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Hypergraph Theory Applied to Quantum Structures

Milan Bubák † and **Mirko Navara** *,†

Department of Cybernetics, Faculty of Electrical Engineering, Czech Technical University in Prague, CZ-166 27 Prague, Czech Republic; bubakmil@fel.cvut.cz

* Correspondence: navara@fel.cvut.cz

† These authors contributed equally to this work.

Abstract

We study the maximum possible number of edges in certain classes of hypergraphs with a given number of vertices. Our approach uses a generalized notion of a path, which need not consist solely of a sequence of vertices and may terminate in an edge. Certain hypergraphs, known as Greechie diagrams, are used to model events in quantum experiments described by orthoalgebras, orthomodular posets, and orthomodular lattices. Previous work has investigated Greechie diagrams with "many" edges. We apply our method to estimate the minimum possible size of such examples.

Keywords: hypergraph; girth; quantum structure; orthoalgebra; orthomodular poset; orthomodular lattice; group-valued measure

MSC: 05C35; 05C15; 05C65; 28B10; 28B15; 81P10

1. Introduction

This paper makes two principal contributions. First, it establishes bounds on the maximum number of edges in certain hypergraphs with a given number of vertices. Second, it applies a similar technique to Greechie diagrams, i.e., hypergraphs used to represent non-distributive algebras of quantum events, including orthomodular lattices, orthomodular posets, and orthoalgebras. In these algebras, some suprema and infima need not exist; consequently, the operations "and" and "or" may be only partially defined. This feature makes it possible to model systems containing events that can be tested separately but not simultaneously. The state of such a system is described by a probability measure defined for every event, although it may not be defined for their conjunction or disjunction. This situation is unavoidable in quantum systems as a consequence of Heisenberg's uncertainty principle. It also arises in other fields, such as sociology, psychology, and artificial intelligence, where it has so far received relatively little attention. Our aim is to explore what graph theory can contribute to the study of quantum structures and, conversely, what the study of quantum structures can contribute to graph theory.

In the final part of the paper, we use these results to derive lower bounds on the size of orthoalgebras, orthomodular posets, and orthomodular lattices that do not admit group-valued measures. We compare these estimates with known examples that provide upper bounds on complexity.

The paper is organized as follows: In Section 2, we introduce hypergraphs and the necessary notions for their study. Then, we define orthoalgebras and formulate their representation by hypergraphs, called Greechie diagrams. The central focus is on extremal problems in hypergraph theory; we determine bounds for the maximal number of edges

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of hypergraphs with a given number of vertices. We derive upper bounds for Greechie diagrams in Section 3 and for less general uniform hypergraphs in Section 4. Lower bounds are specified in Section 5. In Section 6, we discuss the consequences for the minimal complexity of quantum structures without (group-valued) measures. Conclusions in Section 7 discuss some of our assumptions and possibilities of further research.

2. Basic Notions

2.1. Hypergraphs

Let us introduce the terminology and notation on hypergraphs, following [1–4].

Definition 1. *A hypergraph is a couple (V, E), where V is a set of vertices and E is its covering by subsets called hyperedges (here briefly edges). We denote by $n = |V|$ the number of vertices and by $m = |E|$ the number of edges. The degree $d(v)$ of a vertex v is the number of edges containing it. Hypergraph (V, E) is called r-uniform if $|e| = r$ for all $e \in E$.*

Unlike edges in ordinary graphs, hyperedges may contain more than two vertices. As sets, they are not ordered. Here, we shall restrict attention to *finite* hypergraphs.

Our arguments are based on counting paths. In a graph, a *path* is defined as a sequence of vertices, such that subsequent vertices are connected by an edge. The edges are uniquely determined. This need not be the case in hypergraphs, where we must specify the edges as well. Here, we generalize the notion of path so that it may start and end not only with a vertex but also possibly with an edge.

Definition 2. *A generalized path (here briefly a path) in a hypergraph is an alternating sequence of mutually distinct vertices and edges such that the two subsequent entries are adjacent. The type of a path is a couple (a, b), where a is the number of vertices, and b is the number of edges of the path. We denote by $N_{a,b}$ the number of all (directed) paths of type (a, b).*

In particular,

- $N_{1,0} = n$ is the number of vertices;
- $N_{0,1} = m$ is the number of edges;
- $N_{1,1}$ is the number of adjacent pairs vertex-edge.

The notions from Definitions 1 and 2 depend on the hypergraph, which should be mentioned as their argument. We omit it because the corresponding hypergraph will be clear from the context.

Obviously, if (a, b) is the type of an edge, then $|a - b| \leq 1$. If $a = b$, it is not clear whether the path starts with a vertex and ends with an edge, or vice versa, e.g., consider hypergraph $(V, \{e\})$, where $V = e = \{v_1, v_2\}$. We want $N_{1,1} = 2$, the number of paths (v_1, e), (v_2, e). We do not want to count the paths (e, v_1), (e, v_2), which are also of type $(1, 1)$.

Assumption 1. *Paths of type (a, a) start with a vertex.*

Example 1. *For the hypergraph in Figure 1 left, the paths of type $(1, 1)$ are (v_0, e_1), (v_1, e_1), (v_2, e_1), (v_2, e_2), (v_3, e_2), (v_4, e_2), thus $N_{1,1} = 6$; (v_1, e_1, v_2, e_2) is an example of a path of type $(2, 2)$.*

Definition 3. *A cycle of length g in a hypergraph is a path $(v_1, e_1, \ldots, v_g, e_g)$, $g > 1$, such that $v_1 \in e_g$. The girth of a hypergraph is the length of the shortest cycle (or ∞ if there is no cycle). (The girth g is sometimes used in a weaker sense that there is no cycle of length less than g; see, e.g., [4].)*

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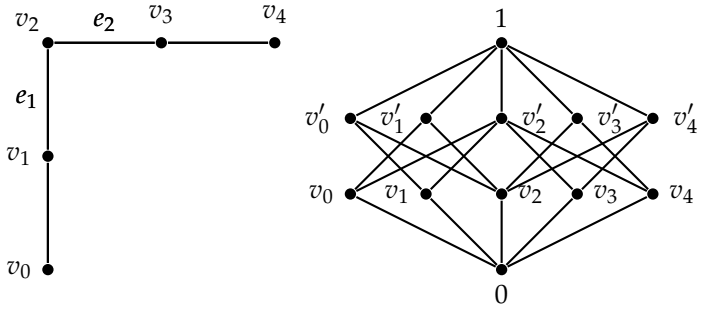


Figure 1. Greechie diagram (**left**) and the corresponding Hasse diagram (**right**) of the orthomodular lattice from Example 2.

Alternatively, a cycle could be defined as a sequence $(v_1, e_1, \ldots, v_g, e_g, v_1)$, $g > 1$, such that $(v_1, e_1, \ldots, v_g, e_g)$ is a path and $v_1 \in e_g$. As we are interested only in the existence of cycles of a given length (hypergraphs of a given girth), the difference between these two definitions is unimportant.

We shall also use M_n to denote the maximal number of edges of a hypergraph with n vertices. This number does not depend on a specific hypergraph, only on n and the given class of hypergraphs (clear from the context of the section).

2.2. Orthoalgebras

Remark 1. *Events of quantum mechanics cannot be classically described as subsets of the set of elementary events. As a consequence of Heisenberg's Uncertainty Principle, for some observable events, their conjunctions or disjunctions need not be observable. As a consequence, distributivity is violated.*

A standard representation of quantum events is by closed subspaces of a Hilbert space (equivalently, projections to these subspaces). Birkhoff and von Neumann [5] proposed an alternative algebraic formalism, now called orthomodular lattices. These appeared to be more general than Hilbert lattices. Since then, orthomodularity has been intensively studied in mathematics, and we contribute to this line.

