdots, s_n \in \{-1,1\}} s_1 \cdots s_n \, M'_\Delta(x_1 + s_1 y_1, \cdots, x_n + s_n y_n) = M'(x_1, y_1, \cdots, x_n, y_n)
\end{aligned}$$

for all $x_1, y_1, \cdots, x_n, y_n \in X$. Hence, $M = M'$.

Conversely, assume there exists a $2n$-additive mapping M satisfying (5) and (7). For any fixed index $i \in \{1, \cdots, n\}$, we have

$$\begin{aligned}
&f(x_1, \cdots, x_{i-1}, x_i + y_i, x_{i+1}, \cdots, x_n) + f(x_1, \cdots, x_{i-1}, x_i - y_i, x_{i+1}, \cdots, x_n) \\
&= M(x_1, x_1 \cdots, x_{i-1}, x_{i-1}, x_i + y_i, x_i + y_i, x_{i+1}, x_{i+1}, \cdots, x_n, x_n) \\
&\quad + M(x_1, x_1 \cdots, x_{i-1}, x_{i-1}, x_i - y_i, x_i - y_i, x_{i+1}, x_{i+1}, \cdots, x_n, x_n) \\
&= 2M(x_1, x_1 \cdots, x_{i-1}, x_{i-1}, x_i, x_i, x_{i+1}, x_{i+1}, \cdots, x_n, x_n) \\
&\quad + 2M(x_1, x_1 \cdots, x_{i-1}, x_{i-1}, y_i, y_i, x_{i+1}, x_{i+1}, \cdots, x_n, x_n) \\
&= 2f(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) + 2f(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n).
\end{aligned}$$

for all $x_1, \cdots, x_n, y_i \in X$. Thus, f satisfies the system (4) of equations. This completes the proof. □

3. Stability Results on Multi-Quadratic Mappings

From now on, Y denotes a Banach abelian group divisible by 4.

We obtain some stability results of the system (4) of functional equations in two different algebraic settings.

Theorem 3. *Suppose that $\varphi : X^{2n} \to [0, \infty)$ is a function such that*

$$\sum_{j=0}^{\infty} \frac{1}{4^{jn}} \varphi(2^j x_1, 2^j y_1, \cdots, 2^j x_n, 2^j y_n) < \infty \tag{9}$$

for all $x_1, y_1, \cdots, x_n, y_n \in X$. *Let* $f : X^n \to Y$ *be a mapping satisfying* $f(x_1, \cdots, x_n) = 0$ *whenever at least one* $x_i = 0$. *For all* $i \in \{1, \cdots, n\}$, *let*

$$\begin{aligned} &\|f(x_1, \cdots, x_{i-1}, x_i + y_i, x_{i+1}, \cdots, x_n) \\ &\quad + f(x_1, \cdots, x_{i-1}, x_i - y_i, x_{i+1}, \cdots, x_n) \\ &\quad - 2[f(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) + f(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)]\|_Y \\ &\leq \varphi\big(x_1, x_1, \cdots, x_{i-1}, x_{i-1}, x_i, y_i, x_{i+1}, x_{i+1}, \cdots, x_n, x_n\big) \end{aligned} \tag{10}$$

for all $x_1, \cdots, x_n, y_i \in X$, *then there exists a unique multi-quadratic mapping* $Q : X^n \to Y$ *such that*

$$\|f(x_1, \cdots, x_n) - Q(x_1, \cdots, x_n)\|_Y \leq \Phi(x_1, \cdots, x_n) \tag{11}$$

for all $x_1, \cdots, x_n \in X$, *where*

$$\Phi(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} \frac{1}{4^{jn}} \psi(2^j x_1, \cdots, 2^j x_n)$$

and

$$\psi(x_1, \cdots, x_n) := \sum_{i=1}^{n} \frac{1}{4^i} \varphi(2x_1, 2x_1, \cdots, 2x_{i-1}, 2x_{i-1}, x_i, x_i, \cdots, x_n, x_n)$$

for all $x_1, \cdots, x_n \in X$.

Proof. Setting $y_i = x_i$ in (10), we obtain for each $i \in \{1, \cdots, n\}$ and any $z_1, \cdots, z_n \in X$:

$$\|f(z_1, \cdots, 2z_i, \cdots, z_n) - 4f(z_1, \cdots, z_i, \cdots, z_n)\|_Y \leq \varphi(z_1, z_1, \cdots, z_i, z_i, \cdots, z_n, z_n). \tag{12}$$

Using a telescoping sum decomposition across coordinates, we express

$$\begin{aligned} &f(2x_1, \cdots, 2x_n) - 4^n f(x_1, \cdots, x_n) \\ &\qquad = \sum_{i=1}^{n} 4^{n-i} [f(2x_1, \cdots, 2x_i, x_{i+1}, \cdots, x_n) - 4f(2x_1, \cdots, 2x_{i-1}, x_i, x_{i+1}, \cdots, x_n)] \end{aligned}$$

for all $x_1, \cdots, x_n \in X$. Applying the triangle inequality and substituting $z_k = 2x_k$ for $k < i$ and $z_k = x_k$ for $k \geq i$ into (12), we deduce

$$\begin{aligned} &\|f(2x_1, \cdots, 2x_n) - 4^n f(x_1, \cdots, x_n)\|_Y \\ &\qquad \leq \sum_{i=1}^{n} 4^{n-i} \varphi(2x_1, 2x_1, \cdots, 2x_{i-1}, 2x_{i-1}, x_i, x_i, x_{i+1}, x_{i+1}, \cdots, x_n, x_n) \end{aligned}$$

for all $x_1, \cdots, x_n \in X$. Dividing both sides by 4^n, we have

$$\left\| \frac{1}{4^n} f(2x_1, \cdots, 2x_n) - f(x_1, \cdots, x_n) \right\|_Y \leq \psi(x_1, \cdots, x_n)$$

for all $x_1, \cdots, x_n \in X$. Replacing x_l with $2^j x_l$ in the above inequality and dividing by 4^{jn}, we get the following for every $j \geq 0$:

$$\left\| \frac{1}{4^{(j+1)n}} f(2^{j+1} x_1, \cdots, 2^{j+1} x_n) - \frac{1}{4^{jn}} f(2^j x_1, \cdots, 2^j x_n) \right\|_Y \leq \frac{1}{4^{jn}} \psi(2^j x_1, \cdots, 2^j x_n). \tag{13}$$

