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A New Preconditioner of the Accelerated Over-Relaxation Iterative Method for Multi-Linear Systems

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Abstract

Multi-linear systems with $\mathcal{Z}$-tensors are more general than those with $\mathcal{M}$-tensors. In this paper, we consider multi-linear systems with $\mathcal{Z}$-tensor coefficients. Theoretically, we provide a convergence analysis and compare the spectral radii of the preconditioned iteration tensor and the original tensor within the $\mathcal{Z}$-tensor framework. Numerical examples are presented to verify the theoretical spectral radius comparisons. Additionally, numerical examples on nonsingular $\mathcal{M}$-tensors demonstrate the effectiveness of the proposed preconditioner.

Keywords: multi-linear system; $\mathcal{Z}$-tensor; preconditioned AOR method; comparison

MSC: 65F10; 15A69

1. Introduction

Let $\mathbb{R}$ be the real field. Denote the set of all m-order n-dimensional real tensors as $\mathbb{R}^{[m,n]}$. All the entries of $\mathcal{A} \in \mathbb{R}^{[m,n]}$ are defined as

$$\mathcal{A} = (a_{i_2 \cdots i_m}), \ \ a_{i_1 i_2 \cdots i_m} \in \mathbb{R}, \ \ 1 \leq i_1, i_2, \cdots, i_m \leq n.$$

We consider the multi-linear system

$$\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b} \tag{1}$$

with respect to the unknown vector $\mathbf{x} = [x_1, x_2, \cdots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$, where $\mathcal{A} \in \mathbb{R}^{[m,n]}$, $\mathbf{b} \in \mathbb{R}^n$, and the product $\mathcal{A}\mathbf{x}^{m-1} \in \mathbb{R}^n$ is defined by

$$(\mathcal{A}\mathbf{x}^{m-1})_i = \sum_{i_2, \cdots, i_m} a_{i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}, \ \ i = 1, 2, \cdots, n.$$

The multi-linear system (1) has received considerable attention as it plays an important role in wide applications, such as numerical partial differential equations [1], tensor complementary problems [1,2], higher order Markov chain [3], image processing [4,5], document analysis [6], etc.

Recently, many theoretical properties and algorithms for solving (1) have been studied. Ding and Wei [1] prove that (1) has a unique positive solution when $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor and $\mathbf{b}$ is a positive vector, and they use the Newton method to solve (1) when $\mathcal{A}$ is still symmetric. When $\mathcal{A}$ is not symmetric, Han [7] proposes a homotopy method to find the unique positive solution and its convergence is proved. He et al. [8] transform (1) with an $\mathcal{M}$-tensor into a corresponding nonlinear system where the function is a P-function. Then a Newton-type algorithm with global and quadratic convergence is proposed.

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Splitting-type methods achieve good computational efficiency by circumventing the explicit computation of the Jacobian matrix [9]. To further enhance their numerical performance, various preconditioners have been proposed in the literature [10–20], where the corresponding comparison theorems are established for different splitting schemes when the coefficient tensor $\mathcal{A}$ is an $\mathcal{M}$-tensor. Anderson acceleration is a classical and effective technique for accelerating fixed-point iterations [21]. This has motivated its application to tensor splitting methods for convergence acceleration [22,23]. In particular, since the multi-linear PageRank problem—which arises in Google's PageRank—can be reformulated as a multi-linear system with an $\mathcal{M}$-tensor coefficient, splitting methods with Anderson acceleration have also been developed for this problem [24]. Furthermore, Bentbib et al. [25] proposed an improved extrapolated higher-order power method for the same problem.

There are also some methods for $\mathcal{M}$-tensor systems, such as continuous time neural network [26] and the nonlinear conjugate gradient method [27]. For more general tensor systems, Lv and Ma [28] propose a Levenberg–Marquardt method for solving semi-symmetric tensor equations and global and quadratic convergence under local error bound condition are proved, and so on. For other multi-linear systems solved by splitting methods, see [29,30].

We use the following definition of the product between the matrix and tensor which can be seen as a special case of the product between two tensors [31,32].

