

Article

A New Density Hierarchy Based on Riesz and Hölder Summability

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Abstract

Let $f(n)$ be an Erdős–Ulam function satisfying $f(n+1)/f(n) \rightarrow 1$ as $n \rightarrow \infty$, and let $\mathbf{A}_f$ denote the associated Riesz matrix. Furthermore, let $\mathbf{A}_{f^*}$ be the Riesz matrix generated by the weights $f(n)/(f(1) + f(2) + \dots + f(n))$, $n \geq 1$. The main result establishes that the matrix density induced by an arbitrary power of $\mathbf{A}_f$ —or, equivalently, by a generalized Hölder mean of arbitrary order—is stronger than the matrix density induced by $\mathbf{A}_{f^*}$.

Keywords: weighted density; matrix density; summation method

MSC: 11B05; 40C05

1. Introduction

1.1. Weighted Densities

Let $\mathbb{N}$ denote the set of positive integers, and let χ_S denote the characteristic function of a given subset $S \subset \mathbb{N}$. Weighted densities were introduced and systematically studied in [1,2], where a common framework encompassing both asymptotic density and logarithmic density was developed.

A positive function $f: \mathbb{N} \rightarrow \mathbb{R}^+$ is called an *Erdős–Ulam function* if it satisfies $f(1) = 1$,

$$\sum_{n=1}^{\infty} f(n) = \infty, \quad (1)$$

and

$$\lim_{n \rightarrow \infty} f^*(n) = 0, \quad \text{where} \quad f^*(n) = \frac{f(n)}{\sum_{j=1}^n f(j)}. \quad (2)$$

With respect to an Erdős–Ulam function f , the corresponding *weighted densities* are defined as follows. For $S \subset \mathbb{N}$, let

$$F_S(n) = \frac{\sum_{j=1}^n f(j) \cdot \chi_S(j)}{\sum_{j=1}^n f(j)}.$$

Clearly, $0 \leq F_S(n) \leq 1$.

The lower and upper f -densities of S are then defined by



Academic Editors: Alvaro Anton-Sancho, Vladimir Mityushev and Baccouch Mahboub

Received: 26 June 2026

Revised: 3 August 2026

Accepted: 19 August 2026

Published: 25 August 2026

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$$\underline{d}_f(S) = \liminf_{n \rightarrow \infty} F_S(n) \quad \text{and} \quad \overline{d}_f(S) = \limsup_{n \rightarrow \infty} F_S(n),$$

respectively. In the case where $\underline{d}_f(S) = \overline{d}_f(S)$, we say that S possesses the f -density $d_f(S)$.

Note that the asymptotic density corresponds to the choice $f(n) = 1$, while the logarithmic density corresponds to $f(n) = \frac{1}{n}$.

In the remainder of this paper, densities will be understood as pairs of functions $\underline{d}, \overline{d}$ defined on all subsets of the natural numbers and satisfying the following conditions:

- For every $S \subset \mathbb{N}$,

$$0 \leq \underline{d}(S) \leq \overline{d}(S) \leq 1.$$

- If S is finite, then $\underline{d}(S) = \overline{d}(S) = 0$.
- For every $S_1, S_2 \subset \mathbb{N}$ with $S_1 \cap S_2 = \emptyset$,

$$\underline{d}(S_1) + \underline{d}(S_2) \leq \underline{d}(S_1 \cup S_2) \leq \underline{d}(S_1) + \overline{d}(S_2) \leq \overline{d}(S_1 \cup S_2) \leq \overline{d}(S_1) + \overline{d}(S_2).$$

We also note that, for every density d and every set $S \subset \mathbb{N}$, the following complementarity relation holds:

$$\overline{d}(S) = 1 - \underline{d}(\mathbb{N} \setminus S). \quad (3)$$

A density d' is said to be *stronger* than the density d (and d is *weaker* than d') if, for every set $S \subset \mathbb{N}$,

$$\underline{d}'(S) \leq \underline{d}(S) \leq \overline{d}(S) \leq \overline{d}'(S).$$

We denote this by $d \preceq d'$. The relation $\preceq$ is obviously a partial order on a set of densities. Two densities are called *comparable* if one is stronger than the other (see [3]). It is known that the logarithmic density is weaker than the asymptotic density. More generally, all n^α -densities with $\alpha \geq -1$ are comparable; indeed, if $-1 \leq \alpha < \beta$, then the n^α -density is weaker than the n^β -density.

We say that the density d *extends* a density d' if every set $S \subset \mathbb{N}$ that possesses density d' also possesses density d , and $d(S) = d'(S)$. Clearly, if d' is stronger than d , then d extends d' . If each of two densities extends the other, then they are said to be *equivalent*.

The densities d and d' are *strongly equivalent* if, for every set $S \subset \mathbb{N}$, we have $\underline{d}(S) = \underline{d}'(S)$ (and consequently $\overline{d}(S) = \overline{d}'(S)$).

Kuipers and Niederreiter [4] observed that all the n^α -densities with $\alpha > -1$ are equivalent to each other. The main tool for comparing weighted densities (see, e.g., [3,5–8]) is the classical result obtained by C. T. Rajagopal (cf. [9], Theorem 3), which, in terms of weighted densities, states the following: if for two Erdős–Ulam functions f and g the function $\frac{f(n)}{g(n)}$ is decreasing, then d_g is stronger than d_f .

The following sufficient condition can be found in the work by Hardy [10] (see also [4]): if $\frac{f(n)}{g(n)}$ is increasing and $\frac{f(n) \sum_{j=1}^n g(j)}{g(n) \sum_{j=1}^n f(j)}$ is bounded, then the f -density extends the g -density.

A survey of weighted densities and their connection to the first-digit problem is given in [11]. It is proved that if $f(n) \sim g(n)$ as $n \rightarrow \infty$, then the corresponding f -density and g -density are strongly equivalent.