Hence, for any integers $m > k \geq 0$,

$$\left\| \frac{1}{4^{mn}} f(2^m x_1, \ldots, 2^m x_n) - \frac{1}{4^{kn}} f(2^k x_1, \ldots, 2^k x_n) \right\|_Y \leq \sum_{j=k}^{m-1} \frac{1}{4^{jn}} \psi(2^j x_1, \ldots, 2^j x_n). \tag{14}$$

By assumption (9), the right-hand side of (14) tends to 0 as $k \to \infty$. Thus, the sequence $\{4^{-kn} f(2^k x_1, \cdots, 2^k x_n)\}_{k\in\mathbb{N}}$ is a Cauchy sequence in the Banach space Y. We define $Q : X^n \to Y$ by

$$Q(x_1, \ldots, x_n) := \lim_{k\to\infty} \frac{1}{4^{kn}} f(2^k x_1, \ldots, 2^k x_n) \tag{15}$$

for all $x_1, \ldots, x_n \in X$. Setting $k = 0$ and taking $m \to \infty$ in (14) yields estimate (11). Next, fixing $i \in \{1, \ldots, n\}$, setting $u_1 = \cdots = u_n = 1$, replacing x_l by $2^k x_l$ ($l = 1, \ldots, n$) and y_i by $2^k y_i$ in (10), and dividing by 4^{kn}, we pass to the limit as $k \to \infty$. By (9) and (13), we obtain

$$\begin{aligned} Q(x_1, \cdots, x_{i-1}, x_i + y_i, x_{i+1}, \cdots, x_n) + Q(x_1, \cdots, x_{i-1}, x_i - y_i, x_{i+1}, \cdots, x_n) \\ = 2Q(x_1, \cdots, x_n) + 2Q(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n), \end{aligned}$$

which shows that Q is quadratic in each variable and satisfies system (4).

Uniqueness follows directly: if Q' is another multi-quadratic mapping satisfying (11), then

$$\begin{aligned} \|Q(x_1, \cdots, x_n) - Q'(x_1, \cdots, x_n)\|_Y &\le \frac{1}{4^{kn}} \Big(\|Q(2^k x_1, \cdots, 2^k x_n) - f(2^k x_1, \cdots, 2^k x_n)\|_Y \\ &\quad + \|f(2^k x_1, \cdots, 2^k x_n) - Q'(2^k x_1, \cdots, 2^k x_n)\|_Y \Big) \\ &\le \frac{2}{4^{kn}} \Phi(2^k x_1, \cdots, 2^k x_n) = 2 \sum_{j=k}^{\infty} \frac{1}{4^{jn}} \psi(2^j x_1, \cdots, 2^j x_n). \end{aligned}$$

As $k \to \infty$, the upper bound vanishes, establishing $Q = Q'$. □

Remark 1. *The stability result established in Theorem 3 is better than the previous results in Theorem 1 of Park [10]. In Park's work, the stability of multi-quadratic mappings in Banach spaces was proved under a condition where the control function required a highly complex, coordinate-dependent infinite series summation (involving nested sums over each coordinate $i = 1, \cdots, d$) to bound the approximation error.*

From now on, $\mathcal{A}$ denotes an unital C^*-algebra with unit $1_{\mathcal{A}}$ and

$$\mathcal{U}(\mathcal{A}) := \{u \in \mathcal{A} : \ u^*u = uu^* = 1_{\mathcal{A}}\}$$

its group of unitary elements.

In 2007, U. Haagerup et. al [18] introduced the following results on unitary elements of an unital C^*-algebra.

Lemma 4. *If $a \in \mathcal{A}$ and $|a| \le 1 - \frac{2}{n} (n = 3, 4, \cdots)$, then*

$$a = \frac{1}{n}(u_1 + \cdots + u_n)$$

with $u_1, \cdots, u_n \in \mathcal{U}(\mathcal{A})$.

Definition 2. *A unital C^*-algebra $\mathcal{A}$ is said to have real rank 0 (see [19]) if the invertible self-adjoint elements are dense in $\mathcal{A}_{sa} := \{a \in \mathcal{A} \mid a^* = a\}$.*

From now on, let $\mathcal{A}$ be of real rank 0 and commutative, let $\mathcal{A}_1 := \{a \in \mathcal{A} \mid |a| = 1\}$, let $\mathcal{X}$ be a left $\mathcal{A}$-module and $\mathcal{Y}$ a Banach left $\mathcal{A}$-module with norm $\|\cdot\|_{\mathcal{Y}}$, let

$$\mathcal{A}_l := \{a \in \mathcal{A} \mid Sp(a) \subset \mathbb{C} \setminus [0, \infty)\}$$

and

$$\mathcal{A}_r := \{a \in \mathcal{A} \mid Sp(a) \subset \mathbb{C} \setminus (-\infty, 0]\},$$

and let $\mathcal{A}_l \cup \mathcal{A}_r$ be dense in $\mathcal{A}_{in} := \{a \in \mathcal{A} \mid a \text{ is invertible in } \mathcal{A}\}$.