Definition 1 ([33], Remark 2.2)**.** *Let $A \in \mathbb{R}^{[2,n]}$ and $\mathcal{B} \in \mathbb{R}^{[m,n]}$. A matrix–tensor product $\mathcal{C} = A\mathcal{B} \in \mathbb{R}^{[m,n]}$ is defined by*

$$c_{ji_2\cdots i_m} = \sum_{j_2}^{n} a_{jj_2} b_{j_2 i_2 \cdots i_m}.$$

This formula can be written as

$$\mathcal{C}_{(1)} = (A\mathcal{B})_{(1)} = A\mathcal{B}_{(1)},$$

where $\mathcal{B}_{(1)}$ and $\mathcal{C}_{(1)}$ are the matrices obtained from $\mathcal{B}$ and $\mathcal{C}$ flattened along the first index.

Remark 1. *Following the notation in [34], the mode-k product between a matrix $P \in \mathbb{R}^{[2,n]}$ and a tensor $\mathcal{A} = (a_{i_1,\ldots,i_m}) \in \mathbb{R}^{[m,n]}$ is defined as*

$$\hat{\mathcal{A}} = \mathcal{A} \times_k P, \tag{2}$$

where $\hat{\mathcal{A}} = (\hat{a}_{i_1,\ldots,i_m})$. Therefore, the tensor $\mathcal{C}$ in Definition 1 is equivalent to the mode 1 product between the matrix $P \in \mathbb{R}^{[2,n]}$ and tensor $\mathcal{A}$. When tensor $\mathcal{A}$ is reduced to a matrix A, this is equivalent to when A is left-multiplied by P.

We recall the definition of the majorization matrix $M(\mathcal{B})$ of the tensor $\mathcal{B}$ as follows.

Definition 2 ([35], Definition 2.1)**.** *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. A majorization matrix $M(\mathcal{A})$ of $\mathcal{A}$ is an $n \times n$ matrix with entries*

$$M(\mathcal{A})_{ij} = a_{ij\cdots j}, \quad i, j = 1, 2, \cdots, n.$$

Note that, if $M(\mathcal{A})$ is a nonsingular matrix and $\mathcal{A} = M(\mathcal{A})\mathcal{I}_m$, then we call $M(\mathcal{A})^{-1}$ as the 2-order left-inverse of $\mathcal{A}$, where $\mathcal{I}_m$ is the m-order n-dimensional identity tensor. An identity tensor $\mathcal{I}_m = (\delta_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ is defined as

$$\delta_{i_1\cdots i_m} = \begin{cases} 1, & i_1 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{E}, \mathcal{F} \in \mathbb{R}^{[m,n]}$ construct a tensor splittinf $\mathcal{A} = \mathcal{E} - \mathcal{F}$. Assuming that $M(\mathcal{E})^{-1}$ exists, Liu et al. [33] introduced the tensor splitting iterative method for solving (1), defined as follows:

$$\mathbf{x}_k = \left[M(\mathcal{E})^{-1}\mathcal{F}\mathbf{x}_{k-1}^{m-1} + M(\mathcal{E})^{-1}\mathbf{b}\right]^{\left[\frac{1}{m-1}\right]},$$

where the tensor $M(\mathcal{E})^{-1}\mathcal{F}$ is called the iterative tensor.

Denote $\mathbf{y} = M(\mathcal{E})^{-1}\mathcal{F}\mathbf{x}_{k-1}^{m-1} + M(\mathcal{E})^{-1}\mathbf{b}$, then $\mathbf{y}^{\left[\frac{1}{m-1}\right]}$ is defined as $\mathbf{y}^{\left[\frac{1}{m-1}\right]} = (y_1^{\frac{1}{m-1}}, \cdots, y_n^{\frac{1}{m-1}})$.

Let $P \in \mathbb{R}^{[2,n]}$ be a nonsingular matrix. Unlike the matrix case, when $\mathcal{A}$ is an m-th order tensor, we consider the mode-k product for $k \in \{2, \ldots, m\}$. For $(\mathcal{A} \times_k (P^{-1}P))\mathbf{x}^{m-1}$, we have

$$\begin{aligned}(\mathcal{A} \times_k (P^{-1}P))\mathbf{x}^{m-1} &= (\mathcal{A} \times_k P^{-1} \times_k P)\mathbf{x}^{m-1} \\ &:= (\widehat{\mathcal{A}} \times_k P)\mathbf{x}^{m-1},\end{aligned}$$

where $\widehat{\mathcal{A}} = \mathcal{A} \times_k P^{-1}$. However, in contrast to the matrix case, the properties of the mode-k tensor–matrix product do not allow us to directly find a vector $\mathbf{y}$ such that

$$(\widehat{\mathcal{A}} \times_k P)\mathbf{x}^{m-1} = \widehat{\mathcal{A}}\mathbf{y}^{m-1}, \qquad k \in \{2, \ldots, m\}. \tag{3}$$

