mathematics

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Curvature Invariants on Partially Totally Real Submanifolds in Complex Space Forms

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Abstract

Kähler manifolds are the most studied complex manifolds. A Kähler manifold with constant holomorphic sectional curvature is said to be a complex space form. There are special classes of submanifolds in Kähler manifolds. Recently, Poyraz et al. defined partially totally real submanifolds in complex space forms. In the present paper, we study curvature invariants adapted to partially totally real submanifolds. First, we define a Riemannian invariant, denoted by $\delta_{PTR}(D)$ extending the definition of Chen's CR δ-invariant and estimate it in terms of the squared mean curvature. Furthermore, we obtain Chen–Ricci inequalities that distinguish between tangent directions in the two distributions. Finally, we prove a Chen first inequality for totally real two-plane sections.

Keywords: Kähler manifold; complex space form; partially totally real submanifold; δ-invariants; curvature inequalities

MSC: 53C25; 53C40; 53B25

1. Introduction

Complex space forms constitute an important class of Riemannian manifolds, providing a natural extension of both Riemannian and Kählerian geometries. Due to their rich geometric structure, they represent an important ambient setting for the study of various classes of submanifolds.

There exist several distinguished classes of submanifolds in complex space forms, including complex submanifolds, totally real submanifolds, slant submanifolds, and CR-submanifolds. Recently, Poyraz et al. [1] studied another class of submanifolds in Kähler manifolds, which they called partially totally real (PTR) submanifolds, as a generalization of CR-submanifolds. The same class of submanifolds has previously been considered by one of the present authors [2] under the name of generalized CR-submanifolds. Obviously, this class contains both totally real submanifolds, CR-submanifolds, hemi-slant submanifolds and pointwise hemi-slant submanifolds.

A fundamental problem in submanifold theory is to understand the relationship between intrinsic and extrinsic geometry. In particular, establishing sharp inequalities

Check for updates

Academic Editor: Bayram Sahin

Received: 24 June 2026
Revised: 5 August 2026
Accepted: 7 August 2026
Published: 9 August 2026

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relating intrinsic invariants, such as scalar and Ricci curvatures, to extrinsic quantities, such as the squared mean curvature, has been one of the central directions of research.

Motivated by the general problem of comparing intrinsic and extrinsic geometric quantities in isometric immersions, B.-Y. Chen introduced the notion of δ-invariants [3,4], which measure the difference between intrinsic curvature quantities and those associated with certain distinguished subspaces. These invariants have led to a large family of optimal inequalities, now known as Chen inequalities, providing important information about the geometry of submanifolds and their ambient spaces.

Since their introduction, Chen-type inequalities have been investigated in a wide variety of ambient spaces, including real and complex space forms, Sasakian, Kenmotsu and cosymplectic space forms, locally conformal Kähler manifolds, locally conformal almost cosymplectic manifolds, generalized complex space forms, generalized Sasakian space forms, as well as other classes of almost Hermitian and contact metric manifolds [3,5–13]. Recent works continue to extend Chen's program to new geometric settings and to investigate curvature inequalities for increasingly general classes of submanifolds [14–17].

However, despite the extensive study of Chen-type inequalities in various geometric settings, curvature inequalities involving invariants specifically adapted to partially totally real submanifolds remain less explored.

The purpose of this paper is to study curvature invariants naturally associated with the canonical orthogonal decomposition of a partially totally real submanifold

$$TM = D \oplus D^{\perp},$$

and to establish new optimal Chen-type inequalities involving these invariants.

More precisely, we define a δ-invariant associated with the distribution D, which reflects the contribution of the scalar curvature arising from the interaction between D and its orthogonal complementary distribution $D^{\perp}$. We then establish an optimal inequality relating this invariant to the squared mean curvature.

Furthermore, we obtain the corresponding Chen–Ricci inequalities by distinguishing between tangent vectors belonging to D and those in $D^{\perp}$. This approach leads to sharper estimates reflecting the different geometric roles of the two distributions.

Finally, we investigate Chen's first inequality for totally real two-plane sections of partially totally real submanifolds.

These results extend and improve several known inequalities in submanifold theory and provide new insights into the intrinsic and extrinsic geometry of partially totally real submanifolds.

2. Preliminaries

In this section, we recall the basic definitions and notations that will be used throughout the paper.

Let $\tilde{M}$ be an m-dimensional complex manifold and denote by J its canonical complex structure. It is an endomorphism of the tangent bundle $T\tilde{M}$ satisfying $J^2 = -I$.

A Riemannian metric g on $\tilde{M}$ is said to be Hermitian if

$$g(JX, JY) = g(X, Y), \qquad \forall X, Y \in \Gamma(T\tilde{M}). \tag{1}$$

If J is parallel with respect to the Levi-Civita connection $\tilde{\nabla}$ of g, then $(\tilde{M}, J, g)$ is a Kähler manifold. Equivalently, the fundamental two-form ω, given by $\omega(X, Y) = g(JX, Y)$, for all vector fields X, Y on $\tilde{M}$, is closed.

A two-plane section $\pi \subset T_p\tilde{M}$ is called holomorphic if $J\pi = \pi$. If the sectional curvature $K(\pi)$ is a constant c for every holomorphic plane section at every point, then $\tilde{M}$ is a complex space form, denoted as $\tilde{M}(c)$.