Theorem 4. *Suppose that $\varphi : \mathcal{X}^{2n} \to [0,\infty)$ is a function satisfying (9). Let $f : \mathcal{X}^n \to \mathcal{Y}$ be a mapping satisfying $f(x_1,\cdots,x_n) = 0$ whenever at least one $x_i = 0$. For all $i \in \{1,\cdots,n\}$, let f satisfy the inequality*

$$\begin{aligned}&\|f(a_1x_1,\cdots,a_{i-1}x_{i-1},a_i(x_i+y_i),a_{i+1}x_{i+1},\cdots,a_nx_n)\\&\quad+f(a_1x_1,\cdots,a_{i-1}x_{i-1},a_i(x_i-y_i),a_{i+1}x_{i+1},\cdots,a_nx_n)\\&\quad-2(a_1\cdots a_n)^2[f(x_1,\cdots,x_{i-1},x_i,x_{i+1},\cdots,x_n)+f(x_1,\cdots,x_{i-1},y_i,x_{i+1},\cdots,x_n)]\|_Y\\&\le\varphi(x_1,x_1,\cdots,x_{i-1},x_{i-1},x_i,y_i,x_{i+1},x_{i+1},\cdots,x_n,x_n)\end{aligned}\tag{16}$$

for all $a_1,\cdots,a_n \in \exp(\mathcal{U}(\mathcal{A})) \cup \{1_{\mathcal{A}}\}$ and all $x_1,\cdots,x_n,y_i \in \mathcal{X}$. For each fixed $x_1,\cdots,x_n \in \mathcal{X}$, let the sequence $\{4^{-kn}f(2^ka_1x_1,\cdots,2^ka_nx_n)\}_{k\in\mathbb{N}}$ converges uniformly on $\mathcal{A}_1^n$. If $f(a_1x_1,\cdots,a_nx_n)$ is continuous in $(a_1,\cdots,a_n) \in (\mathcal{A}_1 \cup \mathbb{R})^n$ for each fixed $x_1,\cdots,x_n \in \mathcal{X}$, then there exists a unique multi-quadratic mapping $Q : \mathcal{X}^n \to \mathcal{Y}$ satisfying Inequality (11) such that

$$Q(a_1x_1,\cdots,a_nx_n) = (a_1\cdots a_n)^2Q(x_1,\cdots,x_n)\tag{17}$$

for all $a_1,\cdots,a_n \in \mathcal{A}$ and all $x_1,\cdots,x_n \in \mathcal{X}$.

Proof. Since f satisfies Inequality (16) for $a_1 = \cdots = a_n = 1$ and all $x_1,\cdots,x_n \in \mathcal{X}$, by the same reasoning as in the proof of Theorem 3, there exists a unique multi-quadratic mapping $Q : \mathcal{X}^n \to \mathcal{Y}$ satisfying (11) for all $x_1,\cdots,x_n \in \mathcal{X}$.

Let $x_1^\circ,\cdots,x_n^\circ \in \mathcal{X}$ be fixed. And let $L : \mathcal{Y} \to \mathbb{R}$ be any continuous linear functional, that is, L is an arbitrary element of the dual space of $\mathcal{Y}$. For $j \in \{1,\cdots,n\}$ and $k \in \mathbb{N}$, consider the functions $\alpha_{jk} : \mathbb{R}^j \to \mathbb{R}$ defined by

$$\alpha_{jk}(t_1,\cdots,t_j) := 4^{-k}L\big(f(2^kt_1x_1^\circ,\cdots,2^kt_jx_j^\circ,2^kx_{j+1}^\circ,\cdots,2^kx_n^\circ)\big)$$

for all $t_1,\cdots,t_j \in \mathbb{R}$. By the continuity assumption, the function α_{jk} is continuous for all $j \in \{1,\cdots,n\}$ and all $k \in \mathbb{N}$. Note that

$$\alpha_{jk}(t_1,\cdots,t_j) = L\big(4^{-k}f(2^kt_1x_1^\circ,\cdots,2^kt_jx_j^\circ,2^kx_{j+1}^\circ,\cdots,2^kx_n^\circ)\big)$$

for all $j \in \{1,\cdots,n\}$, all $k \in \mathbb{N}$ and all $t_1,\cdots,t_n \in \mathbb{R}$. By the convergence assumption, the sequence $\{\alpha_{jk}(t_1,\cdots,t_j)\}_{k\in\mathbb{N}}$ is a Cauchy sequence for all $j \in \{1,\cdots,n\}$ and all $t_1,\cdots,t_j \in \mathbb{R}$. For all $j \in \{1,\cdots,n\}$, define a function $\alpha_j : \mathbb{R}^j \to \mathbb{R}$ by

$$\alpha_j(t_1,\cdots,t_j) := \lim_{k\to\infty}\alpha_{jk}(t_1,\cdots,t_j)$$

for all $t_1,\cdots,t_j \in \mathbb{R}$. Note that

$$\begin{aligned}\alpha_j(t_1,\cdots,t_j) &= L\big(Q(t_1x_1^\circ,\cdots,t_jx_j^\circ,x_{j+1}^\circ,\cdots,x_n^\circ)\big)\\&= L\big(Q(t_1x_1^\circ,\cdots,t_jx_j^\circ,1x_{j+1}^\circ,\cdots,1x_n^\circ)\big) = \alpha_n(t_1,\cdots,t_j,1,\cdots,1)\end{aligned}$$

for all $j \in \{1, \cdots, n\}$ and all $t_1, \cdots, t_j \in \mathbb{R}$. Since Q satisfies (4), we have