Later on, the investigation of orthomodular lattices was extended to more general algebras: orthomodular posets, orthoalgebras, and effect algebras (in increasing order of generality). All of these became called quantum structures, sometimes also quantum logics, see [6–8]. This means that works on quantum structures contribute to the study of algebras listed above.

We deal here with three classes of quantum structures: orthoalgebras, orthomodular posets, and orthomodular lattices. These can be represented by hypergraphs, while the more general effect algebras can be represented only by labeled hypergraphs, and not all allow for that. Orthoalgebras are the most general among the quantum structures studied here; thus, we formulate the definitions for them. On the other hand, the deepest results are obtained for orthomodular posets and lattices; then, we speak of quantum structures to address readers who are not concerned with the more general orthoalgebras.

See [9,10] for details of the following definitions.

Definition 4. *An orthoalgebra is a quadruple $(L, \oplus, \mathbf{0}, \mathbf{1})$, where L is a set, $\mathbf{0}, \mathbf{1} \in L$, and $\oplus$ is a partial binary operation on L satisfying the following conditions for all $a, b, c \in L$:*

- $a \oplus b = b \oplus a$,
- $a \oplus (b \oplus c) = (a \oplus b) \oplus c$,
- $\exists!\, d \in L : a \oplus d = 1$,
- *$a \oplus a$ is defined iff $a = \mathbf{0}$.*

As $\oplus$ is a partial operation, the first two equalities should be read: "If one side of the equality exists, then the other exists, too, and both sides are equal." When there is no risk of confusion, we speak briefly of an orthoalgebra L instead of $(L, \oplus, \mathbf{0}, \mathbf{1})$.

Two elements, $a, b \in L$, are called orthogonal *iff $a \oplus b$ is defined (in symbols $a \perp b$). A subset, $K \subseteq L$, is an orthogonal family if $\bigoplus_{a \in K} a$ is defined. (The order of its elements is irrelevant.)*

We define a partial order on L by setting, for $a, b \in L$, $a \leq b$ iff there is an element $c \in L$, such that $b = a \oplus c$. This partial order induces the partial operations of supremum $\vee$ and infimum $\wedge$ on L.

An orthoalgebra L is called

- *an orthomodular poset iff each orthogonal pair has a join in L;*
- *an orthomodular lattice iff L is a lattice.*

An element, $a \in L$, $a > 0$, is said to be an atom if there is no $b \in L$, such that $0 < b < a$. A block of an orthoalgebra L is a maximal Boolean subalgebra.

Theorem 1 ([6,7]). *Every orthoalgebra is the union of its blocks.*

Proof. Each element, a, belongs to the Boolean subalgebra $\{\mathbf{0}, a, a', \mathbf{1}\}$. By Zorn's Lemma, every Boolean subalgebra can be extended to a maximal one. □

The important fact is that the orthoalgebraic operations defined on blocks coincide on their intersections. Here, we deal only with *finite* orthoalgebras. Their elements can be expressed (non-uniquely) as suprema of orthogonal atoms.

2.3. Description of Orthoalgebras by Hypergraphs

Orthomodular lattices, orthomodular posets, and orthoalgebras are usually described by Hasse diagrams, but this approach is rather ineffective; hence, Greechie diagrams were introduced. Here, we follow [11].

Definition 5 ([12]). *A Greechie diagram is a finite hypergraph (V, E) satisfying the following conditions:*

(G1) $\forall e \in E : |e| \geq 3$ *(each edge has at least three vertices).*

(G2) $\forall e_1, e_2 \in E : |e_1 \cap e_2| \leq 1$ *(every two edges intersect in at most one vertex).*

Condition (G2) is equivalent to the statement that the Greechie diagram does not contain a cycle of length two, so it is of girth at least three.

Theorem 2 ([6,7,12]). *For each Greechie diagram, G, there is a finite orthoalgebra, L, such that the vertices of G are the atoms of L and the edges of G are maximal sets of mutually orthogonal atoms of L (corresponding to blocks of L). We say that G is the Greechie diagram of L. If, moreover, G is of girth at least four (resp. five), it is a Greechie diagram of an orthomodular poset (resp. an orthomodular lattice).*

Proof. The proofs for orthomodular posets and orthomodular lattices form a substantial part of [12]; they are summarized in [7]. The orthoalgebraic version can be found, e.g., in [6]. □

Example 2. *Figure 1 shows the Greechie and Hasse diagram of an orthomodular lattice with five atoms and two blocks (each isomorphic to the Boolean algebra 2^3) with atoms $\{v_0, v_1, v_2\}$, $\{v_2, v_3, v_4\}$, respectively. More details can be found, e.g., in [7,9].*

Remark 2. *Conditions (G1) and (G2) of Definition 5 were used originally in [12]. Later on, it was found that they are not necessary; there are more general hypergraphs representing orthoalgebras in*

the sense of Theorem 2, see [7,13]. They are sometimes also called (generalized) Greechie diagrams. Our motivation to restrict attention to Definition 5 is as follows:

(G1): Edges with two vertices may occur. However, they must be disjoint from all other edges. Adding such an edge increases the number of vertices by 2 and the number of edges by 1. As we shall be interested in hypergraphs with "few" vertices and "many" edges, hypergraphs violating (G1) do not contribute to the extremal results that follow. There can even be an edge with a single vertex. This is possible only when it is the only edge of the hypergraph, which represents the two-element Boolean algebra.

(G2): If each edge has exactly three vertices, condition (G2) is necessary [12,14]. Larger blocks may lead to hypergraphs that contain cycles of length two, three, and four, even for orthomodular lattices. Necessary and sufficient conditions for hypergraphs to represent various quantum structures are known; see [7,13]. However, they are complex and often difficult to verify. Literature on extremal problems usually ignores such cases and brings arguments (not proofs!) why such constructions do not enhance the bounds derived under the restrictions of Definition 5.

3. Upper Bounds on the Number of Edges of Hypergraphs

In this section, we derive a relevant relation between the number of paths in a hypergraph and bounds on the number of edges as a function of the number of vertices. We begin with minimal assumptions on the hypergraph and then progressively introduce additional constraints to obtain tighter bounds. These constraints are especially relevant for modeling the structural features of quantum systems.

3.1. General Hypergraphs

We shall use the function

$$h(x) = x(x-1)$$

which, applied to natural numbers, counts the number of 2-element variations from a set of size x. We shall also use it as a convex function of real arguments $x \geq 1$.

Proposition 1. *In a hypergraph* (V, E),

$$N_{1,1} = \sum_{v \in V} d(v) = \sum_{e \in E} |e| \geq \max(m, n)\,, \tag{1}$$

$$N_{2,1} = \sum_{e \in E} h(|e|)\,, \tag{2}$$

$$N_{1,2} = \sum_{v \in V} h(d(v)) \geq N_{1,1}\left(\frac{N_{1,1}}{n} - 1\right). \tag{3}$$

Proof. $N_{1,1}$: We are counting all pairs (v, e), $v \in V$, $e \in E$, such that $v \in e$. For each $e \in E$, we can choose v from $|e|$ adjacent vertices. Alternatively, for each $v \in V$, we can choose e from $d(v)$ adjacent edges.

$N_{1,2}$: We are counting all paths (v_1, e, v_2). For each $e \in E$, we can choose v_1, v_2 from $|e|$ adjacent vertices in $h(|e|)$ ways. (Notice that the paths are directed.)

$N_{2,1}$: We are counting all paths (e_1, v, e_2). For each $v \in V$, we can choose e_1, e_2 from $d(v)$ adjacent vertices in $h(d(v))$ ways.