The curvature tensor of a complex space form $\tilde{M}(c)$ is given by [18]

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) &= \frac{c}{4}\Big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\\ &\quad + g(Z,JY)g(JX,W) - g(Z,JX)g(JY,W) + 2g(X,JY)g(JZ,W)\Big].\end{aligned}$$

Let M be an n-dimensional submanifold of a Kähler manifold $(\tilde{M}, J, g)$. Denote by $\widetilde{\nabla}$ and ∇ the Levi-Civita connections on $\tilde{M}$ and M, respectively. Then the Gauss and Weingarten formulae are

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \qquad \widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N,$$

for $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where h is the second fundamental form, $\nabla^{\perp}$ the normal connection, and A_N the shape operator. Moreover,

$$g(A_N X, Y) = g(h(X,Y), N).$$

Denote by R and $\widetilde{R}$ the Riemannian curvature tensors of ∇ and $\widetilde{\nabla}$, respectively. Then the Gauss equation is given by

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) - g(h(X,W),h(Y,Z)) + g(h(Y,W),h(X,Z)).$$

For any $p \in M$ and $X \in T_pM$, we decompose the action of J as $JX = PX + FX$, where $PX \in T_pM$ and $FX \in T_p^{\perp}M$.

Let $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2m}\}$ be a local orthonormal frame on $\tilde{M}$ such that $e_1, \ldots, e_n$ are tangent to M. We denote

$$h_{ij}^{\gamma} = g(h(e_i, e_j), e_{\gamma}), \quad i,j = 1, \cdots, n,\ \gamma = n+1, \ldots, 2m.$$

Then the squared norm of the second fundamental form is

$$\|h\|^2 = \sum_{\gamma=n+1}^{2m} \sum_{i,j=1}^{n} (h_{ij}^{\gamma})^2,$$

and the mean curvature vector is given by

$$H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i).$$

The scalar curvature $\tau(p)$ is defined by

$$\tau(p) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j) = \frac{1}{2} \sum_{i,j=1}^{n} R(e_i, e_j, e_j, e_i).$$

Definition 1. *Let M be a submanifold of a Kähler manifold $(\tilde{M}, J, g)$. Then*

- *M is called a complex submanifold if J preserves all tangent spaces T_pM, i.e., $J(T_pM) = T_pM$, $\forall p \in M$;*
- *M is called a totally real submanifold if J maps any tangent space into the normal space, i.e., $J(T_pM) \subset T_p^{\perp}M$, $\forall p \in M$.*

Definition 2 ([19]). *A submanifold M of a Kähler manifold $(\tilde{M}, J, g)$ is called a CR-submanifold if there exist two orthogonal complementary distributions D and $D^{\perp}$ on M such that*

$$TM = D \oplus D^{\perp},$$

and

- *D is invariant, i.e., $J(D) = D$;*
- *$D^{\perp}$ is totally real, i.e., $J(D^{\perp}) \subset T^{\perp}M$.*

It is clear that CR-submanifolds contain holomorphic and totally real submanifolds as special cases.

Definition 3 ([20]). *A submanifold M of a Kähler manifold $(\tilde{M}, J, g)$ is called a slant submanifold if for any $p \in M$ and any nonzero $X \in T_pM$, the angle between JX and T_pM is a constant $\theta \in [0, \pi/2]$, independent of the choice of X. The constant θ is called the slant angle.*

Invariant and totally real submanifolds correspond to the cases $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant submanifold is called *proper* if $0 < \theta < \frac{\pi}{2}$.

Definition 4 ([21]). *Let M be a submanifold of a Kähler manifold $(\tilde{M}, J, g)$. Then M is called a hemi-slant submanifold if its tangent bundle admits an orthogonal decomposition*

$$TM = D_{\theta} \oplus D^{\perp},$$

such that

- *$D^{\perp}$ is totally real, i.e., $J(D^{\perp}) \subset T^{\perp}M$;*
- *D_{θ} is a slant distribution with constant slant angle $\theta \in (0, \frac{\pi}{2})$.*

If the slant angle $\theta = \theta(p)$ depends on the point $p \in M$, then M is called a pointwise hemi-slant submanifold.

Poyraz et al. [1] have considered the following class of submanifolds in Kähler manifolds.

Definition 5 ([1]). *Let M be a submanifold of a Kähler manifold $(\tilde{M}, J, g)$. Then M is called a partially totally real submanifold if there exist two orthogonal complementary distributions D and $D^{\perp}$ such that*

$$TM = D \oplus D^{\perp},$$

and

- *The distribution $D^{\perp}$ is totally real, i.e., $J(D^{\perp}) \subset T^{\perp}M$;*
- *The distribution $D^{\perp}$ has maximal dimension.*

In this case, the tangential component P of J preserves D, namely, $P(D) \subset D$.

We point out that in [1], the second condition was not stated in the definition, but it was assumed during the paper. In the present form, PTR submanifolds and generalized CR-submanifolds (see [2]) constitute the same class of submanifolds.

Remark 1. *The integrability of the distributions involved in the definition of a PTR submanifold were investigated in [1,2].*

Examples of PTR submanifolds can be found in [1].

3. A δ-Invariant Associated with Partially Totally Real Submanifolds

Motivated by Chen's CR-δ invariant [22], we introduce a δ-invariant associated with the distribution D of a partially totally real submanifold in a complex space form. Taking into account the decomposition $TM = D \oplus D^{\perp}$, this invariant refines the scalar curvature by measuring the contribution arising from the totally real distribution $D^{\perp}$. In this section, we establish an optimal Chen-type inequality relating this invariant to the squared mean curvature.