$$\begin{aligned}
&\alpha_j(r_1 + t_1, \cdots, r_j + t_j) + \alpha_j(r_1 - t_1, \cdots, r_j - t_j) \\
&\quad = L\big(Q((r_1 + t_1)x_1^\circ, \cdots, (r_j + t_j)x_j^\circ, x_{j+1}^\circ, \cdots, x_n^\circ)\big) \\
&\qquad + L\big(Q((r_1 - t_1)x_1^\circ, \cdots, (r_j - t_j)x_j^\circ, x_{j+1}^\circ, \cdots, x_n^\circ)\big) \\
&= L\big(Q((r_1 + t_1)x_1^\circ, \cdots, (r_j + t_j)x_j^\circ, x_{j+1}^\circ, \cdots, x_n^\circ) \\
&\qquad + Q((r_1 - t_1)x_1^\circ, \cdots, (r_j - t_j)x_j^\circ, x_{j+1}^\circ, \cdots, x_n^\circ)\big) \\
&= L\big(Q(r_1x_1^\circ + t_1x_1^\circ, \cdots, r_jx_j^\circ + t_jx_j^\circ, x_{j+1}^\circ, \cdots, x_n^\circ) \\
&\qquad + Q(r_1x_1^\circ - t_1x_1^\circ, \cdots, r_jx_j^\circ - t_jx_j^\circ, x_{j+1}^\circ, \cdots, x_n^\circ)\big) \\
&= L\big(2Q(r_1x_1^\circ, r_2x_2^\circ + t_2x_2^\circ, \cdots, r_jx_j^\circ + t_jx_j^\circ, x_{j+1}^\circ, \cdots, x_n^\circ) \\
&\qquad + 2Q(t_1x_1^\circ, r_2x_2^\circ - t_2x_2^\circ, \cdots, r_jx_j^\circ - t_jx_j^\circ, x_{j+1}^\circ, \cdots, x_n^\circ)\big) \\
&= \cdots \\
&= L\big(2^jQ(r_1x_1^\circ, \cdots, r_jx_j^\circ, x_{j+1}^\circ, \cdots, x_n^\circ) \\
&\qquad + 2^jQ(t_1x_1^\circ, \cdots, t_jx_j^\circ, x_{j+1}^\circ, \cdots, x_n^\circ)\big) \\
&= 2^j\alpha(r_1, \cdots, r_j) + 2^j\alpha(t_1, \cdots, t_j)
\end{aligned}$$

for all $j \in \{1, \cdots, n\}$ and all $r_1, \cdots, r_j, t_1, \cdots, t_j \in \mathbb{R}$. For all $j \in \{1, \cdots, n\}$, α_j is a Borel function since it is the pointwise limit of continuous functions. For all $j \in \{2, \cdots, n\}$ and all $v = (t_1, \cdots, t_{j-1}) \in \mathbb{R}^{j-1}$, define a function $\beta_j^v : \mathbb{R} \to \mathbb{R}$ by

$$\beta_j^v(t) := \alpha_j(t_1, \cdots, t_{j-1}, t) = \alpha_n(t_1, \cdots, t_{j-1}, t, 1, \cdots, 1)$$

for all $t \in \mathbb{R}$. For all $j \in \{2, \cdots, n\}$ and all $v = (t_1, \cdots, t_{j-1}) \in \mathbb{R}^{j-1}$, the function β_j^v is also a Borel function. For all $j \in \{2, \cdots, n\}$ and all $v = (t_1, \cdots, t_{j-1}) \in \mathbb{R}^{j-1}$, the function β_j^v as a measurable quadratic function is continuous (see [20]), so it has the form $\beta_j^v(t) = t^2\beta_j^v(1)$ for all $t \in \mathbb{R}$. Thus

$$\alpha_n(t_1, \cdots, t_{j-1}, t, 1, \cdots, 1) = t^2\alpha_n(t_1, \cdots, t_{j-1}, 1, 1, \cdots, 1)$$

for all $j \in \{2, \cdots, n\}$ and all $t_1, \cdots, t_{j-1}, t \in \mathbb{R}$. And thus

$$\begin{aligned}
\alpha_n(t_1, \cdots, t_n) &= t_n^2\alpha_n(t_1, \cdots, t_{n-1}, 1) = (t_{n-1}t_n)^2\alpha_n(t_1, \cdots, t_{n-2}, 1, 1) = \cdots \\
&= (t_1, \cdots, t_n)^2\alpha_n(1, \cdots, 1)
\end{aligned}$$

for all $t_1, \cdots, t_n \in \mathbb{R}$. Hence, we have

$$\begin{aligned}
L\big(Q(t_1x_1^\circ, \cdots, t_nx_n^\circ)\big) &= \alpha(t_1, \cdots, t_n) = t_1^2 \cdots t_n^2\alpha(1, \cdots, 1) \\
&= t_1^2 \cdots t_n^2 L\big(Q(x_1^\circ, \cdots, x_n^\circ)\big) = L\big(t_1^2 \cdots t_n^2 Q(x_1^\circ, \cdots, x_n^\circ)\big)
\end{aligned}$$

for all $t_1, \cdots, t_n \in \mathbb{R}$. Since L is any continuous linear functional,

$$Q(t_1x_1, \cdots, t_nx_n) = t_1^2 \cdots t_n^2 Q(x_1, \cdots, x_n) \tag{18}$$

for all $t_1, \cdots, t_n \in \mathbb{R}$ and all $x_1, \cdots, x_n \in \mathcal{X}$.