Therefore, we adopt the following preconditioning strategy

$$P\mathcal{A}\mathbf{x}^{m-1} = P\mathbf{b},$$

where P is called a preconditioner. Let

$$P\mathcal{A} = \mathcal{E}_P - \mathcal{F}_P.$$

Then a preconditioned tensor splitting iterative method for solving (1) is proposed in [36] as follows:

$$\mathbf{x}_k = \left[M(\mathcal{E}_P)^{-1}\mathcal{F}_P\mathbf{x}_{k-1}^{m-1} + M(\mathcal{E}_P)^{-1}\mathbf{b}\right]^{\left[\frac{1}{m-1}\right]}.$$

Let

$$P = I + S_\alpha + R_\beta,$$

where

$$S_\alpha = \begin{bmatrix} 0 & -\alpha_1 a_{12\cdots 2} & 0 & \cdots & 0 \\ 0 & 0 & -\alpha_2 a_{23\cdots 3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & -\alpha_{n-1} a_{n-1n\cdots n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and

$$R_\beta = \begin{bmatrix} 0 & -\beta_1 a_{12\cdots 2} & -\beta_2 a_{13\cdots 3} & \cdots & -\beta_{n-1} a_{1n\cdots n} \\ 0 & 0 & 0 & 0\cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Cui et al. [13] indicate that the above preconditioner can reduce to some known preconditioner, and they proposed the preconditioned Gauss–Seidel method for solving a nonsingular $\mathcal{M}$-tensor equation; the corresponding comparison results are also given. What's more, Cui et al. [29] give the comparison results for multi-linear systems with a $\mathcal{Z}$-tensor, which $I + S_\alpha + R_\beta$ is the special case of their preconditioner. In addition, Chen and Li [11] propose a preconditioner $I + \bar{S}_{\delta\beta}$, where

$$\bar{S}_{\alpha\beta} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ -\alpha_2 a_{21\cdots 1} - \beta_2 & 0 & 0 & 0\cdots & 0 \\ -\alpha_3 a_{31\cdots 1} - \beta_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_n a_{n1\cdots 1} - \beta_n & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The main contributions of this paper are twofold. First, we consider a new and more effective preconditioner preconditioner $I + S_{\alpha\beta}$, where

$$S_{\alpha\beta} = \begin{bmatrix} 0 & -\alpha_1^1 a_{12\cdots 2} + \beta_1^1 & -\alpha_2^1 a_{13\cdots 3} + \beta_2^1 & \cdots & -\alpha_{n-1}^1 a_{1n\cdots n} + \beta_{n-1}^1 \\ 0 & 0 & -\alpha_2^2 a_{23\cdots 3} + \beta_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \ddots & -\alpha_{n-1}^{n-1} a_{n-1n\cdots n} + \beta_{n-1}^{n-1} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Clearly, if

$$\begin{cases} \beta_i^1 = 0, & i = 1, 2, 3, \cdots, n-1, \\ \beta_i^j = 0, & i = j = 2, 3, \cdots, n-1, \end{cases}$$

then $S_{\alpha\beta}$ reduces to $S_\alpha + R_\beta$. Second, we propose the preconditioned AOR method and compare the spectral radius of the iteration tensors when the coefficient tensor is a weakly irreducible $\mathcal{Z}$-tensor, extending some earlier results in the literature. In particular, we show that the spectral radius is also monotone with respect to the parameters w and γ involved in the AOR method.

The rest of this paper is organized as follows. In Section 2, we introduce some definitions and lemmas which will be used in the sequel. In Section 3, a new preconditioned

AOR method is proposed we also consider two shapes of splitting. In Section 4 the comparison results on spectral radii of different iterative tensors are given. In Section 5, some numerical examples are given to show the efficiency of the proposed preconditioned iterative method. In particular, we show that this preconditioned method is better than the method by Cui et al. [13] and Chen and Li [11] when the coefficient tensor is an $\mathcal{M}$-tensor.

2. Preliminaries

In this section, we recall some definitions and lemmas on tensors, which will be used later to establish spectral radius results for different tensor iterations.