Let M be an n-dimensional partially totally real submanifold of a complex space form $\tilde{M}(c)$, with $TM = D \oplus D^{\perp}$. Let $p \in M$ and $\{e_1, \ldots, e_n\}$ be an orthonormal basis of T_pM adapted to the decomposition, that is $\{e_1, \ldots, e_d\}$ is an orthonormal basis of D_p and $\{e_{d+1}, \ldots, e_n\}$ is an orthonormal basis of $D_p^{\perp}$.

The components of the second fundamental form h are $h_{ij}^{\gamma} = g(h(e_i, e_j), e_\gamma)$, $i, j = 1, \cdots, n$, $n+1 \leq \gamma \leq 2m$. We also denote by $\sigma_{rs}^{t} = g(h(e_r, e_s), Je_t)$, $d+1 \leq r, s, t \leq n$.

The scalar curvature of M at $p \in M$ is given by

$$\tau(p) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j), \tag{2}$$

and the scalar curvature of the distribution D is defined by

$$\tau(D)(p) = \sum_{1 \leq i < j \leq d} K(e_i \wedge e_j). \tag{3}$$

By analogy with the *CR* δ-invariant defined by Chen [22], we define the *PTR* δ-invariant by

$$\delta_{PTR}(D)(p) = \tau(p) - \tau(D)(p). \tag{4}$$

From Equations (2)–(4), it follows that

$$\delta_{PTR}(D)(p) = \sum_{\substack{1 \leq i \leq d \\ d+1 \leq r \leq n}} K(e_i \wedge e_r) + \sum_{d+1 \leq r < s \leq n} K(e_r \wedge e_s). \tag{5}$$

Remark 2. *Although the invariant $\delta_{PTR}(D)$ measures the contribution to the scalar curvature complementary to the distribution D, we keep the notation $\delta_{PTR}(D)$ in accordance with the convention used in [22], where the invariant is indexed by the distinguished distribution rather than by its complement.*

In the next Theorem, we assume that the codimension of the PTR submanifold M is minimum, i.e., $J(D^{\perp}) = T^{\perp}M$. By analogy with CR-submanifolds, we call such a submanifold a generic partially totally real submanifold.

Theorem 1. *Let M be an n-dimensional generic partially totally real submanifold of a complex space form $\tilde{M}(c)$ of real dimension $2m$. Assume that $TM = D \oplus D^{\perp}$, $\dim D = d$, and $\dim D^{\perp} = q, n = d + q$. Then, for any $p \in M$, the PTR δ-invariant satisfies*

$$\delta_{PTR}(D) \leq \frac{n^2}{2}\|H\|^2 + q(2d + q - 1)\frac{c}{8} - \frac{3q^2}{2(q+2)}\|H_{D^{\perp}}\|^2. \tag{6}$$

Moreover, the equality sign of (6) holds identically if and only if the following three conditions are satisfied:

(a) M is D-minimal, i.e., $H_D = 0$;
(b) M is mixed totally geodesic;

(c) *there exists an orthonormal frame* $\{e_{d+1}, \cdots, e_n\}$ *of* $D^\perp$ *such that the second fundamental form of* M *satisfies*

$$\sigma^r_{rr} = 3\sigma^r_{ss}, \text{ for } d+1 \le r \ne s \le n,$$
$$\sigma^r_{st} = 0, \text{ for distinct } r, s, t \in \{d+1, \ldots, n\}.$$

Proof. Let $p \in M$ and $\{e_1, \ldots, e_n\}$ an orthonormal basis T_pM such that $\{e_1, \ldots, e_d\} \subset D_p$ and $\{e_{d+1}, \ldots, e_n\} \subset D_p^\perp$. Put $q = n - d$.

Applying the Gauss equation for submanifolds, the sectional curvature of the plane section spanned by e_a and e_b, $1 \le a < b \le n$ can be expressed as

$$K(e_a \wedge e_b) = \widetilde{R}(e_a, e_b, e_b, e_a) + \sum_{\gamma=n+1}^{2m} \Big(h^\gamma_{aa} h^\gamma_{bb} - (h^\gamma_{ab})^2 \Big).$$

Hence

$$\delta_{PTR}(D)(p) = \sum_{\substack{1 \le i \le d \\ d+1 \le r \le n}} \widetilde{R}(e_i, e_r, e_r, e_i) + \sum_{d+1 \le r < s \le n} \widetilde{R}(e_r, e_s, e_s, e_r)$$
$$+ \sum_{\gamma=n+1}^{2m} \sum_{\substack{1 \le i \le d \\ d+1 \le r \le n}} \Big(h^\gamma_{ii} h^\gamma_{rr} - (h^\gamma_{ir})^2 \Big) + \sum_{\gamma=n+1}^{2m} \sum_{d+1 \le r < s \le n} \Big(h^\gamma_{rr} h^\gamma_{ss} - (h^\gamma_{rs})^2 \Big).$$