The inequality in (3) follows from Jensen's inequality applied to function h,

$$\begin{aligned} N_{1,2} = \sum_{v \in V} h(d(v)) &\geq n\, h\left(\frac{1}{n}\sum_{v \in V} d(v)\right) = n\, h\left(\frac{N_{1,1}}{n}\right) \\ &= N_{1,1}\left(\frac{N_{1,1}}{n} - 1\right). \end{aligned} \tag{4}$$

□

3.2. Greechie Diagrams (Orthoalgebras)

Greechie diagrams of orthoalgebras satisfy conditions (G1), (G2) of Definition 5 that allow us to relate (1)–(3) with m.

Proposition 2. *Let (V, E) be a Greechie diagram (a hypergraph satisfying Definition 5) and let $k \geq 2$. Then,*

$$N_{1,1} = \sum_{v \in V} d(v) = \sum_{e \in E} |e| \geq 3m\,, \tag{5}$$

$$N_{1,2} = \sum_{v \in V} h(d(v)) \geq N_{1,1}\left(\frac{N_{1,1}}{n} - 1\right) \geq 3m\left(\frac{3m}{n} - 1\right), \tag{6}$$

$$N_{k,k-1} \geq 2N_{k-1,k-1}\,, \tag{7}$$

$$N_{k,k} \geq 2N_{k-1,k}\,. \tag{8}$$

Proof. (G1) extends (1) and (3) to (5) and (6). Inequalities (7) (resp. (8)) are obtained by taking each path of type $(k-1, k-1)$ (resp. $(k-1, k)$) with a starting (resp. ending) edge e, and extending it by one of the $|e| - 1 \geq 2$ remaining vertices. □

Let us recall that, for $n \in \mathbb{N}$, we denote by M_n the maximal number of edges of a hypergraph (of a specified type) with n vertices.

Remark 3. *We are going to prove upper bounds on M_n. In the proofs, we shall use m, the number of edges of a specific hypergraph.*

Theorem 3. *For Greechie diagrams,*

$$M_n \leq \frac{n(n-1)}{6}\,. \tag{9}$$

Proof. Cycles of length two are forbidden. Thus, each pair of vertices can be connected by at most one edge and the number $h(n) = n(n-1)$ of such pairs is an upper bound on the number $N_{2,1}$ of paths of type $(2, 1)$; with the lower bound from (7) and (5),

$$n(n-1) \geq N_{2,1} \geq 2N_{1,1} \geq 6m\,, \tag{10}$$

verifying (9). □

The bound (9) was derived in [15]. It is tight for $n = 7$ (see Figure 2). In Theorem 10 (see Section 5.1), we shall show that it is tight also for any $n = 3^k$, $k \in \mathbb{N}$.

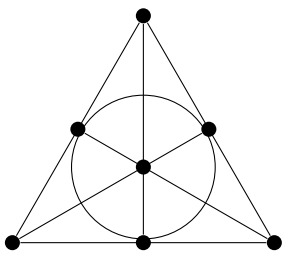


Figure 2. The Fano plane.

3.3. Girth Four Greechie Diagrams (Orthomodular Posets)

Greechie diagrams of orthomodular posets are without cycles of length three; for them, the upper bound on M_n can be improved.

Proposition 3. *Let (V, E) be a Greechie diagram of girth at least four. Let $v_0 \in V$, $e_0 \in E$. There is at most one path of type $(1,1)$ or $(2,2)$ connecting v_0 and e_0, i.e., of the form (v_0, e_0) or (v_0, e_1, v_1, e_0) for some $v_1 \in V$, $e_1 \in E$.*

Proof. If there were two such paths, they would form a cycle of length less than four, like that in Figure 3. □

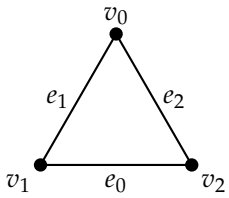


Figure 3. A cycle of length three originating from paths (v_0, e_1, v_1, e_0), (v_0, e_2, v_2, e_0) in the proof of Proposition 3.

Theorem 4. *For Greechie diagrams of girth at least four,*

$$M_n \leq \frac{n(n+3)}{18}. \tag{11}$$

Proof. Let us count the cases of Proposition 3 for all possible choices of $v_0 \in V$, $e_0 \in E$. Among these mn possibilities, there are $N_{1,1}$ paths of type $(1,1)$ and $N_{2,2}$ paths of type $(2,2)$ starting in v_0 and ending in e_0. They do not occur simultaneously for given v_0, e_0, thus

$$mn \geq N_{1,1} + N_{2,2}. \tag{12}$$

We substitute (8), (5), and (6),

$$mn \geq N_{1,1} + 2N_{1,2} \geq 3m + 6m\left(\frac{3m}{n} - 1\right) = 3m\left(\frac{6m}{n} - 1\right).$$

This means

$$6m - n \leq \frac{n^2}{3},$$

verifying (11). □

The bound (11) is an upper bound on the maximal number of edges (blocks) of an orthomodular poset with n atoms. It is not asymptotically optimal; it was shown in [16] that, for any $c > 0$ and n sufficiently large, we have $m < cn^2$.

3.4. Girth Five Greechie Diagrams (Orthomodular Lattices)

Greechie diagrams of orthomodular lattices are of girth at least five; for them, the upper bound on M_n can be further improved.

Proposition 4. *Let (V, E) be a Greechie diagram of girth at least five. Let $v_0, v_1 \in V$. There is at most one path of type $(2,1)$ or $(3,2)$ connecting v_0 and v_1, i.e., of the form (v_0, e_1, v_1) or $(v_0, e_1, v_2, e_2, v_1)$ for some $v_2 \in V$, $e_1, e_2 \in E$.*

Proof. If there were two such paths, they would form a cycle of length less than five. □

Theorem 5. *For Greechie diagrams of girth at least five,*

$$M_n \leq \frac{n}{12}\left(1 + 2\sqrt{n - \tfrac{3}{4}}\right). \tag{13}$$

Proof. Let us count the cases of Proposition 4 for all possible choices of $v_0, v_1 \in V$. Among these $h(n) = n(n-1)$ possibilities, there are $N_{2,1}$ paths of type $(2,1)$ and $N_{3,2}$ paths of type $(3,2)$ starting in v_0 and ending in v_1. They do not occur simultaneously for given v_0, v_1, thus

$$n(n-1) \geq N_{2,1} + N_{3,2}\,. \tag{14}$$

In combination with (7) and (5), we get an upper bound on $N_{3,2}$,

$$N_{3,2} \leq n(n-1) - N_{2,1} \leq n(n-1) - 6m\,. \tag{15}$$

Inequalities (7), (8) and (6) give us a lower bound on $N_{3,2}$,

$$N_{3,2} \geq 2N_{2,2} \geq 4N_{1,2} \geq \frac{36m^2}{n} - 12m\,. \tag{16}$$

We combine (15) and (16),

$$n(n-1) - 6m \geq N_{3,2} \geq \frac{36m^2}{n} - 12m\,.$$

Given n, this is a quadratic inequality for m,

$$m^2 - \frac{n}{6}m - \frac{n^2(n-1)}{36} \leq 0$$

with solution

$$\begin{aligned} m &\geq \frac{n}{12} - \frac{1}{12}\sqrt{4n^3 - 3n^2} = \frac{2n\sqrt{n-\frac{3}{4}} - n}{12}\,, \\ m &\leq \frac{n}{12} + \frac{1}{12}\sqrt{4n^3 - 3n^2} = \frac{2n\sqrt{n-\frac{3}{4}} + n}{12}\,. \end{aligned} \tag{17}$$

Only inequality (17) is relevant and gives us an upper bound M_n on the maximal number of edges. □

The bound (13) is an upper bound on the maximal number of edges (blocks) of an orthomodular lattice with n atoms. The same result was obtained for 3-uniform hypergraphs in [4]. Our proof is shorter, uses a different technique, and does not need the uniformity assumption.

