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An Alternative Proof of λ-Zappa–Szép Products of Bands and Groups

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Abstract

In previous work, Wazzan, together with Gilbert, established a connection between the λ-Zappa–Szép product of bands and groups and Billhardt's λ-semidirect product of inverse semigroups using automata-theoretic methods. Motivated by the need for a structural and transparent interpretation of this correspondence, we present an alternative proof based on the Ehresmann–Schein–Nambooripad theorem. The approach reformulates the construction through the associated inductive groupoid and makes the matched-pair actions explicit, so that the interaction between the band component and the group component can be followed directly at the level of arrows, identities, and composition. Within this framework, several structural features of the resulting product are clarified, including regularity, idempotents, inverse-type behavior, orthodoxness, $\mathcal{L}$-unipotence, and the connection with semidirect products. The paper also consolidates related intermediate results into a unified proof route and illustrates the construction by examples and schematic figures. This groupoid-based viewpoint provides a conceptually motivated treatment of the λ-Zappa–Szép product and highlights its role as a bridge between matched-pair decompositions, bands and groups, and inverse semigroup theory.

Keywords: Zappa–Szép groupoid; inverse semigroup; λ-semidirect product

MSC: 18B40; 20F05; 20M05; 20M18; 20L17

1. Introduction

The theory and applications of Zappa–Szép products have seen substantial development in recent years, particularly within the domains of algebra, category theory, and operator algebras. These advancements highlight the structural richness and versatility of Zappa–Szép constructions, positioning them as a unifying framework across multiple areas of modern mathematics . In semigroup theory, Zenab [1] studied the algebraic properties of Zappa–Szép products with semilattices and restriction semigroups, demonstrating that the restriction semigroup can be intrinsically embedded into the product structure under certain assumptions. Another foundational workload has been produced by Özalan [2], in which the author Introduces a new semigroup structure, which is a generalization of a Rees matrix semigroup and that of a completely zero-simple semigroup. This framework sets the stage for the exploration of new theoretical developments in the study of Zappa–Szép products. In the C^* -algebra category, Brownlowe et al. [3] studied Zappa–Szép products of étale groupoids, where the groupoid C^*-algebra of the product can also be realized as a a C^*-blend, subsuming all of the C^*-algebras of the groupoids. In another direction, Duwenig and Li [4] defined the concept of Zappa–Szép product twists on paired groupoids and

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established the connection between this intermediate representation and twisted groupoid C^*-algebras. Categorical viewpoints also appear in Mundey and Sims' work [5] where they constructed a spectral sequence to compute the homology of Zappa–Szép products of small categories to combine homological algebra and self-similar groupoid dynamics. Zaman [6] studies the organization of left P-Ehresmann and right Ehresmann semigroups, presents the concept of λ-semidirect product as a generalization of the semidirect product. The article represents a link between above mentioned families of semigroups, and Zappa–Szép products, and brings new information about semigroups to semigroup and category theory. In a closely related development, DeWolf and Pronk [7] generalize the Ehresmann–Schein–Nambooripad (ESN) Theorem to inverse categories and prove that inverse categories are equivalent as categories to locally complete inductive groupoids. Their research emphasizes general combinations of the ESN theorem that originally connected inverse semigroups and groupoids, that is inverse categories: so the importance of the ESN theorem is growing in algebraic structures, topology and computer science. Altogether, these works exemplify the increased relevance of Zappa–Szép products as a unifying notion in algebraic structures, category theory, and functional analysis, of which categorical techniques, particularly those that employ monoids (semigroups), seem to be the main focus.

1.1. Literature Position and Contribution

Classical Zappa–Szép products originate in the study of group factorisations developed by Zappa [8] and Szép [9]. They generalise both direct products and semidirect products: in a direct product both components act trivially, while in a semidirect product only one component acts non-trivially. In a Zappa–Szép product both components act on each other through compatible actions, and this two-sided interaction is precisely what makes the construction useful for decompositions beyond the semidirect case. The semigroup version was developed systematically by Kunze [10], and Brin [11] later emphasized the flexibility of the construction for groups, monoids, and categories.

For the present paper, the central point of departure is the work of Gilbert and Wazzan [12], where Zappa–Szép products of bands and groups were related to Billhardt's λ-semidirect products of inverse semigroups. Their approach uses automata-theoretic methods and identifies a suitable inverse-semigroup component inside the Zappa–Szép product. Subsequent work of Gould and Zenab [13] placed λ-Zappa–Szép constructions in the broader setting of restriction semigroups, while Zenab [1], Brownlowe et al. [3], and recent groupoid and categorical treatments [4,5,14] show that Zappa–Szép products remain an active tool across semigroup theory, groupoids, category theory, and operator-algebraic constructions.

1.2. Automata-Theoretic Origin of the Construction

Since the construction studied here traces back to the automata-theoretic approach of Gilbert and Wazzan [12], we record more explicitly how the Zappa–Szép compatibility axioms are related to Mealy-type automata and to self-similar actions. This also clarifies the relationship between the earlier automata-theoretic construction and the inductive-groupoid approach used in the present paper.

Let

$$\wp = A \bowtie S$$

be a Zappa–Szép product of semigroups A and S, with structure maps

$$\triangleright : S \times A \longrightarrow A, \qquad \triangleleft : S \times A \longrightarrow S.$$

Following the interpretation used in [12], one may regard S as the state set of a Mealy-type algebraic automaton whose input and output set is A. Define

$$\delta : S \times A \longrightarrow S, \qquad \delta(s,a) = s \triangleleft a,$$

and

$$\lambda : S \times A \longrightarrow A, \qquad \lambda(s,a) = s \triangleright a.$$

Thus, when the automaton reads an input $a \in A$ from a state $s \in S$, it produces the output $s \triangleright a$ and moves to the new state $s \triangleleft a$. Symbolically, the transition is

$$s \xrightarrow{a \,|\, s \triangleright a} s \triangleleft a.$$

The Zappa–Szép axioms express the compatibility of this state/output mechanism with multiplication in A and S. To see this, suppose that two inputs $a, a' \in A$ are processed successively from the state s. Processing a first produces the output $s \triangleright a$ and changes the state to $s \triangleleft a$. Processing a' from this new state then produces

$$(s \triangleleft a) \triangleright a'$$

and changes the state to

$$(s \triangleleft a) \triangleleft a'.$$

Consequently, compatibility between successive processing and the single input aa' is expressed precisely by

$$s \triangleright (aa') = (s \triangleright a)\big((s \triangleleft a) \triangleright a'\big), \tag{SS2}$$

and

$$s \triangleleft (aa') = (s \triangleleft a) \triangleleft a'. \tag{SS3}$$

Thus (SS2) and (SS3) describe what Gilbert and Wazzan call the serial processing of inputs.

There is an analogous compatibility with multiplication of states. For $s, t \in S$ and $a \in A$, the identities

$$(st) \triangleright a = s \triangleright (t \triangleright a) \tag{SS1}$$

and

$$(st) \triangleleft a = \big(s \triangleleft (t \triangleright a)\big)(t \triangleleft a) \tag{SS4}$$

show that processing an input by the composite state st is consistent with successive processing by t and s, together with the corresponding update of the state component. These are the sequential-processing identities in the automata-theoretic description of [12]. Hence the four central Zappa–Szép axioms may be interpreted as the algebraic consistency conditions for the input/output and state-transition operations.

In the band–group situation considered in the present paper, we specialize to

$$A = B, \qquad S = G.$$

Accordingly, an element $g \in G$ plays the role of a state and $b \in B$ plays the role of an input, so that

$$g \xrightarrow{b \,|\, g \triangleright b} g \triangleleft b.$$

The two matched-pair maps therefore have distinct automata-theoretic roles: $g \triangleright b$ records the transformed input, while $g \triangleleft b$ records the state retained after the interaction.

This recursive structure is closely related to self-similar group and semigroup actions. In the classical self-similar setting, where the underlying inputs are words in a free monoid,

the action of a state on a word is determined recursively by its action on the first letter together with the restriction of that state to the remaining word. The roles of these two operations are played here by $\triangleright$ and $\triangleleft$, respectively. This connection between automata, rooted-tree actions and self-similar groups is developed in [15,16], while the relationship between self-similar actions and Zappa–Szép-type monoid factorisations is treated in [17,18]. More general Zappa–Szép constructions associated with self-similar actions and categorical structures are also considered in [5,11].

For the inverse-semigroup construction, Gilbert and Wazzan [12] subsequently use a second, equivalent automata description adapted to the Billhardt construction. In the semilattice–group case $P = E \bowtie G$, they pass to a (G, E)-automaton whose state set is E and whose input/output set is G. With their notation, a transition from $e \in E$ has the form

$$e \xrightarrow{g \,|\, e*g} e^{|g},$$

where

$$e * g = (g^{-e})^{-1}, \qquad e^{|g} = g^{-1} \triangleright e.$$

The Billhardt subautomaton consists precisely of those transitions for which input and output coincide,

$$e * g = g.$$

Using the relation above, this is equivalent to

$$g^{-e} = g^{-1},$$

which, in the matched-pair notation adopted in the present paper, is the condition

$$g^{-1} \triangleleft e = g^{-1}.$$

Thus the λ-condition defining

$$\mathrm{B}_{\triangleleft\triangleright}(\wp) = \left\{ (e, g) \in E \bowtie G : g^{-1} \triangleleft e = g^{-1} \right\}$$

has a direct automata-theoretic interpretation: it selects the transitions of the Billhardt automaton for which the input and output group elements agree.

The present paper starts from the same matched-pair data but reorganizes them in a different way. Rather than using the automaton to establish the inverse-semigroup multiplication directly, we regard the elements satisfying the λ-condition as arrows of a groupoid. The source and target maps, inverses, partial order, restrictions and corestrictions are then determined explicitly. After showing that the ordered groupoid is inductive, the Ehresmann–Schein–Nambooripad correspondence yields the inverse-semigroup multiplication as the associated pseudoproduct. Thus the automata-theoretic and groupoid approaches do not define two different products; rather, they provide two complementary descriptions of the same matched-pair interaction. The former emphasizes state transitions and recursive processing, whereas the latter emphasizes arrows, restrictions, order and the categorical origin of the inverse-semigroup structure. In the automata-theoretic construction of [12], the Billhardt subautomaton retains precisely those transitions for which input and output coincide. Under the matched-pair notation used here, this condition becomes

$$g^{-1} \triangleleft e = g^{-1}.$$

Thus the defining λ-condition should be understood as an input–output equality condition, rather than as a return-to-identity condition.

Table 1 summarizes the correspondence relevant to the Billhardt subautomaton and the groupoid construction used below.

Table 1. Correspondence between the automata-theoretic description of the Billhardt construction and the ordered-groupoid description of the λ-Zappa–Szép construction.

Automata-Theoretic Description	Groupoid Description in the Present Paper
State $e \in E$ in the Billhardt automaton	Identity $(e, 1_G)$
Transition $e \xrightarrow{g \mid e*g} e^{\mid g}$	Pair (e, g) before imposing the λ-condition
Input-output condition $e * g = g$	λ-condition $g^{-1} \triangleleft e = g^{-1}$
Transition satisfying $e * g = g$	Arrow $(e, g) \in \mathrm{B}_{\triangleleft\triangleright}(\wp)$
Updated state $e^{\mid g} = g^{-1} \triangleright e$	Target identity $(g^{-1} \triangleright e, 1_G)$ of the arrow (e, g)
Automata composition and recursive state/output rules	Matched-pair identities used in the composition and restriction formulas

The present paper goes beyond a reformulation of the results in [12,14] and provides a distinct structural extension of those works. In [12], the λ-Zappa–Szép construction for bands and groups is obtained mainly through automata-theoretic and direct algebraic arguments, whereas the present approach reconstructs the inverse-semigroup structure from an explicitly defined ordered groupoid and derives the multiplication through the Ehresmann–Schein–Nambooripad correspondence. In particular, we introduce the relevant partial order, verify the ordered-groupoid axioms and restriction properties, and show that the resulting structure is an inductive groupoid; the inverse-semigroup multiplication then arises naturally as the associated pseudoproduct. This changes the nature of the argument from a direct verification of formulas to a structural explanation of why the construction has inverse-semigroup behavior. Moreover, the matched-pair formulation makes the two actions $\triangleright$ and $\triangleleft$ explicit throughout the construction and allows the natural order, restrictions, idempotents, inverses, Green relations, factorisability, orthodoxness, unit-regularity, and L-unipotency to be studied within one coherent framework. Thus, the contribution of this paper is not merely an alternative computation of previously known formulas, but a new ESN-based proof route and a unified groupoid-theoretic interpretation that connects, extends, and clarifies the earlier approaches of [12,14].

A key challenge in the structural theory of inverse semigroups is constructing them from simpler algebraic components. In this context, traditional methods, such as the classical semidirect product, often fall short, as the product of two inverse semigroups does not generally yield another inverse semigroup. Bernd [19] addressed this issue by reformulating semidirect products in a complementary manner, introducing a new construction he termed the λ-semidirect product. This innovative transformation ensures that the resulting structure retains its inverse properties.

Definition 1 ([19] *Billhardt's $\lambda-$semidirect product*)**.** *Let T operate on S by endomorphisms and define $S \rhd^{\lambda} T = \{(s,t) \in S \times T : s = tt^{-1} \rhd s\}$ equipped with the binary operation defined by*

$$(s,t)(s',t') = \Big(\Big((tt')(tt')^{-1} \rhd s\Big)(t \rhd s'), tt'\Big).$$

The result which justifies this definition is the following.

Proposition 1 ([19])**.** *Let S and T be inverse semigroups such that T operates on S by endomorphisms. Then $S \rhd^{\lambda} T$ is an inverse semigroup.*

A semidirect product of semigroups is a specific case of a Zappa–Szép product where one of the structure maps is trivial. Semidirect products involving bands and groups are crucial in the structural theory of inverse semigroups. As such, the study of Zappa–Szép products formed from bands and groups is particularly significant, given their role in understanding the algebraic framework of inverse semigroups. Consequently, this opens the door to a broader theoretical framework. Motivated by these findings, our objective in [12] was to establish a connection between the construction of Zappa–Szép products of bands and groups (as derived from automata theory) and the λ-semidirect product of inverse semigroups introduced by Billhardt. Specifically, we aimed to generalize Billhardt's λ-semidirect products to encompass Zappa–Szép product structures. This involves identifying a component of a Zappa–Szép product of a band and a group, whose binary operation can be transformed into a proper operation, thereby qualifying it as an inverse semigroup.

Theorem 1 ([12]). *If $\wp = B \lhd\rhd G$, where B is a band and G is a group, then*

$$\mathbf{B}_{\lhd\rhd}(\wp) = \left\{(b,g) \in \wp : \left(g^{-1} \lhd b\right) = g^{-1}\right\}$$

yields an inverse semigroup equipped with the following multiplication:

$$(b,g)(b',g') = \left(b \cdot (g \rhd b'), (g \lhd b')\left(g' \lhd \left(g'^{-1}g^{-1} \rhd b\right)\right)\right).$$

The Ehresmann–Schein–Nambooripad (ESN) Theorem [20], (Theorem 4.1.8) is fundamental in the investigation of inverse semigroups and asserts that there is a categorical equivalence between inverse semigroups and inductive groupoids. This correspondence not only gives a deep understanding of the structure of partial symmetries, but also establishes a link between the algebraic and categorical points of view. The theorem has wide applicability in many fields of mathematics, such as topology, category theory, and theoretical computer science (in particular in modeling partial transformations). On the basis of the rich content and practical usability of this result many generalizations have been worked out. For instance, Hollings [21] generalized the theorem to inverse categories, which are connected to locally complete inductive groupoids, and Stokes [22] obtained similar results for bi-unary semigroups related to biaction categories. Later contributions by Wang [23], and Lawson [24], also considered Ehresmann semigroups, and their relationship with Ehresmann categories. Furthermore, ESN has played a key role in applications as the construction of étale groupoids associated to inverse semigroups [25] (Section 4.2), and the study of Galois-type correspondences between semigroup operations on rings [26] (Chapter 5). Such varied applications attest to the theorem's versatility and foundational nature as well as its driving force in both theoretical and applied research on algebraic and categorical structures.

1.3. Relation to the Recent One-Sided ESN Framework

A recent development relevant to the present construction is the Ehresmann–Schein–Nambooripad-type theorem for left restriction semigroupoids established by Haag, Lautenschlaeger and Tamusiunas [27]. Their approach develops a one-sided version of the ESN philosophy by relating left restriction semigroupoids to locally inductive constellations. This provides a categorical framework for structures having a partially defined multiplication together with a one-sided restriction operation, and also includes inverse-semigroupoid structures as an important special case.

The relation with the construction considered in the present paper can be made explicit. Let

$$\mathcal{G} = \mathrm{B}_{\triangleleft\triangleright}(\wp)$$

denote the groupoid obtained from the λ-Zappa–Szép data in Section 2. Since $\mathcal{G}$ is a groupoid, every arrow $x \in \mathcal{G}$ possesses an inverse x^{-1}. Consequently, one may define

$$x^{+} = xx^{-1}.$$

With the convention for composition adopted in the present paper, x^{+} is the identity associated with the source of x, and

$$x^{+}x = x.$$

Thus the groupoid arising from the λ-Zappa–Szép construction carries the canonical one-sided restriction structure associated with an inverse semigroupoid. In this sense, the groupoid constructed here lies within the inverse-semigroupoid specialization of the framework developed in [27].

This observation shows that the present construction is compatible with the recent one-sided ESN framework once the λ-Zappa–Szép groupoid has been established. The natural order, identities, and restriction operations appearing in our ordered-groupoid construction may therefore also be viewed through the corresponding one-sided categorical structure.

There is, however, an important distinction between the two approaches. The present paper begins with the matched-pair maps

$$\triangleright : G \times E \longrightarrow E, \qquad \triangleleft : G \times E \longrightarrow G,$$

constructs the corresponding groupoid, introduces its natural partial order, proves that the identities form a meet-semilattice, and then shows that the resulting ordered groupoid is inductive. The classical Ehresmann–Schein–Nambooripad correspondence is subsequently applied to recover the inverse-semigroup multiplication as the associated pseudoproduct. In particular, the multiplication obtained in Theorem 4 is derived from the inductive-groupoid structure itself.

By contrast, the framework of [27] starts with a left restriction semigroupoid already equipped with a unary restriction operation

$$x \longmapsto x^{+}$$

and a partially defined multiplication satisfying the corresponding one-sided restriction axioms. Therefore, the ESN-based proof developed in the present paper does not extend verbatim to an arbitrary left restriction semigroupoid. A genuine extension of the λ-Zappa–Szép construction to that setting would require additional compatibility conditions relating the matched-pair actions $\triangleright$ and $\triangleleft$ to the unary restriction operation and to the partially defined semigroupoid composition.