Definition 3 ([37,38]). *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. A pair $(\lambda, \mathbf{x})$ is called an eigenvalue–eigenvector (or simply eigenpair) of $\mathcal{A}$ if it satisfies*

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]},$$

where $\mathbf{x}^{[m-1]} = (x_1^{m-1}, \cdots, x_n^{m-1})$. The largest modulus among all eigenvalues of $\mathcal{A}$ is defined as the spectral radius of $\mathcal{A}$, denoted as $\rho(\mathcal{A})$.

Definition 4 ([39,40]). *A tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called a $\mathcal{Z}$-tensor if its off-diagonal entries are nonpositive. A $\mathcal{Z}$-tensor is called an $\mathcal{M}$-tensor if there exist a nonnegative tensor $\mathcal{B} \in \mathbb{R}^{[m,n]}$ and a positive real scalar η such that*

$$\mathcal{A} = \eta \mathcal{I} - \mathcal{B}, \quad \eta \geq \rho(\mathcal{B}).$$

If $\eta > \rho(\mathcal{B})$, then $\mathcal{A}$ is called nonsingular $\mathcal{M}$-tensor.

Definition 5 ([33]). *Let $\mathcal{A}, \mathcal{E}, \mathcal{F} \in \mathbb{R}^{[m,n]}$. Then $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is said to be a splitting of $\mathcal{A}$ if $\mathcal{E}$ is a left-nonsingular; a regular splitting if $\mathcal{E}$ is a left-nonsingular with $M(\mathcal{E})^{-1} \geq 0$ and $\mathcal{F} \geq 0$; a weak regular splitting if $\mathcal{E}$ is a left-nonsingular with $M(\mathcal{E})^{-1} \geq 0$ and $M(\mathcal{E})^{-1}\mathcal{F} \geq 0$; and a convergent splitting if $\rho(M(\mathcal{E})^{-1}\mathcal{F}) < 1$.*

Lemma 1 ([33]). *Let $\mathcal{A}$ be a $\mathcal{Z}$-tensor. Then the following statements are equivalent:*

(1) $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor;
(2) There exists some $\mathbf{x} \geq 0$, such that $\mathcal{A}\mathbf{x}^{m-1} > 0$;
(3) $\mathcal{A}$ has a convergent (weak) regular splitting;
(4) All the (weak) regular splittings of $\mathcal{A}$ are convergent.

Lemma 2 ([33]). *If $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor, then the matrix $M(\mathcal{A})$ is a nonsingular $\mathcal{M}$-matrix.*

Lemma 3 ([34]). *If $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor, then all the diagonal entries of $\mathcal{A}$ are positive.*

Definition 6 ([34]). *The representative matrix of $\mathcal{A}$ is defined by $\hat{M} = (\hat{m}_{ij}) \in \mathbb{R}^{n \times n}$ with*

$$\hat{m}_{ij} = \sum_{j \in \{i_2, \cdots, i_m\}} |a_{ii_2 \cdots i_m}|.$$

Based on this definition, we have the following property for the tensor $\mathcal{A}$.

Lemma 4 ([34]). *Tensor $\mathcal{A}$ is weakly irreducible if and only if its representative matrix $\hat{M}$ is irreducible.*

Lemma 5 ([34]). *If $\mathcal{A}$ is nonnegative, then $\rho(\mathcal{A})$ is an H-eigenvalue with a nonnegative H-eigenvector* $\mathbf{x}$. *If $\mathcal{A}$ is weakly irreducible nonnegative, then $\rho(\mathcal{A})$ is a positive H-eigenvalue with a positive H-eigenvector* $\mathbf{x}$.

The following lemma can be easily deduced from the results in [34].