4. Upper Bounds on the Number of Edges of Uniform Hypergraphs

In this section, we will describe how constraints on girth and uniformity at the same time affect the maximal number of edges of the hypergraph. It will be through the revision of formulas that were derived without uniformity. We will assume the girth to be at least three. For $r \geq 3$, r-uniform hypergraphs are a special case of Greechie diagrams studied in the previous section. The case of $r = 2$, ordinary graphs, can be proved similarly.

4.1. Girth Three r-Uniform Hypergraphs

We strengthen Formulas (5)–(8).

Proposition 5. *Let $r \geq 2$. Let (V, E) be an r-uniform hypergraph of girth at least three and let $k \geq 2$. Then,*

$$N_{1,1} = \sum_{v \in V} d(v) = \sum_{e \in E} |e| = rm\,, \tag{18}$$

$$N_{1,2} = \sum_{v \in V} h(d(v)) \geq N_{1,1}\left(\frac{N_{1,1}}{n} - 1\right) = rm\left(\frac{rm}{n} - 1\right), \tag{19}$$

$$N_{k,k-1} = (r-1)N_{k-1,k-1}\,, \tag{20}$$

$$N_{k,k} = (r-1)N_{k-1,k}\,. \tag{21}$$

Proof. The proof follows the arguments used in Proposition 2, refined by the fact that each edge has *exactly* r vertices. This allows us to derive equalities, not only inequalities. Although Proposition 2 assumed that each edge has at least three vertices, it can be easily verified that Formulas (18)–(21) are also valid for $r = 2$, i.e., for ordinary graphs. □

Theorem 6. *Let $r \geq 2$. For r-uniform hypergraphs of girth at least three,*

$$M_n \leq \frac{n(n-1)}{r(r-1)}\,. \tag{22}$$

Proof. Each of $n(n-1)$ pairs of vertices (v_0, v_1) can be connected by at most one edge e_0, creating a path (v_0, e_0, v_1) of type $(2,1)$. Thus

$$n(n-1) \geq N_{2,1}\,.$$

Combining it with (20) and (18) gives

$$n(n-1) \geq N_{2,1} = (r-1)N_{1,1} = (r-1)rm\,,$$

verifying (22). □

The bound (22) is an improvement of (9). It is tight for the Fano plane for $r = 3$, as mentioned in Section 3.2, and also for any $r \geq 3$ and infinitely many values of n, as we shall show in Section 5.1. Thus, the bound is asymptotically tight. For $r = 2$, (22) is the tight bound $\frac{n(n-1)}{2}$, corresponding to the complete (ordinary) graph.

4.2. Girth Four r-Uniform Hypergraphs

The restriction on girth allows us to improve the upper bound on M_n.

Theorem 7. *Let $r \geq 2$. For r-uniform hypergraphs of girth at least four,*

$$M_n \leq \frac{n(n + r(r-2))}{r^2(r-1)}\,. \tag{23}$$

Proof. Similar to the proof of Proposition 3, we use (12). Now, the numbers of paths $N_{1,1}$ and $N_{2,2}$ can be specified more.

We apply (21), (18) and (19),

$$mn \geq N_{1,1} + N_{2,2} = N_{1,1} + (r-1)N_{1,2} \geq rm + (r-1)rm\left(\frac{rm}{n} - 1\right).$$

We multiply the inequality by $\frac{n}{m}$,

$$n^2 \geq rn - (r-1)rn + (r-1)r^2m = -(r-2)rn + (r-1)r^2m\,,$$

simplified to

$$n(n + r(r-2)) \geq (r-1)r^2m\,.$$

As an inequality for m, it leads to the bound (23). □

The bound (23) is an improvement of (11). For $r = 2$, (23) is the asymptotically tight bound $\frac{n^2}{4}$, corresponding to a complete (ordinary) bipartite graph for n even.

4.3. Girth Five r-Uniform Hypergraphs

The restriction on girth allows us to further improve the upper bound on M_n.

Theorem 8. *Let $r \geq 2$. For r-uniform hypergraphs of girth at least five,*

$$M_n \leq \frac{n}{r(r-1)} \left(\frac{r}{2} - 1 + \sqrt{n + \frac{r(r-4)}{4}} \right). \tag{24}$$

Proof. Similarly to the proof of Proposition 4, we use (14) in the form of

$$N_{3,2} \leq n(n-1) - N_{2,1} = n(n-1) - (r-1)N_{1,1} = n(n-1) - (r-1)rm\,, \tag{25}$$

where we substituted from (20) and (18). A lower bound on $N_{3,2}$ follows from (20), (21) and (19),

$$N_{3,2} = (r-1)\,N_{2,2} = (r-1)^2\,N_{1,2} \geq (r-1)^2\,rm\Big(\frac{rm}{n} - 1\Big)\,. \tag{26}$$

If we combine (25) and (26), we arrive at

$$(r-1)^2\,rm\Big(\frac{rm}{n} - 1\Big) \leq N_{3,2} \leq n(n-1) - (r-1)rm\,.$$

We multiply it by n,

$$(r-1)^2 r^2 m^2 - (r-1)^2 rnm \leq n^2(n-1) - (r-1)rnm\,,$$

and interpret it as a quadratic inequality for m,

$$\big((r-1)^2 r^2\big)m^2 - \big(nr(r-1)(r-2)\big)m - n^2(n-1) \leq 0\,. \tag{27}$$

Dividing by the leading coefficient,

$$m^2 - \frac{n(r-2)}{r(r-1)}m - \frac{n^2(n-1)}{r^2(r-1)^2} \leq 0\,.$$

Solving this quadratic inequality with respect to m, we obtain

$$m \leq \frac{(r-2)}{2r(r-1)}\,n + \frac{1}{r(r-1)}\,n\,\sqrt{n + \frac{r(r-4)}{4}}\,, \tag{28}$$

$$m \geq \frac{(r-2)}{2r(r-1)}\,n - \frac{1}{r(r-1)}\,n\,\sqrt{n + \frac{r(r-4)}{4}}\,.$$

Only the first inequality is relevant. It is equivalent to the claim of the theorem. □

The bound (24) is an improvement of (13). The special case for $r = 3$,

$$M_n \leq \frac{n}{6}\left(\frac{1}{2} + \sqrt{n - \frac{3}{4}}\right) = \frac{n}{12} + \frac{n}{6}\sqrt{n - \frac{3}{4}}\,, \tag{29}$$

was obtained in [4]. For $r = 2$, i.e., for ordinary graphs, (24) reduces to

$$M_n \leq \frac{n}{2}\sqrt{n-1}\,.$$

This bound was established in [17] and tested against graphs generated by a brute-force algorithm, showing that the bound is not tight. In the same paper, a lower bound is also derived, and it is not tight either.

Corollary 1. *Using Taylor expansion of the square root, we may derive the asymptotic behavior of the upper estimates* (24) *and* (29) *(see [18] for details):*

$$M_n \leq \frac{1}{r(r-1)} n^{3/2} + \frac{r-2}{2r(r-1)} n + O(n^{1/2}) .$$

A comparison with ([4] Theorem 2.3) will be given in Remark 4.

Girth Five r-Uniform r-Partite Hypergraphs

Definition 6. *Hypergraph (V, E) is called r-partite if r is the smallest number of disjoint subsets $V_1, \ldots, V_r$ such that:*

$$\bigcup_{i=1}^{r} V_i = V ,$$
$$\forall e \in E \; \forall i \in \{1, \ldots, r\} : |e \cap V_i| \leq 1 . \tag{30}$$

E.g., the hypergraph in Figure 1 is 3-partite, while that in Figure 2 is not. For r-partite *r-uniform* hypergraphs, inequality (30) becomes an equality, and it is possible to strengthen the above results.