Since the ambient space is a complex space form, we have

$$\widetilde{R}(e_a, e_b, e_b, e_a) = \frac{c}{4}[1 + 3g(e_a, Je_b)^2].$$

Because M is a generic partially totally real submanifold, $J(D^\perp) = T^\perp M$. Thus, for $e_i \in D_p \subset T_pM$ and $e_r, e_s \in D_p^\perp$, we obtain $Je_r \in T_p^\perp M$, and hence

$$g(e_i, Je_r) = 0, \quad g(e_s, Je_r) = 0.$$

Then

$$\widetilde{R}(e_i, e_r, e_r, e_i) = \frac{c}{4}, \qquad \widetilde{R}(e_r, e_s, e_s, e_r) = \frac{c}{4}.$$

Consequently,

$$\sum_{\substack{1 \le i \le d \\ d+1 \le r \le n}} \widetilde{R}(e_i, e_r, e_r, e_i) = \frac{c}{4} dq, \qquad \sum_{d+1 \le r < s \le n} \widetilde{R}(e_r, e_s, e_s, e_r) = \frac{c}{4} \frac{q(q-1)}{2}.$$

Substituting into the above expression, we get

$$\begin{aligned} \delta_{PTR}(D)(p) &= \frac{c}{8} q(2d + q - 1) \\ &+ \sum_{\gamma=n+1}^{2m} \sum_{\substack{1 \le i \le d \\ d+1 \le r \le n}} \Big(h^\gamma_{ii} h^\gamma_{rr} - (h^\gamma_{ir})^2 \Big) + \sum_{\gamma=n+1}^{2m} \sum_{d+1 \le r < s \le n} \Big(h^\gamma_{rr} h^\gamma_{ss} - (h^\gamma_{rs})^2 \Big). \end{aligned}$$

Since $(h^\gamma_{ir})^2 \ge 0$, we obtain

$$\delta_{PTR}(D)(p) \le \frac{c}{8} q(2d + q - 1) + \sum_{\gamma=n+1}^{2m} \sum_{\substack{1 \le i \le d \\ d+1 \le r \le n}} h^\gamma_{ii} h^\gamma_{rr} + \sum_{\gamma=n+1}^{2m} \sum_{d+1 \le r < s \le n} [h^\gamma_{rr} h^\gamma_{ss} - (h^\gamma_{rs})^2]. \tag{7}$$

We denote by

$$H_D = \frac{1}{d}\sum_{i=1}^{d} h(e_i, e_i), \quad H_{D^\perp} = \frac{1}{q}\sum_{r=d+1}^{n} h(e_r, e_r),$$

the mean curvature vectors of D and $D^\perp$, and

$$\|h_{D^\perp}\|^2 = \sum_{r,s=d+1}^{n} \|h(e_r, e_s)\|^2.$$

The above inequality can be written in the form

$$\delta_{PTR}(D) \leq \frac{c}{8}q(2d+q-1) + \frac{n^2}{2}\|H\|^2 - \frac{d^2}{2}\|H_D\|^2 - \frac{1}{2}\|h_{D^\perp}\|^2. \tag{8}$$

From the Kähler condition and $D^\perp$ being totally real, it follows that $A_{JZ}W = A_{JW}Z$, for all $Z, W \in D^\perp$.

Denote by $\sigma_{rs}^t = g(h(e_r, e_s), Je_t)$, where $d+1 \leq r, s, t \leq n$. Then the above equality implies that $\sigma_{rs}^t = \sigma_{rt}^s = \sigma_{st}^r$, for every $d+1 \leq r, s, t \leq n$.

By analogy with [23], we can write

$$\begin{aligned}
(q+2)\|h_{D^\perp}\|^2 - 3q^2\|H_{D^\perp}\|^2 &= (q-1)\sum_{r=d+1}^{n}\left(\sum_{s=d+1}^{n}\sigma_{ss}^r\right)^2 \\
&+3(q+1)\sum_{d+1\leq r\neq s\leq n}(\sigma_{ss}^r)^2 + 6(q+2)\sum_{d+1\leq r<s<t\leq n}(\sigma_{st}^r)^2 \\
&+2(q+2)\sum_{r=d+1}^{n}\sum_{d+1\leq s<t\leq n}\sigma_{ss}^r\sigma_{tt}^r \\
&= (q-1)\sum_{r=d+1}^{n}(\sigma_{rr}^r)^2 + 3(q+1)\sum_{d+1\leq r\neq s\leq n}(\sigma_{ss}^r)^2 \\
&+6(q+2)\sum_{d+1\leq r<s<t\leq n}(\sigma_{st}^r)^2 - 6\sum_{r=d+1}^{n}\sum_{d+1\leq s<t\leq n}\sigma_{ss}^r\sigma_{tt}^r \\
&= 6(q+2)\sum_{d+1\leq r<s<t\leq n}(\sigma_{st}^r)^2 + \sum_{d+1\leq s\neq r\leq n}(\sigma_{rr}^r - 3\sigma_{ss}^r)^2 \\
&+3\sum_{r\neq s,t}\sum_{d+1\leq s<t\leq n}(\sigma_{ss}^r - \sigma_{tt}^r)^2 \geq 0,
\end{aligned} \tag{9}$$

which implies

$$\|h_{D^\perp}\|^2 \geq \frac{3q^2}{q+2}\|H_{D^\perp}\|^2. \tag{10}$$

Equality holds in (10) if and only if

$$\sigma_{rr}^r = 3\sigma_{ss}^r, \text{ for } d+1 \leq r \neq s \leq n, \tag{11}$$

$$\sigma_{st}^r = 0, \text{ for distinct } r, s, t \in \{d+1, \cdots, n\}. \tag{12}$$

Substituting (10) in (8), it follows that

$$\delta_{PTR}(D) \leq \frac{n^2}{2}\|H\|^2 + q(2d+q-1)\frac{c}{8} - \frac{3q^2}{2(q+2)}\|H_{D^\perp}\|^2,$$

i.e., the desired inequality.