The recent one-sided ESN framework nevertheless suggests a natural direction in which the present construction could be generalized. In particular, it is closely related in spirit to the restriction-semigroup treatment of λ-Zappa–Szép products developed by Gould and Zenab [13]. Their work places the λ-Zappa–Szép construction in a broader restriction setting, whereas the present paper works in the inverse-semigroup case obtained from an inductive groupoid. It would therefore be natural to investigate whether the matched-pair construction considered here can be formulated directly within the locally inductive constellation framework.

Thus the one-sided ESN theorem does not replace the classical ESN correspondence required for the inverse-semigroup construction proved here. Rather, it provides a broader categorical setting in which the groupoid obtained in the present paper can be interpreted and suggests a natural framework for future one-sided and multi-object generalizations of λ-Zappa–Szép products. Driven by the significance of the Ehresmann–Schein–Nambooripad (ESN) Theorem, we offer an alternative proof and conceptual framework for the λ-Zappa Szép product involving bands and groups—arising via automata theory. Our method adopts a categorical viewpoint grounded in the ESN theorem.

The paper is organized as follows. Section 1 introduces the matched-pair notation and recalls the basic definitions of Zappa–Szép products needed throughout the paper. Section 2 develops the main algebraic properties of Zappa–Szép products of bands and groups and establishes the structural results required for the inverse-semigroup construction. In particular, we construct the associated groupoid, introduce the relevant partial order, prove that the resulting ordered groupoid is inductive, and then apply the Ehresmann–Schein–Nambooripad theorem to obtain the inverse-semigroup structure. Section 2 also includes examples illustrating the construction in both finite and wreath-product settings and showing explicitly how the abstract theory can be applied. Finally, Section 3 summarizes the main conclusions of the groupoid-theoretic approach, discusses its relationship with the earlier results, and indicates possible directions for further research.

We now present the key definitions that serve as the basis for the rest of the article.

Definition 2. *Suppose that A and B are monoids. The Zappa–Szép product $\wp = A \lhd\rhd B$ is defined on the set $A \times B$ equipped with:*

- A left structure map $\rhd : B \times A \longrightarrow A$, written $b \rhd a$,
- A right structure map $\lhd : B \times A \longrightarrow B$, written $b \lhd a$,

Which satisfy the following compatibility axioms first formulated by G. Zappa [8] for all $a, a' \in A$ and for all $b, b' \in B$.

(SS1) $(bb') \rhd a = b \rhd (b' \rhd a)$.

(SS2) $b \rhd (aa') = (b \rhd a)((b \lhd a) \rhd a')$.

(SS3) $b \lhd (aa') = (b \lhd a) \lhd a'$.

(SS4) $(bb') \lhd a = (b \lhd (b' \rhd a))\left(b' \lhd a\right)$.

(SS5) $1_B \rhd a = a$.

(SS6) $b \rhd 1_A = 1_A$.

(SS7) $b \lhd 1_A = b$.

(SS8) $1_B \lhd a = 1_B$.

This structure defines a Zappa–Szép product $\wp = B \lhd\rhd G$, where the multiplication is given by:

$$(a, b)(a', b') = \left(a\left(b \rhd a'\right), \left(b \lhd a'\right)b'\right).$$

Definition 3. *Let ϑ be a groupoid with partial order $\leq$. Then $(\vartheta, \leq)$ is said to be an ordered groupoid if the following conditions are satisfied:*

(OG1) *Whenever $g \leq h$, the inverses satisfy $g^{-1} \leq h^{-1}$; that is, the ordering is preserved under inversion.*

(OG2) *If $g \leq h$ and $k \leq l$, and both compositions gk and hl are defined, then it follows that $gk \leq hl$; in other words, the order is compatible with composition.*

(OG3) *Let $g \in \vartheta$ and suppose there exists an identity element $b \in S$ such that $b \leq gg^{-1}$. Then, there exists a unique element, denoted by $g \lhd b$, satisfying $(g \lhd b) \leq g$ and $(g \lhd b)(g \lhd b)^{-1} = b$.*

*(OG3)** *Similarly, for any $g \in \vartheta$ and identity b such that $b \leq g^{-1}g$, there exists a unique element, written $b \rhd g$ satisfying $(b \rhd g) \leq g$ and $(b \rhd g)^{-1}(b \rhd g) = b$.*
It is shown in [20] that $(OG3)^*$ is a consequence of $(OG1)$ and $(OG3)$, and may be expressed as: $\left(b \rhd g^{-1}\right)^{-1}$. An ordered groupoid is said to be inductive c if the set of identity elements $E(S)$ forms a meet-semilattice under $\leq$.

Definition 4. *Let $\wp = B \lhd \rhd G$ be a Zappa–Szép product, where A is a monoid and B is a group, with structure maps*

$$\rhd : B \times A \longrightarrow A, \qquad \lhd : B \times A \longrightarrow B,$$

satisfying the compatibility conditions (SS1)–(SS8). *The λ-Zappa–Szép product is the subset*

$$\mathbf{B}_{\lhd\rhd}(\wp) = \left\{(a, b) \in \wp : b^{-1} \lhd a = b^{-1}\right\},$$

equipped with the restricted multiplication inherited from $\wp$.

We consider the following special case of λ–Zappa–Szép product construction introduced in [14], where $S = B$ a band and $T = G$ a group, and subset $\mathbf{B}_{\lhd\rhd}(\wp)$ of the Zappa–Szép product $\wp = B \lhd \rhd G$

$$\mathbf{B}_{\lhd\rhd}(\wp) = \left\{(b, g) \in \wp : \left(g^{-1}\right) \lhd b = g^{-1}\right\}$$

This subset forms the arrow set of a groupoid constructed from the structure map of G on B with the following structure and display in Figure 1 :

- Vertex set: $E(\mathbf{B}_{\lhd\rhd}(\wp)) = \{(b, 1) : b \in B\} \cong B$;
- Arrow set: $\mathbf{B}_{\lhd\rhd}(\wp)$;
- Source and target: an arrow (b, g) starts at $(b, g)(b, g)^{-1} = (b, 1)$, finishes at $(b, g)^{-1}(b, g) = \left(g^{-1} \rhd b, 1\right)$;
- Inverse: the inverse of the arrow (b, g) is $\left(g^{-1} \rhd b, g^{-1}\right)$;
- Identity: the identity arrow at b is $(b, 1)$; and
- Composition: arrows (b_1, g) and (b_2, h) are compassable if and only if $g^{-1} \rhd b_1 = b_2$, which produces the composite arrow $(b_1(g \rhd b_2), (g \lhd b_2)h)$.

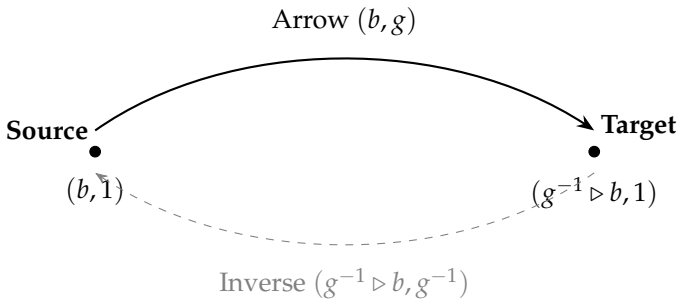


Figure 1. Visualization of an arrow in the Zappa–Szép groupoid.

The set of identities of the groupoid is

$$E(\mathbf{B}_{\lhd\rhd}(\wp)) = \{(b, 1_G) : b \in B\} \cong B.$$

We denote by

$$B \triangleleft G = \{(b, g) \in \wp : g \triangleleft b = 1_G\}.$$

We note that as in our general λ–Zappa–Szép product $\mathbf{B}_{\triangleleft\triangleright}(\wp)$, the composition of compassable arrows in $\mathbf{B}_{\triangleleft\triangleright}(\wp)$ is just the Zappa–Szép composition. This perspective recuperates Theorem 4.3 in [12] and provides insight into numerous induced identities {like Equation (4.8)} in terms of groupoids. Since these results can be somewhat abstract or nonintuitive, we complement the above general procedure with a specific example that demonstrates how the Zappa–Szép interaction rules are retrieved from the underlying structure of B and G, which gives a simpler way to form the inverse semigroup. We refer the reader to [14] for a comprehensive treatment of the correspondence between inductive groupoids and inverse semigroups, including methods for constructing one from the other. In Section 2 some arithmetic of Zappa–Szép product $\wp = B \triangleleft \triangleright G$ of a band B and a group G are given: It is regular semigroup with $E(\wp) = \{(b, g) : g \triangleleft b = 1\} \cong B \triangleleft G$; with $V((b, g_1)) = \left\{\left(g_1^{-1} \triangleright b, g_2\right) : g_2 \triangleleft b = \left(g_1^{-1}\right) \triangleleft b\right\}$, if B has an identity then $\wp$ is unit-regular, and if B is a semilattice then $\wp$ is orthodox and $\mathcal{L}$-unipotent; $\wp = B \triangleleft \triangleright G$ is E-unitary inverse if and only if $\wp$ is the semidirect product of the band B and the group G; If B is a semilattice with identity then $\wp$ is covered by $E(\wp) \triangleleft G$. We refer to [28,29] for some of the properties of $\mathcal{L}$-unipotent semigroups, noting that a *right* inverse semigroup in [29] is a *left* inverse semigroup in [28]. For this reason we have preferred the term $\mathcal{L}$-unipotent following Clifford [30].

The main construction followed in the sequel is summarized in Figure 2. The diagram emphasizes that the proof does not rely only on element-wise computations; instead, the matched-pair actions are used to pass from the band–group interaction to the associated Zappa–Szép product and then to the corresponding inverse semigroup through the groupoid viewpoint.

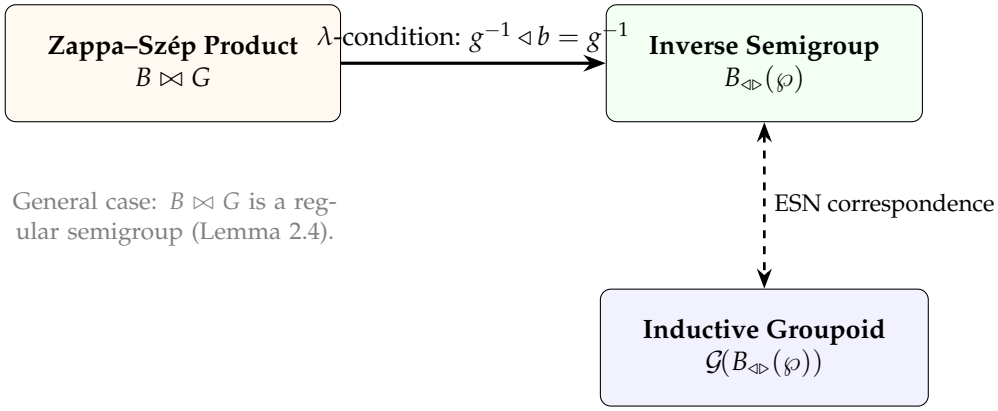


Figure 2. Structural transition from a Zappa–Szép product to the λ-Zappa–Szép inverse semigroup. Imposing the λ-condition selects the relevant subset, while the Ehresmann–Schein–Nambooripad correspondence identifies the resulting inverse-semigroup structure with its associated inductive groupoid.

2. The Arithmetic of Zappa–Szép Products of Bands and Groups

This section examines the algebraic structure of the Zappa–Szép product Zappa–Szép product $\wp = B \triangleleft \triangleright G$ formed from a band B and a group G. We consider left and right structure maps of G on B, denoted by $g \triangleright b$ and $g \triangleleft b$, respectively, and assume that they satisfy the compatibility conditions (SS1)–(SS7). At this stage, condition (SS8) is not assumed; Lemma 1 (1) shows that it follows automatically in the present group setting. Our focus is on how the structural features of the band and the group interact within the combined product $\wp$. We analyze properties such as element regularity, the behavior of inverses and

idempotents, $\mathcal{L}$-unipotent, and unit-regular. The results established here form a foundational toolkit for further exploration of Zappa–Szép constructions in semigroup theory. Recall that a band B is a semigroup of idempotents. The natural partial order on a band is defined by $e \leq f$ if and only if $ef = fe = e$. This order is fundamental in understanding the structure of bands and their interaction with groups via Zappa–Szép products. We begin with a key lemma that will be referenced throughout the sections.

Lemma 1. *Let B be a band and G a group, and suppose that there are structure maps*

$$\triangleright : G \times B \longrightarrow B, \qquad \triangleleft : G \times B \longrightarrow G,$$

satisfying the Zappa–Szép compatibility conditions required below, with (SS8) *not assumed. Suppose in particular that*

$$1_G \triangleright b = b \qquad \text{for all } b \in B.$$

Then the following hold:

(1) $1_G \triangleleft b = 1_G$ *for every $b \in B$. Hence condition* (SS8) *follows automatically in this setting.*

(2) *For all $g \in G$, $b \in B$:* $(g \triangleleft b)^{-1} = \left(g^{-1}\right) \triangleleft (g \triangleright b)$,

(3) *Let $b_1, b_2 \in B$ with $b_1 \leq b_2$, where $\leq$ denotes the natural partial order on the band B, that is,*

$$b_1 \leq b_2 \iff b_1 b_2 = b_2 b_1 = b_1.$$

Then, for every $g \in G$,

$$g \triangleright b_1 = (g \triangleright b_2)\big((g \triangleleft b_2) \triangleright b_1\big),$$

and

$$(g \triangleleft b_2) \triangleright b_1 = \big((g \triangleleft b_2) \triangleright b_1\big)(g \triangleright b_1).$$

In particular, if

$$g \triangleleft b_2 = 1_G,$$

then

$$b_1 = b_1(g \triangleright b_1).$$

(4) *Suppose, in addition, that B is a semilattice. Then, for every $g \in G$ and $b_1, b_2 \in B$ with $b_1 \leq b_2$,*

$$(g \triangleleft b_2) \triangleright b_1 = g \triangleright b_1.$$

In particular, if

$$g \triangleleft b_2 = 1_G,$$

then

$$g \triangleright b_1 = b_1.$$

Proof. For part (1), let $g \in G$ and $b \in B$. Using the compatibility condition for the right structure map and the identity $1_G \triangleright b = b$, we obtain

$$g \triangleleft b = (g1_G) \triangleleft b = \big(g \triangleleft (1_G \triangleright b)\big)(1_G \triangleleft b) = (g \triangleleft b)(1_G \triangleleft b).$$

Since G is a group, cancellation gives

$$1_G \triangleleft b = 1_G.$$

Thus (SS8) follows from the remaining assumptions.

(2) Using part (1), we compute

$$\begin{aligned} 1_G = 1_G \triangleleft b &= \left(g^{-1}g\right) \triangleleft b \\ &= \left(g^{-1}\right) \triangleleft (g \triangleright b) \cdot (g \triangleleft b), \end{aligned}$$

and hence

$$(g \triangleleft b)^{-1} = \left(g^{-1}\right) \triangleleft (g \triangleright b).$$

Thus (2) is achieved.

(3) Let $b_1, b_2 \in B$ with $b_1 \leq b_2$, and let $g \in G$. Since $b_1 \leq b_2$, we have

$$b_2 b_1 = b_1.$$

Using axiom (SS2), we obtain

$$\begin{aligned} g \triangleright b_1 &= g \triangleright (b_2 b_1) \\ &= (g \triangleright b_2)\big((g \triangleleft b_2) \triangleright b_1\big). \end{aligned}$$

This proves the first identity.

To prove the second identity, put

$$h = g \triangleleft b_2 \in G, \qquad x = g \triangleright b_1 \in B.$$

By (SS1) and (SS5),

$$g^{-1} \triangleright x = g^{-1} \triangleright (g \triangleright b_1) = (g^{-1}g) \triangleright b_1 = b_1.$$

Hence, again by (SS1),

$$(hg^{-1}) \triangleright x = h \triangleright (g^{-1} \triangleright x) = h \triangleright b_1.$$

Since B is a band, $x^2 = x$. Therefore, by (SS2),

$$\begin{aligned} h \triangleright b_1 &= (hg^{-1}) \triangleright x \\ &= (hg^{-1}) \triangleright (x^2) \\ &= \big((hg^{-1}) \triangleright x\big)\Big(\big((hg^{-1}) \triangleleft x\big) \triangleright x\Big). \end{aligned}$$

We now compute the group element $(hg^{-1}) \triangleleft x$. By (SS4),

$$\begin{aligned} (hg^{-1}) \triangleleft x &= \big(h \triangleleft (g^{-1} \triangleright x)\big)\big(g^{-1} \triangleleft x\big) \\ &= (h \triangleleft b_1)\big(g^{-1} \triangleleft (g \triangleright b_1)\big). \end{aligned}$$

Since $h = g \triangleleft b_2$ and $b_2 b_1 = b_1$, axiom (SS3) gives

$$h \triangleleft b_1 = (g \triangleleft b_2) \triangleleft b_1 = g \triangleleft (b_2 b_1) = g \triangleleft b_1.$$

Moreover, by part (2),

$$g^{-1} \triangleleft (g \triangleright b_1) = (g \triangleleft b_1)^{-1}.$$

Consequently,

$$(hg^{-1}) \triangleleft x = (g \triangleleft b_1)(g \triangleleft b_1)^{-1} = 1_G.$$

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It follows from (SS5) that

$$\begin{aligned} h \triangleright b_1 &= (h \triangleright b_1)\big(1_G \triangleright x\big) \\ &= (h \triangleright b_1)x. \end{aligned}$$

Substituting $h = g \triangleleft b_2$ and $x = g \triangleright b_1$ yields

$$(g \triangleleft b_2) \triangleright b_1 = \big((g \triangleleft b_2) \triangleright b_1\big)(g \triangleright b_1),$$

which is the second required identity.

Finally, if $g \triangleleft b_2 = 1_G$, then

$$(g \triangleleft b_2) \triangleright b_1 = 1_G \triangleright b_1 = b_1.$$

Therefore the second identity becomes

$$b_1 = b_1(g \triangleright b_1),$$

as required.

(4) Assume that B is a semilattice and let $b_1 \leq b_2$. Set

$$x = (g \triangleleft b_2) \triangleright b_1, \qquad y = g \triangleright b_1.$$

By part (3),

$$y = (g \triangleright b_2)x$$

and

$$x = xy.$$

Since B is a semilattice, its multiplication is commutative and idempotent. The equality $x = xy$ implies

$$x \leq y.$$

On the other hand,

$$y = (g \triangleright b_2)x = x(g \triangleright b_2),$$

so

$$y \leq x.$$

Hence, by antisymmetry of the natural order,

$$x = y.$$

Therefore,

$$(g \triangleleft b_2) \triangleright b_1 = g \triangleright b_1.$$

If, in addition, $g \triangleleft b_2 = 1_G$, then

$$(g \triangleleft b_2) \triangleright b_1 = 1_G \triangleright b_1 = b_1.$$

Consequently,

$$g \triangleright b_1 = b_1.$$

□

Remark 1. *Lemma 1 (1) shows that, when the second component is a group, axiom* (SS8) *is redundant under the stated compatibility assumptions: it follows from* $1_G \triangleright b = b$ *by cancellation in G. Thus, in this particular setting,* (SS8) *need not be imposed independently.*

We examine the arithmetic of Zappa–Szép products of bands and groups, focusing on how the structure of each component influences the product through their structure maps. A simple portmanteau construction introduces the structural analysis that follows. For each $b \in B$, define the stabilizer of b under the left structure map by

$$G_b = \{g \in G : g \triangleright b = b\}.$$

By (SS1) and (SS5), G_b is a subgroup of G. Indeed, $1_G \triangleright b = b$, and if $g, h \in G_b$, then

$$(gh) \triangleright b = g \triangleright (h \triangleright b) = g \triangleright b = b.$$

Moreover, if $g \in G_b$, then

$$g^{-1} \triangleright b = g^{-1} \triangleright (g \triangleright b) = (g^{-1}g) \triangleright b = b,$$

so $g^{-1} \in G_b$.