Theorem 9. *Let $r \geq 2$. For r-partite r-uniform hypergraphs of girth at least five,*

$$M_n \leq \frac{n}{2r} + \frac{n}{r}\sqrt{\frac{\frac{n}{r}-1}{r-1} + \frac{1}{4}} = \frac{n}{r}\left(\frac{1}{2} + \sqrt{\frac{n-r}{r(r-1)} + \frac{1}{4}}\right) . \tag{31}$$

Proof. Let us count the maximum number of paths (v_1, e_1, v, e_2, v_2) of type $(3, 2)$ that have ends v_1, v_2 in the same part V_i. For each vertex $v \in V_j$ in the middle, we can choose $V_i \neq V_j$ in $r-1$ ways. There are $h(d(v))$ combinations of edges to choose, which gives us

$$(r-1) \sum_{v \in V} h(d(v))$$

possibilities. This number cannot be bigger than the number of couples of vertices from V_j, that is $\frac{n}{r}(\frac{n}{r} - 1)$. If it were bigger, there would have to be two paths with the same ends, forming a cycle of length four. Thus,

$$(r-1) \sum_{v \in V} h(d(v)) \leq \sum_{i=1}^{r} |V_i|(|V_i| - 1) = n\Big(\frac{n}{r} - 1\Big) . \tag{32}$$

We apply Jensen's inequality to the left-hand side in the same way as in (4) and substitute from (18),

$$(r-1)N_{1,1}\left(\frac{N_{1,1}}{n} - 1\right) = (r-1)rm\Big(\frac{rm}{n} - 1\Big) \leq n\Big(\frac{n}{r} - 1\Big) .$$

We multiply it by n,

$$(r-1)r^2m^2 - (r-1)rnm \leq n^2\Big(\frac{n}{r} - 1\Big)$$

and obtain a quadratic inequality for m (divided by the leading coefficient),

$$m^2 - \frac{n}{r}m - \frac{n^2\left(\frac{n}{r}-1\right)}{(r-1)r^2} \le 0\,,$$

with solution

$$m \ge \frac{n}{2r} - \frac{n}{r}\sqrt{\frac{\frac{n}{r}-1}{r-1}+\frac{1}{4}}\,,$$
$$m \le \frac{n}{2r} + \frac{n}{r}\sqrt{\frac{\frac{n}{r}-1}{r-1}+\frac{1}{4}}\,.$$

Only the second inequality is relevant. □

Corollary 2. *In* (31)*, we can perform a Taylor expansion of the square root in the numerator,*

$$M_n \le \frac{n}{2r} + \frac{n}{r}\sqrt{\frac{\frac{n}{r}-1}{r-1}+\frac{1}{4}} = \frac{n}{2r} + \left(\frac{n}{r}\right)^{3/2}\frac{\sqrt{1+r\frac{r-5}{4n}}}{\sqrt{r-1}}\,,$$
$$M_n \le \frac{n}{2r} + \left(\frac{n}{r}\right)^{3/2}\frac{\left(1+O(n^{-1})\right)}{\sqrt{r-1}}\,,$$
$$M_n \le \frac{1}{\sqrt{r-1}}\left(\frac{n}{r}\right)^{3/2} + \frac{n}{2r} + O(n^{1/2})\,.$$

Remark 4. *Corollary 2 corresponds to ([4] Theorem 2.3). However, there are several issues in this source. It claims that* $\sum_{v\in V} d(v) = m$ *instead of the correct* rm*; this mistake does not carry on, so the final result is true. It proves two asymptotic formulas,*

$$M_n \le \frac{1}{r(r-1)}n^{3/2} + \frac{r-2}{2r(r-1)}n + O(n^{-1/2})\,,$$
$$M_n \le \frac{1}{\sqrt{r-1}}n^{3/2} + \frac{1}{2}n + O(n^{1/2})\,.$$

In the first inequality, there is a wrong sign in $O(n^{-1/2})$*; cf.* (29)*. In the second, which assumes an r-partite hypergraph, n denotes* $|V_i|$*, not* $|V|$*, i.e., r times less. (In ([4] Section 2), the authors skip between the two meanings of n.) The proof contains a self-reference, which should have referred to ([4] Theorem 2.1).*

For hypergraphs of girth three or four, we did not find any strengthening of Theorems 6 and 7 for *r-partite r*-uniform hypergraphs. This remains a possible topic for future research.

4.4. Summary of Upper Bounds

We summarize some upper bounds on M_n for r-uniform hypergraphs with a small n and r in Table 1. They just demonstrate the proved formulas; some of the bounds for small n can be easily verified to be non-tight because the effect of girth occurs only for larger numbers of vertices. Upper bounds for Greechie diagrams are not included because they are the same as those for 3-uniform hypergraphs.

Table 1. Some proved upper bounds on the number M_n of edges of uniform hypergraphs with n vertices (G is the minimal girth), compared to the tight bounds.

Hypergraphs	G	M_n	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}	M_{11}	M_{12}
3-uniform	3	$\frac{n(n-1)}{6}$	1	2	3	5	7	9	12	15	18	22
		tight bounds	1	1	2	4	7	8	12	13	17	20

Table 1. *Cont.*

Hypergraphs	G	M_n	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}	M_{11}	M_{12}
3-uniform	4	$\frac{n(n+3)}{18}$	1	1	2	3	3	4	6	7	8	10
		tight bounds	1	1	2	2	3	4	6	6	7	9
3-uniform	5	$\frac{n}{12}\left(1+2\sqrt{n-\frac{3}{4}}\right)$	1	1	2	2	3	4	5	5	6	7
		tight bounds	1	1	2	2	3	3	4	5	5	6
3-partite 3-uniform	5	$\frac{n}{6}\left(1+\sqrt{\frac{2n}{3}-1}\right)$	1	1	2	2	3	4	4	5	6	7
		tight bounds	1	1	2	2	3	3	4	5	5	6
4-uniform	3	$\frac{n(n-1)}{12}$	0	1	1	2	3	4	6	7	9	11
		tight bounds	0	1	1	1	2	2	3	5	6	9
4-uniform	4	$\frac{n(n+8)}{48}$	0	1	1	1	2	2	3	3	4	5
		tight bounds	0	1	1	1	2	2	2	3	3	4
4-uniform	5	$\frac{n}{12}\left(1+\sqrt{n}\right)$	0	1	1	1	2	2	3	3	3	4
		tight bounds	0	1	1	1	2	2	2	3	3	3
4-partite 4-uniform	5	$\frac{n}{8}\left(1+\sqrt{\frac{n-1}{3}}\right)$	0	1	1	1	2	2	2	3	3	4
		tight bounds	0	1	1	1	2	2	2	3	3	3

5. Lower Bounds on the Maximal Number of Edges of Hypergraphs

In this section, we present lower bounds on M_n, witnessed by examples of hypergraphs from the given class. These bounds will often be obtained for uniform hypergraphs.

5.1. Construction of Girth Three r-Uniform Hypergraphs

Theorem 10. *Let p be a prime number and $i, k \in \mathbb{N}$. Let $r = p^i$ and $n = r^k$. Then, there is an r-uniform hypergraph with girth at least three that has n vertices and m edges, where*

$$m = \frac{n(n-1)}{r(r-1)}.$$

Proof. We represent the n vertices as k-dimensional vectors over the field $\mathbb{Z}_r$. The edges are all sets of the form

$$\{\vec{x} + j\vec{y} : j = 0, \ldots, r-1\}, \tag{33}$$

where $\vec{x}, \vec{y} \in \mathbb{Z}_r^k$, $\vec{y} \neq \vec{0}$. Non-zero multiples of $\vec{y}$ (there are $r-1$ of them) lead to the same edge. Thus only

$$\frac{n-1}{r-1} = \frac{r^k-1}{r-1}$$

choices of $\vec{y}$ give a chance to generate different edges of the form (33). However, each edge is obtained for r vertices from all $n = r^k$ choices of $\vec{x}$. In total, (33) produces

$$\frac{n(n-1)}{r(r-1)} = r^{k-1}\frac{r^k-1}{r-1}$$

different edges. This hypergraph satisfies the required properties. □

Theorem 10 with Theorem 6 proves that the bound (22) is tight if r is a power of a prime. To our knowledge, no construction is known that would work for arbitrary r. However, there exists a result that allows us to prove that, for each r, the upper bound on the number of edges is *asymptotically* tight. In [10], Václav Voráček observed that uniform

Greechie diagrams of orthoalgebras are Steiner systems. He used results from [15], but did not develop this idea any further.