The equality case of (6) holds identically if and only if $h(e_i, e_r) = 0$, i.e., M is mixed totally geodesic, $H_D = 0$, or equivalently, M is D-minimal, and the conditions (11) and (12) are satisfied, i.e., M is quasi-umbilical. □

4. Chen–Ricci Inequalities

In this section, we will investigate the Chen–Ricci inequality for partially totally real submanifolds.

In the classical Chen–Ricci inequality for n-dimensional submanifolds M in space forms $\tilde{M}(c)$ (see [24]), the estimate is expressed in terms of an arbitrary unit tangent vector $X \in TM$, without distinguishing its position relative to the underlying structure. More precisely, one has:

$$\mathrm{Ric}(X) \leq (n-1)c + \frac{n^2}{4}\|H\|^2.$$

However, in the case of partially totally real submanifolds, it depends on whether X belongs to D or to $D^\perp$.

The aim of this section is to refine the Chen–Ricci inequality by analyzing separately these two situations. This approach provides a more accurate description of the interplay between the intrinsic geometry of the submanifold and the ambient structure.

Let M be an n-dimensional partially totally real submanifold of a complex space form $\tilde{M}(c)$, endowed with the decomposition $TM = D \oplus D^\perp$, where D and $D^\perp$ are orthogonal complementary distributions.

The definition of a partially totally real submanifold implies $PX = 0$, for all $X \in D^\perp$.

This relation will be used in the sequel to simplify the curvature expressions arising in the Gauss equation and to derive Chen–Ricci inequalities adapted to each distribution.

Theorem 2. *Let M be an n-dimensional partially totally real submanifold of a complex space form $\tilde{M}(c)$ of real dimension $2m$. Assume that $TM = D \oplus D^\perp$. Then:*

(i) For any unit vector $X \in D$, one has

$$\mathrm{Ric}(X) \leq (n-1)\frac{c}{4} + \frac{3c}{4}\|PX\|^2 + \frac{n^2}{4}\|H\|^2.$$

(ii) For any unit vector $X \in D^\perp$, one has

$$\mathrm{Ric}(X) \leq (n-1)\frac{c}{4} + \frac{n^2}{4}\|H\|^2.$$

Proof. (i) Let $p \in M$ and $X \in D_p$ be a unit tangent vector. We choose $\{e_1, \ldots, e_n\}$ an orthonormal basis of T_pM adapted to the decomposition $TM = D \oplus D^\perp$, such that $e_1 = X$, $\{e_1, \ldots, e_d\} \subset D_p$, $\{e_{d+1}, \ldots, e_n\} \subset D_p^\perp$. We also choose an orthonormal basis $\{e_{n+1}, \ldots, e_{2m}\}$ of the normal space $T_p^\perp M$.

By definition, the Ricci curvature in the direction $X = e_1$ is given by

$$\mathrm{Ric}(X) = \sum_{j=2}^{n} K(e_1 \wedge e_j).$$

Using the Gauss equation, we obtain

$$K(e_1 \wedge e_j) = \tilde{R}(e_1, e_j, e_j, e_1) + \sum_{\gamma=n+1}^{2m} \left(h_{11}^\gamma h_{jj}^\gamma - (h_{1j}^\gamma)^2 \right).$$

Since $\tilde{M}(c)$ is a complex space form, its curvature tensor gives

$$\tilde{R}(e_1, e_j, e_j, e_1) = \frac{c}{4}[1 + 3g(Je_1, e_j)^2].$$

Therefore,

$$K(e_1 \wedge e_j) = \frac{c}{4}[1 + 3g(Je_1, e_j)^2] + \sum_{\gamma=n+1}^{2m} \Big(h_{11}^{\gamma} h_{jj}^{\gamma} - (h_{1j}^{\gamma})^2 \Big).$$

Summing over $j = 2, \ldots, n$, we get

$$\mathrm{Ric}(X) = (n-1)\frac{c}{4} + \frac{3c}{4} \sum_{j=2}^{n} g(Je_1, e_j)^2 + \sum_{\gamma=n+1}^{2m} \sum_{j=2}^{n} \Big(h_{11}^{\gamma} h_{jj}^{\gamma} - (h_{1j}^{\gamma})^2 \Big).$$

Since $X = e_1 \in D_p$, the tangential component PX belongs to D_p. Moreover,

$$g(JX, e_j) = g(PX, e_j),$$

for every tangent vector e_j. Hence

$$\sum_{j=2}^{n} g(JX, e_j)^2 = \|PX\|^2.$$

Thus,

$$\mathrm{Ric}(X) = (n-1)\frac{c}{4} + \frac{3c}{4}\|PX\|^2 + \sum_{\gamma=n+1}^{2m} \sum_{j=2}^{n} \Big(h_{11}^{\gamma} h_{jj}^{\gamma} - (h_{1j}^{\gamma})^2 \Big).$$