Lemma 6. *If $\mathcal{A}$ is a tensor with positive diagonal entries, then $\rho(\mathcal{A}) > 0$.*

Lemma 7 ([36]). *Let $\mathcal{A}$ be a nonnegative tensor. Then*

$$\mu \mathbf{x}^{[m-1]} \leq (<) \mathcal{A}\mathbf{x}^{m-1},\ \mathbf{x} \geq 0,\ \mathbf{x} \neq 0 \ \textit{implies}\ \ \mu \leq (<)\rho(\mathcal{A})$$

and

$$\mathcal{A}\mathbf{x}^{m-1} \leq \nu \mathbf{x}^{[m-1]},\ \mathbf{x} > 0 \ \textit{implies}\ \ \rho(\mathcal{A}) \leq \nu.$$

3. Preconditioned AOR Method

In this section, we present a preconditioned AOR method and introduce two different tensor splitting methods for the $\mathcal{Z}$-tensor equation. We consider the coefficient tensor $\mathcal{A}$ of (1) with positive diagonal entries. Then, without loss of generality, we always assume that each diagonal entry of the tensor $\mathcal{A}$ is 1. Therefore, let

$$\mathcal{A} = \mathcal{I}_m - \mathcal{L} - \mathcal{F},$$

where $\mathcal{L} = L\mathcal{I}_m$, $-L$ is the strictly lower triangle matrix of $M(\mathcal{A})$. We split $\mathcal{A}$ into $\mathcal{A} = \mathcal{U} - \mathcal{V}$ with

$$\mathcal{U} = \frac{1}{\omega}(\mathcal{I}_m - \gamma\mathcal{L}),\ \ \mathcal{V} = \frac{1}{\omega}[(1-\omega)\mathcal{I}_m + (\omega - \gamma)\mathcal{L} + \omega\mathcal{F}],$$

where ω and γ are parameters with $\omega \neq 0$. Its AOR iterative tensor is

$$\mathcal{T} = M(\mathcal{U})^{-1}\mathcal{V}. \tag{4}$$

For our new preconditioner $P_{\alpha\beta} = I + S_{\alpha\beta}$, we get a new preconditioned multilinear system

$$\mathcal{A}^{(\alpha\beta)}\mathbf{x}^{m-1} = \mathbf{b}^{(\alpha\beta)}$$

with $\mathcal{A}^{(\alpha\beta)} = P_{\alpha\beta}\mathcal{A}$, and $\mathbf{b}^{(\alpha\beta)} = P_{\alpha\beta}\mathbf{b}$. Then we have

$$\mathcal{A}^{(\alpha\beta)} = \mathcal{I}_m - \mathcal{L} - \mathcal{F} + S_{\alpha\beta}\mathcal{I}_m - S_{\alpha\beta}\mathcal{L} - S_{\alpha\beta}\mathcal{F} = \mathcal{D}^{(\alpha\beta)} - \mathcal{L}^{(\alpha\beta)} - \mathcal{F}^{(\alpha\beta)},$$

where $\mathcal{D}^{(\alpha\beta)} = D^{(\alpha\beta)}\mathcal{I}_m$, $\mathcal{L}^{(\alpha\beta)} = L^{(\alpha\beta)}\mathcal{I}_m$, and $D^{(\alpha\beta)}, -L^{(\alpha\beta)}$ are the diagonal matrix and the strictly lower triangle matrix of $M(\mathcal{A}^{(\alpha\beta)})$. Specifically, the tensor $S_{\alpha\beta}\mathcal{L}$ can be expressed as

$$S_{\alpha\beta}\mathcal{L} = \mathcal{D}_{\alpha\beta} - \mathcal{L}_{\alpha\beta} - \mathcal{F}_{\alpha\beta},$$

here $\mathcal{D}_{\alpha\beta} = D_{\alpha\beta}\mathcal{I}_m$, $\mathcal{L}_{\alpha\beta} = L_{\alpha\beta}\mathcal{I}_m$, and $D_{\alpha\beta}$, $-L_{\alpha\beta}$ are the diagonal matrix and the strictly lower triangle matrix of $M(S_{\alpha\beta}\mathcal{L})$. Then we have

$$\begin{cases} \mathcal{D}^{(\alpha\beta)} = \mathcal{I}_m - \mathcal{D}_{\alpha\beta}, \\ \mathcal{L}^{(\alpha\beta)} = \mathcal{L} - \mathcal{L}_{\alpha\beta}, \\ \mathcal{F}^{(\alpha\beta)} = \mathcal{F} - S_{\alpha\beta}\mathcal{I}_m - \mathcal{F}_{\alpha\beta} + S_{\alpha\beta}\mathcal{F}. \end{cases}$$