Definition 7. *A Steiner system with parameters t, r, n, denoted $S(t, r, n)$, is an n-element set S together with a collection of r-element subsets of S called blocks (edges), with the property that each t-element subset of S is contained in exactly one block.*

For us, only Steiner systems of type $S(2, r, n)$ are interesting. We require that any two-element subset can belong to *at most one* block. Steiner system $S(2, r, n)$ requires that any two-element subset belongs to *exactly one* block. Note that the number of blocks is

$$m = \frac{\binom{n}{2}}{\binom{r}{2}} = \frac{n(n-1)}{r(r-1)},$$

since there are $\binom{n}{2}$ pairs of vertices in total and $\binom{r}{2}$ pairs in each edge. This is the bound (22); thus, a Steiner system $S(2, r, n)$ (if it exists) is an r-uniform hypergraph on n vertices with the maximal number of edges. However, it is a long-standing question (since 1853) when this construction is possible. There are necessary divisibility conditions from the requirement that the bound (22) must be integer. In 2014, in [19], Peter Keevash showed that these conditions are also sufficient if n is large enough.

5.2. Construction of Girth Four r-Uniform Hypergraphs

Theorem 11. *Let p be a prime number and $i, k \in \mathbb{N}$. Let $r = p^i$ and $n = r^k$. Then, there is an r-uniform hypergraph with girth at least four that has n vertices and m edges, where*

$$m = (2r)^{k-1} = \frac{2^{k-1}}{r} n = \frac{n^{1+\log_r 2}}{2r}. \tag{34}$$

Proof. We slightly modify the proof of Theorem 10. The n vertices are all k-dimensional vectors over the field $\mathbb{Z}_r$. The edges are of the form (33) for all $\vec{x} \in \mathbb{Z}_r^k$, but only for all $\vec{y} \in \mathbb{Z}_r^k$ such that an odd number of entries is 1 and the remaining entries are 0. There are 2^{k-1} choices of $\vec{y}$ and (as before) r^{k-1} choices of $\vec{x}$, leading to different edges of the form (33).

It remains to verify that this construction ensures the girth of at least four. Suppose that there is a cycle of length three. Its vertices are of the form

$$\begin{aligned} \vec{v}_0 &= \vec{x}_0 + j_0 \vec{y}_0, \\ \vec{v}_1 &= \vec{v}_0 + j_1 \vec{y}_1, \\ \vec{v}_2 &= \vec{v}_1 + j_2 \vec{y}_2, \end{aligned}$$

where j_0, j_1, j_2 are non-zero and $\vec{y}_1 \neq \vec{y}_2$. To form a cycle, we need to find $j_3 \in \mathbb{Z} \setminus \{0\}$ and $\vec{y}_3 \in \mathbb{Z}^k$ satisfying

$$\vec{v}_2 - \vec{v}_0 = j_3 \vec{y}_3.$$

However, the equation

$$j_1 \vec{y}_1 + j_2 \vec{y}_2 = j_3 \vec{y}_3$$

has no solution. Indeed, the right-hand side has an *odd* number of entries j_3, with the others being zero. To have only these entries on the left-hand side, we need $j_1, j_2, j_1 + j_2 \in \{0, j_3\}$. This is possible only for $j_3 = j_2 = j_1 = -j_1$, which implies $k = 2j_1$, but there is an *even* number of non-zero entries (j_1) on the left-hand side; a contradiction derived from the assumed existence of a cycle of length three. □

Comparison with (23) shows that the bound (34) is not asymptotically tight, except for the case of ordinary graphs, where $r = 2$ and $M_n = \frac{n^2}{4}$.

5.3. Constructions of Girth Five r-Uniform Hypergraphs

For comparison, in this section, we summarize known results about asymptotic behavior of the lower estimates for girth at least five. The approach used in Sections 5.1 and 5.2 is not applicable.

Example 3 ([20]). *Take a hypergraph with vertices a_i, b_i, edges $\{a_i, b_i, a_{i+1}\}$ and $\{b_i, a_i + 4, b_{i+10}\}$, $i \in \{0, \ldots, 21\}$ (indices modulo 22). This hypergraph has 44 vertices, 44 edges, and girth five. A similar construction can be arbitrarily big.*

This gives a linear asymptotical lower bound. All remaining estimates in this section hold for an n that is sufficiently large. To our knowledge, the first superlinear bound was published in [4]:

$$M_n \geq \frac{1}{4} r^{-4r/3} n^{4/3}.$$

It was improved in ([21] Corollary 15), for $r \geq 4$, to

$$M_n \geq n^{\frac{4}{3}+\delta}, \quad \delta \in \left(0, \frac{1}{36}\right),$$

and in ([22] Theorem 1.2), for $r \geq 3$, to

$$M_n \geq c_r n^{10/7}$$

for some c_r. We included this review to demonstrate the complexity and recent progress in this problem.

6. Consequences for the Existence of States on Hypergraphs of Quantum Structures

6.1. Motivation and Formulation of the Problem

Quantum states may be represented by probability measures on orthoalgebras [6–8,11,23,24]. In this context, a *state* serves as an acronym for a *probability measure*. Measures on orthoalgebras generalize measures on Boolean algebras to non-distributive structures.

Definition 8. *Let $(L, \oplus, \mathbf{0}, \mathbf{1})$ be an orthoalgebra and $(G, +, 0)$ a commutative group. A G-valued measure on L is a mapping $s\colon L \to G$, which satisfies additivity:*

If $a, b \in L$ and $a \oplus b$ exists, then $s(a \oplus b) = s(a) + s(b)$.

A state is a measure with values in the real interval $[0, 1]$ and satisfying $s(\mathbf{1}) = 1$.

Here, we deal only with finite orthoalgebras; thus, finite additivity suffices. In *finite* orthoalgebras, all elements are sums of orthogonal elements; thus, each measure is uniquely determined by its restriction to atoms. As each Greechie diagram uniquely determines the corresponding orthoalgebra, the study of measures can be performed on the Greechie diagrams. For this purpose, we define measures and states on (general) hypergraphs.

Definition 9. *Let (V, E) be a hypergraph and $(G, +, 0)$ a commutative group. A G-valued measure on (V, E) is a mapping $s\colon V \to G$ if there is a constant $c \in G$ such that*

$$\forall e \in E : \sum_{v \in e} s(e) = c. \tag{35}$$

A state on (V, E) is a mapping $s\colon V \to [0, 1]$ satisfying (35) for $c = 1$.

Proposition 6. *Let (V, E) be a Greechie diagram of an orthoalgebra $(L, \oplus, \mathbf{0}, \mathbf{1})$ and $(G, +, 0)$ a commutative group. A mapping $s\colon L \to G$ is a G-valued measure (resp. a state) on the orthoalgebra $(L, \oplus, \mathbf{0}, \mathbf{1})$ iff its restriction $s|_V$ is a G-valued measure (resp. a state) on the hypergraph (V, E).*

Proof. We take $s(\mathbf{1})$ for the constant c in Definition 9. Finite additivity can be expressed by any of the following equivalent forms:

$$\begin{aligned} s(a \oplus b) &= s(a) + s(b)\,, \\ s\Big(\bigoplus_{i=1}^{k} a_i\Big) &= \sum_{i=1}^{k} s(a_i)\,, \\ s(\mathbf{1}) &= \sum_{i=0}^{k} s(a_i) \qquad \text{if } \bigoplus_{i=0}^{k} a_i = \mathbf{1}\,. \end{aligned} \tag{36}$$

In (36), $a_0 = \Big(\bigoplus_{i=1}^{k} a_i\Big)'$.