Since $(h_{1j}^{\gamma})^2 \geq 0$, we obtain

$$\mathrm{Ric}(X) \leq (n-1)\frac{c}{4} + \frac{3c}{4}\|PX\|^2 + \sum_{\gamma=n+1}^{2m} h_{11}^{\gamma} \sum_{j=2}^{n} h_{jj}^{\gamma}.$$

Using the elementary inequality

$$ab \leq \frac{1}{4}(a+b)^2,$$

with equality if and only if $a = b$, we obtain

$$h_{11}^{\gamma} \sum_{j=2}^{n} h_{jj}^{\gamma} \leq \frac{1}{4} \left(\sum_{j=1}^{n} h_{jj}^{\gamma} \right)^2.$$

Summing over $\gamma = n+1, \ldots, 2m$, it follows that

$$\sum_{\gamma=n+1}^{2m} h_{11}^{\gamma} \sum_{j=2}^{n} h_{jj}^{\gamma} \leq \frac{1}{4} \sum_{\gamma=n+1}^{2m} \left(\sum_{j=1}^{n} h_{jj}^{\gamma} \right)^2.$$

Since the mean curvature vector is

$$H = \frac{1}{n} \sum_{j=1}^{n} h(e_j, e_j),$$

we have

$$\|H\|^2 = \frac{1}{n^2} \sum_{\gamma=n+1}^{2m} \left(\sum_{j=1}^{n} h_{jj}^{\gamma} \right)^2.$$

https://doi.org/10.3390/math14162880

Therefore,

$$\sum_{\gamma=n+1}^{2m} h_{11}^{\gamma} \sum_{j=2}^{n} h_{jj}^{\gamma} \leq \frac{n^2}{4}\|H\|^2.$$

Consequently,

$$\mathrm{Ric}(X) \leq (n-1)\frac{c}{4} + \frac{3c}{4}\|PX\|^2 + \frac{n^2}{4}\|H\|^2.$$

(ii) Let $p \in M$ and let $X \in D_p^{\perp}$ be a unit tangent vector. In this case, JX is normal to M; then $PX = 0$.

Substituting $PX = 0$ in the proof of the previous item, we get the inequality from (ii). □

Remark 3. *A unit tangent vector X at p satisfies the equality case if and only if*

1. $h(X, Y) = 0$, *for all tangent vectors $Y \in T_pM$ orthogonal to X;*
2. $2h(X, X) = H(p)$.

Remark 4. *Since $P(D) \subset D$, the term $\|PX\|^2$ depends only on the distribution D, being independent of the totally real distribution $D^{\perp}$.*

Theorem 2 generalizes the Chen–Ricci inequalities for Lagrangian submanifolds in complex space forms (see [25]) and slant submanifolds in complex space forms (see [26]), respectively.

5. A Chen First Inequality

The study of Chen invariants was started by B.Y. Chen in [3], where he proved the Chen first inequality for submanifolds in Riemannian space forms. Recall that the Chen first invariant of a Riemannian manifold M is defined by $\delta_M = \tau - \inf K$, where τ and K are the scalar and sectional curvatures of M, respectively.

Chen first inequality states [3]:

Theorem 3. *Let $R(c)$ be an m-dimensional Riemannian space form and M an arbitrary n-dimensional submanifold of $R(c)$, $p \in M$ and π a two-plane section at p. Then, one has:*

$$\tau(p) - (\inf K)(p) \leq \frac{n-2}{2}\left[\frac{n^2}{n-1}\|H\|^2 + (n+1)c\right].$$

Moreover, the equality holds for a two-plane section π at $p \in M$ if and only if, with respect to a suitable orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_m\}$, the shape operators take the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & \mu - a & 0 & \dots & 0 \\ 0 & 0 & \mu & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mu \end{pmatrix},$$

$$A_r = \begin{pmatrix} h^r_{11} & h^r_{12} & 0 & ... & 0 \\ h^r_{12} & -h^r_{11} & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}, \quad r = n+2, ..., m.$$

In this section, we establish the Chen first inequality for two-plane sections tangent to $D^{\perp}$.

Let $\tilde{M}(c)$ be an m-dimensional complex space form, M an n-dimensional partially totally real submanifold, $p \in M$ and $\pi \subset D^{\perp}_p$ a two-plane section at p.

Let $e_1, e_2 \in \pi$ be two orthonormal vectors, $\{e_1, \cdots, e_n\}$ an orthonormal basis of T_pM and $\{e_{n+1}, \cdots, e_{2m}\}$ an orthonormal basis of $T^{\perp}_p M$.

The scalar curvature $\tau(p)$ is given by

$$\tau(p) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j). \tag{13}$$

We choose $e_1, \cdots, e_q \in D^{\perp}_p, e_{q+1}, \cdots, e_n \in D_p$.

We use the Gauss equation for computing the scalar curvature τ and the sectional curvature $K(\pi)$.

We denote by $\|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j)$.