Consider now the following sequence of functions. Let $f_0(n)$ be an Erdős–Ulam function, and define $f_k(n) = f_{k-1}^*(n)$ for $k \geq 1$. It follows from [12] that, for every $k \geq 1$, the function f_k is again an Erdős–Ulam function, and the corresponding f_{k+1} -density strictly extends the f_k -density. As a consequence of a more general theorem in [12], we know that for arbitrary real numbers

$$0 \leq \alpha_0 \leq \alpha_1 \leq \dots \leq \beta_1 \leq \beta_0 \leq 1$$

there exists a set $S \subset \mathbb{N}$ such that

$$\underline{d}_{f_0}(S) = \alpha_0, \quad \underline{d}_{f_1}(S) = \alpha_1, \quad \dots, \quad \bar{d}_{f_1}(S) = \beta_1, \quad \bar{d}_{f_0}(S) = \beta_0.$$

In [13], the relationships between weighted densities defined by Erdős–Ulam functions f and g were investigated. The authors showed that the relationship between the f -density and the g -density depends mainly on the asymptotic behavior of the function

$$\frac{f^*(n)}{g^*(n)} = \frac{\frac{f(n)}{\sum_{j=1}^n f(j)}}{\frac{g(n)}{\sum_{j=1}^n g(j)}}.$$

Depending on the asymptotic behavior of $\frac{f^*(n)}{g^*(n)}$, the following results can be obtained (see [13]):

- If $\frac{f^*(n)}{g^*(n)} \rightarrow 1$, then the d_f -density and the d_g -density are strongly equivalent.
- If $\frac{f^*(n)}{g^*(n)} \rightarrow p > 1$, then the f -density is stronger than the g -density, and the two densities are equivalent (but not strongly equivalent in general). Moreover, for every $S \subset \mathbb{N}$, the following sharp bounds hold:

$$\underline{d}_f(S) \geq \frac{\underline{d}_g(S)^p}{\underline{d}_g(S)^{p-1}},$$

and, if $\underline{d}_g(S) < 1$,

$$\bar{d}_f(S) \leq 1 - \frac{(1 - \bar{d}_g(S))^p}{(1 - \underline{d}_g(S))^{p-1}}.$$

These bounds are most possible in the sense of Theorem 4 in [13].

- If $\frac{f^*(n)}{g^*(n)} \rightarrow \infty$, and, moreover, $f(n)$ and $g(n)$ are monotone, then for arbitrary real numbers $0 \leq \alpha \leq \beta \leq \gamma \leq \delta \leq 1$, there exists a set $S \subset \mathbb{N}$ such that

$$\underline{d}_f(S) = \alpha, \quad \underline{d}_g(S) = \beta, \quad \bar{d}_g(S) = \gamma, \quad \bar{d}_f(S) = \delta.$$

1.2. Infinite Triangular Matrices

A matrix summation method is a method for summing sequences by means of an infinite matrix.

Consider a real infinite triangular matrix $\mathbf{A} = (a_{n,k})$, meaning that $a_{n,k} \in \mathbb{R}$ for all $n, k \in \mathbb{N}$ and $a_{n,k} = 0$ whenever $k > n$. Then, for a sequence $\bar{x} = (x_n)$ of real numbers, the sequence $\bar{y} = \mathbf{A}\bar{x} = (y_n)$ is given by

$$y_n = \sum_{k=1}^{\infty} a_{n,k} x_k,$$

Note that the series on the right-hand side always converges. If $\lim \mathbf{A}\bar{x} = L$ whenever $\lim \bar{x} = L$, then $\mathbf{A}$ is said to be *regular*.

The following lemma, known as the Toeplitz theorem, gives necessary and sufficient conditions for the regularity of a matrix (see [14]).

Lemma 1. *An infinite triangular matrix $\mathbf{A}$ is regular if and only if*

$$\sup_n \sum_{k=1}^{\infty} |a_{n,k}| < \infty,$$

$$\lim_{k \rightarrow \infty} a_{n,k} = 0 \quad \text{for each } n = 1, 2, 3, \dots,$$

and

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} a_{n,k} = 1.$$

Lemma 2 ([10], Theorem 9 on p. 52). *Let $\mathbf{A}$ be a non-negative regular infinite triangular matrix, and let $\bar{x}$ be a bounded sequence of real numbers. Then,*

$$\liminf \bar{x} \leq \liminf \mathbf{A}\bar{x} \leq \limsup \mathbf{A}\bar{x} \leq \limsup \bar{x}. \quad (4)$$

As in the finite-dimensional case, if $\mathbf{A} = (a_{n,k})$ and $\mathbf{B} = (b_{n,k})$ are real infinite triangular matrices, then their product $\mathbf{AB}$ is the matrix $\mathbf{C} = (c_{n,k})$, where

$$c_{n,k} = \sum_{i=1}^{\infty} a_{n,i} b_{i,k}.$$

The associative law of matrix multiplication does not generally hold for infinite matrices. However, if $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are infinite triangular matrices, then the following theorem holds.

Theorem 1 ([15], Theorems 2.2.5 and 2.2.6). *For arbitrary real infinite triangular matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, and for any sequence $\bar{x}$, the following statements hold:*

1. $\mathbf{A}(\mathbf{B}\bar{x}) = (\mathbf{AB})\bar{x}$
2. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$

It can easily be verified that the product of regular triangular matrices is again regular and triangular.

As in the finite case, we can define the notions of identity matrix and inverse matrix. The matrix $\mathbf{I} = (i_{n,k})$, where

$$i_{n,k} = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$$

is called the identity matrix. Clearly, for every matrix $\mathbf{A}$ and every sequence $\bar{x}$,

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A} \quad \text{and} \quad \mathbf{I}\bar{x} = \bar{x}.$$

An infinite matrix $\mathbf{A}$ is called *invertible* if there exists an infinite matrix $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I},$$

where $\mathbf{I}$ denotes the infinite identity matrix. The matrix $\mathbf{B}$ is then called the inverse of $\mathbf{A}$ and is denoted by $\mathbf{A}^{-1}$.

1.3. Matrix Densities

Definition 1.

$$\underline{d}_{\mathbf{A}}(S) = \liminf \mathbf{A}\bar{\chi}_S \quad \text{and} \quad \bar{d}_{\mathbf{A}}(S) = \limsup \mathbf{A}\bar{\chi}_S$$

are called the lower and upper $\mathbf{A}$ -density of the set S . In the case where $\underline{d}_{\mathbf{A}}(S) = \bar{d}_{\mathbf{A}}(S)$, we say that S has $\mathbf{A}$ -density $d_{\mathbf{A}}(S)$.