If s is a state on the orthoalgebra, (36) applied to an edge $\{a_0, a_1, \ldots, a_k\}$ verifies (35).

If s is a state on the Greechie diagram and $\bigoplus_{i=0}^{k} a_i = \mathbf{1}$, we may express each a_i as an orthosum of atoms,

$$a_i = \bigoplus_{j=1}^{\ell_i} b_{i,j}\,. \tag{37}$$

Then,

$$\begin{aligned} e = \{&b_{0,1}, \ldots, b_{0,\ell_0}, \\ &b_{1,1}, \ldots, b_{1,\ell_1}, \\ &\vdots \\ &b_{k,1}, \ldots, b_{k,\ell_k}\} \end{aligned}$$

is a maximal orthogonal set of atoms, i.e., an edge, and

$$\bigoplus_{v \in e} v = \bigoplus_{i=0}^{k} \bigoplus_{j=1}^{\ell_i} b_{i,j} = \bigoplus_{i=0}^{k} a_i = \mathbf{1}\,.$$

Application of (35) verifies (36):

$$c = \sum_{v \in e} s(v) = \sum_{i=0}^{k} s(a_i) = s(\mathbf{1})\,.$$

Thus, s extends from V to a measure on L by

$$s(a_i) = \sum_{j=1}^{\ell_i} s(b_{i,j})\,. \tag{38}$$

The decomposition (37) is not unique. However, for another decomposition, we obtain another edge fulfilling the same equality; thus, the extension of from V to L is correctly defined by (38). □

While Boolean algebras admit an abundance of (finitely additive) states, orthoalgebras need not. It was found by Greechie in [12] that there are even finite orthomodular lattices without any state. This result was strengthened in two independent papers [20,25]: There are orthomodular lattices without any non-zero measure in any group.

It is rather easy to construct a hypergraph of girth four (Greechie diagram of an orthomodular poset) without states, see Figure 4. All its vertices can be disjointly covered by three horizontal, resp. four vertical edges, contradicting the requirement that the sum over each edge must be the same non-zero constant, 1. A similar principle was also used for orthomodular lattices in [12] and optimized in [26]. However, such constructions admit many group-valued measures, even *signed measures* with values in the set of reals (without the requirement of non-negativity).

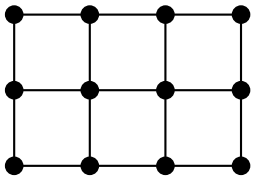


Figure 4. Greechie diagram of an orthomodular poset without states.

Each of m blocks (edges of the Greechie diagram) represents one linear equation for $n+1$ variables: the n values of the measure on atoms (vertices) and the value $c = s(\mathbf{1})$ at the unit element. In order to admit only the zero measure, we need at least $n+1$ equations, thus $m \geq n+1$. (See paper [10] for detailed arguments.) This is only possible if n is sufficiently large; we present lower estimates in the sequel.

6.2. General Results on Greechie Diagrams

We ask which Greechie diagrams admit more edges than vertices, $m > n$. Previous sections presented upper bounds on the number of edges of a Greechie diagram with a given number of vertices. This approach allows one to put lower bounds on the complexity of quantum structures without group-valued states. For an upper bound on complexity, we refer to known constructions, summarized in [10] (see also [27]). The lower bounds can be improved due to the following observations, which will be summarized in Assumption 2.

Consider a Greechie diagram (V, E). Suppose that there is a vertex, $v \in V$, of degree one, i.e., contained in a single edge, $e \in E$. We can construct a smaller Greechie diagram (V_0, E_0), defined by

$$E_0 = E \setminus \{e\}\,,$$
$$V_0 = \bigcup E_0\,.$$

Each measure on (V_0, E_0) has an extension to (V, E). This means that a Greechie diagram (V, E) with this property could not be a minimal Greechie diagram without non-zero group-valued measures. From now on, we shall be interested only in Greechie diagrams whose all vertices have degree at least two.

If the Greechie diagram is r-uniform, it admits a measure attaining a constant value at each vertex. In order to avoid this trivial non-zero solution of the corresponding system of linear equations, we need non-uniform Greechie diagrams. The smallest ones (subject to conditions (G1), (G2) of Definition 5) should be among those whose edges are "small", i.e., mostly with three vertices, but at least one edge must have more than three vertices.

Assumption 2. *In Section 6, we restrict attention to finite Greechie diagrams (V, E) that satisfy conditions (G1), (G2) of Definition 5 and the following:*

(G3) $\forall v \in V : d(v) \geq 2$ *(each vertex belongs to at least two edges).*
(G4) $\exists e_0 \in E : |e_0| \geq 4$ *(there is an edge e_0 with at least four vertices).*

Remark 5. *Here, we also see the motivation for condition (G1) of Definition 5. Let (V, E) be a Greechie diagram of an orthoalgebra. Suppose that there is a two-element edge, $e = \{v_1, v_2\} \in E$.*

According to [12], it must be disjoint from all other edges. We can construct a smaller Greechie diagram (V_0, E_0)*, defined by*

$$V_0 = V \setminus e\,,$$
$$E_0 = E \setminus \{e\}\,.$$

Each measure on (V_0, E_0) *has an extension to* (V, E)*, with value* 0 *at* v_1 *and* c *at* v_2*. This means that a Greechie diagram* (V, E) *with this property could not be a minimal Greechie diagram without non-zero group-valued measures.*

In contrast to (G1), condition (G2) is introduced solely as a necessary condition for the Greechie diagram to represent an orthoalgebra, according to [12] and its generalization in [14].

Conditions (G3) and (G4) allow us to improve the bounds derived in previous sections.

Proposition 7. *Let* (V, E) *be a Greechie diagram satisfying Assumption 2 (a hypergraph satisfying (G1)–(G4)) and let* $k \geq 2$*. Then,*

$$N_{1,1} = \sum_{v\in V} d(v) = \sum_{e\in E} |e| \geq 3m+1\,, \tag{39}$$
$$N_{2,1} = \sum_{e\in E} h(|e|) \geq 2\sum_{e\in E} |e| + 4 = 2N_{1,1} + 4 \geq 6m+6\,, \tag{40}$$
$$N_{1,2} = \sum_{v\in V} h(d(v)) \geq N_{1,1}\left(\frac{N_{1,1}}{n} - 1\right) \geq \frac{(3m+1)^2}{n} - 3m - 1\,, \tag{41}$$
$$N_{2,2} \geq 2N_{1,2} + 4 \geq \frac{2(3m+1)^2}{n} - 6m + 2\,, \tag{42}$$
$$N_{3,2} \geq 2N_{2,2} + 8 \geq 4N_{1,2} + 16 \geq \frac{4(3m+1)^2}{n} - 12m + 12\,. \tag{43}$$

Proof. We follow the proof of Proposition 2 with modifications allowed by (G3), (G4).

Inequality (39) is obvious and (41) is its direct consequence.

For the first inequality in (40), we use the same idea as for (2). We choose an edge, e, and two of its vertices can be chosen in $h(|e|)$ ways. If $|e| = 3$, then $h(|e|) = 6 = 2|e|$. If $|e| = 4$, then $h(|e|) = 12 = 2|e| + 4$. In all other cases, the difference $h(|e|) - 2|e|$ is even bigger.