Fix $i \in \{1, \cdots, n\}$. Let k be an arbitrary positive integer. Replacing $(x_1, \cdots, x_n)$ by $(2^k x_1, \cdots 2^k x_n)$ and letting $y_i = 0$ in Inequality (16), we have

$$\begin{aligned}&\|2f(2^k a_1 x_1, \cdots, 2^k a_n x_n) - 2(a_1 \cdots a_n)^2 f(2^k x_1, \cdots, 2^k x_n)\|_{\mathcal{Y}} \\ &\qquad \leq \varphi(2^k x_1, 2^k x_1, \cdots, 2^k x_{i-1}, 2^k x_{i-1}, 2^k x_i, 0, 2^k x_{i+1}, 2^k x_{i+1}, \cdots, 2^k x_n, 2^k x_n)\end{aligned}$$

for all $a_1, \cdots, a_n \in \exp(\mathcal{U}(\mathcal{A})) \cup \{1_{\mathcal{A}}\}$ and all $x_1, \cdots, x_n \in \mathcal{X}$. Dividing by $2 \cdot 4^{kn}$, we obtain

$$\begin{aligned}&\left\| \frac{1}{4^{kn}} f(2^k a_1 x_1, \cdots, 2^k a_n x_n) - (a_1 \cdots a_n)^2 \frac{1}{4^{kn}} f(2^k x_1, \cdots, 2^k x_n) \right\|_{\mathcal{Y}} \\ &\leq \frac{1}{2 \cdot 4^{kn}} \varphi(2^k x_1, 2^k x_1, \cdots, 2^k x_{i-1}, 2^k x_{i-1}, 2^k x_i, 0, 2^k x_{i+1}, 2^k x_{i+1}, \cdots, 2^k x_n, 2^k x_n)\end{aligned}$$

for all $a_1, \cdots, a_n \in \exp(\mathcal{U}(\mathcal{A})) \cup \{1_{\mathcal{A}}\}$ and all $x_1, \cdots, x_n \in \mathcal{X}$. Taking $k \to \infty$, the right-hand side tends to 0 by (9). Thus, we have

$$\begin{aligned}&Q(a_1 x_1, \cdots, a_n x_n) = \lim_{k \to \infty} 4^{-kn} f(2^k a_1 x_1, \cdots, 2^k a_n x_n) \\ &= (a_1 \cdots a_n)^2 \lim_{k \to \infty} 4^{-kn} f(2^k x_1, \cdots, 2^k x_n) = (a_1 \cdots a_n)^2 Q(x_1, \cdots, x_n)\end{aligned}$$

for all $a_1, \cdots, a_n \in \exp(\mathcal{U}(\mathcal{A})) \cup \{1_{\mathcal{A}}\}$ and all $x_1, \cdots, x_n \in \mathcal{X}$.

For every element $a_1, \cdots, a_n \in \mathcal{A}_l$, there are positive integers $m_1, \cdots, m_n > 2$ such that

$$\left| \frac{1}{m_i} (1_{\mathcal{A}} + \log a) \right| \leq 1 - \frac{2}{m_i}$$

for all $i \in \{1, \cdots, n\}$. By Lemma 4, there are $u_{i1}, \cdots, u_{im} \in \mathcal{U}(\mathcal{A})$ satisfying

$$1_{\mathcal{A}} + \log a_i = u_{i1} + \cdots + u_{im}$$

for all $i \in \{1, \cdots, n\}$. For $e = e^{1_{\mathcal{A}}} \in \mathcal{A}$, we have

$$\begin{aligned}Q(ea_1 x_1, \cdots, ea_n x_n) &= Q\left(e^{1_{\mathcal{A}} + \log a_1} x_1, \cdots, e^{1_{\mathcal{A}} + \log a_n} x_n\right) \\ &= Q\left(e^{u_{11} + \cdots + u_{1m}} x_1, \cdots, e^{u_{n1} + \cdots + u_{nm}} x_n\right) \\ &= \left(e^{u_{11} + \cdots + u_{1m}}\right)^2 \cdots \left(e^{u_{n1} + \cdots + u_{nm}}\right)^2 Q(x_1, \cdots, x_n) \\ &= \left(e^{u_{11} + \cdots + u_{1m} + \cdots + u_{n1} + \cdots + u_{nm}}\right)^2 Q(x_1, \cdots, x_n) \\ &= \left(e^{1_{\mathcal{A}} + \log a_1 + \cdots + 1_{\mathcal{A}} + \log a_n}\right)^2 Q(x_1, \cdots, x_n) \\ &= (ea_1 \cdots ea_n)^2 Q(x_1, \cdots, x_n) = e^{2n} (a_1 \cdots a_n)^2 Q(x_1, \cdots, x_n)\end{aligned}$$

for all $a_1, \cdots, a_n \in \mathcal{A}_l$ and all $x_1, \cdots, x_n \in \mathcal{X}$. Since

$$1_{\mathcal{A}}^* = 1_{\mathcal{A}} 1_{\mathcal{A}}^* = (1_{\mathcal{A}}^*)^* 1_{\mathcal{A}}^* = (1_{\mathcal{A}} 1_{\mathcal{A}}^*)^* = (1_{\mathcal{A}}^*)^* = 1_{\mathcal{A}},$$

we have $1_{\mathcal{A}} 1_{\mathcal{A}}^* = 1_{\mathcal{A}}^* 1_{\mathcal{A}} = 1_{\mathcal{A}}$. So, $1_{\mathcal{A}}$ is unitary. Thus, $e = e^{1_{\mathcal{A}}} \in \exp(\mathcal{U}(\mathcal{A}))$. Hence,

$$Q(ea_1 x_1, \cdots, ea_n x_n) = e^{2n} Q(a_1 x_1, \cdots, a_n x_n)$$

for all $a_1, \cdots, a_n \in \mathcal{A}_l$ and all $x_1, \cdots, x_n \in \mathcal{X}$. Therefore, we obtain that Q satisfies Equation (17) for all $a_1, \cdots, a_n \in \mathcal{A}_l$ and all $x_1, \cdots, x_n \in \mathcal{X}$. By the same process as the above argument, one can show that Q satisfies Equation (17) for all $a_1, \cdots, a_n \in \mathcal{A}_r$ and all $x_1, \cdots, x_n \in \mathcal{X}$. Since $\mathcal{A}_l \cup \mathcal{A}_r$ is dense in $\mathcal{A}_{in}$, for each $i \in \{1, \cdots, n\}$, there is a sequence $\{b_{ik}\}_k$ in $\mathcal{A}_l \cup \mathcal{A}_r \setminus \{0\}$ such that $b_{ik} \to a_i$ as $k \to \infty$. By the assumption that