The next theorem follows immediately from Theorem 2.

Lemma 3. Let $\mathbf{A}$ and $\mathbf{B}$ be non-negative regular infinite matrices. Thus, the density $d_{\mathbf{AB}}$ is weaker than the density $d_{\mathbf{B}}$ ($d_{\mathbf{AB}} \preceq d_{\mathbf{B}}$); i.e., for each set $S \subset \mathbb{N}$,

$$0 \leq \underline{d}_{\mathbf{B}}(S) \leq \underline{d}_{\mathbf{AB}}(S) \leq \overline{d}_{\mathbf{AB}}(S) \leq \overline{d}_{\mathbf{B}}(S) \leq 1.$$

Proof. Let $S \subset \mathbb{N}$. If we substitute $\mathbf{B}\overline{\chi_S}$ instead of $\overline{x}$ into (4), we obtain the inequalities

$$\begin{aligned} \liminf \mathbf{B}\overline{\chi_S} &\leq \liminf \mathbf{A}(\mathbf{B}\overline{\chi_S}) = \liminf (\mathbf{AB})\overline{\chi_S} \\ &\leq \limsup (\mathbf{AB})\overline{\chi_S} = \limsup \mathbf{A}(\mathbf{B}\overline{\chi_S}) \leq \limsup \mathbf{B}\overline{\chi_S}, \end{aligned}$$

which are equivalent to the statement of the lemma. $\square$

Weighted densities are special cases of matrix densities since if $(f(n))$ is an Erdős–Ulam sequence, then for the matrix $\mathbf{A}_f = (a_{n,k})$, where

$$a_{n,k} = \begin{cases} \frac{f(k)}{\sum_{i=1}^n f(i)} & \text{if } k \leq n, \\ 0 & \text{if } k > n, \end{cases}$$

i.e.,

$$\mathbf{A}_f = \begin{pmatrix} \frac{f(1)}{\sum_{i=1}^1 f(i)} & 0 & 0 & \dots \\ \frac{f(1)}{\sum_{i=1}^2 f(i)} & \frac{f(2)}{\sum_{i=1}^2 f(i)} & 0 & \dots \\ \frac{f(1)}{\sum_{i=1}^3 f(i)} & \frac{f(2)}{\sum_{i=1}^3 f(i)} & \frac{f(3)}{\sum_{i=1}^3 f(i)} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

it obviously holds that $\underline{d}_f(S) = \underline{d}_{\mathbf{A}_f}(S)$ and $\overline{d}_f(S) = \overline{d}_{\mathbf{A}_f}(S)$ for all $S \subset \mathbb{N}$. The matrix $\mathbf{A}_f$ is called the Riesz matrix associated with the sequence $(f(n))$. It is known that $\mathbf{A}_f$ is regular if $\sum_{n=1}^{\infty} f(n) = \infty$.

The matrix $\mathbf{A}_f$ is invertible, and its inverse is

$$\mathbf{A}_f^{-1} = \begin{pmatrix} \frac{\sum_{i=1}^1 f(i)}{f(1)} & 0 & 0 & 0 & \dots \\ 1 - \frac{\sum_{i=1}^2 f(i)}{f(2)} & \frac{\sum_{i=1}^2 f(i)}{f(2)} & 0 & 0 & \dots \\ 0 & 1 - \frac{\sum_{i=1}^3 f(i)}{f(3)} & \frac{\sum_{i=1}^3 f(i)}{f(3)} & 0 & \dots \\ 0 & 0 & 1 - \frac{\sum_{i=1}^4 f(i)}{f(4)} & \frac{\sum_{i=1}^4 f(i)}{f(4)} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{f^*(1)} & 0 & 0 & 0 & \cdots \\ 1 - \frac{1}{f^*(2)} & \frac{1}{f^*(2)} & 0 & 0 & \cdots \\ 0 & 1 - \frac{1}{f^*(3)} & \frac{1}{f^*(3)} & 0 & \cdots \\ 0 & 0 & 1 - \frac{1}{f^*(4)} & \frac{1}{f^*(4)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The aim of this paper is to study the relationship between the matrix densities defined by the powers of $\mathbf{A}_f$ and the matrix density defined by the Riesz matrix associated with the sequence

$$\frac{f(n)}{f(1) + f(2) + \cdots + f(n)}, \quad n = 1, 2, \dots$$

Matrix limitation methods were applied to the study of weighted densities and their corresponding ideals in [16–20]. We also note that the connection between matrix summability methods and ideals has been extensively studied. In particular, Tryba [21] proved that an ideal on $\mathbb{N}$ is a matrix summability ideal if and only if it is a nonpathological generalized density ideal. This result provides a strong theoretical justification for the study of matrix densities and their hierarchies.

2. Main Results

For the reader's convenience, we summarize the main notation used in the proofs in Table 1.

Table 1. Summary of the main notation.

Notation	Description
$\mathbf{A} = (a_{n,k})$	Riesz matrix associated with f (see (8))
$\mathbf{B} = (b_{n,k}) = \mathbf{A}^{-1}$	Inverse of $\mathbf{A}$ (see (9))
$\mathbf{C} = (c_{n,k}) = \mathbf{A}_{f^*}$	Riesz matrix associated with f^* (see (10))
$\mathbf{C}_l = (c_{n,k}^{(l)})$	$\mathbf{C}$ with first l columns removed (see (11))
$\mathbf{B}_l = (b_{n,k}^{(l)})$	$\mathbf{B}$ with first l rows and columns removed (see (12))
$\mathbf{D}_l = (d_{n,k}^{(l)})$	Diagonal matrices (see (13)–(14))
$\mathbf{E} = (e_{n,k})$	Shift matrix representing column deletion
$f^*(n)$	$\frac{f(n)}{\sum_{j=1}^n f(j)}$
$\overline{x^{(k)}}$	$\mathbf{A}^k \bar{x}$

Let $\bar{x}$ be an arbitrary real sequence. Then, the sequence $\overline{h^{k+1}} = (h_n^{k+1})$, defined by the sequence-to-sequence transformation

$$h_n^{k+1} = \frac{h_1^k + h_2^k + \cdots + h_n^k}{n},$$

where $\overline{h^0} = \bar{x}$, is called the Hölder transform of $\bar{x}$ of the order $k + 1$. Obviously,

$$\overline{h^k} = \mathbf{A}_{(1)}^k \bar{x},$$

where $\mathbf{A}_{(1)}$ is the Riesz matrix corresponding to the asymptotic density.