Proposition 2. *Let* $\wp = B \bowtie G$ *be a Zappa–Szép product of a band B and a group G, with structure maps*

$$\triangleright : G \times B \longrightarrow B, \qquad \triangleleft : G \times B \longrightarrow G.$$

For each $b \in B$, *define*

$$G_b = \{g \in G : g \triangleright b = b\}.$$

Then G_b *is a subgroup of G. Define*

$$\phi_b : G_b \longrightarrow G, \qquad \phi_b(g) = g \triangleleft b.$$

Then the following statements hold.

1. *For every* $b \in B$, *the map*
$$\phi_b : G_b \longrightarrow G$$
is a group homomorphism.
2. *Let* $b \in B$, $g \in G$, *and put*
$$c = g \triangleright b.$$
Then
$$G_c = gG_bg^{-1}.$$
Moreover, for every $h \in G_c$,
$$\phi_c(h) = (g \triangleleft b)\, \phi_b(g^{-1}hg)\, (g \triangleleft b)^{-1}.$$
3. *If* ϕ_b *is injective, then*
$$\phi_{g \triangleright b}$$
is injective for every $g \in G$.
4. For *each* $b \in B$, *define*
$$\rho_b : G \longrightarrow G, \qquad \rho_b(g) = g \triangleleft b.$$

Then the maps ρ_b are injective for all $b \in B$ if and only if

$$g \triangleleft b = 1_G \quad \Longrightarrow \quad g = 1_G$$

for every $g \in G$ and every $b \in B$.

Proof. First, we show that G_b is a subgroup of G. By (SS5),

$$1_G \triangleright b = b,$$

so $1_G \in G_b$. If $g, h \in G_b$, then by (SS1),

$$(gh) \triangleright b = g \triangleright (h \triangleright b) = g \triangleright b = b.$$

Thus $gh \in G_b$. Moreover, if $g \in G_b$, then

$$g^{-1} \triangleright b = g^{-1} \triangleright (g \triangleright b) = (g^{-1}g) \triangleright b = 1_G \triangleright b = b.$$

Hence $g^{-1} \in G_b$, and therefore $G_b \leq G$.
(1) Let $g, h \in G_b$. Since $h \triangleright b = b$, axiom (SS4) gives

$$\begin{aligned} \phi_b(gh) &= (gh) \triangleleft b \\ &= \big(g \triangleleft (h \triangleright b)\big)(h \triangleleft b) \\ &= (g \triangleleft b)(h \triangleleft b) \\ &= \phi_b(g)\phi_b(h). \end{aligned}$$

Also, byLemma 1 (1),

$$\phi_b(1_G) = 1_G \triangleleft b = 1_G.$$

Thus ϕ_b is a group homomorphism.
(2) Let

$$c = g \triangleright b.$$

For $h \in G$, we have

$$\begin{aligned} h \in G_c &\Longleftrightarrow h \triangleright c = c \\ &\Longleftrightarrow h \triangleright (g \triangleright b) = g \triangleright b \\ &\Longleftrightarrow (g^{-1}hg) \triangleright b = b \\ &\Longleftrightarrow g^{-1}hg \in G_b. \end{aligned}$$

Consequently,

$$G_c = gG_bg^{-1}.$$

Now let $h \in G_c$ and put

$$k = g^{-1}hg.$$

Then $k \in G_b$, and hence

$$k \triangleright b = b.$$

Since $h = gkg^{-1}$, repeated use of (SS4) gives

$$\begin{aligned} \phi_c(h) &= h \triangleleft c \\ &= (gkg^{-1}) \triangleleft (g \triangleright b) \\ &= (g \triangleleft b)(k \triangleleft b)\big(g^{-1} \triangleleft (g \triangleright b)\big). \end{aligned}$$

By Lemma 1 (2),

$$g^{-1} \triangleleft (g \triangleright b) = (g \triangleleft b)^{-1}.$$

Therefore,

$$\begin{aligned} \phi_c(h) &= (g \triangleleft b)(k \triangleleft b)(g \triangleleft b)^{-1} \\ &= (g \triangleleft b)\, \phi_b(k)\, (g \triangleleft b)^{-1} \\ &= (g \triangleleft b)\, \phi_b(g^{-1}hg)\, (g \triangleleft b)^{-1}. \end{aligned}$$

(3) Suppose that ϕ_b is injective, and put

$$c = g \triangleright b.$$

Let $h_1, h_2 \in G_c$ and suppose that

$$\phi_c(h_1) = \phi_c(h_2).$$

By part (2),

$$(g \triangleleft b)\phi_b(g^{-1}h_1g)(g \triangleleft b)^{-1} = (g \triangleleft b)\phi_b(g^{-1}h_2g)(g \triangleleft b)^{-1}.$$

Canceling in G gives

$$\phi_b(g^{-1}h_1g) = \phi_b(g^{-1}h_2g).$$

Since

$$g^{-1}h_1g,\ g^{-1}h_2g \in G_b$$

and ϕ_b is injective, we obtain

$$g^{-1}h_1g = g^{-1}h_2g.$$

Hence

$$h_1 = h_2.$$

Therefore ϕ_c is injective.
(4) Suppose first that ρ_b is injective for every $b \in B$. Let $g \in G$ and $b \in B$ satisfy

$$g \triangleleft b = 1_G.$$

By Lemma 1 (1),

$$1_G \triangleleft b = 1_G.$$

Hence

$$g \triangleleft b = 1_G \triangleleft b.$$

Since ρ_b is injective, it follows that

$$g = 1_G.$$

Conversely, suppose that

$$g \triangleleft b = 1_G \quad \Longrightarrow \quad g = 1_G$$

for every $g \in G$ and $b \in B$.

Fix $b \in B$ and let $g, h \in G$ satisfy

$$g \triangleleft b = h \triangleleft b.$$

Set

$$d = h \triangleright b \in B.$$

Using axiom (SS4), we obtain

$$(gh^{-1}) \triangleleft d = \big(g \triangleleft (h^{-1} \triangleright d)\big)\big(h^{-1} \triangleleft d\big).$$

Since

$$d = h \triangleright b,$$

axiom (SS1) gives

$$h^{-1} \triangleright d = h^{-1} \triangleright (h \triangleright b) = (h^{-1}h) \triangleright b = 1_G \triangleright b = b.$$

Moreover, by Lemma 1 (2),

$$h^{-1} \triangleleft (h \triangleright b) = (h \triangleleft b)^{-1}.$$

Therefore,

$$\begin{aligned}(gh^{-1}) \triangleleft d &= (g \triangleleft b)(h \triangleleft b)^{-1} \\ &= 1_G.\end{aligned}$$

By the assumed condition, we obtain

$$gh^{-1} = 1_G,$$

and hence

$$g = h.$$

Thus ρ_b is injective. Since $b \in B$ was arbitrary, every map ρ_b is injective. □

Our main structural results are that a Zappa–Szép product of a band and a group is an orthodox semigroup (Theorem 3) and inverse if and only if we have a semidirect product (Theorem 2). As a first step in this direction we have:

Lemma 2. *The Zappa–Szép product $\wp = B \triangleleft \triangleright G$ of a band B and a group G is a regular semigroup.*

Proof. We prove that for each $(b, g) \in \wp = B \triangleleft \triangleright G$, $\big(g^{-1} \triangleright b, g^{-1} \triangleleft b\big)$ is an inverse of (b, g). We calculate by $(SS1)$ and $(SS4)$

$$\begin{aligned}&(b, g)\Big(g^{-1} \triangleright b, g^{-1} \triangleleft b\Big)(b, g) \\ &= \Big(b \cdot \Big(g \triangleright (g^{-1} \triangleright b)\Big), (g \triangleleft \Big(g^{-1} \triangleright b\Big)) \cdot \Big(g^{-1} \triangleleft b\Big)\Big)(b, g) \\ &= \Big(b \cdot \Big(gg^{-1} \triangleright b\Big), \Big(gg^{-1}\Big) \triangleleft b\Big)(b, g) \\ &= (b, 1)(b, g) \\ &= (b \cdot (1 \triangleright b), (1 \triangleleft b) \cdot g) = (b, g).\end{aligned}$$

and since $(b, g)\big(g^{-1} \triangleright b, g^{-1} \triangleleft b\big) = (b, 1)$, we have by $(SS2)$ and $(SS3)$

$$\begin{aligned}&\Big(g^{-1} \triangleright b, g^{-1} \triangleleft b\Big)(b, g)\Big(g^{-1} \triangleright b, g^{-1} \triangleleft b\Big) \\ &= \Big(g^{-1} \triangleright b, g^{-1} \triangleleft b\Big)(b, 1) \\ &= \Big(\Big(g^{-1} \triangleright b\Big)\Big(\Big(g^{-1} \triangleleft b\Big) \triangleright b\Big), \Big(\Big(g^{-1} \triangleleft b\Big) \triangleleft b\Big) \cdot 1\Big) \\ &= \Big(g^{-1} \triangleright b^2, g^{-1} \triangleleft b^2\Big) \\ &= \Big(g^{-1} \triangleright b, g^{-1} \triangleleft b\Big).\end{aligned}$$

Then (b,g) is regular element and $\left(g^{-1} \triangleright b, g^{-1} \triangleleft b\right) \in V(b,g)$. Thus $\wp = B \triangleleft \triangleright G$ is regular. □

Remark 2. *The class of E-unitary semigroups is one of the most important in inverse semigroup theory. The simplest examples of E-unitary inverse semigroups are the semidirect products of bands by groups and of course they are also Zappa–Szép product of bands by groups.*

Question: *Are there other Zappa–Szép product of bands by groups that are inverse and E-unitary? We shall see (Theorem 2) that the answer is no.*

Lemma 3. *If Zappa–Szép product $\wp = B \triangleleft \triangleright G$ of a band B and a group G. Then the subset of idempotents is $E(\wp) = \{(b,g) : g \triangleleft b = 1\}$.*

Proof. Suppose that (b,g) is an idempotent. Then

$$(b,g)^2 = (b \cdot (g \triangleright b), (g \triangleleft b) \cdot g) = (b,g).$$

Thus $(b,g) \in E(\wp)$ if and only if $b \cdot (g \triangleright b) = b$ and $g \triangleleft b = 1$. By Lemma 1 part (3), if $g \triangleleft b = 1$ then $b \cdot (g \triangleright b) = b$. Therefore

$$E(\wp) = \{(b,g) : g \triangleleft b = 1\}.$$

□

Theorem 2. *Let $\wp = B \triangleleft \triangleright G$ of a band B and a group G. Then $\wp = B \triangleleft \triangleright G$ is inverse if and only if B has a trivial structure map on G, so that $\wp$ is the semidirect product $\wp = B \rtimes G$. Hence if $\wp = B \triangleleft \triangleright G$ is inverse then it is E−unitary.*

Proof. Let $\wp = B \triangleleft \triangleright G$ is inverse Zappa–Szép products. Then for all $b \in B$ and $g \in G$ we have by Lemma 1 parts (4) and (1):

$$\begin{aligned} \left(g^{-1} \cdot (g \triangleleft b)\right) \triangleleft b &= \left(g^{-1}\right) \triangleleft ((g \triangleleft b) \triangleright b) \cdot ((g \triangleleft b) \triangleleft b) \\ &= \left(g^{-1}\right) \triangleleft (g \triangleright b) \cdot (g \triangleleft b) = \left(g^{-1} g\right) \triangleleft b = 1 \triangleleft b = 1 \end{aligned}$$

Thus $g^{-1} \cdot (g \triangleleft b) = 1$, so that $g = g \triangleleft b$. Hence B has a trivial structure map on G, so that $\wp$ is the semidirect product $\wp = B \triangleleft G$. □

Remark 3. *We conclude that B has a trivial structure map on G if and only if B is closed under conjugation, in this case we have a semidirect product. Suppose $\wp$ is the uniquely factorisable product of a submonoid of idempotents $E \subseteq E(\wp)$; subgroup $G \subseteq \wp$. So each element in $\wp$ is uniquely expressible as bg and so we write $gb = (g \triangleright b)(g \triangleleft b) \in \wp$. But $gb = (gbg^{-1})g$. So if B is closed under conjugation by G, then $g \triangleright b = gbg^{-1}$ and $g \triangleleft b = g$. So B has a trivial structure map on G and $\wp$ is the semidirect product $\wp = B \triangleleft G$. Now consider Zappa–Szép products $\wp$ of a band B (B has an identity 1 acting trivially on G) and a group G, then $\wp$ contains subsemigroups subsemillatice $\bar{B} = \{(b,1) : b \in B\} \cong B$; subgroup $\bar{G} = \{(1,g) : g \in G\} \cong G$. and each element of $\wp$ is uniquely expressible as $(b,g) = (b,1)(1,g)$. So $\wp$ is uniquely factorizable products of $\bar{B}$ and $\bar{G}$. But $\bar{B}$ is not necessarily closed under conjugation by $\bar{G}$ since*

$$(1,g)(b,1)(1,g)^{-1} = (g \triangleright b, g \triangleleft b)\left(1, g^{-1}\right) = \left(g \triangleright b, (g \triangleleft b)g^{-1}\right) \notin \bar{B} \text{ if } g \triangleleft b \neq g.$$

Relation to Szendrei's full restricted semidirect products.

The recent work of Szendrei [31] provides a useful comparison with the preceding semidirect product specialization. Szendrei's full restricted semidirect product is defined

for inverse semigroups K and T from an action of T on K by endomorphisms together with a surjective homomorphism

$$\epsilon : K \longrightarrow E(T)$$

satisfying the full-restriction condition

$$e \cdot a = a \quad \Longleftrightarrow \quad \epsilon(a) \leq e, \qquad e \in E(T),\ a \in K.$$

The corresponding product is defined on

$$K \bowtie T = \{(a,t) \in K \times T : \epsilon(a) = \mathbf{r}(t)\},$$

with multiplication

$$(a,t)(b,u) = \big(a(t \cdot b), tu\big).$$

The relation with the matched-pair formulation used in the present paper becomes particularly transparent in the semidirect product specialization. If the right Zappa–Szép action is trivial,

$$g \triangleleft b = g,$$

then the multiplication in $B \bowtie G$ reduces to

$$(b,g)(c,h) = \big(b(g \triangleright c), gh\big),$$

so that only the left action $\triangleright$ remains. This is exactly the ordinary semidirect product form. Moreover, the λ-condition

$$g^{-1} \triangleleft b = g^{-1}$$

is then automatically satisfied. Thus, when the band B is a semilattice E, and hence is itself an inverse semigroup, this semidirect specialization is compatible with Szendrei's framework with

$$K = E, \qquad T = G, \qquad g \cdot e = g \triangleright e.$$

Since G is a group, $E(G) = \{1_G\}$, and the full-restriction condition reduces to the usual group action situation.

The genuinely two-sided case considered in the present paper is, however, different. In general the right action $\triangleleft$ is nontrivial, and the multiplication contains the factor

$$(g \triangleleft c)h.$$

Szendrei's full restricted semidirect product does not contain such an independent right action. Consequently, the general matched-pair arithmetic of Section 2 should not be identified with a full restricted semidirect product.

Nevertheless, Szendrei's construction indicates how the present notation can be adapted in the one-sided case. For a general inverse semigroup T, one retains the left action by setting

$$t \triangleright a = t \cdot a,$$

takes the right action to be trivial,

$$t \triangleleft a = t,$$

and replaces the unrestricted Cartesian product by the restricted domain

$$\{(a,t) : \epsilon(a) = \mathbf{r}(t)\}.$$

The additional compatibility required in this setting is therefore Szendrei's full-restriction condition on ϵ and the action, rather than an additional matched-pair identity. Extending the genuinely two-sided Zappa–Szép construction to a corresponding "full restricted" matched-pair product for two inverse semigroups would require new compatibility axioms and is left for future work.

Preservation of the semidirect specialization under the ESN construction.

The preceding comparison also clarifies the extent to which the ESN-based construction preserves the semidirect products occurring in Szendrei's framework. Assume now that $B = E$ is a semilattice and that the right matched-pair action is trivial, that is,

$$g \triangleleft e = g \qquad (g \in G,\ e \in E).$$

Then the Zappa–Szép multiplication reduces to

$$(e, g)(f, h) = \big(e(g \triangleright f), gh\big),$$

which is precisely the usual semidirect product multiplication associated with the action of G on E.

Moreover, in this case the defining λ-condition is automatic, since

$$g^{-1} \triangleleft e = g^{-1}$$

for every $e \in E$ and $g \in G$. Consequently,

$$\mathrm{B}_{\triangleleft\triangleright}(P) = P = E \rtimes G.$$

Substituting the trivial right action into the pseudoproduct multiplication of Theorem 4 gives

$$(e, g) \otimes (f, h) = \big(e(g \triangleright f), gh\big).$$

Hence the ESN construction does not alter the semidirect product specialization: the pseudoproduct reconstructed from the inductive groupoid is exactly the original semidirect product multiplication.

This agrees with the group-factor specialization of Szendrei's full restricted semidirect product. Indeed, when the acting inverse semigroup is the group G, one has

$$E(G) = \{1_G\}.$$

Thus the relevant homomorphism to $E(G)$ is necessarily

$$\epsilon : E \longrightarrow E(G), \qquad \epsilon(e) = 1_G,$$

and the full-restriction condition reduces to the ordinary group action situation. Therefore the semidirect product $E \rtimes G$ obtained above is compatible with Szendrei's construction.

It is important, however, not to identify the general λ-Zappa–Szép product with a full restricted semidirect product. When the right action $\triangleleft$ is nontrivial, the Zappa–Szép multiplication contains the term

$$(g \triangleleft f)h$$

in its second component rather than gh. Thus the natural projection onto G is not, in general, the homomorphism occurring in an ordinary or full restricted semidirect product. Consequently, there is no general inclusion of the two-sided λ-Zappa–Szép products considered here into Szendrei's class merely from the defining matched-pair axioms.