And the splitting is $\mathcal{A}^{(\alpha\beta)} = \mathcal{U}^{(\alpha\beta)} - \mathcal{V}^{(\alpha\beta)}$ with

$$\mathcal{U}^{(\alpha\beta)} = \frac{1}{\omega}(\mathcal{D}^{(\alpha\beta)} - \gamma\mathcal{L}^{(\alpha\beta)}), \ \ \mathcal{V}^{(\alpha\beta)} = \frac{1}{\omega}[(1-\omega)\mathcal{D}^{(\alpha\beta)} + (\omega-\gamma)\mathcal{L}^{(\alpha\beta)} + \omega\mathcal{F}^{(\alpha\beta)}],$$

and the preconditioned AOR iterative tensor is

$$\mathcal{T}^{(\alpha\beta)} = M(\mathcal{U}^{(\alpha\beta)})^{-1}\mathcal{V}^{(\alpha\beta)} \tag{5}$$

In particular, when $\beta_i^j = 0$, $P_{\alpha\beta}$ is reduced to P_α, this is the preconditioner in Cui et al. [13] and the special case in Cui et al. [29]. The corresponding preconditioned multi-linear system is

$$\mathcal{A}^{(\alpha)}\mathbf{x}^{m-1} = \mathbf{b}^{(\alpha)}$$

with $\mathcal{A}^{(\alpha)} = P_\alpha\mathcal{A}$, and $\mathbf{b}^{(\alpha)} = P_\alpha\mathbf{b}$. Similarly, the tensor $S_{\alpha\beta}\mathcal{L}$ is also reduced to $S_\alpha\mathcal{L} = \mathcal{D}_\alpha - \mathcal{L}_\alpha - \mathcal{F}_\alpha$, then we have

$$\mathcal{A}^{(\alpha)} = \mathcal{D}^{(\alpha)} - \mathcal{L}^{(\alpha)} - \mathcal{F}^{(\alpha)},$$

where

$$\begin{cases} \mathcal{D}^{(\alpha)} = \mathcal{I}_m - \mathcal{D}_\alpha, \\ \mathcal{L}^{(\alpha)} = \mathcal{L} - \mathcal{L}_\alpha, \\ \mathcal{F}^{(\alpha)} = \mathcal{F} - S_\alpha\mathcal{I}_m - \mathcal{F}_\alpha + S_\alpha\mathcal{F}. \end{cases}$$

Consequently, we get the splitting way $\mathcal{A}^{(\alpha)} = \mathcal{U}^{(\alpha)} - \mathcal{V}^{(\alpha)}$ with

$$\mathcal{U}^{(\alpha)} = \frac{1}{\omega}(\mathcal{D}^{(\alpha)} - \gamma\mathcal{L}^{(\alpha)}), \ \ \mathcal{V}^{(\alpha)} = \frac{1}{\omega}[(1-\omega)\mathcal{D}^{(\alpha)} + (\omega-\gamma)\mathcal{L}^{(\alpha)} + \omega\mathcal{F}^{(\alpha)}],$$

and preconditioned AOR iterative tensor is

$$\mathcal{T}^{(\alpha)} = M(\mathcal{U}^{(\alpha)})^{-1}\mathcal{V}^{(\alpha)} \tag{6}$$

We further consider another splitting of $\tilde{\mathcal{A}}$, which will be used to illustrate how the new preconditioner improved the effectiveness compared with the ones in Cui et al. [29]. The tensor $\mathcal{A}^{(\alpha\beta)}$ is split into $\mathcal{A}^{(\alpha\beta)} = \check{\mathcal{U}}^{(\alpha\beta)} - \check{\mathcal{V}}^{(\alpha\beta)}$ with

$$\check{\mathcal{U}}^{(\alpha\beta)} = \frac{1}{\omega}(\mathcal{I}_m - \gamma\mathcal{L}^{(\alpha\beta)}), \ \ \check{\mathcal{V}}^{(\alpha\beta)} = \frac{1}{\omega}[(1-\omega)\mathcal{I}_m + (\omega-\gamma)\mathcal{L}^{(\alpha\beta)} + \omega(\mathcal{D}_{\alpha\beta} + \mathcal{F}^{(\alpha\beta)})],$$

then the preconditioned AOR iterative tensor is

$$\check{\mathcal{T}}^{(\alpha\beta)} = M(\check{\mathcal{U}}^{(\alpha\beta)})^{-1}\check{\mathcal{V}}^{(\alpha\beta)} \tag{7}$$

Next, we will show that tensor $\tilde{\mathcal{A}}$ and $\bar{\mathcal{A}}$ are $\mathcal{M}$-tensors when the parameters α_i^j, β_i^j meet certain conditions.