$f(a_1x_1, \cdots, a_nx_n)$ is continuous in $(a_1, \cdots, a_n) \in (\mathcal{A}_1 \cup \mathbb{R})^n$ for each fixed $x_1, \cdots, x_n \in \mathcal{X}$, we have

$$\begin{aligned} Q(a_1x_1, \cdots, a_nx_n) &= Q\Big(\lim_{k\to\infty} |b_{1k}|\frac{1}{|b_{1k}|}b_{1k}x_1, \cdots, \lim_{k\to\infty} |b_{nk}|\frac{1}{|b_{nk}|}b_{nk}x_n\Big) \\ &= \lim_{k\to\infty} Q\Big(|b_{1k}|\frac{1}{|b_{1k}|}b_{1k}x_1, \cdots, |b_{nk}|\frac{1}{|b_{nk}|}b_{nk}x_n\Big) \\ &= \lim_{k\to\infty} (|b_{1k}|\cdots|b_{nk}|)^2\, Q\Big(\frac{1}{|b_{1k}|}b_{1k}x_1, \cdots, \frac{1}{|b_{nk}|}b_{nk}x_n\Big) \\ &= \lim_{k\to\infty} (|b_{1k}|\cdots|b_{nk}|)^2\Big(\frac{1}{|b_{1k}|}b_{1k}\cdots\frac{1}{|b_{nk}|}b_{nk}\Big)^2 Q(x_1, \cdots, x_n) \\ &= \lim_{k\to\infty} (b_{1k}\cdots b_{nk})^2 Q(x_1, \cdots, x_n) = (a_1\cdots a_n)^2 Q(x_1, \cdots, x_n) \end{aligned} \tag{19}$$

for all $a_1, \cdots, a_n \in \mathcal{A}_{in}$ and all $x_1, \cdots, x_n \in \mathcal{X}$.

Since the sequence $\{4^{-kn} f(2^k a_1x_1, \cdots, 2^k a_nx_n)\}_{k\in\mathbb{N}}$ converges uniformly on $\mathcal{A}_1^n$, and $f(a_1x_1, \cdots, a_nx_n)$ is continuous in $(a_1, \cdots, a_n) \in (\mathcal{A}_1 \cup \mathbb{R})^n$ for each fixed $x_1, \cdots, x_n \in \mathcal{X}$, we see that $Q(a_1x_1, \cdots, a_nx_n)$ is also continuous in $(a_1, \cdots, a_n) \in (\mathcal{A}_1 \cup \mathbb{R})^n$ for each fixed $x_1, \cdots, x_n \in \mathcal{X}$. Let $c_1, \cdots, c_n \in \mathcal{A} \setminus (\mathcal{A}_{in} \cup \{0\})$. Since $\mathcal{A}_{in} \cap \mathcal{A}_{sa}$ is dense in $\mathcal{A}_{sa}$, there are sequences $\{c_{1k}\}_k, \cdots, \{c_{1k}\}_k$ in $\mathcal{A}_{in} \cap \mathcal{A}_{sa}$ such that $c_{ik} \to c_i$ as $k \to \infty$ for all $i \in \{1, \cdots, n\}$. Put $d_{ik} := \frac{1}{|c_{ik}|}c_{ik}$ for all $i \in \{1, \cdots, n\}$ and all $k \in \mathbb{N}$. Then, $d_{ik} \to \frac{1}{|c_i|}c_i$ as $k \to \infty$ and $d_{ik} \in \mathcal{A}_{in} \cap \mathcal{A}_{sa}$ for all $i \in \{1, \cdots, n\}$. By the continuity of Q, we have

$$\begin{aligned} \lim_{k\to\infty} Q(d_{1k}x_1, \cdots, d_{nk}x_n) &= Q\Big(\lim_{k\to\infty} d_{1k}x_1, \cdots, \lim_{k\to\infty} d_{nk}x_n\Big) \\ &= Q\Big(\frac{1}{|c_1|}c_1x_1, \cdots, \frac{1}{|c_n|}c_nx_n\Big) \end{aligned} \tag{20}$$

for all $x_1, \cdots, x_n \in \mathcal{X}$. By Equality (19), we obtain that

$$\begin{aligned} &\Big\|Q(d_{1k}x_1, \cdots, d_{nk}x_n) - \Big(\frac{1}{|c_1|}c_1\cdots\frac{1}{|c_n|}c_n\Big)^2 Q(x_1, \cdots, x_n)\Big\| \\ &\quad = \Big\|(d_{1k}\cdots d_{nk})^2 Q(x_1, \cdots, x_n) - \Big(\frac{1}{|c_1|}c_1\cdots\frac{1}{|c_n|}c_n\Big)^2 Q(x_1, \cdots, x_n)\Big\| \\ &\quad \to \Big\|\Big(\frac{1}{|c_1|}c_1\cdots\frac{1}{|c_n|}c_n\Big)^2 Q(x_1, \cdots, x_n) - \Big(\frac{1}{|c_1|}c_1\cdots\frac{1}{|c_n|}c_n\Big)^2 Q(x_1, \cdots, x_n)\Big\| = 0 \end{aligned}$$