Accordingly, the precise overlap is the semidirect specialization $\triangleleft$ trivial. A representation of a genuinely two-sided λ-Zappa–Szép product within Szendrei's normal-extension framework would require additional hypotheses ensuring the relevant normal-extension and full-restriction conditions, and is not asserted in the present paper.

Theorem 3. *A Zappa–Szép product $\wp = B \triangleleft \triangleright G$ of a band B and a group G is an orthodox semigroup.*

Proof. By Lemma 2 $\wp$ is regular and by Proposition 3

$$E(\wp) = \{(b, g) : g \triangleleft b = 1\}.$$

Thus it remains to prove that $E(\wp)$ is a subsemigroup of $\wp$. Suppose that (b_1, g_1) and (b_2, g_2) are idempotents in $\wp$. Then

$$(b_1, g_1)(b_2, g_2) = (b_1 \cdot (g_1 \triangleright b_2), (g_1 \triangleleft b_2) \cdot g_2)$$

is an idempotent if and only if

$$((g_1 \triangleleft b_2) \cdot g_2) \triangleleft (b_1 \cdot (g_1 \triangleright f)) = 1.$$

We calculate

$$\begin{aligned} \left((g_1 \triangleleft b_2) \cdot g_2 \cdot (g_1)^{-1}\right) \triangleleft b_1 &= \left(((g_1 \triangleleft b_2) \cdot g_2) \triangleleft \left((g_1)^{-1} \triangleright b_1\right).(g_1)^{-1} \triangleleft b_1\right) \\ &= ((g_1 \triangleleft b_2).g_2) \triangleleft b_1. \end{aligned}$$

Using Lemma 1 part (4) we have that $g \triangleright b = b$, since $g \triangleleft b = 1$, and so $g^{-1} \triangleright b = b$. We now have that $\left(g^{-1}\right) \triangleleft b = (g^{-1}) \triangleleft \left(g^{-1} \triangleright b\right) = (g \triangleleft b)^{-1} = 1$. Using Lemma 1 part (2). Then calculate by $(SS4)$

$$\begin{aligned} &\left((g_1 \triangleleft b_2) \cdot g_2 \cdot g_1^{-1}\right) \triangleleft (g_1 \triangleright b_2) \\ &= ((g_1 \triangleleft b_2) \cdot g_2) \triangleleft \left(g_1^{-1} \triangleright (g_1 \triangleright b_2)\right) \cdot \left(g_1^{-1} \triangleleft (g_1 \triangleright b_2)\right) \\ &= ((g_1 \triangleleft b_2) \cdot g_2) \triangleleft b_2 \cdot \left(g_1^{-1} \triangleleft (g_1 \triangleright b_2)\right) \\ &= (g_1 \triangleleft b_2)(g_2 \triangleleft b_2)\left(g_1^{-1} \triangleleft (g_1 \triangleright b_2)\right) \\ &= (g_1 \triangleleft b_2)(g_2 \triangleleft b_2)(g_1 \triangleleft b_2)^{-1} = 1. \end{aligned}$$

Therefore

$$\begin{aligned} ((g_1 \triangleleft b_2) \cdot g_2) \triangleleft (b_1 \cdot (g_1 \triangleright b_2)) &= [((g_1 \triangleleft b_2) \cdot g_2) \triangleleft b_1] \triangleleft (g_1 \triangleright b) \\ &= \left((g_1 \triangleleft b_2) \cdot g_2 \cdot g_1^{-1}\right) \triangleleft (b_1 \cdot (g_1 \triangleright b_2)) \\ &= \left((g_1 \triangleleft b_2) \cdot g_2 \cdot g_1^{-1}\right) \triangleleft ((g_1 \triangleright b_2) \cdot b_1) \\ &= \left(\left((g_1 \triangleleft b_2) \cdot g_2 \cdot g_1^{-1}\right) \triangleleft (g_1 \triangleright b_2)\right) \triangleleft b_1 \\ &= 1 \triangleleft b_1 = 1. \end{aligned}$$

Thus $\wp = B \triangleleft \triangleright G$ is orthodox monoid. □

Proposition 3. *If $\wp = B \triangleleft \triangleright G$, where B is a band with identity $1 \in B$, then $\wp$ is a unit-regular monoid.*

Proof. We first compute the group of units $U(\wp)$. If (b_1, g_1) is a unit with inverse (b_2, g_2). Then$(b_1, g_1)(b_2, g_2) = (b_2, g_2)(b_1, g_1) = (1_B, 1_G)$. Therefore

$$(b_1 \cdot (g_1 \triangleright b_2), (g_1 \triangleleft b_2) \cdot g_2) = (b_2 \cdot (g_2 \triangleright b_1), (g_2 \triangleleft b_1)g_1) = (1_B, 1_G).$$

Hence $b_1 \cdot (g_1 \triangleright b_2) = b_2 \cdot (g_2 \triangleright b_1) = 1_B$, and $(g_1 \triangleleft b_2) \cdot g_2 = (g_2 \triangleleft b_1) \cdot g_1 = 1_G$. Therefore $b_1 = 1_B$, and $g_1 \cdot g_2 = 1_G$ which implies $g_2 = g_1^{-1}$. Thus $U(\wp) \subseteq \{(1_B, g) : g \in G\}$. Conversely, $\{(1_B, g) : g \in G\} \subseteq U(\wp)$, since $\left(1_B, g^{-1}\right) \in \{(1_B, g) : g \in G\}$ such that $(1_B, g)\left(1_B, g^{-1}\right) = \left(1_B, g^{-1}\right)(1_B, g) = (1_B, 1_G)$ for each $(1_B, g)$. Thus $U(\wp) = \{(1_B, g) : g \in G\}$. Now let $(b, g) \in U(\wp)$. Then there exists a unit $\left(1_B, g^{-1}\right) \in U(\wp)$ such that

$$\begin{aligned}(b, g)\left(1_B, g^{-1}\right)(b, g) &= \left(b \cdot (g \triangleright 1_B), (g \triangleleft 1_B).g^{-1}\right)(b, g) = (b, 1_G)(b, g) \\ &= (b \cdot (1_G \triangleright b), (1_G \triangleleft b).g) = (b, g).\end{aligned}$$

Thus $\wp = B \triangleleft \triangleright G$ is a unit-regular monoid. □

Proposition 4. *$\wp = B \triangleleft \triangleright G$, where B is a band with identity $1 \in B$ and G is a group, is covered by $E(\wp) \rtimes G$.*

Proof. We have

$$E(\wp) = \{(e, g) : g \triangleleft e = 1\}.$$

The group G acts on $E(\wp)$ by conjugation in $\wp$; that is, for each $x \in G$ and $(e, g) \in E(\wp)$ we define

$$x * (e, g) = (1, x)(e, g)\left(1, x^{-1}\right).$$

Using the Zappa–Szép multiplication with the actions $\triangleright$ and $\triangleleft$, we obtain

$$\begin{aligned}x * (e, g) &= (1, x)(e, g)\left(1, x^{-1}\right) \\ &= (x \triangleright e, (x \triangleleft e)g)\left(1, x^{-1}\right) \\ &= \left((x \triangleright e)(((x \triangleleft e)g) \triangleright 1), ((x \triangleleft e)g) \triangleleft 1\ x^{-1}\right) \\ &= \left(x \triangleright e, (x \triangleleft e)gx^{-1}\right).\end{aligned}$$

This element belongs to $E(\wp)$, since

$$\begin{aligned}\left((x \triangleleft e)gx^{-1}\right) \triangleleft (x \triangleright e) &= ((x \triangleleft e)g) \triangleleft \left(x^{-1} \triangleright (x \triangleright e)\right)\ \left(x^{-1} \triangleleft (x \triangleright e)\right) \\ &= (x \triangleleft (e(g \triangleright e)))(g \triangleleft e)\left(x^{-1} \triangleleft (x \triangleright e)\right) \\ &= (x \triangleleft e)\left(x^{-1} \triangleleft (x \triangleright e)\right) \\ &= \left(xx^{-1}\right) \triangleleft (x \triangleright e) \\ &= 1.\end{aligned}$$

Thus, we may form the semidirect product $E(\wp) \rtimes G$ of the semigroup $E(\wp)$ by the group G, where

$$\begin{aligned}((e, g), x)((f, h), y) &= ((e, g)(x * (f, h)), xy) \\ &= \left((e, g)\left(x \triangleright f, (x \triangleleft f)hx^{-1}\right), xy\right)\end{aligned}$$

$$= \quad \Big(\Big(e((gx)\triangleright f), (g\triangleleft(x\triangleright f))(x\triangleleft f)hx^{-1}\Big), xy\Big).$$

Define

$$\theta : E(\wp)\rtimes G \to B\triangleleft\triangleright G, \qquad \theta((e,g),x) = (e,gx).$$

Then, θ is a covering map; namely, θ is a surjective homomorphism and an isomorphism on idempotents. It is clear that θ is surjective. The idempotents in $E(\wp)\rtimes G$ are precisely of the form $((e,g),1)$, and

$$\theta((e,g),1) = (e,g),$$

so θ is bijective on idempotents. Moreover, θ is a semigroup homomorphism, since

$$\begin{aligned}\theta((e,g),x)((f,h),y) &= \theta\Big(\Big(e((gx)\triangleright f), (g\triangleleft(x\triangleright f))(x\triangleleft f)hx^{-1}\Big), xy\Big)\\ &= \Big(e((gx)\triangleright f), (g\triangleleft(x\triangleright f))(x\triangleleft f)hx^{-1}xy\Big)\\ &= (e((gx)\triangleright f), (g\triangleleft(x\triangleright f))(x\triangleleft f)hy)\end{aligned}$$

and

$$\begin{aligned}\theta((e,g),x)\theta((f,h),y) &= (e,gx)(f,hy)\\ &= (e((gx)\triangleright f), (g\triangleleft(x\triangleright f))(x\triangleleft f)hy).\end{aligned}$$

Therefore $E(\wp)\rtimes G$ covers $\wp = B\triangleleft\triangleright G$. □

Remark 4. *However, we get the same results from the following two facts due to MacFadden: for every orthodox unit regular monoid P there is an orthodox uniquely unit regular monoid F and a surjective idempotent separating homomorphism $\theta : F \to S$; thus every orthodox unit regular monoid has an orthodox uniquely unit regular idempotent separating cover and the second fact says that every orthodox, uniquely unit regular monoid is isomorphic to the semidirect product of a band (with identity) and a group and from Theorem 3, Proposition 3.*

We now consider the structure of Zappa–Szép products of semilattices and groups that are not necessarily semidirect products $E\rtimes G$. The following consolidated proposition records the inverse-set formula and the main consequences needed later. This presentation avoids separating closely related facts into several short corollaries and makes the logical dependence of the results clearer.

Proposition 5. *Let $P = B\bowtie G$, where B is a band and G is a group. Then, for every $(e,g)\in P$,*

$$V((e,g)) = \Big\{\Big(g^{-1}\triangleright e, s\Big) : s\triangleleft e = g^{-1}\triangleleft e\Big\}. \tag{1}$$

If, in addition, $B = E$ is a semilattice, then the following hold.

1. *If $(e,g)\in E(P)$, then*
$$V((e,g)) = \{(e,s) : s\triangleleft e = 1\}. \tag{2}$$
2. *If $(e,g)\in E(P)$, then for every $f\in E$,*
$$ef = e(g\triangleright f). \tag{3}$$
3. *The product $P = E\bowtie G$ is orthodox.*
4. *The idempotent-generated part satisfies*
$$E(P)\cong E\ltimes G, \tag{4}$$

where the multiplication is induced by

$$(e,g)(f,h) = (ef, (g \triangleleft f)h), \qquad (e,g),(f,h) \in E(P).$$

5. *The product $P = E \bowtie G$ is $\mathcal{L}$-unipotent.*

Proof. We first prove (1). Let $(p,s) \in V((e,g))$. Then,

$$(e,g) = (e,g)(p,s)(e,g) = (e(g \triangleright p)((g \triangleleft p)s \triangleright e), (((g \triangleleft p)s) \triangleleft e)g).$$

Thus

$$e = e(g \triangleright p)((g \triangleleft p)s \triangleright e), \qquad ((g \triangleleft p)s) \triangleleft e = 1.$$

By Lemma 1 part (4), this implies

$$e = e(g \triangleright p).$$

Similarly,

$$(p,s) = (p,s)(e,g)(p,s) = (p(s \triangleright e)((s \triangleleft e)g \triangleright p), (((s \triangleleft e)g) \triangleleft p)s),$$

and hence

$$\begin{aligned} p &= p(s \triangleright e)((s \triangleleft e)g \triangleright p), \\ s &= (((s \triangleleft e)g) \triangleleft p)s. \end{aligned}$$

Therefore $((s \triangleleft e)g) \triangleleft p = 1$ and $p = p(s \triangleright e)$. Moreover,

$$1 = ((g \triangleleft p)s) \triangleleft e = (g \triangleleft (p(s \triangleright e)))(s \triangleleft e) = (g \triangleleft p)(s \triangleleft e),$$

so that

$$(g \triangleleft p)^{-1} = s \triangleleft e.$$

Consequently, $(p,s) \in V((e,g))$ if and only if

$$e = e(g \triangleright p), \qquad p = p(s \triangleright e), \qquad (g \triangleleft p)^{-1} = s \triangleleft e.$$

Using Lemma 1 parts (2) and (4), we obtain

$$\begin{aligned} g^{-1} \triangleright e &= g^{-1} \triangleright (e(g \triangleright p)) \\ &= g^{-1} \triangleright ((g \triangleright p)e) \\ &= p\Big(\Big(g^{-1} \triangleleft (g \triangleright p)\Big) \triangleright e\Big) \\ &= p\Big((g \triangleleft p)^{-1} \triangleright e\Big) \\ &= p((s \triangleleft e) \triangleright e) \\ &= p(s \triangleright e) = p. \end{aligned}$$

Also,

$$s \triangleleft e = (g \triangleleft p)^{-1} = g^{-1} \triangleleft (g \triangleright p) = g^{-1} \triangleleft e.$$

Thus,

$$V((e,g)) \subseteq \Big\{(g^{-1} \triangleright e, s) : s \triangleleft e = g^{-1} \triangleleft e\Big\}.$$

Conversely, suppose that $s \triangleleft e = g^{-1} \triangleleft e$. Then

$$s \triangleright e = (s \triangleleft e) \triangleright e = (g^{-1} \triangleleft e) \triangleright e = g^{-1} \triangleright e.$$

Furthermore,

$$1 = (gg^{-1}) \triangleleft e = \Big(g \triangleleft (g^{-1} \triangleright e)\Big)(g^{-1} \triangleleft e) = \Big(g \triangleleft (g^{-1} \triangleright e)\Big)(s \triangleleft e).$$

It follows that

$$(g^{-1} \triangleright e, s)(e, g) = \Big((g^{-1} \triangleright e)(s \triangleright e), (s \triangleleft e)g\Big) = \Big(g^{-1} \triangleright e, (s \triangleleft e)g\Big).$$

Hence

$$(e, g)(g^{-1} \triangleright e, s)(e, g) = (e, g),$$

and similarly

$$(g^{-1} \triangleright e, s)(e, g)(g^{-1} \triangleright e, s) = (g^{-1} \triangleright e, s).$$

This proves (1).

Now assume that $B = E$ is a semilattice. If $(e, g) \in E(P)$, then $g \triangleleft e = 1$ and $g \triangleright e = e$ by Lemma 1 part (4). Moreover,

$$1 = (g^{-1}g) \triangleleft e = (g^{-1} \triangleleft (g \triangleright e))(g \triangleleft e) = (g^{-1} \triangleleft e)(g \triangleleft e),$$

so $g^{-1} \triangleleft e = 1$. Formula (2) follows from (1).

For $(e, g) \in E(P)$ and $f \in E$, since $ef \leq e$, Lemma 1 gives

$$g \triangleleft (ef) = (g \triangleleft e) \triangleleft f = 1.$$

Then

$$\begin{aligned} ef &= 1 \triangleright (ef) \\ &= (g \triangleleft (ef)) \triangleright (ef) \\ &= g \triangleright (ef) \\ &= g \triangleright (fe) \\ &= (g \triangleright f)((g \triangleleft f) \triangleright e) \\ &= (g \triangleright f)((g \triangleleft (fe)) \triangleright e) \\ &= (g \triangleright f)((g \triangleleft (ef)) \triangleright e) \\ &= (g \triangleright f)e = e(g \triangleright f), \end{aligned}$$

which proves (3).

Let $(e, g), (f, h) \in E(P)$. By (3),

$$(e, g)(f, h) = (e(g \triangleright f), (g \triangleleft f)h) = (ef, (g \triangleleft f)h).$$

This product is idempotent, because

$$\begin{aligned} ((g \triangleleft f)h) \triangleleft (ef) &= ((g \triangleleft f)h) \triangleleft (fe) \\ &= (g \triangleleft (f(h \triangleright f)))(h \triangleleft f) \triangleleft e \\ &= (g \triangleleft f) \triangleleft e \\ &= (g \triangleleft e) \triangleleft f = 1. \end{aligned}$$

Thus, P is orthodox, and the displayed multiplication gives $E(P) \cong E \ltimes G$.

Finally, let $(e,g) \in P$ and suppose that $(g^{-1} \triangleright e, s)$ and $(g^{-1} \triangleright e, t)$ both belong to $V((e,g))$. By (1),

$$s \triangleleft e = g^{-1} \triangleleft e = t \triangleleft e.$$

We compute

$$\begin{aligned}(g^{-1} \triangleright e, s)(e,g) &= \Big((g^{-1} \triangleright e)(s \triangleright e), (s \triangleleft e)g\Big) \\ &= \Big((g^{-1} \triangleright e)((s \triangleleft e) \triangleright e), (s \triangleleft e)g\Big) \\ &= \Big((g^{-1} \triangleright e)((g^{-1} \triangleleft e) \triangleright e), (g^{-1} \triangleleft e)g\Big).\end{aligned}$$

The same expression is obtained with t in place of s. Hence $(g^{-1} \triangleright e, s)(e,g) = (g^{-1} \triangleright e, t)(e,g)$, proving that P is $\mathcal{L}$-unipotent. □

2.1. λ–Zappa–Szép Product of Band and Group

In this section, we construct—from the Zappa–Szép product $\wp = B \triangleleft \triangleright G$ of a band B and a group G—an inverse semigroup by constructing an inductive groupoid.

Lemma 4. *If* $(b,g) \in \mathbf{B}_{\triangleleft\triangleright}(\wp)$, *then* $g = g \triangleleft \left(g^{-1} \triangleright b\right)$.