Taking into account the nature of $e_1, \cdots, e_n$, we find the following:

$$\tau(p) = \frac{c}{4}\left(\frac{n(n-1)}{2} + \frac{3}{2}\|P\|^2\right) + \sum_{\gamma=n+1}^{2m} \sum_{1 \le i < j \le n} (h^{\gamma}_{ii} h^{\gamma}_{jj} - (h^{\gamma}_{ij})^2). \tag{14}$$

Since $\pi \subset D^{\perp}_p$, its sectional curvature is

$$K(\pi) = \frac{c}{4} + \sum_{\gamma=n+1}^{2m} (h^{\gamma}_{11} h^{\gamma}_{22} - (h^{\gamma}_{12})^2). \tag{15}$$

Subtracting Equation (15) from Equation (14), we obtain

$$\begin{aligned} \tau(p) - K(\pi) &= \frac{c}{4}\left[\frac{(n+1)(n-2)}{2} + \frac{3}{2}\|P\|^2\right] + \sum_{\gamma=n+1}^{2m}\left[\sum_{1 \le i < j \le n} h^{\gamma}_{ii} h^{\gamma}_{jj} - h^{\gamma}_{11} h^{\gamma}_{22}\right] \\ &- \sum_{\gamma=n+1}^{2m}\left[\sum_{1 \le i < j \le n} (h^{\gamma}_{ij})^2 - (h^{\gamma}_{12})^2\right]. \end{aligned}$$

It follows that

$$\tau(p) - K(\pi) \le \frac{c}{4}\left[\frac{(n+1)(n-2)}{2} + \frac{3}{2}\|P\|^2\right] + \sum_{\gamma=n+1}^{2m}\left[\sum_{1 \le i < j \le n} h^{\gamma}_{ii} h^{\gamma}_{jj} - h^{\gamma}_{11} h^{\gamma}_{22}\right]. \tag{16}$$

For obtaining the Chen first inequality, we will apply the following steps:

Lemma 1 ([27]). *Let $n \ge 3$ be an integer and let $a_1, ..., a_n$ be n real numbers. Then one has*

$$\sum_{1 \le i < j \le n} a_i a_j - a_1 a_2 \le \frac{n-2}{2(n-1)}\left(\sum_{i=1}^{n} a_i\right)^2.$$

Moreover, the equality holds if and only if we have $a_1 + a_2 = a_3 = ... = a_n$.

https://doi.org/10.3390/math14162880

In addition, the equality case of (17) holds if and only we have equalities in (16) and the above Lemma, i.e.,

$$\begin{cases} h_{ij}^r = 0, 3 \leq i \neq j \leq n, r = n+2, \cdots 2m, \\ h_{11}^r + h_{22}^r = h_{33}^r = \cdots h_{nn}^r, r = n+2, \cdots 2m. \end{cases}$$

If we choose e_{n+1} parallel with $H(p)$, one also has Trace $A_r = 0$, for $r = n+2, \cdots 2m$. Summing up, we state the following result.

Theorem 4. *Let $\tilde{M}(c)$ be an m-dimensional complex space form, M an n-dimensional ($n \geq 3$) partially totally real submanifold, $q = \dim D^{\perp} \geq 2$, $p \in M$ and $\pi \subset D_p^{\perp}$ a two-plane section at p. Then, we have:*

$$\tau(p) - K(\pi) \leq \frac{c}{4}\left[\frac{(n+1)(n-2)}{2} + \frac{3}{2}\|P\|^2\right] + \frac{n^2(n-2)}{2(n-1)}\|H\|^2. \tag{17}$$

Moreover, a plane section $\pi \subset D_p^{\perp}$ at $p \in M$ satisfies the equality case of (17) if and only if, with respect to a suitable orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_{2m}\}$, the shape operators take the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & ... & 0 \\ 0 & \mu - a & 0 & ... & 0 \\ 0 & 0 & \mu & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & \mu \end{pmatrix},$$

$$A_r = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & ... & 0 \\ h_{12}^r & -h_{11}^r & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}, \quad r = n+2, ..., 2m.$$

6. Conclusions

In this article, we have investigated curvature invariants adapted to the study of partially totally real submanifolds in complex space forms. By exploiting the natural decomposition of the tangent bundle, we introduced a new δ-invariant associated with the distribution orthogonal to the totally real distribution. This invariant refines the classical scalar curvature by isolating the contribution arising from the interaction between the two complementary distributions.

Using this invariant, we established an optimal inequality relating the δ-invariant to the squared mean curvature. We also derived Chen–Ricci inequalities by distinguishing between tangent vectors belonging to the totally real distribution and its complementary orthogonal distribution.

In addition, we established Chen's first inequality for totally real two-plane sections, a natural problem in the geometry of partially totally real submanifolds of complex space forms. Moreover, we characterized the equality case completely in terms of the corresponding shape operators.

Consequently, these results extend and unify a number of well-known inequalities for totally real submanifolds, CR-submanifolds and slant submanifolds in complex space forms [5,11,12,23,28], thereby contributing to a broader understanding of submanifold geometry.

This study can be continued by investigating other Riemannian invariants, for instance the Casorati curvature, of partially totally real submanifolds in complex space forms. Also, one can consider corresponding inequalities for other classes of submanifolds in complex space forms, for example partially slant submanifolds.