as $k \to \infty$. By Equality (20) and the above convergence, we see that

$$\begin{aligned} &\Big\|Q\Big(\frac{1}{|c_1|}c_1x_1, \cdots, \frac{1}{|c_n|}c_nx_n\Big) - \Big(\frac{1}{|c_1|}c_1\cdots\frac{1}{|c_n|}c_n\Big)^2 Q(x_1, \cdots, x_n)\Big\| \\ &\le \Big\|Q\Big(\frac{1}{|c_1|}c_1x_1, \cdots, \frac{1}{|c_n|}c_nx_n\Big) - Q(d_{1k}x_1, \cdots, d_{nk}x_n)\Big\| \\ &\quad + \Big\|Q(d_{1k}x_1, \cdots, d_{nk}x_n) - \Big(\frac{1}{|c_1|}c_1\cdots\frac{1}{|c_n|}c_n\Big)^2 Q(x_1, \cdots, x_n)\Big\| \to 0 \end{aligned}$$

as $k \to \infty$ for all $x_1, \cdots, x_n \in \mathcal{X}$. By the above convergence and Equation (18), we have

$$\begin{aligned} Q(c_1x_1, \cdots, c_nx_n) &= Q\Big(\frac{1}{|c_1|}c_1x_1, \cdots, \frac{1}{|c_n|}c_nx_n\Big) \\ &= |c_1\cdots c_n|^2\Big(\frac{1}{|c_1|}c_1\cdots\frac{1}{|c_n|}c_n\Big)^2 Q(x_1, \cdots, x_n) = (c_1\cdots c_n)^2 Q(x_1, \cdots, x_n) \end{aligned}$$

for all $x_1, \cdots, x_n \in \mathcal{X}$. By Equality (19) and the above equality,

$$Q(a_1 x_n \cdots, a_n x_n) = (a_1 \cdots a_n)^2 Q(x_1, \cdots, x_n)$$

for all $a_1, \cdots, a_n \in \mathcal{A} \setminus \{0\}$ and all $x_1, \cdots, x_n \in \mathcal{X}$. By Equation (19), we have

$$Q(0x_1, \cdots, 0x_n) = (0 \cdots 0)^2 Q(x_1, \cdots, x_n)$$

for all $x_1, \cdots, x_n \in \mathcal{X}$. Therefore, Equation (17) holds for all $a_1, \cdots, a_n \in \mathcal{A}$ and all $x_1, \cdots, x_n \in \mathcal{X}$. □

From now on, X denotes an abelian group divisible by 2 and Y denotes a Banach abelian group divisible by 4.

For a mapping $f : X^n \to Y$, we write

$$\begin{aligned} &Df(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}) \\ &:= \sum_{s_1, \cdots, s_n \in \{-1,1\}} f(x_{11} + s_1 x_{12}, \cdots, x_{n1} + s_n x_{n2}) - 2^n \sum_{i_1, \cdots, i_n \in \{1,2\}} f(x_{1i_1}, \cdots, x_{ni_n}) \end{aligned}$$

for all $x_{11}, x_{12}, \cdots, x_{n1}, x_{n2} \in X$.

We obtain a stability result of functional Equation (2) using the fixed point alternative. The next theorem is the fixed point counterpart of Theorem 1.

Theorem 5. *Suppose that $\varphi : X^{2n} \to [0, \infty)$ is a function for which there exists a constant L with $0 < L < 1$ such that*

$$\varphi(2x_1, 2y_1, \cdots, 2x_n, 2y_n) \leq 4^n L \, \varphi(x_1, y_1, \cdots, x_n, y_n) \tag{21}$$

for all $x_1, y_1, \cdots, x_n, y_n \in X$. If a mapping $f : X^n \to Y$ with $f(x_1, \cdots, x_n) = 0$ whenever at least one $x_i = 0$ satisfies Inequality (3), then there exists a unique multi-quadratic mapping $Q : X^n \to Y$, which satisfies the inequality

$$\left\| f(x_1, \cdots, x_n) - Q(x_1, \cdots, x_n) \right\|_Y \leq \frac{1}{4^n(1-L)} \varphi(x_1, x_1, \cdots, x_n, x_n) \tag{22}$$

for all $x_1, \cdots, x_n \in X$.

Proof. Let $S := \{ g : X^n \to Y \mid g(x_1, \cdots, x_n) = 0 \text{ whenever at least one } x_i = 0 \}$ and define a generalized metric d on S by

$$\begin{aligned} d(g,h) := \inf \{ c \in [0, \infty] : &\| g(x_1, \cdots, x_n) - h(x_1, \cdots, x_n) \|_Y \\ &\leq c \, \varphi(x_1, x_1, \cdots, x_n, x_n) \text{ for all } x_1, \cdots, x_n \in X \}. \end{aligned}$$

It is well known that (S, d) is a generalized complete metric space. Define the linear operator $J : S \to S$ by

$$Jg(x_1, \cdots, x_n) := \frac{1}{4^n} g(2x_1, \cdots, 2x_n)$$

for all $x_1, \cdots, x_n \in X$. Note that $Jg \in S$ for all $g \in S$.

Let $g, h \in S$ and let $c \in [0, \infty]$ be such that $d(g,h) \leq c$. Then

$$\| g(x_1, \cdots, x_n) - h(x_1, \cdots, x_n) \|_Y \leq c \, \varphi(x_1, x_1, \cdots, x_n, x_n)$$

for all $x_1, \cdots, x_n \in X$. By Inequality (21), we have

$$\begin{aligned}\|Jg(x_1, \cdots, x_n) - Jh(x_1, \cdots, x_n)\|_Y &= \frac{1}{4^n}\|g(2x_1, \cdots, 2x_n) - h(2x_1, \cdots, 2x_n)\|_Y \\ &\leq \frac{c}{4^n}\,\varphi(2x_1, 2x_1, \cdots, 2x_n, 2x_n) \\ &\leq cL\,\varphi(x_1, x_1, \cdots, x_n, x_n).\end{aligned}$$

for all $x_1, \cdots, x_n \in X$. Hence, $d(Jg, Jh) \leq L\,d(g, h)$ for all $g, h \in S$.