Proof. From the assumption that $(b,g) \in \mathbf{B}_{\triangleleft\triangleright}(\wp)$, then $\left(g^{-1}\right) \triangleleft b = g^{-1}$.Now consider the following calculation: $1 = 1 \triangleleft b = \left(gg^{-1}\right) \triangleleft b = g \triangleleft \left(g^{-1} \triangleright b\right) \cdot g^{-1} \triangleleft b$. But since $\left(g^{-1}\right) \triangleleft b = g^{-1}$ by assumption we get: $1 = g \triangleleft \left(g^{-1} \triangleright b\right) \cdot g^{-1}$. Now multiply both sides on the right by g, which is invertable: $g = g \triangleleft \left(g^{-1} \triangleright b\right)$. This concludes the proof. □

Lemma 5. *Suppose that* $(b,g) \in \mathbf{B}_{\triangleleft\triangleright}(\wp)$. *Then*

$$\left(g^{-1} \triangleright b, g^{-1} \triangleleft b\right) \in V(b,g),$$

that is, $\left(g^{-1} \triangleright b, g^{-1} \triangleleft b\right)$ *is the inverse of* (b,g) *and* $\mathbf{B}_{\triangleleft\triangleright}(\wp)$ *is regular.*

Proof. First we compute:

$$\begin{aligned}&(b,g)\left(g^{-1} \triangleright b, g^{-1} \triangleleft b\right)(b,g) \\ &= \left(b \cdot \left(g \triangleright \left(g^{-1} \triangleright b\right)\right), \left(g \triangleleft \left(g^{-1} \triangleright b\right)\right) \cdot \left(g^{-1} \triangleleft b\right)\right)(b,g) \\ &= \left(b \cdot \left(gg^{-1} \triangleright b\right), gg^{-1} \triangleleft b\right)(b,g) \\ &= \left(b, \left(gg^{-1} \triangleleft b\right)\right)(b,g) \\ &= (b,1)(b,g) \\ &= (b \cdot (1 \triangleright b), (1 \triangleleft b) \cdot g) = (b,g).\end{aligned}$$

Second we verify

$$\left(g^{-1} \triangleright b, g^{-1} \triangleleft b\right)(b,g)\left(g^{-1} \triangleright b, g^{-1} \triangleleft b\right) = \left(g^{-1} \triangleright b, g^{-1} \triangleleft b\right).$$

We compute

$$\begin{aligned}&\left(g^{-1} \triangleright b, g^{-1} \triangleleft b\right)(b,g)\left(g^{-1} \triangleright b, g^{-1} \triangleleft b\right) \\ &= \left(\left(g^{-1} \triangleright b\right) \cdot \left(\left(g^{-1} \triangleleft b\right) \triangleright b\right), \left((g^{-1} \triangleleft b) \triangleleft b\right) \cdot g\right)\left(g^{-1} \triangleright b, g^{-1} \triangleleft b\right)\end{aligned}$$

$$
\begin{aligned}
&= \left(\left(g^{-1} \triangleright b\right) \cdot \left(\left(g^{-1} \triangleleft b\right) \triangleright \left(g^{-1} \triangleright b\right)\right), \left((g^{-1} \triangleleft b) \cdot g\right) \triangleleft \left(g^{-1} \triangleright b\right) \cdot \left(g^{-1} \triangleleft b\right)\right) \\
&= \left(\left(g^{-1} \triangleright b\right) \cdot \left((g \triangleleft b) \triangleright \left(g^{-1} \triangleright b\right)\right), \left((g^{-1} \triangleleft b) \cdot g\right) \triangleleft \left(g^{-1} \triangleright b\right) \cdot \left(g^{-1} \triangleleft b\right)\right) \\
&= \left(g^{-1} \triangleright (bb), \left(\left(g^{-1} \triangleleft b\right) \triangleleft (1 \triangleright b)\right) \cdot \left(g \triangleleft \left(g^{-1} \triangleright b\right)\right) \cdot \left(g^{-1} \triangleleft b\right)\right) \\
&= \left(g^{-1} \triangleright b, \left(g^{-1} \triangleleft b\right)\left(gg^{-1} \triangleleft b\right)\right) \\
&= \left(g^{-1} \triangleright b, g^{-1} \triangleleft b\right).
\end{aligned}
$$

Therefore, $\left(g^{-1} \triangleright b, g^{-1} \triangleleft b\right)$ is the invese of (b, g), and $\mathbf{B}_{\triangleleft\triangleright}(\wp)$ is regular. This is illustrates in Figure 3. □

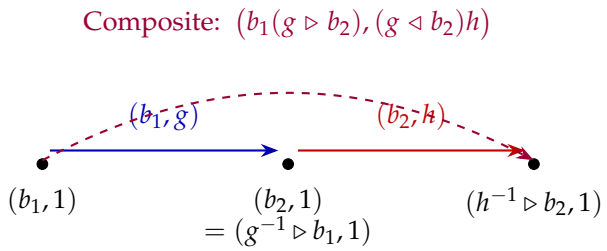


Figure 3. Schematic of arrow composition. This illustrates the partial nature of the groupoid multiplication: elements can be multiplied only when they form a continuous path.

Proposition 6. *Let $\wp = B \triangleleft\triangleright G$, where B is a band and G is a group vie left and right structure maps, satisfying the Zappa–Szép compatibility conditions. Then the subset*

$$\mathbf{B}_{\triangleleft\triangleright}(\wp) = \left\{(b, g) \in \wp : \left(g^{-1} \triangleleft b\right) = g^{-1}\right\}$$

forms a groupoid, with composition defined for pairs $(b_1, g_1), (b_2, g_2) \in \mathbf{B}_{\triangleleft\triangleright}(\wp)$ such that $(g^{-1} \triangleright b_1, 1) = (b_2\, , 1)$ for some $b_2 \in B$. In this case, the product is given by

$$(b_1, g_1)(b_2, g_2) = (b_1 \cdot (g_1 \triangleright b_2), (g_1 \triangleleft b_2) \cdot g_2)$$

If the identity element $1 \in G$ has a trivial structure map on B then $\mathbf{B}_{\triangleleft\triangleright}(\wp)$ is a groupoid.

Proof. We must verify that $\mathbf{B}_{\triangleleft\triangleright}(\wp)$ is closed under inversion and partial composition, and that source and target maps are well-behaved. Now $\left(g^{-1} \triangleright b, g^{-1} \triangleleft b\right) \in \mathbf{B}_{\triangleleft\triangleright}(\wp)$ that is $\mathbf{B}_{\triangleleft\triangleright}(\wp)$ is closed under inverses. This follows from Lemma 5.

◀ $(e(g \triangleright f), (g \triangleleft f)h) \in B_{\bowtie}(P)$.

$$(e(g \triangleright f), (g \triangleleft f)h) \in B_{\bowtie}(P) \Leftrightarrow \left[((g \triangleleft f)h)^{-1}\right] \triangleleft (e(g \triangleright f)) = ((g \triangleleft f)h)^{-1}.$$

Now

$$
\begin{aligned}
&\left[((g \triangleleft f)h)^{-1}\right] \triangleleft (e(g \triangleright f)) \\
&= \left[h^{-1}(g \triangleleft f)^{-1}\right] \triangleleft (e(g \triangleright f)) \\
&= \left[h^{-1}\right] \triangleleft \left((g \triangleleft f)^{-1} \triangleright e(g \triangleright f)\right) \left[(g \triangleleft f)^{-1}\right] \triangleleft (e(g \triangleright f)).
\end{aligned}
$$

Since $g^{-1} \triangleright e = f$ implies that $\left(gg^{-1}\right) \triangleright e = g \triangleright f$, this implies that $e = g \triangleright f$, therefore

$$\begin{aligned} (g \triangleleft f)^{-1} &= \left(g^{-1}\right) \triangleleft (g \triangleright f) \\ \left(g^{-1}\right) \triangleleft (g \triangleright f) &= \left(g^{-1}\right) \triangleleft e \\ &= g^{-1}. \end{aligned}$$

Then

$$\begin{aligned} \left[((g \triangleleft f)h)^{-1}\right] \triangleleft (e(g \triangleright f)) &= \left[h^{-1}\right] \triangleleft \left((g \triangleleft f)^{-1} \triangleright e(e)\right) \left[(g \triangleleft f)^{-1}\right] \triangleleft (e(e)) \\ &= \left[h^{-1}\right] \triangleleft \left((g \triangleleft f)^{-1} \triangleright e\right) \left[(g \triangleleft f)^{-1}\right] \triangleleft e \\ &= \left[h^{-1}\right] \triangleleft \left(g^{-1} \triangleright e\right) \left[g^{-1}\right] \triangleleft e \\ &= h^{-1}(g \triangleleft f)^{-1} \\ &= ((g \triangleleft f)h)^{-1}. \end{aligned}$$

- $(e(g \triangleright f), (g \triangleleft f)h)$ starts at $(e, 1)$.

  But $(e(g \triangleright f), (g \triangleleft f)h)$ starts at $(e(g \triangleright f), 1)$ and $g \triangleright f = e$, so $e(g \triangleright f) = e$,
- $(e(g \triangleright f), (g \triangleleft f)h)$ ends at $\left(h^{-1} \triangleright f, 1\right)$.

  But $(e(g \triangleright f), (g \triangleleft f)h)$ ends at $\left(h^{-1}(g \triangleleft f)^{-1} \triangleright e(g \triangleright f), 1\right)$

$$\begin{aligned} h^{-1}(g \triangleleft f)^{-1} \triangleright e(g \triangleright f) &= h^{-1} \triangleright \left(g^{-1} \triangleright e^2\right) \\ &= h^{-1} \triangleright \left(g^{-1} \triangleright e\right) \\ &= h^{-1} \triangleright f. \end{aligned}$$

Thus $B_{\bowtie}(P)$ is a groupoid. □

We now introduce the order used in the inductive-groupoid construction:

$$(e, g) \leq (f, h) \iff e \leq f \text{ and } g = h \triangleleft (h^{-1} \triangleright e).$$

This extends the ordering on $E(B_{\bowtie}(P))$ from Proposition 6.

Proposition 7. *The relation above is a partial order on* $B_{\bowtie}(P)$.

Proof. Reflexivity follows as follows. Suppose $(e, g) \in B_{\bowtie}(P)$. Then $(g^{-1}) \triangleleft e = g^{-1}$ and, by Lemma 5 and Lemma 1 part (2), $g = g \triangleleft (g^{-1} \triangleright e)$. Since trivially $e \leq e$, we obtain $(e, g) \leq (e, g)$.

For transitivity, we keep the full calculation. We prove that if $(e, g) \leq (f, h) \leq (l, k)$, then $(e, g) \leq (l, k)$.

We have

$$(e, g) \leq (f, h) \iff e \leq f \text{ and } g = h \triangleleft (h^{-1} \triangleright e).$$

and

$$(f, h) \leq (l, k) \iff f \leq l \text{ and } h = k \triangleleft (k^{-1} \triangleright f).$$

By Lemma 1 Part (2) we conclude

$$g = h \triangleleft (h^{-1} \triangleright e) = (h^{-1} \triangleleft e)^{-1}$$

and

$$h = k \triangleleft (k^{-1} \triangleright f) = \left(k^{-1} \triangleleft f\right)^{-1}.$$

Now, $e \leq f \leq l$ implies that $e \leq l$ and

$$g = (h^{-1} \triangleleft e)^{-1} = (\left(k^{-1} \triangleleft f\right) \triangleleft e)^{-1} = \left(k^{-1} \triangleleft (fe)\right)^{-1} = \left(k^{-1} \triangleleft e\right)^{-1} = k \triangleleft (k^{-1} \triangleright e).$$

Thus, the relation is transitive.

For antisymmetry, the calculation is as follows. Assume that $(e,g) \leq (f,h)$ and $(f,h) \leq (e,g)$, then $(e,g) = (f,h)$.

Now

$$(e,g) \leq (f,h) \iff e \leq f \text{ and } g = h \triangleleft (h^{-1} \triangleright e)$$

and

$$(f,h) \leq (e,g) \iff f \leq e \text{ and } h = g \triangleleft (g^{-1} \triangleright f).$$

Thus $e = f$ and

$$g = (h^{-1} \triangleleft e)^{-1} = \left((g^{-1} \triangleleft f) \triangleleft e\right)^{-1} = \left(g^{-1} \triangleleft f\right)^{-1} = g \triangleleft (g^{-1} \triangleright f) = h.$$

Thus the relation is antisymmetric.

Hence the relation is reflexive, transitive, and antisymmetric. Therefore, $(B_{\bowtie}(P), \leq)$ is a partially ordered set. □

We next verify the ordered-groupoid axioms and the restriction property in a single consolidated result.

Proposition 8. *The partially ordered groupoid $(B_{\bowtie}(P), \leq)$ is an inductive groupoid. More precisely:*

(OG1) *if $(e,g) \leq (f,h)$, then $(e,g)^{-1} \leq (f,h)^{-1}$;*

(OG2) *if $(p,s) \leq (e,g)$ and $(q,t) \leq (f,h)$, and both products are defined, then $(p,s)(q,t) \leq (e,g)(f,h)$;*

(OG3) *if $(f,1) \leq (e,1)$, then $\left(f, g \triangleleft (g^{-1} \triangleright f)\right)$ is the restriction of (e,g) to $(f,1)$.*

Proof. (OG1). Suppose that $(e,g) \leq (f,h)$, so that $e \leq f$ and $g = h \triangleleft (h^{-1} \triangleright e)$. Now, by Lemma 1 Part (2) and (4) we have

$$g^{-1} \triangleright e = \left(h \triangleleft (h^{-1} \triangleright e)\right)^{-1} \triangleright e = h^{-1} \triangleleft e \triangleright e = h^{-1} \triangleright e.$$

Thus,

$$\begin{aligned}\left(g^{-1} \triangleright e\right)\left(h^{-1} \triangleright f\right) &= \left(h^{-1} \triangleright e\right)\left(h^{-1} \triangleright f\right) \\ &= \left(h^{-1} \triangleright f\right)\left(h^{-1} \triangleright e\right) \\ &= \left(h^{-1} \triangleright f\right)\left(((h^{-1}) \triangleleft f) \triangleright e\right) \\ &= h^{-1} \triangleright fe = h^{-1} \triangleright e = g^{-1} \triangleright e.\end{aligned}$$

and

$$\begin{aligned}\left(h^{-1} \triangleright f\right)\left(g^{-1} \triangleright e\right) &= \left(h^{-1} \triangleright f\right)\left(h^{-1} \triangleright e\right) \\ &= \left(h^{-1} \triangleright f\right)\left(((h^{-1}) \triangleleft f) \triangleright e\right) \\ &= h^{-1} \triangleright fe = h^{-1} \triangleright e = g^{-1} \triangleright e.\end{aligned}$$

Therefore, $g^{-1} \triangleright e \leq h^{-1} \triangleright f$. Also,

$$\begin{aligned}(h^{-1}) \triangleleft (hg^{-1} \triangleright e) &= (h^{-1}) \triangleleft (hh^{-1} \triangleright e) = (h^{-1}) \triangleleft (hh^{-1} \triangleright e) \\ &= (h^{-1}) \triangleleft (1 \triangleright e) = (h^{-1}) \triangleleft e = g^{-1}.\end{aligned}$$

Hence $(e,g)^{-1} \leq (f,h)^{-1}$, proving (OG1).

(OG2). Suppose that $(p,s)(q,t) = (p,u)$ and $(e,g)(f,h) = (e,k)$ are defined Then, we have

$$p \leq e \text{ and } s = g \triangleleft (g^{-1} \triangleright p)$$

and

$$q \leq f \text{ and } t = h \triangleleft (h^{-1} \triangleright q)$$

and we have the following where $u = (s \triangleleft q)t = (s \triangleleft (s^{-1} \triangleright p))t = st$ by Lemma 1 part (2). where $k = (g \triangleleft f)h = (g \triangleleft (g^{-1} \triangleright e))h = gh$ by Lemma 1 part (2). Now,

$$\begin{aligned}u &= st = \Big(g \triangleleft (g^{-1} \triangleright p)\Big)\Big(h \triangleleft (h^{-1} \triangleright q)\Big) = \Big(g \triangleleft (g^{-1} \triangleright p)\Big)\Big(h \triangleleft (h^{-1}g^{-1} \triangleright p)\Big) \\ &= \Big(g \triangleleft (g^{-1} \triangleright p)\Big)\Big(h \triangleleft (h^{-1}g^{-1} \triangleright p)\Big) = \Big(g \triangleleft (g^{-1} \triangleright p)\Big)\Big(h \triangleleft (h^{-1}g^{-1} \triangleright p)\Big)\end{aligned}$$

and

$$\begin{aligned}k \triangleleft (k^{-1} \triangleright p) &= g\Big(h \triangleleft ((gh)^{-1} \triangleright p)\Big) \\ &= \Big(g \triangleleft (h \triangleright (h^{-1}g^{-1} \triangleright p))\Big)\Big(h \triangleleft (h^{-1}g^{-1} \triangleright p)\Big) \\ &= \Big(g \triangleleft (g^{-1} \triangleright p)\Big)\Big(h \triangleleft (h^{-1}g^{-1} \triangleright p)\Big).\end{aligned}$$

Then, $u = k \triangleleft (k^{-1} \triangleright p)$. Moreover, $p \leq e$. Thus, $(p,u) \leq (e,k)$ as required.

(OG3). Suppose that $(e,g) \in B_{\bowtie}(P)$ and $(f,1)$ is an identity such that $(f,1) \leq (e,1)$, since

$(g \triangleleft (g^{-1} \triangleright f))^{-1} = g^{-1} \triangleleft f = (g^{-1} \triangleleft f) \triangleleft f$, then $\big(f, g \triangleleft (g^{-1} \triangleright f)\big) \in B_{\bowtie}(P)$. Also

$$\begin{aligned}&\Big(f, g \triangleleft (g^{-1} \triangleright f)\Big)\Big(f, g \triangleleft (g^{-1} \triangleright f)\Big)^{-1} \\ &= \Big(f, g \triangleleft (g^{-1} \triangleright f)\Big)\Big((g^{-1} \triangleleft f) \triangleright f, g^{-1} \triangleleft f\Big) \\ &= \Big(f\Big[((g^{-1} \triangleleft f)^{-1}) \triangleright ((g^{-1} \triangleleft f) \triangleright f)\Big], \Big([(g^{-1} \triangleleft f)^{-1}] \triangleleft ((g^{-1} \triangleleft f) \triangleright f)\Big)(g^{-1} \triangleleft f)\Big) \\ &= (f,1).\end{aligned}$$

Moreover, $\big(f, g \triangleleft (g^{-1} \triangleright f)\big) \leq (e,g)$ and unique since if $(x,y) \in B_{\bowtie}(P)$ such that $(x,y)(x,y)^{-1} = (f,1)$ then

$$\begin{aligned}(x,y)(x,y)^{-1} &= (x,y)\Big(y^{-1} \triangleright x, y^{-1}\Big) = \Big(x\Big(yy^{-1} \triangleright x\Big), y \triangleleft (y^{-1} \triangleright x)y^{-1}\Big) \\ &= \Big(x, \Big(yy^{-1}\Big) \triangleleft x\Big) = (x,1) = (f,1).\end{aligned}$$

Thus $x = f$ and $(x,y) \leq (e,g)$ if and only if $x \leq e$ and $y = g \triangleleft (g^{-1} \triangleright x) = g \triangleleft (g^{-1} \triangleright f)$. Hence $\big(f, g \triangleleft (g^{-1} \triangleright f)\big)$ is unique whereupon $\big(f, g \triangleleft (g^{-1} \triangleright f)\big)$ is the restriction of (e,g) to $(e,1)$.