Author Contributions: Conceptualization, M.M., I.M. and O.P.; methodology, I.M. and O.P.; validation, M.M. and A.O.; formal analysis, M.M., I.M. and O.P.; investigation, I.M. and A.O.; resources, I.M., M.M. and A.O.; writing—original draft preparation, M.M., I.M. and O.P.; writing—review and editing, M.M., A.O., I.M. and O.P.; visualization, I.M. and A.O.; supervision, I.M.; project administration, I.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Poyraz, D.; Şahin, B.; Yerlikaya, F. PTR-submanifolds of a Kähler manifold. *Results Math.* **2025**, *80*, 151. [CrossRef]
2. Mihai, I. Certain submanifolds of a Kaehler manifold. In *Geometry and Topology of Submanifolds*; World Scientific: Singapore, 1994; Volume VII, pp. 186–188.
3. Chen, B.Y. Some pinching and classification theorems for minimal submanifolds. *Arch. Math.* **1993**, *60*, 568–578. [CrossRef]
4. Chen, B.Y. Some new obstructions to minimal and Lagrangian isometric immersions. *Jpn. J. Math.* **2000**, *26*, 105–127. [CrossRef]
5. Chen, B.Y. A general inequality for submanifolds in complex space forms and its applications. *Arch. Math.* **1996**, *67*, 519–528. [CrossRef]
6. Defever, F.; Mihai, I.; Verstraelen, L. B.-Y. Chen's inequalities for submanifolds of Sasakian space forms. *Boll. Unione Mat. Ital.* **2001**, *4-B*, 521–529.
7. Arslan, K.; Ezentas, R.; Mihai, I.; Murathan, C.; Özgür, C. B.Y. Chen inequalities for submanifolds in locally conformal almost cosymplectic manifolds. *Bull. Inst. Math. Acad. Sin.* **2001**, *29*, 231–242.
8. Kim, J.S.; Choi, J. A basic inequality for submanifolds in a cosymplectic space form. *Int. J. Math. Math. Sci.* **2003**, *2003*, 573089. [CrossRef]
9. Mihai, A. Warped product submanifolds in generalized complex space forms. *Acta Math. Acad. Paedagog. Nyhazi.* **2005**, *21*, 79–87.
10. Chen, B.Y.; Mihai, I. Isometric immersions of contact Riemannian manifolds in real space forms. *Houst. J. Math.* **2005**, *31*, 743–764.
11. Alegre, P.; Carriazo, A.; Kim, Y.; Yoon, D. B.-Y. Chen's inequality for submanifolds of generalized space forms. *Indian J. Pure Appl. Math.* **2007**, *38*, 185.
12. Tripathi, M.M. Chen-Ricci inequality for submanifolds of contact metric manifolds. *J. Adv. Math. Stud.* **2008**, *1*, 111–135.
13. Vîlcu, G.E. B.-Y. Chen inequalities for slant submanifolds in quaternionic space forms. *Turk. J. Math.* **2010**, *34*, 115–128.
14. Mohammed, M.; Massamba, F.; Mihai, I.; Elamin, A.E.A.M.A.; Aldien, M.S. Some Chen inequalities for submanifolds in trans-Sasakian manifolds admitting a semi-symmetric non-metric connection. *Axioms* **2024**, *13*, 195. [CrossRef]
15. Mihai, I.; Olteanu, A. Chen-like inequalities for submanifolds in Kähler manifolds admitting semi-symmetric non-metric connections. *Symmetry* **2024**, *16*, 1401. [CrossRef]
16. Bag, B.; Khan, M.A.; Pal, T.; Hui, S.K. First Chen inequality for biwarped product submanifold of a Riemannian space form. *AIMS Math.* **2025**, *10*, 9917–9932. [CrossRef]
17. Aquib, Md. Chen-type inequalities for PS-submanifolds in complex space forms. *Axioms* **2026**, *15*, 339. [CrossRef]
18. Yano, K.; Kon, M. *Structures on Manifolds*; World Scientific: Singapore, 1985.
19. Bejancu, A. CR-submanifolds of a Kähler manifold. *Proc. Am. Math. Soc.* **1978**, *69*, 135–142. [CrossRef]
20. Chen, B.Y. *Geometry of Slant Submanifolds*; Katholieke Universiteit Leuven, Lecture Notes from June–July 1990; Katholieke Universiteit Leuven: Leuven, Belgium, 1990; pp. 1–124.
21. Sahin, B. Warped product submanifolds of Kaehler manifolds with a slant factor. *Ann. Pol. Math.* **2009**, *95*, 207–226. [CrossRef]
22. Chen, B.Y. An optimal inequality for CR-warped products in complex space forms involving CR δ-invariant. *Int. J. Math.* **2012**, *23*, 1250045. [CrossRef]

23. Al-Solamy, F.; Chen, B.Y.; Deshmukh, S. Two optimal inequalities for anti-holomorphic submanifolds and their applications. *Taiwan. J. Math.* **2014**, *18*, 199–217. [CrossRef]
24. Chen, B.Y. Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions. *Glasg. Math. J.* **1999**, *41*, 33–41. [CrossRef]
25. Chen, B.Y. On Ricci curvature of isotropic and Lagrangian submanifolds in complex space forms. *Arch. Math.* **2000**, *74*, 154–160. [CrossRef]
26. Matsumoto, K.; Mihai, I.; Tazawa, Y. Ricci tensor of slant submanifolds in complex space forms. *Kodai Math. J.* **2003**, *26*, 85–94. [CrossRef]
27. Chen, B.Y.; Mihai, A.; Mihai, I. A Chen first inequality for statistical submanifolds in Hessian manifolds of constant Hessian curvature. *Results Math.* **2019**, *74*, 165. [CrossRef]
28. Oiaga, A.; Mihai, I. B. Y. Chen inequalities for slant submanifolds in complex space forms. *Demonstr. Math.* **1999**, *32*, 835–846. [CrossRef]

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