Definition 2. Let $\mathbf{A}_{(1)}$ denote the Riesz matrix associated with the sequence $(f(n) = 1)$. Then, for every $k \in \mathbb{N}$, the density $h^{(k)}$ determined by the matrix $\mathbf{A}_{(1)}^k$ — i.e., for every $S \subset \mathbb{N}$,

$$\underline{h}^{(k)}(S) = \liminf \mathbf{A}_{(1)}^k \overline{\chi_S} \quad \text{and} \quad \overline{h}^{(k)}(S) = \limsup \mathbf{A}_{(1)}^k \overline{\chi_S},$$

— is called the k -th order Hölder density.

The classical Hölder mean of the order k is obtained by iterating the arithmetic mean k times. In matrix form, this corresponds precisely to the k -th power of the Riesz matrix $\mathbf{A}_{(1)}$, where $(f(n) = 1)$. Thus, $\mathbf{A}_{(1)}^k$ represents the k -th order Hölder transform. This observation motivates the following generalization.

Since the product of positive regular triangular matrices is again a positive regular triangular matrix, the powers of the $\mathbf{A}_f$ matrices also determine densities. On this basis, the Hölder densities can be generalized as follows.

Definition 3. Let $(f(n))$ be an Erdős–Ulam sequence, and let $\mathbf{A}_f$ denote the Riesz matrix associated with $(f(n))$. Then, for every $k \in \mathbb{N}$, the matrix $\mathbf{A}_f^k$ is called the k -th order Hölder matrix associated with $(f(n))$, and the density $d_{\mathbf{A}_f^k}$ determined by it is called the k -th order Hölder density associated with $(f(n))$.

According to Lemma 3, the sequence of densities $(d_{\mathbf{A}_f^k})_{k=1}^\infty$ is decreasing—i.e.,

$$d_{\mathbf{A}_f^1} \succeq d_{\mathbf{A}_f^2} \succeq \cdots \succeq d_{\mathbf{A}_f^n} \succeq d_{\mathbf{A}_f^{n+1}} \succeq \cdots \quad (5)$$

In this paper, we will also give a lower bound for the above sequence in terms of weighted density under certain conditions.

Theorem 2. Let $\{f(n)\}_{n=1}^\infty$ be an Erdős–Ulam sequence such that

$$\lim_{n \rightarrow \infty} \frac{f(n+1)}{f(n)} = 1. \quad (6)$$

Then, for every $l \in \mathbb{N}$,

$$d_{\mathbf{A}_f^l} \succeq d_{\mathbf{A}_{f^*}}.$$

The next theorem is a stronger formulation of Theorem 2.

Theorem 3. Let $(f(n))_{n=1}^\infty$ be an Erdős–Ulam sequence such that

$$\lim_{n \rightarrow \infty} \frac{f(n+1)}{f(n)} = 1.$$

Then, for every bounded sequence $\bar{x}$ and for every $l \in \mathbb{N}$,

$$\liminf \mathbf{A}_f^l \bar{x} \leq \liminf \mathbf{A}_{f^*} \bar{x} \leq \limsup \mathbf{A}_{f^*} \bar{x} \leq \limsup \mathbf{A}_f^l \bar{x}. \quad (7)$$

To prove this theorem, we will need some lemmas, for which we use the following notation.

Let $\mathbf{A} = (a_{n,k})$ denote $\mathbf{A}_f$ —i.e.,

$$a_{n,k} = \begin{cases} \frac{f(k)}{\sum_{j=1}^n f(j)} & \text{if } k \leq n, \\ 0 & \text{if } k > n, \end{cases} \quad (8)$$

and set $\mathbf{B} = (b_{n,k}) = \mathbf{A}_f^{-1}$, where

$$b_{n,k} = \begin{cases} \frac{1}{f^*(k)} & \text{if } k = n, \\ 1 - \frac{1}{f^*(k+1)} & \text{if } k = n-1, \\ 0 & \text{otherwise,} \end{cases} \quad (9)$$

And finally let $\mathbf{C} = (c_{n,k}) = \mathbf{A}_{f^*}$, where

$$c_{n,k} = \begin{cases} \frac{f^*(k)}{\sum_{j=1}^n f^*(j)} & \text{if } k \leq n, \\ 0 & \text{if } k > n. \end{cases} \quad (10)$$

From [13] (Lemma 1), it follows that $\lim_{n \rightarrow \infty} f^*(n) = 0$ and $\sum_{j=1}^{\infty} f^*(j) = \infty$; therefore, $\mathbf{C}$ is a regular Riesz matrix.

We will also need the following matrices. Let $\mathbf{C}_0 = \mathbf{C}$, and for each $l \in \mathbb{N}$, let $\mathbf{C}_l = (c_{n,k}^{(l)})$ be the matrix obtained by omitting the first column from $\mathbf{C}_{l-1}$ —i.e.,

$$c_{n,k}^{(l)} = c_{n,k+1}^{(l-1)} = c_{n,k+l},$$

so

$$c_{n,k}^{(l)} = \begin{cases} \frac{f^*(k+l)}{\sum_{j=1}^n f^*(j)} & \text{if } k \leq n-l, \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

The operation consisting of deleting the first column in an infinite matrix can be viewed as a linear transformation represented by the infinite regular matrix $\mathbf{E} = (e_{n,k})$, where

$$e_{n,k} = \begin{cases} 1 & \text{if } k = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Using this notation, we have $\mathbf{C}_l = \mathbf{C}\mathbf{E}^l$.