To prove the first inequality in (42), take, by (G4), an edge e_0 with at least four vertices, v_1, v_2, v_3, v_4. By (G3), there are edges e_i different from e_0 and containing v_i, $i \in \{1,2,3,4\}$. We formed 4 paths (e_0, v_i, e_i) of type $(1,2)$. These can be completed to paths (v_j, e_0, v_i, e_i) of type $(2,2)$, $j \in \{1,2,3,4\}$, $j \neq i$. This can be done in 3 ways for a given i. This is a gain in comparison with the derivation of (7), where we calculated 2 ways, summing up to the total of $2N_{1,2}$. This unit gain is obtained for all choices of $i \in \{1,2,3,4\}$, thus $N_{2,2} \geq 2N_{1,2} + 4$.

To prove the first inequality in (43), we start from a path of type $(2,2)$ and complete it to a path of type $(3,2)$, choosing a vertex from the last edge. Previous results assumed its size at least 3. If this edge is e_0, with $|e_0| \geq 4$, we get at least one additional vertex to choose. The question is how many times this must happen. Assume that e_0 contains vertices v_i, $i \in \{1,2,3,4\}$. By (G3), there are edges e_i different from e_0 and containing v_i, $i \in \{1,2,3,4\}$. We start with path (e_i, v_i, e_0), which can be chosen in 4 ways. There are at least $|e_i| - 1 \geq 2$ choices of a vertex $v_i^* \in e_i \setminus \{v_i\}$, extending it to a path (v_i^*, e_i, v_i, e_0). Further, we extend it to a path $(v_i^*, e_i, v_i, e_0, v_j)$, $j \in \{1,2,3,4\}$, $j \neq i$. This can be done in at least 3 ways; only 2 of them were calculated in the previous bound without (G3), (G4). Before this last extension, we had at least $4 \cdot 2$ options in the choice of i and v_i^*, explaining the summand 8. □

6.3. Orthoalgebras (Non-Uniform Greechie Diagrams of Girth Three)

For Greechie diagrams of orthoalgebras, we have Formula (9). Substituting $m \geq n+1$, we get

$$n+1 \leq \frac{n(n-1)}{6},$$

This is a quadratic inequality

$$n^2 - 7n - 6 \geq 0$$

with the positive solution $n \geq \frac{7+\sqrt{73}}{2} \doteq 7.8$, thus $n \geq 8$.

This bound can be improved by taking into account (G4). Combining (10) with (40) and $m \geq n+1$, we obtain

$$n(n-1) \geq N_{2,1} \geq 2N_{1,1} + 4 = 6m + 6 \geq 6n + 12\,.$$

This is a quadratic inequality

$$n^2 - 7n - 12 \geq 0$$

with the positive solution $n \geq \frac{7+\sqrt{97}}{2} \doteq 8.42$; thus, $n \geq 9$.

This bound on n is not tight. For $n = 10$ (which is rather close to the lower bound derived here), an orthoalgebra without non-zero group-valued states was presented in [28]. In [10], it was proven to be a minimal possible construction. The proof is based on a detailed discussion of a possible structure of the orthoalgebra, while our simpler arguments were quantitative.

6.4. Orthomodular Posets (Non-Uniform Greechie Diagrams of Girth Four)

For Greechie diagrams of orthomodular posets, we have Formula (11). With $m \geq n+1$, we get

$$n+1 \leq \frac{n(n+3)}{18}\,.$$

This is a quadratic inequality,

$$n^2 - 15n - 18 \geq 0$$

with the positive solution $n \geq \frac{15+\sqrt{297}}{2} \doteq 16.1$; thus, $n \geq 17$.

With (G4) being taken into account, this bound can be improved by substituting (39) and (42) into (12),

$$mn \geq N_{1,1} + N_{2,2} \geq 3m + 1 + \frac{2(3m+1)^2}{n} - 6m + 2 = \frac{2(3m+1)^2}{n} - 3m + 3\,.$$

For $m := n+1$, we obtain

$$(n+1)n \geq \frac{2(3n+4)^2}{n} - 3n\,,$$

which can be expanded to

$$n^3 - 14n^2 - 48n - 32 \geq 0\,.$$

This inequality has an approximate positive solution $n \geq 16.9443$, thus $n \geq 17$.

The smallest known orthomodular poset without non-zero group-valued measures, constructed in [10], achieves $n = 21$. This is not far from the lower bound. Constructions done for other purposes (e.g., an r-uniform orthomodular poset with 18 vertices and

18 edges constructed in [29]) indicate that we approached the limit of quantitative methods in the derivation of bounds.

6.5. Orthomodular Lattices (Non-Uniform Greechie Diagrams of Girth Five)

Inequality (17) with $m \geq n+1$ gives

$$n+1 \leq \frac{2n\sqrt{n-\frac{3}{4}}+n}{12}.$$

The approximate numerical solution is $n \geq 33.0311$, thus $n \geq 34$.

This can be improved by substituting (40) and (43) in (14),

$$n(n-1) \geq N_{2,1}+N_{3,2} \geq 6m+6+\frac{4(3m+1)^2}{n}-12m+12 = \frac{4(3m+1)^2}{n}-6m+18\,.$$

For $m := n+1$, we obtain

$$n(n-1) \geq \frac{4(3n+4)^2}{n}-6n+12\,.$$

This can be expanded to

$$n^3-31n^2-108n-64 \geq 0\,,$$

with an approximate positive solution, $n \geq 34.2115$; thus, $n \geq 35$.

As upper bounds, orthomodular lattices without non-zero group-valued measures were found independently in [20,25] (see [30] for an overview). The latter one is smaller, with $n = 73$. (In [10,27], it was claimed to be 74. The mistake was corrected in [30].) The smallest such construction known now is that from [10], for $n = 67$. This is still far from our lower bound, which suggests that working with Greechie diagrams of orthomodular lattices is not an easy task.

For comparison, there are 3-uniform Greechie diagrams of orthomodular lattices with $m = n = 35$, and with the brute force algorithm, it was proven that they are the smallest possible Greechie diagrams of orthomodular lattices with $m \geq n$ edges; see [31,32].

7. Conclusions

We studied bounds on the maximum number of edges in hypergraphs with a given number of vertices subject to additional conditions. We established upper bounds, sought constructions yielding corresponding lower bounds, and compared the two. Some of our upper bounds recover earlier results, whereas others are new, more general, or obtained through simpler proofs. We present them within a unified proof framework based on our generalized definition of a path.

We considered uniform hypergraphs, as well as hypergraphs subject to restrictions on their girth, i.e., the minimum cycle length. We also investigated Greechie diagrams, i.e., hypergraphs used to represent posets of events in quantum systems, particularly orthoalgebras, orthomodular posets, and orthomodular lattices. We applied our results to a question examined in earlier work: how large must a quantum structure without a group-valued measure be? Although the bounds for more general hypergraphs were typically attained by 3-uniform hypergraphs, Greechie diagrams without group-valued measures are necessarily non-uniform. We, therefore, extended our analysis to the non-uniform case. For orthoalgebras and orthomodular posets, the lower and upper bounds on the size of the smallest structures without group-valued measures are nearly tight. By contrast, for

orthomodular lattices, a substantial gap remains between the proven lower bound on the number of vertices and the size of the smallest known example.

As in the previous studies on which our work builds, we restricted our attention to a class of Greechie diagrams that does not encompass all finite orthoalgebras. First, we excluded isolated two-vertex edges, as these appear not to affect the extremal results. Second, we excluded pairs of edges that intersect in more than one vertex. Such hypergraphs have received little attention in graph theory. Although their relevance to extremal questions is uncertain, it has not been systematically ruled out. Investigating them may, therefore, be a fruitful direction for future research. This line of inquiry is likely to be substantially more complex because the conditions defining such generalized Greechie diagrams are intricate.

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