Finally, the partially ordered set of identities forms a meet semilattice, $E(B_{\bowtie}(P)) \cong E$. Hence $(B_{\bowtie}(P), \leq)$ is an inductive groupoid. □

Theorem 4. *If $P = E \bowtie G$, where E is a semilattice and G is a group, then*

$$B_{\bowtie}(P) = \left\{(e,g) \in P : \left(g^{-1}\right) \triangleleft e = g^{-1}\right\}$$

is an inverse semigroup with multiplication defined by

$$(e,g)(f,h) = \left(e(g \triangleright f), g \triangleleft fh \triangleleft \left(h^{-1}g^{-1} \triangleright e\right)\right).$$

Proof. Let $(e,g),(f,h) \in B_{\bowtie}(P)$ since $(e,g)^{-1}(e,g) = \left(g^{-1} \triangleright e, 1\right)$ and $(f,h)(f,h)^{-1} = (f,1)$, we form the pseudoproduct $(e,g) \otimes (f,h)$ using the greatest lower bound

$$\begin{aligned} \ell &= (e,g)^{-1}(e,g) \wedge (f,h)(f,h)^{-1} = \left(g^{-1} \triangleright e, 1\right) \wedge (f,1) \\ &= \left(g^{-1} \triangleright e, 1\right)(f,1) = \left(\left(g^{-1} \triangleright e\right)f, 1\right) \end{aligned}$$

and

$$\begin{aligned} (\ell \mid (f,h)) &= \left(\left(\left(g^{-1} \triangleright e\right)f, 1\right) \mid (f,h)\right) \\ &= \left(\left(g^{-1} \triangleright e\right)f, h \triangleleft \left(\left(h^{-1}g^{-1} \triangleright e\right)\left(\left(h^{-1}\right)^{g^{-1} \triangleright e} \triangleright f\right)\right)\right). \end{aligned}$$

and

$$\begin{aligned} ((e,g) \mid \ell) &= \left(\ell \mid \left(g^{-1} \triangleright e, g^{-1}\right)\right)^{-1} \\ &= \left((g^{-1} \triangleright e)f, (g^{-1}) \triangleleft \left(g \triangleright \left(g^{-1} \triangleright e\right)f\right)\right)^{-1} \\ &= \left((g^{-1} \triangleright e)f, (g^{-1}) \triangleleft \left(\left(g \triangleright g^{-1} \triangleright e\right)\left(g \triangleleft \left(g^{-1} \triangleright e\right) \triangleright f\right)\right)\right)^{-1} \\ &= \left((g^{-1} \triangleright e)f, (g^{-1}) \triangleleft (e(g \triangleright f))\right)^{-1} \\ &= \left(\left[(g^{-1}) \triangleleft (e(g \triangleright f))\right]^{-1} \triangleright (g^{-1} \triangleright e)f, \left[(g^{-1}) \triangleleft (e(g \triangleright f))\right]^{-1}\right). \end{aligned}$$

Now, since $(e,g) \in B_{\bowtie}(P)$ then $g^{-e} = g^{-1}$ and so,

$$\left[(g^{-1}) \triangleleft (e(g \triangleright f))\right]^{-1} = \left[(g^{-1}) \triangleleft (g \triangleright f)\right]^{-1}.$$

By Part (2) Lemma 1 we have

$$\left[(g^{-1}) \triangleleft (g \triangleright f)\right]^{-1} = g \triangleleft f.$$

Then,

$$((e,g) \mid \ell) = \left(g \triangleleft f \triangleright (g^{-1} \triangleright e)f, g \triangleleft f\right).$$

Therefore

$$(e,g) \otimes (f,h) = ((e,g) \mid \ell)(\ell \mid (f,h))$$

$$
\begin{aligned}
&= \begin{pmatrix} \\ \\ \\ \end{pmatrix} \\
&\quad \cdot \left(\left(g^{-1} \triangleright e\right) f, h \triangleleft \left(\left(h^{-1} g^{-1} \triangleright e\right) \left(\left(h^{-1}\right)^{g^{-1} \triangleright e} \triangleright f \right) \right) \right) \\
&= \left(g \triangleleft f \triangleright (g^{-1} \triangleright e) f, (g \triangleleft f) \triangleleft \left(\left(g^{-1} \triangleright e\right) f \right) h \triangleleft \left(\left(h^{-1} g^{-1} \triangleright e\right) \left(\left(h^{-1}\right)^{g^{-1} \triangleright e} \triangleright f \right) \right) \right) \\
&= \begin{pmatrix} \\ \\ \\ \end{pmatrix}.
\end{aligned}
$$

Now, we have in the ordering defined on $B_{\bowtie}(P)$, $(g^{-1} \triangleright e) f \leq g^{-1} \triangleright e$ and $g^{-1} \triangleright e$ has a trivial structure map on g this implies that $g \triangleleft \left((g^{-1} \triangleright e) f\right) = g \triangleleft f$.

Then by Part (4) Lemma 1 we have

$$
\begin{aligned}
g \triangleleft f \triangleright (g^{-1} \triangleright e) f &= g \triangleleft \left((g^{-1} \triangleright e) f\right) \triangleright \left[(g^{-1} \triangleright e) f\right] \\
&= g \triangleright \left[(g^{-1} \triangleright e) f\right] \\
&= \left(g \triangleright g^{-1} \triangleright e\right) \left(g \triangleleft \left(g^{-1} \triangleright e\right) \triangleright f\right) \\
&= \left(g g^{-1} \triangleright e\right) (g \triangleright f) = e(g \triangleright f).
\end{aligned}
$$

and

$$
\begin{aligned}
g \triangleleft f h \triangleleft \left(h^{-1} \triangleright \left[\left(g^{-1} \triangleright e\right) f\right]\right) &= g \triangleleft f h \triangleleft \left(h^{-1} \triangleright \left[f \left(g^{-1} \triangleright e\right)\right]\right) \\
&= g \triangleleft f h \triangleleft \left(\left(h^{-1} \triangleright f\right) \left(\left(h^{-1}\right) \triangleleft f \triangleright \left(g^{-1} \triangleright e\right)\right)\right) \\
&= g \triangleleft f \left(h \triangleleft \left(h^{-1} \triangleright f\right)\right) \triangleleft \left(h^{-1} \triangleright \left(g^{-1} \triangleright e\right)\right) \\
&= g \triangleleft f h \triangleleft \left(h^{-1} g^{-1} \triangleright e\right).
\end{aligned}
$$

Therefore

$$
(e, g)(f, h) = \left(e(g \triangleright f), g \triangleleft f h \triangleleft \left(h^{-1} g^{-1} \triangleright e\right)\right).
$$

That this binary operation makes $B_{\bowtie}(P)$ an inverse semigroup follows from its construction as the pseudoproduct on an inductive groupoid. □

Remark on the pseudoproduct. The arithmetic developed above is valid at the level of the classical ESN pseudoproduct. Indeed, for composable arrows the partial groupoid composition agrees with the original Zappa–Szép multiplication, whereas for arbitrary arrows the pseudoproduct is obtained by restricting and corestricting the two arrows to the meet of the corresponding identities and then composing them. Thus the pseudoproduct should be distinguished from both the ordinary semidirect product and the unrestricted Zappa–Szép product. It is a total inverse-semigroup multiplication reconstructed from the inductive-groupoid structure, while still retaining the matched-pair data through the actions $\triangleright$ and $\triangleleft$. Only in the special case in which the right action is trivial,

$$g \triangleleft e = g,$$

does the formula reduce to the ordinary semidirect product multiplication

$$(e,g)(f,h) = \big(e(g \triangleright f), gh\big).$$

The one-sided restriction-semigroupoid framework provides a broader possible extension of this construction. However, the pseudoproduct used in the present paper is specifically the classical ESN pseudoproduct associated with the inductive groupoid constructed in Proposition 8.

A direct proof of the associativity of the multiplication given in Theorem 4 is presented in [12]. We record the main structural consequences of this multiplication in one consolidated result.

Proposition 9. *For the inverse semigroup $B_{\bowtie}(P)$ constructed above, the following hold.*

(i) The set of idempotents is

$$E(B_{\bowtie}(P)) = \{(e,1) : e \in E\} \cong E.$$

(ii) For $(e,g),(f,h) \in \mathbf{B}_{\triangleleft\triangleright}(\wp)$,

$$(e,g)\mathcal{L}(f,h) \iff g^{-1} \triangleright e = h^{-1} \triangleright f, \qquad (e,g)\mathcal{R}(f,h) \iff e = f.$$

Moreover,

$$H_{(e,1)} = \{(e,g) : g \triangleright e = e \text{ and } g \triangleleft e = g\}.$$

(iii) If B has an identity 1, then $\mathbf{B}_{\triangleleft\triangleright}(\wp)$ is factorisable.

Proof. (i) Idempotents. We have $E(B_{\bowtie}(P)) = \Big\{(e,g) \in B_{\bowtie}(P) : (e,g)^2 = (e,g)\Big\}$. Then

$$(e,g)^2 = (e,g)(e,g) = \Big(e(g \triangleright e), g \triangleleft eg \triangleleft \Big(g^{-1}g^{-1} \triangleright e\Big)\Big) = (e,g)$$

Hence $e = e(g \triangleright e)$ and $g = g \triangleleft eg \triangleleft \big(g^{-1}g^{-1} \triangleright e\big)$. Then

$$\begin{aligned} g^{-1} \triangleright e &= g^{-1} \triangleright e(g \triangleright e) = \Big(g^{-1} \triangleright e\Big)\Big(\Big(g^{-1}\Big) \triangleleft e \triangleright g \triangleright e\Big) \\ &= \Big(g^{-1} \triangleright e\Big)e. \end{aligned}$$

Therefore

$$g = g \triangleleft \Big(g^{-1} \triangleright e\Big) = g \triangleleft \Big(\Big(g^{-1} \triangleright e\Big)e\Big) = \Big(g \triangleleft \Big(g^{-1} \triangleright e\Big)\Big) \triangleleft e = g \triangleleft e.$$

It follows that

$$\begin{aligned} g &= g \triangleleft eg \triangleleft \left(g^{-1}g^{-1} \triangleright e\right) = gg \triangleleft \left(g^{-1}g^{-1} \triangleright e\right) \\ g \triangleleft \left(g^{-1}g^{-1} \triangleright e\right) &= 1. \end{aligned}$$

Hence, by parts (2) and (4) Lemma 1

$$\begin{aligned} g^{-1} \triangleright e &= \left(g \triangleleft \left(g^{-1}g^{-1} \triangleright e\right)\right)^{-1} \triangleright \left(g^{-1} \triangleright e\right) = (g^{-1}) \triangleleft \left(g^{-1} \triangleright e\right) \triangleright \left(g^{-1} \triangleright e\right) \\ &= \left(g^{-1}g^{-1}\right) \triangleleft e \triangleright e = g^{-1}g^{-1} \triangleright e. \end{aligned}$$

Therefore,

$$1 = g \triangleleft \left(g^{-1}g^{-1} \triangleright e\right) = g \triangleleft \left(g^{-1} \triangleright e\right) = (\left(g^{-1}\right) \triangleleft e)^{-1} = g.$$

Thus $E(B_{\bowtie}(P)) = \{(e,1) : e \in E\}$, and using the product given on $B_{\bowtie}(P)$ defined by

$$(e,g)(f,h) = \left(e(g \triangleright f), g \triangleleft fh \triangleleft \left(h^{-1}g^{-1} \triangleright e\right)\right),$$

we have

$$(e,1)(f,1) = (e(1 \triangleright f), 1 \triangleleft f1 \triangleleft (1 \triangleright 1 \triangleright e)) = (ef,1).$$

Thus $E(B_{\bowtie}(P)) \cong E$ and so it is a semilattice.

(ii) Green relations. (a) By definition $(e,g)\mathcal{L}(f,h)$ if and only if $(e,g)^{-1}(e,g) = (f,h)^{-1}(f,h)$ if and only if

$$\begin{aligned} \left(g^{-1} \triangleright e, g^{-1}\right)(e,g) &= \left(h^{-1} \triangleright f, h^{-1}\right)(f,h) \\ \left((g^{-1} \triangleright e)\left(g^{-1} \triangleright e\right), \left(g^{-1}\right) \triangleleft eg\right) &= \left(\left(h^{-1} \triangleright f\right)\left(h^{-1} \triangleright f\right), \left(h^{-1}\right) \triangleleft fh\right) \end{aligned}$$

if and only if

$$\begin{aligned} \left((g^{-1} \triangleright e)((g^{-1}) \triangleleft e \triangleright e), g^{-1}g\right) &= \left(\left(h^{-1} \triangleright f\right)\left((h^{-1}) \triangleleft f \triangleright f\right), h^{-1}h\right) \\ \left(g^{-1} \triangleright e, 1\right) &= \left(h^{-1} \triangleright f, 1\right). \end{aligned}$$

(b) Similar to part (a).

(c) Since by definition $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, then

$$H_{(e,1)} = \left\{(e,g) : g^{-1} \triangleright e = e \text{ and } \left(g^{-1}\right) \triangleleft e = g^{-1}\right\}.$$

But then $g^{-1} \triangleright e = e$ if and only if $g \triangleright e = e$

and

$$g = (\left(g^{-1}\right) \triangleleft e)^{-1} = g \triangleleft \left(g^{-1} \triangleright e\right) = g \triangleleft e.$$

Thus

$$H_{(e,1)} \subseteq \{(e,g) : g \triangleright e = e \text{ and } g \triangleleft e = g\},$$

the reverse inclusion is clear.

(iii) Factorisability. Let $(e,g) \in \mathbf{B}_{\triangleleft\triangleright}(\wp)$, then $(e,1) \in E(B_{\bowtie}(P))$ and $(1,g) \in U(\mathbf{B}_{\triangleleft\triangleright}(\wp))$ such that

$$\begin{aligned} (e,1)(1,g) &= \left(e(1 \triangleright 1), 1^1g \triangleleft \left(1 \triangleright g^{-1} \triangleright e\right)\right) \\ &= (e,g). \end{aligned}$$

We conclude $\mathbf{B}_{\triangleleft\triangleright}(\wp)$ is factorisable. □

Effective and automata-theoretic interpretation.

We emphasize that "inductive" in Proposition 8 refers to the inductive-groupoid structure and does not by itself assert an algorithmic or recursive procedure. Nevertheless, the constructions above are effective whenever the semilattice E, the group G, and the structure maps $\triangleright$ and $\triangleleft$ are finite or effectively computable.

Indeed, Proposition 8 gives explicit formulas for restrictions in the ordered groupoid; hence these restrictions can be computed from the order on E, the group operations in G, and the two matched-pair maps. Likewise, the multiplication in Theorem 4 is obtained by a finite sequence of evaluations of the meet, group operations, and the maps $\triangleright$ and $\triangleleft$. Proposition 9 then gives effective tests for idempotents and Green relations using the same data.

From the automata viewpoint of Section 1.2, each evaluation $g \triangleright e$ and $g \triangleleft e$ corresponds respectively to an output and a state transition. Thus the ESN pseudoproduct may be interpreted as a finite algebraic composition of computable transition data. In particular, for finite E and G, all these operations can be implemented directly. For arbitrary infinite structures, however, no general decidability or complexity result is claimed without additional assumptions on their effective presentations.

2.2. Examples

1. Let $B = \{e, f, ef\}$ be a commutative band and $G = \{1, a, b, ab\}$ be the Klein four-group. Define a structure map from G to transformations of B by: $a \triangleright e = f, b \triangleright e = f, a \triangleright ef = ef, a \triangleright f = e, b \triangleright f = e$, and $b \triangleright ef = ef, ab \triangleright e = e, ab \triangleright f = f, ab \triangleright ef = ef$. and $1 \triangleright x = x$ for all $x \in B$. Define the corresponding structure map from B to transformations of G using: $a \triangleleft e = b, b \triangleleft e = b, a \triangleleft f = b, b \triangleleft f = b, a \triangleleft ef = b$, and $b \triangleleft ef = b, (ab) \triangleleft e = 1, (ab) \triangleleft f = 1, (ab) \triangleleft ef = 1$. Then the Zappa-Szép product $\wp = B \triangleleft \triangleright G$ of B and G is defined with multiplication and display in Figure 4:

$$(x, g)(y, h) = (x \cdot (g \triangleright y), (g \triangleleft y)(h \triangleleft x)).$$

	$(e,1)$	(e,a)	(e,b)	(e,ab)	$(f,1)$	(f,a)	(f,b)	(f,ab)	$(ef,1)$	(ef,a)	(ef,b)	(ef,ab)
$(e,1)$	$(e,1)$	(e,a)	(e,b)	(e,ab)	$(f,1)$	(f,a)	(f,b)	(f,ab)	$(ef,1)$	(ef,a)	(ef,b)	(ef,ab)
(e,a)	(e,a)	(ef,ab)	$(ef,1)$	(ef,a)	(e,b)	(e,ab)	$(e,1)$	(e,a)	(ef,b)	(ef,ab)	$(ef,1)$	(ef,a)
(e,b)	(e,b)	(ef,ab)	$(ef,1)$	(ef,a)	(e,b)	(e,ab)	$(e,1)$	(e,a)	$(ef,1)$	(ef,ab)	$(ef,1)$	(ef,a)
(e,ab)	(e,ab)	(e,a)	(e,b)	(e,ab)	$(ef,1)$	(ef,a)	(ef,b)	(ef,ab)	$(ef,1)$	(ef,a)	(ef,b)	(ef,ab)
$(f,1)$	$(f,1)$	(ef,a)	(ef,b)	(ef,ab)	$(f,1)$	(f,a)	(f,b)	(f,ab)	$(ef,1)$	(ef,a)	(ef,b)	(ef,ab)
(f,a)	(f,a)	(f,ab)	$(f,1)$	(f,a)	(ef,b)	(ef,ab)	$(ef,1)$	(ef,ab)	(ef,b)	(ef,ab)	$(ef,1)$	(ef,a)
(f,b)	(f,b)	(f,ab)	$(f,1)$	(f,a)	(ef,b)	(ef,ab)	$(ef,1)$	(ef,a)	(ef,b)	(ef,ab)	$(ef,1)$	(ef,a)
(f,ab)	(f,ab)	(ef,a)	(ef,b)	(ef,ab)	$(f,1)$	(f,a)	(f,b)	(f,ab)	$(ef,1)$	(ef,a)	(ef,b)	(ef,ab)
$(ef,1)$	$(ef,1)$	(ef,a)	(ef,b)	(ef,ab)	$(ef,1)$	(ef,a)	(ef,b)	(ef,ab)	$(ef,1)$	(ef,a)	(ef,b)	(ef,ab)
(ef,a)	(ef,a)	(ef,ab)	$(ef,1)$	(ef,a)	(ef,b)	(ef,ab)	$(ef,1)$	(ef,a)	(ef,b)	(ef,ab)	$(ef,1)$	(ef,a)
(ef,b)	(ef,b)	(ef,ab)	$(ef,1)$	(ef,a)	(ef,b)	(ef,ab)	$(ef,1)$	(ef,a)	(ef,b)	(ef,ab)	$(ef,1)$	(ef,a)
(ef,ab)	(ef,ab)	(ef,a)	(ef,b)	(ef,ab)	$(ef,1)$	(ef,a)	(ef,b)	(ef,ab)	$(ef,1)$	(ef,a)	(ef,b)	(ef,ab)

Figure 4. Zappa–Szép product arising from the interaction between the commutative band $B = \{e, f, ef\}$ and the Klein four-group $G = \{1, a, b, ab\}$. The diagram records how the two structure maps determine the multiplication in the product.