Furthermore, let $\mathbf{B}_0 = \mathbf{B}$, and for each $l \in \mathbb{N}$, let $\mathbf{B}_l = (b_{n,k}^{(l)})$ be the matrix obtained from $\mathbf{B}_{l-1}$ by omitting its first row and first column—i.e.,

$$b_{n,k}^{(l)} = b_{n+1,k+1}^{(l-1)} = b_{n+l,k+l},$$

so

$$b_{n,k}^{(l)} = \begin{cases} \frac{1}{f^*(k+l)} & \text{if } k = n, \\ 1 - \frac{1}{f^*(k+l+1)} & \text{if } k = n-1, \\ 0 & \text{otherwise.} \end{cases} \quad (12)$$

Finally, let $\mathbf{D}_1 = (d_{n,k}^{(1)})$, where

$$d_{n,k}^{(1)} = \begin{cases} \frac{1}{\sum_{j=1}^n f^*(j)} = c_{n,1} & \text{if } k = n, \\ 0 & \text{otherwise,} \end{cases} \quad (13)$$

and for each $l \in \mathbb{N}$ with $l \geq 2$, let $\mathbf{D}_l = (d_{n,k}^{(l)})$ be the matrix obtained from $\mathbf{D}_{l-1}$ by omitting the first column—i.e.

$$d_{n,k}^{(l)} = d_{n,k+1}^{(l-1)} = d_{n,k+l-1}^{(1)},$$

so

$$d_{n,k}^{(l)} = \begin{cases} c_{n,1} & \text{if } k = n-l+1, \\ 0 & \text{otherwise.} \end{cases} \quad (14)$$

Lemma 4. Let l be a positive integer. For the matrices defined in (11)–(14), we have

$$\mathbf{C}_{l-1} \mathbf{B}_{l-1} = \mathbf{C}_l + \mathbf{D}_l.$$

Proof. Set $\mathbf{H}_l = (h_{n,k}^{(l)}) = \mathbf{C}_{l-1} \mathbf{B}_{l-1}$. Then,

$$\begin{aligned} h_{n,k}^{(l)} &= \sum_{i=1}^{\infty} c_{n,i}^{(l-1)} b_{i,k}^{(l-1)} = \sum_{i=1}^{\infty} c_{n,i+l-1} b_{i+l-1,k+l-1} \\ &= c_{n,k+l-1} b_{k+l-1,k+l-1} + c_{n,k+l} b_{k+l,k+l-1}. \end{aligned}$$

We distinguish three cases.

1. If $k > n-l+1$, then

$$h_{n,k}^{(l)} = 0.$$

2. If $k = n-l+1$, then

$$h_{n,k}^{(l)} = h_{n,n-l+1}^{(l)} = c_{n,k+l-1} b_{k+l-1,k+l-1} = c_{n,n} b_{n,n} = c_{n,1} = d_{n,n-l+1}^{(l)}.$$

3. If $k < n-l+1$, then

$$\begin{aligned} h_{n,k}^{(l)} &= c_{n,k+l-1} b_{k+l-1,k+l-1} + c_{n,k+l} b_{k+l,k+l-1} \\ &= \frac{f^*(k+l-1)}{\sum_{i=1}^n f^*(i)} \cdot \frac{1}{f^*(k+l-1)} + \frac{f^*(k+l)}{\sum_{i=1}^n f^*(i)} \left(1 - \frac{1}{f^*(k+l)}\right) \\ &= \frac{f^*(k+l)}{\sum_{i=1}^n f^*(i)} = c_{n,k+l} = c_{n,k}^{(l)}. \end{aligned}$$

It follows that

$$\mathbf{C}_{l-1} \mathbf{B}_{l-1} = \mathbf{C}_l + \mathbf{D}_l.$$

□

Lemma 5. Let $l \geq 2$ be a positive integer. For the matrices defined in (9)–(14), we have

$$\mathbf{CB}^l = \mathbf{C}_l + \sum_{k=1}^{l-1} \mathbf{C}_k(\mathbf{B} - \mathbf{B}_k)\mathbf{B}^{l-k-1} + \sum_{k=1}^l \mathbf{D}_k\mathbf{B}^{l-k}. \quad (15)$$

Proof. The proof will be obtained via mathematical induction.

1. The case $l = 2$: Through Lemma 4, $\mathbf{CB} = \mathbf{C}_0\mathbf{B}_0 = \mathbf{C}_1 + \mathbf{D}_1$, so

$$\begin{aligned} \mathbf{CB}^2 &= (\mathbf{C}_1 + \mathbf{D}_1)\mathbf{B} \\ &= \mathbf{C}_1\mathbf{B}_1 + \mathbf{C}_1(\mathbf{B} - \mathbf{B}_1) + \mathbf{D}_1\mathbf{B} \\ &= \mathbf{C}_2 + \mathbf{C}_1(\mathbf{B} - \mathbf{B}_1) + \mathbf{D}_2 + \mathbf{D}_1\mathbf{B}. \end{aligned}$$

2. Suppose that (15) holds for some natural number $l \geq 2$. Then, via Lemma 4,

$$\begin{aligned} \mathbf{CB}^{l+1} &= \left(\mathbf{C}_l + \sum_{k=1}^{l-1} \mathbf{C}_k(\mathbf{B} - \mathbf{B}_k)\mathbf{B}^{l-k-1} + \sum_{k=1}^l \mathbf{D}_k\mathbf{B}^{l-k} \right) \mathbf{B} \\ &= \mathbf{C}_l\mathbf{B} + \sum_{k=1}^{l-1} \mathbf{C}_k(\mathbf{B} - \mathbf{B}_k)\mathbf{B}^{l-k} + \sum_{k=1}^l \mathbf{D}_k\mathbf{B}^{l-k+1} \\ &= \mathbf{C}_l\mathbf{B}_l + \mathbf{C}_l(\mathbf{B} - \mathbf{B}_l) + \sum_{k=1}^{l-1} \mathbf{C}_k(\mathbf{B} - \mathbf{B}_k)\mathbf{B}^{l-k} + \sum_{k=1}^l \mathbf{D}_k\mathbf{B}^{l-k+1} \\ &= \mathbf{C}_{l+1} + \mathbf{D}_{l+1} + \sum_{k=1}^l \mathbf{C}_k(\mathbf{B} - \mathbf{B}_k)\mathbf{B}^{l-k} + \sum_{k=1}^l \mathbf{D}_k\mathbf{B}^{l-k+1} \\ &= \mathbf{C}_{l+1} + \sum_{k=1}^{(l+1)-1} \mathbf{C}_k(\mathbf{B} - \mathbf{B}_k)\mathbf{B}^{(l+1)-k-1} + \sum_{k=1}^{l+1} \mathbf{D}_k\mathbf{B}^{(l+1)-k}. \end{aligned}$$