Putting $(y_1, \cdots, y_n) = (x_1, \cdots, x_n)$ in (3), we obtain

$$\left\|f(2x_1, \cdots, 2x_n) - 4^n f(x_1, \cdots, x_n)\right\|_Y \leq \varphi(x_1, x_1, \cdots, x_n, x_n).$$

for all $x_1, \cdots, x_n \in X$. Dividing the above inequality by 4^n, we have

$$\|Jf(x_1, \cdots, x_n) - f(x_1, \cdots, x_n)\|_Y \leq \frac{1}{4^n}\,\varphi(x_1, x_1, \cdots, x_n, x_n)$$

for all $x_1, \cdots, x_n \in X$. Thus, we have $d(Jf, f) \leq \frac{1}{4^n} < \infty$.

By the fixed point alternative of Diaz and Margolis [21], since J is a strict contraction on the generalized complete metric space (S, d) and $d(Jf, f) < \infty$, the operator J possesses a unique fixed point Q in the set $S^* := \{g \in S : d(f, g) < \infty\}$, and moreover the sequence $\{J^k f\}$ converges to Q, explicitly,

$$Q(x_1, \cdots, x_n) = \lim_{k\to\infty} \left(J^k f\right)(x_1, \cdots, x_n)$$

for all $x_1, \cdots, x_n \in X$. That is, Q satisfies Equation (15). Since $Q = JQ$, the mapping Q satisfies $Q(2x_1, \cdots, 2x_n) = 4^n Q(x_1, \cdots, x_n)$ for all $x_1, \cdots, x_n \in X$. The fixed point alternative also provides

$$d(f, Q) \leq \frac{1}{4^n(1 - L)}.$$

Thus, we obtain Inequality (22).

By Inequality (3), we have

$$\frac{1}{4^{kn}}\,\varphi\left(2^k x_1, 2^k y_1, \cdots, 2^k x_n, 2^k y_n\right) \leq L^k\,\varphi(x_1, y_1, \cdots, x_n, y_n)$$

for all $x_1, y_1, \cdots, x_n, y_n \in X$ and all $k \geq 0$. Since $0 < L < 1$, we have

$$\lim_{k\to\infty} \frac{1}{4^{kn}}\,\varphi\left(2^k x_1, 2^k y_1, \cdots, 2^k x_n, 2^k y_n\right) = 0 \tag{23}$$

for all $x_1, y_1, \cdots, x_n, y_n \in X$. Now fix $u_1 = \cdots = u_n = 1$. Replacing each x_i by $2^k x_i$ and each y_i by $2^k y_i$ in (3) and dividing by 4^{kn}, we obtain

$$\frac{1}{4^{kn}}\|Df\left(2^k x_1, 2^k y_1, \cdots, 2^k x_n, 2^k y_n\right)\|_{\mathcal{Y}} \leq \frac{1}{4^{kn}}\,\varphi\left(2^k x_1, 2^k y_1, \cdots, 2^k x_n, 2^k y_n\right)$$

for all $x_1, y_1, \cdots, x_n, y_n \in X$. Taking the $k \to \infty$ in the above inequality, by (15) and (23), we obtain that Q satisfies Equation (2). By Theorem 3 in [12], Q satisfies the system (4) of equations and is a multi-quadratic mapping.

Suppose $Q' : X^n \to Y$ is a multi-quadratic mapping satisfying (22). Then, $Q' \in S$ and $d(f, Q') \leq \frac{1}{4^n(1-L)} < \infty$, so $Q' \in S^*$. Since Q' is multi-quadratic in each variable, we have

$$Q'(2x_1, \cdots, 2x_n) = 4^n Q'(x_1, \cdots, x_n)$$

for all $x_1, \cdots, x_n \in X$. That is, $JQ' = Q'$. Thus, Q' is a fixed point of J lying in S^*, and by the uniqueness clause of the fixed point alternative, $Q' = Q$. This completes the proof. □

Remark 2. *The stability result established in Theorem 5 is better than the previous results in Theorem 1 of Ciepliński [11]. Ciepliński's approach relied on evaluating the stability coordinate-by-coordinate, requiring a cumbersome system of limit conditions and separate contractive hypotheses for each individual coordinate index i.*

4. Perspective and Future Directions

While the current study comprehensively establishes the general solution and stability of multi-quadratic functional equations in classical spaces and Banach modules over C^*-algebras, exploring these results within generalized topological settings presents a compelling avenue for future research. As noted in the recent literature, extending stability theorems to random normed spaces introduces new analytical frameworks for multi-quadratic mappings [15].

Furthermore, extending our main results and techniques to strong probabilistic b-metric spaces offers significant theoretical depth. Recently, Achtoun et al. introduced the concept of controlled strong probabilistic b-metric spaces and established fixed-point theorems for probabilistic ω-contractions [22]. Investigating the Hyers–Ulam stability of our proposed multi-variable functional equation within such generalized probabilistic spaces would require adapting the fixed-point alternative method to accurately accommodate probabilistic distances and distance distribution functions.

In these non-classical settings, several additional assumptions and analytical difficulties might arise. For instance, the control functions and convergence criteria would need to be rigorously redefined in a probabilistic sense. Additionally, the contractive conditions must be carefully modified to align with the relaxed or controlled triangle inequalities inherent in b-metric structures. We believe that incorporating these generalized topological structures will substantially broaden the potential scope of our approach and provide a clear, robust direction for the future development of stability theory in multi-variable functional equations.

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