Thus

$$\begin{aligned} B_{\bowtie}(\wp) &= \left\{(e, g) : g^{-e} = g^{-1}\right\} \\ &= \{(e,1), (f,1), (ef,1), (e,b), (f,b), (ef,b)\}. \end{aligned}$$

Hence $B_{\bowtie}(\wp)$ has multiplication

$$(e,g)(f,h) = \Big(e(g \rhd f), (g \lhd f)\Big(h \lhd \Big(h^{-1}g^{-1} \rhd e\Big)\Big)\Big).$$

We have the following multiplication table

	$(e,1)$	$(f,1)$	$(ef,1)$	(e,b)	(f,b)	(ef,b)
$(e,1)$	$(e,1)$	$(ef,1)$	$(ef,1)$	(e,b)	(ef,b)	(ef,b)
$(f,1)$	$(ef,1)$	$(f,1)$	$(ef,1)$	(ef,b)	(f,b)	(ef,b)
$(ef,1)$	$(ef,1)$	$(ef,1)$	$(ef,1)$	(ef,b)	(ef,b)	(ef,b)
(e,b)	(ef,b)	(e,b)	(ef,b)	$(ef,1)$	$(e,1)$	(ef,b)
(f,b)	(f,b)	(ef,b)	(ef,b)	$(f,1)$	$(ef,1)$	$(ef,1)$
(ef,b)	(ef,b)	(ef,b)	(ef,b)	$(ef,1)$	$(ef,1)$	$(ef,1)$

Thus $E(B_{\bowtie}(\wp)) = \{(e,1),(ef,1),(f,1)\}$ which is closed subsemigroup of $B_{\bowtie}(\wp)$, so $B_{\bowtie}(\wp)$ is orthodox, but it is not an E-unitary semigroup.

2. This example presents a construction that illustrates self-similarity given in [18] through the interaction of a group and a band via a wreath-product operation, and relates it to the notion of Zappa Szép product. This example is discussed in [12], but we expand upon it by providing additional figures and a more detailed explanation. Let B be a band with automorphism group G. Fix a positive integer n, and consider n disjoint copies of B denoted by $B_1, B_2, \ldots, B_n$, we form a new band $\tilde{B}$ by taking the adjoint union of these copies and adjoining two new elements: a greatest element 1 and a least element 0. Elements from the $\lhd$th copy of B are labeled as $b^{(j)}$, where $b \in B$. We represent $\tilde{B}$ in Figure 5:

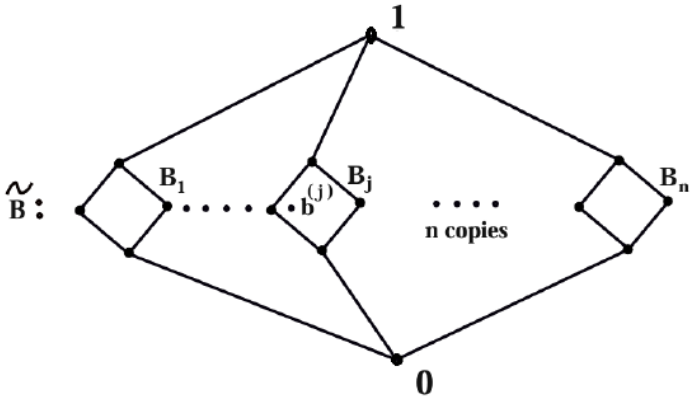


Figure 5. The enlarged band $\tilde{B}$ obtained from the disjoint copies $B_1, \ldots, B_n$ by adjoining a least element 0 and a greatest element 1. Products between elements in different copies collapse to 0, while each copy retains the band multiplication of B.

Let $\omega = G \wr S_n$ of G with the symmetric group S_n. Each element $\gamma \in \omega$ is a pair $(\mathbf{g}, \sigma)$, where:

- $\sigma \in S_n$ is a permutation.
- $\mathbf{g} : [n] \longmapsto G$ is a function assigning group elements to indices. The group S_n operates on $G^{[n]}$(the set of functions from $[n] \to G$) on the left by permuting components: $(\sigma \rhd \mathbf{f})(i) = \mathbf{f}(\sigma^{-1}(i)), \quad \forall i \in [n]$

This gives a well-defined group operation since $\tau \rhd (\sigma \rhd \mathbf{f}) = (\tau\sigma) \rhd \mathbf{f}$. With pointwise multiplication, $G^{[n]} \cong G \times G \times \ldots \times G$ (n times), and $\omega \cong G^{[n]} \rhd S_n$. The left structure map of ω on $\tilde{B}$ is defined as follows:

- The elements 0 and 1 are fixed under all $\gamma \in \omega$,
- For any $b \in B$, the structure map on $b^{(j)} \in \tilde{B}$ is given by:

$$\gamma \rhd b^{(j)} = (\mathbf{g}(\sigma(j))(b))^{\sigma(j)}. \tag{1}$$

In other words, we move $b^{(j)}$ to the $\sigma(j)$-th copy, apply the automorphism $\mathbf{g}(\sigma(j)) \in G$, and remain in the new copy $B_{\sigma(J)}$. This is well defined since if $\sigma \in S_n$ and $\mathbf{f} : [n] \longmapsto G$ then $(\sigma \cdot \mathbf{f})(i) = \mathbf{f}\big(\sigma^{-1}(i)\big)$ and if $\tau \in S_n$. Then

$$\begin{aligned} \tau \cdot (\sigma \cdot \mathbf{f})(i) &= (\sigma \cdot \mathbf{f})\Big(\tau^{-1}(i)\Big) = \mathbf{f}\Big(\sigma^{-1}\tau^{-1}(i)\Big) \\ &= \mathbf{f}\Big((\tau\sigma)^{-1}(i)\Big) = (\tau\sigma \cdot \mathbf{f})(i) \end{aligned}$$

We represent the structure map of ω on $\tilde{B}$ in Figure 6.

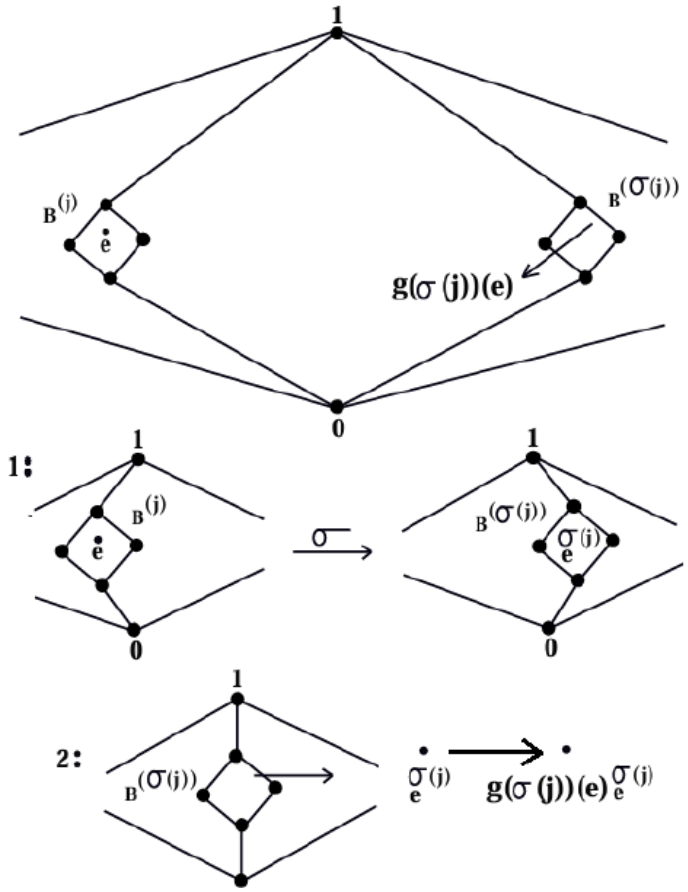


Figure 6. Left structure map of the wreath product $\omega = G \wr S_n$ on $\tilde{B}$. The permutation component moves an element to the corresponding copy, and the group component then applies the appropriate automorphism inside that copy.

Under this structure map we take $b^{(j)}$ which is copy of element $b \in B$ in the jth-copy in $\tilde{B}$ we move $b^{(j)}$ to $b^{\sigma(j)}$ we change the copy as in 1, and then find the automorphism in $\mathbf{g}$ that applies to copy $B^{\sigma(j)}$ this is $\mathbf{g}(\sigma(j))$ apply it to $b^{\sigma(j)}$ we stay in the same copy as in 2. We denote the identity element of $G^{[n]}$ by $\mathbf{1}$, and the element of $G^{[n]}$ that maps $i \longmapsto \mathbf{g} \in G$ and $j \longmapsto \mathbf{1}_G$ if $j \neq i$ will be denoted $(g)_i$.
Now define the structure map of $\tilde{B}$ on ω as follows:

- $\mathbf{1} \triangleleft \gamma = \gamma$,
- $(\mathbf{g}, \sigma) \triangleleft \mathbf{0} = (\mathbf{1}, \sigma)$,
- and for each $b^{(j)} \in B_j$

$$(\mathbf{g}, \sigma) \triangleleft b^{(j)} = \Big((\mathbf{g}(\sigma(j)))_{\sigma(j)}, \sigma \Big). \tag{2}$$

We represent the structure map of $\tilde{B}$ on ω in Figure 7.

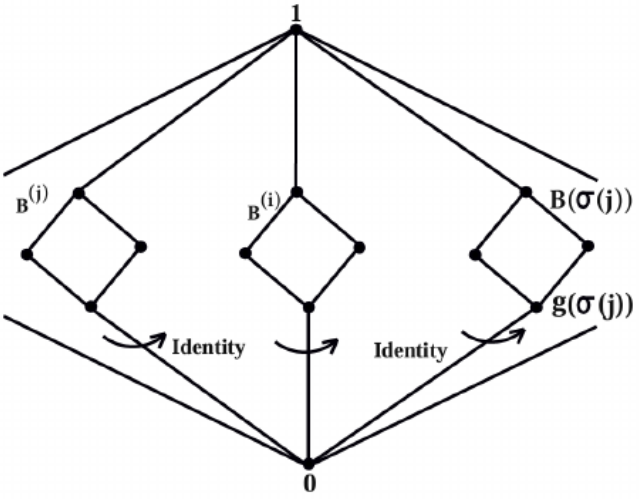


Figure 7. Right structure map of $\tilde{B}$ on ω. Acting by an element $b^{(j)}$ retains the component indexed by $\sigma(j)$ and replaces the remaining components by the identity, thereby encoding the local nature of the interaction.

we retain out σ and $\mathbf{g}(\sigma(j))$ while replacing all other $\mathbf{g}(i)$ to the identity automorphism and every element in the same copy B_j operates in the same way.

$$(\mathbf{g}(\sigma(j)))_{\sigma(j)} = (\mathbf{1}, \ldots, \mathbf{g}(\sigma(j)), \ldots, \mathbf{1})$$

Now we check that these structure maps satisfy $(SS1) - (SS8)$ and so we have a Zappa–Szép product $\Im = \tilde{B} \bowtie \omega$.
$(SS1)$: let $\beta = (\mathbf{h}, \delta), \gamma = (\mathbf{g}, \sigma) \in \omega$. Let $e^{(j)} \in \tilde{B}$. We want to verify $\beta \triangleright (\gamma \triangleright e^{(j)}) = (\beta \cdot \gamma) \triangleright e^{(j)}$. From the definition of the left structure map:

$$\Big(\gamma \triangleright e^{(j)} \Big) = \mathbf{g}(\sigma(j))(e)^{(\sigma(j))}$$

Then applying $\beta = (\mathbf{h}, \delta)$ to this gives:

$$\beta \triangleright (\gamma \triangleright e^{(j)}) = \beta \cdot \Big(\mathbf{g}(\sigma(j))(e)^{(\sigma(j))} \Big) = \mathbf{h}(\delta\sigma(j))(\mathbf{g}(\sigma(j))(e))^{(\delta\sigma(j))}. \tag{*}$$

Now cosider the group product $\beta \cdot \gamma = (\mathbf{h}(\delta \triangleright \mathbf{g}), \delta\sigma)$, where $(\delta \triangleright \mathbf{g})(k) = \mathbf{g}\big(\sigma^{-1}\big(\delta^{-1}(k)\big)\big)$. So

$$(\beta \cdot \gamma) \triangleright e^{(j)} = (\mathbf{h}(\delta \triangleright \mathbf{g}))(\delta\sigma(j))(e)^{\delta\sigma(j)}.$$

Now note:

$$\begin{aligned} (\mathbf{h}(\delta \triangleright \mathbf{g}))(\delta\sigma(j)) &= \mathbf{h}(\delta\sigma(j))(\delta \triangleright \mathbf{g})(\delta\sigma(j)) \\ &= \mathbf{h}(\delta\sigma(j))\mathbf{g}\Big(\delta^{-1}\delta\sigma(j)\Big) \\ &= \mathbf{h}(\delta\sigma(j)) \cdot \mathbf{g}(\sigma(j)). \quad ** \end{aligned} \tag{5}$$

So in $(*)$ and $(**)$ we apply the same automorphism to e and the output lies in the same copy of $\tilde{B}$. Therefore, for all $\beta, \gamma \in \omega$ and $e \in B$: $\beta \rhd (\gamma \rhd e^{(j)}) = (\beta \cdot \gamma) \rhd e^{(j)}$. Since this clearly holds if $e^{(j)}$ is replaced by 0 or 1, $(SS1)$ holds universally.
$(SS2)$: let $\beta = (\mathbf{h}, \delta) \in \omega$, and consider elements $x, y \in \tilde{B}$. If either of x or y is equal to the zero or identity element, then the axiom holds trivially. We focus on the case where $x = e^{(j)}$ and $y = f^{(k)}$, with $e, f \in B$.

- Case 1: $j = k$. In this case, the product $xy = (ef)^{(j)}$, and thus $\beta \rhd (xy) = \beta \rhd \left(e^{(j)} f^{(j)}\right) = \beta \rhd (ef)^{(j)} = \mathbf{h}(\delta(j))(ef)^{\delta(j)}$. Now compute the separate structure maps:

$$\begin{aligned} (\beta \rhd x) &= \left(\beta \rhd e^{(j)}\right) = (\mathbf{h}(\delta(j))(e))^{\delta(j)}, \\ \left(\beta^x \lhd e^{(j)} \rhd f^{(j)}\right) &= \mathbf{h}(\delta(j))\left(f^{(\delta(j))}\right) \end{aligned}$$

So their product is: $[\mathbf{h}(\delta(j))(e)\mathbf{h}(\delta(j))(f)]^{\delta(j)} = \mathbf{h}(\delta(j))(ef)^{(\delta(j))}$. Because $\mathbf{h}(\delta(j))$ preserves multiplication (it is an automorphism). Therefore, both sides of the identity match.

- Case 2: $j \neq k$. In this situation $xy = e^{(j)} f^{(k)} = 0$, and thus $\beta \rhd (xy) = \beta \rhd \left(e^{(j)} f^{(k)}\right) = \beta \rhd 0 = 0$. Now examine the product:

$$(\mathbf{h}, \delta) \rhd e^{(j)} \in B_{\delta(j)}, \quad .(\mathbf{h}, \delta) \rhd f^{(k)} \in B_{\delta(k)}$$

Which lies in $B_{\delta(j)}\, B_{\delta(k)}$ with $\delta(j) \neq \delta(k)$, the result is 0, by the construction of $\tilde{B}$ (copies are disjoint and annihilate each other). So the equality still holds. This confirms that Axiom (ZS2) is satisfied in all cases.
$(SS3)$: let $\beta = (\mathbf{h}, \delta)$ clearly if either e or $f \in \tilde{B}$ equals 0 or 1 we have $(\beta \lhd e) \lhd f = \beta \lhd (ef)$. So assume $e = e^{(j)}$ and $f = f^{(k)}$.

- Case 1: $j \neq k$. Then $e^{(j)} f^{(k)} = 0$, and $\beta \lhd \left(e^{(j)} f^{(k)}\right) = \beta^{\mathbf{0}} = (\mathbf{1}, \delta)$. Now $\beta \lhd e^{(j)} = \left(\mathbf{h}(\delta(j))_{\delta(j)}, \delta\right)$. So

$$\begin{aligned} \left(\beta \lhd e^{(j)}\right) \lhd f^{(k)} &= \left(\mathbf{h}(\delta(j))_{\delta(j)}, \delta\right)^{f(k)} \\ &= \left(\left(\left(\mathbf{h}(\delta(j))_{\delta(j)}\right)(\delta(k))\right)_{\delta(k)}, \delta\right). \quad {}^{*'} \end{aligned} \tag{6}$$

But since $\delta(j) \neq \delta(k)$, $\left(\mathbf{h}(\delta(j))_{\delta(j)}\right)(\delta(k)) = \mathbf{1}$. Hence the first component in $(*')$ takes value $\mathbf{1}$ for all i, and so $\left(\beta \lhd e^{(j)}\right) \lhd f^{(k)} = (\mathbf{1}, \delta)$.