□

Lemma 6. Let $l \geq 2$ be a positive integer. For the matrices defined in (8)–(14), we have

$$\mathbf{C} = \mathbf{C}_l\mathbf{A}^l + \sum_{k=1}^{l-1} \mathbf{C}_k(\mathbf{B} - \mathbf{B}_k)\mathbf{A}^{k+1} + \sum_{k=1}^l \mathbf{D}_k\mathbf{A}^k. \quad (16)$$

Proof. It is enough to multiply Equation (15) on the right by $\mathbf{A}^l$. □

Lemma 7. For the matrices defined in (13) and (14), for every natural number l and every bounded sequence $\bar{x} = (x_n)$, it holds that

$$\lim \mathbf{D}_l \bar{x} = 0.$$

Proof. Set $\bar{y} = (y_n) = \mathbf{D}_l \bar{x}$. Then, if $n \geq l$,

$$y_n = \sum_{i=1}^{\infty} d_{n,i}^{(l)} x_i = d_{n,n-l+1} x_{n-l+1} = c_{n,1} x_{n-l+1}.$$

From the regularity of $\mathbf{C}$, it follows that $\lim_{n \rightarrow \infty} c_{n,1} = 0$. Hence, through the boundedness of the sequence $\bar{x}$ and the preceding equation, we obtain $\lim_{n \rightarrow \infty} y_n = 0$. □

Lemma 8. Let $(f(n))_{n=1}^{\infty}$ be a weight sequence such that

$$\lim_{n \rightarrow \infty} \frac{f(n+1)}{f(n)} = 1.$$

Then,

$$\lim_{n \rightarrow \infty} \frac{f^*(n+1)}{f^*(n)} = 1.$$

Proof. This follows immediately from the identity

$$\frac{f^*(n+1)}{f^*(n)} = \frac{f(n+1)}{\sum_{i=1}^{n+1} f(i)} \cdot \frac{\sum_{i=1}^{n+1} f(i) - f(n+1)}{f(n)} = \frac{f(n+1)}{f(n)} (1 - f^*(n+1))$$

and the fact that $\lim_{n \rightarrow \infty} f^*(n) = 0$. $\square$

Lemma 9. Let $(f(n))_{n=1}^{\infty}$ be a weight sequence such that

$$\lim_{n \rightarrow \infty} \frac{f(n+1)}{f(n)} = 1. \quad (17)$$

Then, for every natural number k and for every bounded sequence $\bar{x} = (x_n)_{n \in \mathbb{N}}$, for the matrices defined in (8), (9) and (12), it holds that

$$\lim (\mathbf{B} - \mathbf{B}_k) \mathbf{A} \bar{x} = 0.$$

Proof. Set $\mathbf{G} = (g_{n,m}) = \mathbf{B} - \mathbf{B}_k$. From the definition of $\mathbf{B}$ and $\mathbf{B}_k$,

$$g_{n,m} = \begin{cases} \frac{1}{f^*(n)} - \frac{1}{f^*(n+k)} & \text{if } m = n, \\ -\left(\frac{1}{f^*(n)} - \frac{1}{f^*(n+k)}\right) & \text{if } m = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $\bar{y} = (y_n) = \mathbf{A} \bar{x}$ and $\bar{z} = (z_n) = (\mathbf{B} - \mathbf{B}_k) \mathbf{A} \bar{x} = (\mathbf{B} - \mathbf{B}_k) \bar{y}$. Then, from the definition of $\mathbf{G}$,

$$z_n = \sum_{i=1}^{\infty} g_{n,i} y_i = g_{n,n-1} y_{n-1} + g_{n,n} y_n = \left(\frac{1}{f^*(n)} - \frac{1}{f^*(n+k)} \right) (y_n - y_{n-1}). \quad (18)$$

On the other hand,

$$\begin{aligned} y_n - y_{n-1} &= \sum_{i=1}^n a_{n,i} x_i - \sum_{i=1}^{n-1} a_{n-1,i} x_i \\ &= a_{n,n} x_n - \sum_{i=1}^{n-1} (a_{n-1,i} - a_{n,i}) x_i \\ &= \frac{f(n)}{\sum_{j=1}^n f(j)} x_n - \sum_{i=1}^{n-1} \left(\frac{f(i)}{\sum_{j=1}^{n-1} f(j)} - \frac{f(i)}{\sum_{j=1}^n f(j)} \right) x_i \\ &= \frac{f(n)}{\sum_{j=1}^n f(j)} x_n - \sum_{i=1}^{n-1} \frac{f(n)f(i)}{\sum_{j=1}^{n-1} f(j) \sum_{j=1}^n f(j)} x_i \\ &= \frac{f(n)}{\sum_{j=1}^n f(j)} \left(x_n - \sum_{i=1}^{n-1} \frac{f(i)}{\sum_{j=1}^{n-1} f(j)} x_i \right) \\ &= \frac{f(n)}{\sum_{j=1}^n f(j)} \left(x_n - \sum_{i=1}^{n-1} a_{n-1,i} x_i \right) \\ &= f^*(n) (x_n - y_{n-1}). \end{aligned}$$

Since the sequence $\bar{x}$ is bounded, the sequence $\bar{y} = \mathbf{A}\bar{x}$ is also bounded (by the regularity of $\mathbf{A}$). Hence, there exists a real number L such that $|x_n - y_{n-1}| < L$, so

$$|y_n - y_{n-1}| < f^*(n)L. \quad (19)$$

Using (17)–(19), and Lemma 8, we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} |z_n| &\leq \limsup_{n \rightarrow \infty} \left| \frac{1}{f^*(n)} - \frac{1}{f^*(n+k)} \right| f^*(n)L = L \limsup_{n \rightarrow \infty} \left| 1 - \frac{f^*(n)}{f^*(n+k)} \right| \\ &= L \limsup_{n \rightarrow \infty} \left| 1 - \prod_{i=0}^{k-1} \frac{f^*(n+i)}{f^*(n+i+1)} \right| = L \left(1 - \prod_{i=0}^{k-1} \lim_{n \rightarrow \infty} \frac{f^*(n+i)}{f^*(n+i+1)} \right) = 0. \end{aligned}$$

□

Lemma 10. For every natural number l , the matrix $\mathbf{C}_l$ defined in (11) is non-negative and regular.

Proof. The assertion follows immediately from the fact that $\mathbf{C}_l = \mathbf{C}\mathbf{E}^l$, being a product of non-negative regular matrices. □

Now, we can proceed to the proof of Theorem 3.

Proof of Theorem 3. Since the sequence of densities $(d_{\mathbf{A}_f^k})_{k=1}^\infty$ is decreasing by (5), it is enough to prove the statement for $l \geq 2$; the case $l = 1$ then follows immediately from the case $l = 2$ and (5). With the previous notation, (7) can be written as

$$\liminf \mathbf{A}^l \bar{x} \leq \liminf \mathbf{C} \bar{x} \leq \limsup \mathbf{C} \bar{x} \leq \limsup \mathbf{A}^l \bar{x}.$$

We first show that for any bounded sequence $\bar{x}$,

$$\liminf \mathbf{C} \bar{x} = \liminf \mathbf{C}_l \mathbf{A}^l \bar{x} \quad \text{and} \quad \limsup \mathbf{C} \bar{x} = \limsup \mathbf{C}_l \mathbf{A}^l \bar{x}. \quad (20)$$

Through Lemma 6,

$$\begin{aligned} \mathbf{C} \bar{x} &= \mathbf{C}_l \mathbf{A}^l \bar{x} + \sum_{k=1}^{l-1} \mathbf{C}_k (\mathbf{B} - \mathbf{B}_k) \mathbf{A}^{k+1} \bar{x} + \sum_{k=1}^l \mathbf{D}_k \mathbf{A}^k \bar{x} \\ &= \mathbf{C}_l \mathbf{A}^l \bar{x} + \sum_{k=1}^{l-1} \mathbf{C}_k (\mathbf{B} - \mathbf{B}_k) \mathbf{A} \overline{x^{(k)}} + \sum_{k=1}^l \mathbf{D}_k \overline{x^{(k)}}, \end{aligned} \quad (21)$$

where $\overline{x^{(k)}} = \mathbf{A}^k \bar{x}$. Since $\mathbf{A}$ is non-negative and regular, the sequence $\overline{x^{(k)}}$ is bounded. Through Lemma 7,

$$\lim \sum_{k=1}^l \mathbf{D}_k \overline{x^{(k)}} = \sum_{k=1}^l \lim \mathbf{D}_k \overline{x^{(k)}} = 0. \quad (22)$$

Using the regularity of $\mathbf{C}_k$ and Lemma 9,

$$\lim \sum_{k=1}^{l-1} \mathbf{C}_k (\mathbf{B} - \mathbf{B}_k) \mathbf{A} \overline{x^{(k)}} = \sum_{k=1}^{l-1} \lim \mathbf{C}_k (\mathbf{B} - \mathbf{B}_k) \mathbf{A} \overline{x^{(k)}} = 0. \quad (23)$$

Using (21)–(23) immediately gives (20).

Since $\mathbf{C}_l$ is non-negative and regular via Lemma 10, the statement of the theorem follows immediately from Lemma 3. □

If the Erdős–Ulam sequence $f(n) = 1$ is chosen, then the matrix $\mathbf{A}_f$ determines the asymptotic density. In this case, $f^*(n) = 1/n$ (up to normalization), and hence $\mathbf{A}_{f^*}$ is

precisely the Riesz matrix generating the logarithmic density. Furthermore, Condition (6) is obviously satisfied. Therefore, through Theorem 2 and (5), we may conclude that the Hölder density of arbitrary order is stronger than the logarithmic density and weaker than the asymptotic density.

3. Examples

In this section, we illustrate the scope of Theorem 2 with several concrete examples.

First, if we choose the Erdős–Ulam sequence $f_0(n) = 1$, then $f_1(n) = f_0^*(n) = 1/n$. Theorem 2 recovers the classical hierarchy between asymptotic density, Hölder densities, and logarithmic density:

$$d_{\text{asymptotic}} = d_{f_0} \succeq d_{\mathbf{A}_{f_0}^1} \succeq d_{\mathbf{A}_{f_0}^2} \succeq \cdots \succeq d_{f_1} = d_{\text{logarithmic}}.$$

Next, for $g_0(n) = n^\alpha$ with $\alpha > -1$, Condition (6) is satisfied, and

$$g_1(n) = g_0^*(n) = \frac{n^\alpha}{\sum_{j=1}^n j^\alpha} \sim \frac{\alpha + 1}{n}.$$

Hence, the density induced by $\mathbf{A}_{g_1}$ is strongly equivalent to the logarithmic density. Therefore, Theorem 2 shows that every generalized Hölder density generated by $g_0(n) = n^\alpha$ is stronger than the logarithmic density.