- Case 2: $j = k$. Then $e^{(j)} f^{(j)} = (ef)^j \in B_J$. We compute $\beta \lhd (ef)^{(j)} = \left(\mathbf{h}(\delta(j))_{\delta(j)}, \delta\right)$ and

$$\left(\beta \lhd e^{(j)}\right) \lhd f^{(j)} = \left(\mathbf{h}(\delta(j))_{\delta(j)}, \delta\right)^{f(j)} = \left(\mathbf{h}(\delta(j))_{\delta(j)}, \delta\right).$$

Hence, both sides match and (SS3) is satisfied.
$(SS4)$: let $\beta = (\mathbf{h}, \delta)$ and $\gamma = (\mathbf{g}, \sigma)$. Now $\beta \cdot \gamma = (\mathbf{h}(\delta \cdot \mathbf{g}), \delta\sigma)$ and

$$\begin{aligned} (\beta \cdot \gamma)^1 &= \beta \cdot \gamma, \\ \beta^{\gamma \rhd 1} \cdot \gamma^1 &= \beta^1 \cdot \gamma^1 = \beta \cdot \gamma. \end{aligned}$$

Also,

$$(\beta\cdot\gamma)^0=(\mathbf{1},\delta\sigma),\ \ \beta^{\gamma\triangleright 0}\cdot\gamma^0=\beta^0\cdot\gamma^0=(\mathbf{1},\delta)(\mathbf{1},\sigma)=(\mathbf{1},\delta\sigma).$$

Now take $e=e^{(j)}$. Then:

$$(\beta\cdot\gamma)^{e^{(j)}}=(\mathbf{h}(\delta\cdot\mathbf{g}),\delta\sigma)^{e^{(j)}}=\Big((\mathbf{h}(\delta\cdot\mathbf{g})(\delta\sigma(j)))_{\delta\sigma(j)},\delta\sigma\Big)=(\mathbf{f},\delta\sigma).$$

where $\mathbf{f}:[n]\longmapsto G$ as defined by:

$$\mathbf{f}(k)=\left\{\begin{array}{ll}\mathbf{1} & \ldots\ldots\text{if } k\neq\delta\sigma(j)\\ \mathbf{h}(\delta\sigma(j))\mathbf{g}(\sigma(j)) & \ldots\ldots\text{if } k=\delta\sigma(j)\end{array}\right|$$

Now $\gamma^{e^{(j)}}=(\mathbf{g},\sigma)^{e^{(j)}}=\Big(\mathbf{g}(\sigma(j))_{\sigma(j)},\sigma\Big)$,

$$\beta^{\gamma\triangleright e^{(j)}}=\beta^{\mathbf{g}(\sigma(j))(e)^{(\sigma(j))}}=(\mathbf{h},\delta)^{q^{(\sigma(j))}}=\Big(\mathbf{h}(\delta\sigma(j))_{\delta\sigma(j)},\delta\Big).$$

Therefore:

$$\begin{aligned}\beta^{\gamma\triangleright e^{(j)}}\cdot\gamma^{e^{(j)}} &= \Big(\mathbf{h}(\delta\sigma(j))_{\delta\sigma(j)},\delta\Big)\Big(\mathbf{g}(\sigma(j))_{\sigma(j)},\sigma\Big)\\ &= \Big(\mathbf{h}(\delta\sigma(j))_{\delta\sigma(j)}[\delta\cdot(\mathbf{g}(\sigma(j))_{\sigma(j)}],\delta\sigma\Big)\\ &= (\mathbf{f}',\delta\sigma).\end{aligned}$$

where $\mathbf{f}':[n]\longmapsto G$ as defined as:

$$\delta\cdot\Big(\mathbf{g}(\sigma(j))_{\sigma(j)}\Big)(k) = \left\{\begin{array}{ll}1 & \text{if } \delta^{-1}(k)=\sigma(j),\\ \mathbf{g}(\sigma(j)) & \text{if } k=\ \delta\sigma(j)\end{array}\right|$$

and thus:

$$\mathbf{f}'(k)=\left\{\begin{array}{ll}1 & \text{if } k\neq\delta\sigma(j),\\ \mathbf{h}(\delta\sigma(j))\mathbf{g}(\sigma(j)) & \text{if } k=\ \delta\sigma(j)\end{array}\right|$$

So $\mathbf{f}'=\mathbf{f}$ and axiom $(SS4)$ verified. $(SS5)$ and $(SS6)$ hold by definition. For $(SS7)$: $1_\omega\triangleright e^{(j)}=(1,1)\triangleright e^{(j)}=e^{(j)}$. $(SS8):1_\omega^{e^{(j)}}=(1,1)^{e^{(j)}}=\Big(1(j)_j,1\Big)=(1,1)$.
As a special Case with $n=3$, let $B_1=\{a,b,c,d\}$, $B_2=\{e,f,g,h\}$, $B_3=\{j,k,l,m\}$, and $G=C_2=\{1,\tau\}$. We represent the band $\tilde{B}$ visually in Figure 8.

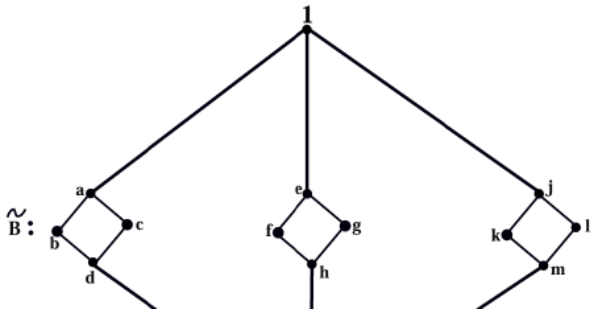


Figure 8. Special case $n=3$ and $G=C_2$. The picture displays the three copies of the band together with 0 and 1, making explicit how copy permutations and local C_2-actions combine in the wreath-product example.

Obvious automorphisms of $\tilde{B}$ either operate via G on a fixed copy of B_j, $j = 1,2,3$ e.g., $k \longmapsto l, l \longmapsto k$ fixing others, or permute the copies of cyclically:

$$\begin{aligned} a &\longmapsto e \longmapsto j \longmapsto a;\ b \longmapsto f \longmapsto k \longmapsto b;\ 0 \longmapsto 0, \\ c &\longmapsto g \longmapsto l \longmapsto c;\ d \longmapsto h \longmapsto m \longmapsto d;\ 1 \longmapsto 1. \end{aligned}$$

These automorphisms generates a group $\omega = C_2 \wr S_3 \cong C_2^{[n]} \rhd S_3$. For example, if $\mathbf{g} : \{1,2,3\} \longmapsto C_2$, with $\mathbf{g}(1) = \tau, \mathbf{g}(2) = 1, \mathbf{g}(3) = \tau$ and $\sigma = (1\,2\,3) \in S_3$. Then: $(\mathbf{g},\sigma)(k) = \mathbf{g}(\sigma(3))(b)^{\sigma(3)} = \tau(b) = c$, $(\mathbf{g},\sigma)(f) = \mathbf{g}(\sigma(2))(f)^{\sigma(2)} = \tau(k) = l$, $(\mathbf{g},\sigma)(b) = \mathbf{g}(\sigma(1))(b)^{\sigma(1)} = 1(f) = f$. Now $\tilde{B}$ operates on ω as: $(\mathbf{g},\sigma)^a = ((\mathbf{g}(2))_2, \sigma)$ since $j = 1, \sigma(j) = 2$, where $(\mathbf{g}(2))_2 : 1 \mapsto 1,\ 2 \mapsto 1,\ 3 \mapsto 1$. $(\mathbf{g},\sigma)^f = ((\mathbf{g}(3))_3, \sigma)$ where where $j = 2; \sigma(j) = 3$, and $(g(3))_2 : 1 \mapsto 1, 2 \mapsto 1, 3 \mapsto \tau$. Then the λ–Zappa–Szép product is:

$$B(\Im) = \left\{(b,g) : g^{-b} = g^{-1}\right\}.$$

So: $B(\Im) = \left\{\left(b^{(i)}, (\mathbf{g},\sigma)\right) : (\mathbf{g},\sigma)^{-e^{(i)}} = (\mathbf{g},\sigma)^{-1}\right\}$. Here $(\mathbf{g},\sigma)^{-e^{(i)}}$ is the map sending $\sigma(i) \mapsto \mathbf{g}(\sigma(i))^{-1}$ and other values to 1. $\{1,\ldots,n\}$ to 1.Meanwhile, $(\mathbf{g},\sigma)^{-1} = \left(\sigma^{-1}\cdot\mathbf{g}^{-1}, \sigma^{-1}\right)$ where $\sigma^{-1}\cdot\mathbf{g}^{-1}$ is a function takes $\sigma(j) \mapsto \mathbf{g}(\sigma(j))^{-1}$ for all j. So the condition matches only if $\mathbf{g}(\sigma(j)) = 1$ for all $j \neq i$. Hence

$$\begin{aligned} B(\Im) = (\{1\} \times \omega) &\\ \cup \left\{\left(b^{(i)}, ((g)_i, \sigma)\right) : b \in B,\ g \in G,\ \sigma \in S_n\right\} &\\ \cup \{(0, (\mathbf{1}, S_n))\}. & \end{aligned}$$

The final decomposition obtained from this construction is summarized in Figure 9. This additional diagram clarifies how the global object $B(\Im)$ is assembled from local semidirect product components and how the symmetric group permutes those components.

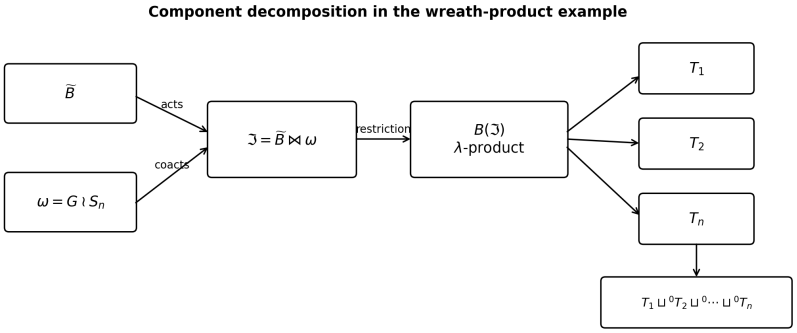


Figure 9. Component decomposition of the wreath-product example. The λ-Zappa–Szép product $B(\Im)$ contains local inverse subsemigroups T_i isomorphic to classical semidirect products, and these components combine through a 0-direct union together with the induced permutation action of S_n.

To structure the internal components of $B(\Im)$, fix an index $i \in \{1,2,3\}$, and define the subset: $T_i := \left\{\left(b^{(i)}, ((g)_i, id)\right) : b \in B, g \in G\right\}$. Each T_i is an inverse subsemigroup of $B(\Im)$, and it is isomorphic to the classical semidirect product $B \lhd G$, where G operates on B via automorphisms. Now let: $T := T_1 \sqcup^0 T_2 \sqcup^0 T_3$ denote the 0-dirct union of the $T_i's$-that is, the disjoint union of the $T_i's$ where cross-products between elements of different vanish. This construction is well-known in the study of bands and semilattices with local units [20,32]. Then the complement: $B(\Im) \setminus (\{1\} \times \omega)$

forms a two sided ideals of $B(\Im)$ and we have $I := T \lhd S_n$, where the symmetric group S_n operates on T by permuting the indices of the underlying components T_i. This kind of hierarchical decomposition is typical in semigroup theory and aligns with standard constructions of Zappa–Szép-type products over stratified bands and group structure maps.

3. Conclusions

This paper has presented an alternative route to the construction of λ-Zappa–Szép products of bands and groups. Starting from a matched pair of structure maps, we developed the arithmetic of the Zappa–Szép product and then used the groupoid perspective to recover the corresponding inverse-semigroup structure. The approach highlights the role of the two actions $\triangleright$ and $\triangleleft$ in controlling regularity, idempotents, inverses, unit-regularity, orthodox behaviour, and $\mathcal{L}$-unipotency.

A central point of the paper is that the passage from the band–group data to the inverse semigroup is not merely a direct elementwise algebraic verification; rather, the ESN framework provides a structural explanation through the associated inductive groupoid and pseudoproduct. The consolidated structural results show that several properties of the product can be read directly from the compatibility of the actions and from the internal structure of the band.

The examples further demonstrate that the construction is flexible. In the finite example involving a commutative band and the Klein four-group, the multiplication table makes the idempotent and orthodox structure explicit. In the wreath-product example, the enlarged band $\widetilde{B}$ and the group $G \wr S_n$ illustrate how local copies of a band can be combined with permutation symmetries to produce a richer λ-Zappa–Szép product. The component decomposition of $B(\Im)$ also shows that the resulting object contains natural inverse subsemigroups arranged as a 0-direct union, with the symmetric group acting by permuting the components.

In this sense, the present treatment complements the automata-theoretic approach of Gilbert and Wazzan [12] and connects it with the later restriction-semigroup viewpoint of Gould and Zenab [13]. The matched-pair notation also makes clear where the construction reduces to the semidirect or direct product cases, and where genuinely two-sided Zappa–Szép behaviour occurs. A further possible extension concerns the multi-object situation. Zappa–Szép products of categorical and self-similar structures have already appeared in related settings, for example in the work of Mundey and Sims [5], while ESN-type correspondences for inverse categories have been studied by DeWolf and Pronk [7]. These developments suggest that a semigroupoid or categorical analogue of the present matched-pair construction may be possible. Establishing such a generalization would, however, require additional axioms and lies beyond the scope of the present paper. Similarly, the one-sided ESN theorem of Haag, Lautenschlaeger and Tamusiunas [27] provides a broader categorical setting in which the groupoid obtained in the present paper can be interpreted and suggests a natural framework for future one-sided and multi-object generalizations of λ-Zappa–Szép products.

These observations suggest several directions for further work. One may investigate when the present construction preserves stronger finiteness, order-theoretic, or categorical properties; compare the resulting products with other semidirect and wreath-type constructions; and study how the graphical and groupoid descriptions can be used to classify examples arising from finite bands, semilattices, and self-similar group actions. Such developments would further clarify the place of λ-Zappa–Szép products within the broader theory of inverse semigroups and inductive groupoids.

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References

1. Zenab, R.E. Algebraic properties of Zappa–Sz ép products of semigroups and monoids. *Semigroup Forum* **2018**, *96*, 316–332. [CrossRef]
2. Özalan, N.U. New results over Zappa-Szép products via a recent semigroup. *Hacet. J. Math. Stat.* **2024**, 1–12. [CrossRef]
3. Brownlowe, N.; Pask, D.; Ramagge, J.; Robertson, D Whittaker, M.F. Zappa–Szép product groupoids and C^*-blends. *Semigroup Forum* **2017**, *94*, 500–519. [CrossRef]
4. Duwenig, A.; Li, B. The Zappa-Szép product of twisted groupoids. *arXiv* **2024**, arXiv:2406.00466.
5. Mundey, A.; Sims, A. Homology and twisted C^*-algebras for self-similar actions and Zappa–Szép products. *Results Math.* **2025**, *80*, 9. [CrossRef]
6. Zaman, B.U. On left P-Ehresmann and right Ehresmann semigroups: λ-lambda-semidirect products and Zappa-Szé p products. *Hacet. J. Math. Stat.* **2024**, *53*, 1039–1059. [CrossRef]
7. DeWolf, D.; Pronk, D. The Ehresmann-Schein-Nambooripad theorem for inverse categories. *arXiv* **2015**, arXiv:1507.08615.
8. Zappa, G. Sulla costruzione dei gruppi prodotto di due dati sottogruppi permutabili tra loro. In *Atti del Secondo Congresso dell'Unione Matematica Italiana, Bologna, Italy, 1940*; Edizioni Cremonese: Rome, Italy, 1942; pp. 119–125.
9. Szép, J. Über die als Produkt zweier Untergruppen darstellbaren endlichen Gruppen. *Comment. Math. Helv.* **1949**, *22*, 31–33. [CrossRef]
10. Kunze, M. Zappa products. *Acta Math. Hung.* **1983**, *41*, 225–239. [CrossRef]
11. Brin, M.G. On the Zappa–Szép product. *Commun. Algebra* **2005**, *33*, 393–424. [CrossRef]
12. Gilbert, N.D.; Wazzan, S. Zappa–Sz ép products of bands and groups. *Semigroup Forum* **2008**, *77*, 438–455. [CrossRef]
13. Gould, V.; Zenab, R.E. Restriction semigroups and λ-Zappa–Szép products. *Period. Math. Hung.* **2016**, *73*, 179–207. [CrossRef]
14. Wazzan, S. Zappa–Szép Groupoids of Inverse Semigroups and an Alternative Proof of Billhardt's λ-Semidirect Products. *Mathematics* **2025**, *13*, 1122. [CrossRef]
15. Nekrashevych, V. *Mathematical Surveys and Monographs*; American Mathematical Society: Providence, RI, USA, 2005; Volume 117.
16. Grigorchuk, R.I.; Nekrashevych, V.V.; Sushchanskii, V.I. Automata, dynamical systems, and groups. *Proc. Steklov Inst. Math.* **2000**, *231*, 128–203.
17. Lawson, M.V. A correspondence between a class of monoids and self-similar group actions I. *Semigroup Forum* **2008**, *76*, 489–517. [CrossRef]
18. Lawson, M.V.; Wallis, A.R. A correspondence between a class of monoids and self-similar group actions II. *Int. Algebra Comput.* **2015**, *25*, 633–668. [CrossRef]
19. Billhardt, B. On a wreath product embedding and idempotent pure congruences on inverse semigroups. *Semigroup Forum* **1992**, *45*, 45–54. [CrossRef]
20. Lawson, M.V. *Inverse Semigroups: The Theory of Partial Symmetries*; World Scientific: Singapore, 1998.
21. Hollings, C. Extending the Ehresmann-Schein-Nambooripad Theorem. *Semigroup Forum* **2010**, *80*, 453–476. [CrossRef]
22. Stokes, T. Ehresmann–Schein–Nambooripad theorems for classes of biunary semigroups. *Semigroup Forum* **2023**, *107*, 751–779. [CrossRef]
23. Wang, S. An Ehresmann–Schein–Nambooripad theorem for locally Ehresmann P-Ehresmann semigroups. *Period. Math.* **2020**, *80*, 108–137. [CrossRef]
24. Lawson, M.V. On Ehresmann semigroups. *Semigroup Forum* **2021**, *103*, 953–965. [CrossRef]

25. Steinberg, B.; Szakács, N. Simplicity of inverse semigroup and étale groupoid algebras. *Adv. Math.* **2021**, *380*, 107611. [CrossRef]
26. Amrutam, T.; Karmakar, P. Fejér property and Galois correspondence for groupoid C^*-algebras. *arXiv* **2025**, arXiv:2511.14474.
27. Haag, R.; Lautenschlaeger, W.G.; Tamusiunas, T. An Ehresmann–Schein–Nambooripad-type theorem for left restriction semigroupoids. *Semigroup Forum* **2026**, *112*, 820–860. [CrossRef]
28. Bailes, G.L., Jr. Right inverse semigroups. *J. Algebra* **1973**, *26*, 492–507. [CrossRef]
29. Venkatesan, P.S. On right unipotent semigroups. *Pac. J. Math.* **1976**, *63*, 555–561. [CrossRef]
30. Clifford, A.H. The structure of bisimple left unipotent semigroups as ordered pairs. *Semigroup Forum* **1972**, *5*, 137–144. [CrossRef]
31. Szendrei, M.B. Normal extensions and full restricted semidirect products of inverse semigroups. *Semigroup Forum* **2025**, *111*, 820–837. [CrossRef]
32. Clifford, A.H.; Preston, G.B. *The Algebraic Theory of Semigroups, Volume 1*; American Mathematical Society: Providence, RI, USA, 1961; Volume 2.
33. Wazzan, S. The Zappa-Szép Product of Semigroups. Ph.D. Thesis, Heriot-Watt University, Edinburgh, UK, 2008.

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