More generally, let f_0 be any Erdős–Ulam sequence satisfying Condition (6), and define $f_{k+1} = f_k^*$ for $k \geq 0$. By Lemma 8, every f_k is again an Erdős–Ulam sequence satisfying (6). From the results in the introduction, the corresponding weighted densities form a decreasing sequence:

$$d_{f_0} \succeq d_{f_1} \succeq d_{f_2} \succeq \cdots.$$

Theorem 2 now implies that for each $k \geq 0$ and each $l \geq 1$,

$$d_{\mathbf{A}_{f_k}^l} \succeq d_{f_{k+1}}.$$

Thus, between every two consecutive terms of the weighted density hierarchy, we may insert an entire decreasing sequence of generalized Hölder densities, with equality at the first step:

$$d_{f_k} = d_{\mathbf{A}_{f_k}^1} \succeq d_{\mathbf{A}_{f_k}^2} \succeq d_{\mathbf{A}_{f_k}^3} \succeq \cdots \succeq d_{f_{k+1}}.$$

A further observation reveals a deeper structural phenomenon. The asymptotic density is equivalent to all weighted densities induced by sequences of the form $g_{0,\alpha}(n) = n^\alpha$ with $\alpha > -1$, in the sense that every set possessing asymptotic density has the same $g_{0,\alpha}$ -density. However, this equivalence is not strong in general: the lower and upper $g_{0,\alpha}$ -densities of a set may differ from its lower and upper asymptotic densities when the set does not possess an asymptotic density. Similarly, the logarithmic density is equivalent to any density generated by $g_{1,\alpha}(n) = \max\{1, (\log n)^\alpha\}/n$ with $\alpha > -1$ but again not strongly equivalent in general.

More generally, at each level of the iteration, there is a whole continuum of equivalent weighted densities for every $\alpha > -1$:

$$g_{0,\alpha}(n) = n^\alpha, \quad g_{1,\alpha}(n) \sim \frac{(\log n)^\alpha}{n}, \quad g_{2,\alpha}(n) \sim \frac{(\log \log n)^\alpha}{n \log n}, \quad \dots$$

All sequences in the same row (i.e., with the same k) generate equivalent densities (in the sense that sets possessing one density have the same value for the other), while Theorem 2 ensures that any generalized Hölder density generated from a sequence in row k is stronger

than any density in row $k + 1$. Thus, the hierarchy is stable under these equivalences, and the parameter α does not affect the comparison; it merely reflects the freedom to choose different weights within the same density class.

For the equivalent classes $g_{k,\alpha}$ introduced above, the hierarchy takes the form

$$d_{g_{k,\alpha}} = d_{\mathbf{A}_{g_{k,\alpha}}^1} \succeq d_{\mathbf{A}_{g_{k,\alpha}}^2} \succeq d_{\mathbf{A}_{g_{k,\alpha}}^3} \succeq \cdots \succeq d_{g_{k,\alpha}^*} \sim d_{f_{k+1}}.$$

Here, $g_{k,\alpha}^*$ denotes the transformed sequence, which is strongly equivalent to f_{k+1} ; hence, the lower bound is—up to strong equivalence—the density induced by f_{k+1} .

The Table 2 summarizes these examples.

Table 2. Equivalent weighted density classes and the resulting hierarchy. Sequences in the same row generate equivalent densities in the sense that every set possessing one of the densities has the same value for all others in the class; however, the equivalence is not necessarily strong. Theorem 2 implies that every generalized Hölder density generated from a sequence in row k is stronger than every density in row $k + 1$.

Level	Typical Sequence	Equivalent Class ($\alpha > -1$)	Lower Bound
0	$f_0(n) = 1$	$g_{0,\alpha}(n) = n^\alpha$	d_{f_1} (logarithmic)
1	$f_1(n) = \frac{1}{n}$	$g_{1,\alpha}(n) \sim \frac{(\log n)^\alpha}{n}$	d_{f_2}
2	$f_2(n) \sim \frac{1}{n \log n}$	$g_{2,\alpha}(n) \sim \frac{(\log \log n)^\alpha}{n \log n}$	d_{f_3}
3	$f_3(n) \sim \frac{1}{n \log n \log \log n}$	$g_{3,\alpha}(n) \sim \frac{(\log \log \log n)^\alpha}{n \log n \log \log n}$	d_{f_4}
$\vdots$	$\vdots$	$\vdots$	$\vdots$

4. Discussion

Theorem 2 establishes that, under the asymptotic regularity condition $\lim_{n \rightarrow \infty} f(n+1)/f(n) = 1$, every Hölder-type density generated by a power of the Riesz matrix $\mathbf{A}_f$ is stronger than the weighted density induced by $\mathbf{A}_{f^*}$. Consequently, the hierarchy of densities in (5) admits a nontrivial lower bound within the class of weighted densities.

The result extends the known relationship between weighted densities and matrix densities and provides a unified framework for studying density hierarchies arising from summability methods. In particular, it clarifies the role of the transformation $f \mapsto f^*$ in connecting Riesz densities with generalized Hölder densities.

A noteworthy feature of the proof is that the density comparison is derived from the stronger sequence inequality stated in Theorem 3. This shows that the result is not merely set-theoretic but valid for arbitrary bounded sequences. Such a formulation may be useful in further investigations of matrix summability methods, statistical convergence, and density-based notions of convergence.

Condition (6) is not merely technical. If the limit $\lim_{n \rightarrow \infty} f(n+1)/f(n)$ exists for an Erdős–Ulam sequence, then it must necessarily be equal to 1. Indeed, if the limit were $p > 1$, then $f^*(n)$ would not tend to 0; if $p < 1$, then $\sum f(n)$ would converge, contradicting the Erdős–Ulam property. Thus, within the class of Erdős–Ulam sequences for which this limit exists, the condition of the theorem is optimal.

We note, however, that there exist Erdős–Ulam sequences for which the ratio $f(n+1)/f(n)$ does not converge (e.g., oscillating sequences). For such sequences, Condition (6) is not applicable (and hence not necessary for the validity of the theorem); their study lies beyond the scope of the present paper.

Several directions for future research remain open. It would be natural to determine whether Condition (6) can be weakened in other ways, characterize the cases in which

the comparison becomes an equivalence, and investigate analogous hierarchies for more general classes of regular matrices.

Author Contributions: Conceptualization, J.T.T.; methodology, F.F. and J.T.T.; formal analysis, F.F.; investigation, F.F. and J.B.; writing—original draft preparation, F.F.; writing—review and editing, J.B.; visualization, F.F. and J.B.; supervision, J.T.T.; project administration, J.T.T.; funding acquisition, J.T.T. All authors have read and agreed to the published version of the manuscript.

Funding: The authors gratefully acknowledge the support provided by the Scientific Grant Agency of the Ministry of Education, Research, Development and Youth of the Slovak Republic (VEGA) under projects VEGA 1/0493/25 (J.T.T. and F.F.).

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflicts of interest.

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