Condition 1.

(1) $0 \leq -\alpha_i^1 a_{1i+1\cdots i+1} + \beta_i^1 < -a_{1i+1\cdots i+1}$, for $i = 1, 2, \cdots, n-1$;
(2) $0 \leq -\alpha_i^j a_{ji+1\cdots i+1} + \beta_i^j < -a_{ji+1\cdots i+1}$, for $i = j = 2, \cdots, n-1$.

Theorem 1. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a nonsingular $\mathcal{M}$-tensor. If Condition 1 is satisfied, then $\mathcal{A}^{(\alpha\beta)}$ is also a nonsingular $\mathcal{M}$-tensor.*

Proof. We denote the entries of $\mathcal{A}^{(\alpha\beta)}$ as $a^{(\alpha\beta)}_{ji_2\cdots i_m}$, then from $\mathcal{A}^{(\alpha\beta)} = (I + S_{\alpha\beta})\mathcal{A}$, we have

$$a^{(\alpha\beta)}_{ji_2\cdots i_m} = \begin{cases} a_{ji_2\cdots i_m} + \sum_{t=1}^{n-1}(-\alpha_t^j a_{jt+1\cdots t+1} + \beta_t^j)a_{t+1i_2\cdots i_m}, & j = 1 \\ a_{ji_2\cdots i_m} + (-\alpha_i^j a_{ji+1\cdots i+1} + \beta_i^j)a_{i+1i_2\cdots i_m}, & i = j = 2, \cdots, n-1 \\ a_{ji_2\cdots i_m}, & j = n \end{cases}$$

Since $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor, $a_{ji_2\cdots i_m} \leq 0$ for all $(j, i_2, \cdots, i_m) \neq (j, j, \cdots, j)$, $j = 1, 2, \cdots, n$. Consider the case $j = 1$, and when $(i_2, \cdots, i_m) = (i+1, \cdots, i+1)$, $i = 1, \cdots, n-1$,

$$\begin{aligned} a^{(\alpha\beta)}_{1i+1\cdots i+1} = \; & a_{1i+1\cdots i+1} + (-\alpha_i^1 a_{1i+1\cdots i+1} + \beta_i^1)a_{i+1i+1\cdots i+1} \\ & + \textstyle\sum_{t=1,t\neq i}^{n-1}(-\alpha_t^1 a_{1t+1\cdots t+1} + \beta_t^1)a_{t+1i+1\cdots i+1} \end{aligned}$$

if $0 \leq -\alpha_i^1 a_{1i+1\cdots i+1} + \beta_i^1 \leq -a_{1i+1\cdots i+1}$, for all $i \in \{1, 2, \cdots, n-1\}$, we have $a^{(\alpha\beta)}_{1i+1\cdots i+1} \leq 0$. When $(i_2, \cdots, i_m) \neq (i+1, \cdots, i+1)$, $i = 1, \cdots, n-1$, it is obvious that if $0 \leq -\alpha_i^1 a_{1i+1\cdots i+1} + \beta_i^1$, for all $i \in \{1, 2, \cdots, n-1\}$, leads to $a^{(\alpha\beta)}_{1i_2\cdots i_m} \leq 0$. Similarly, consider the case $j = 2, \cdots, n-1$, if $0 \leq -\alpha_i^j a_{1i+1\cdots i+1} + \beta_i^j \leq -a_{ji+1\cdots i+1}$, we have $a^{(\alpha\beta)}_{ji_2\cdots i_m} \leq 0$. Consequently, $\mathcal{A}^{(\alpha\beta)}$ is a $\mathcal{Z}$-tensor.

Since $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor, according to Lemma 1, there exists $\mathbf{x} \geq 0$, such that $\mathcal{A}\mathbf{x}^{m-1} > 0$. From Condition 1, we have $I + S_{\alpha\beta} \geq \mathbf{0}$, then $\mathcal{A}^{(\alpha\beta)}\mathbf{x}^{m-1} > 0$. Therefore, by Lemma 1 again, $\mathcal{A}^{(\alpha\beta)}$ is a nonsingular $\mathcal{M}$-tensor. □

In particular, when $\beta_i^j = 0$, we can directly have Corollary 1 from Theorem 1.

Condition 2.

(1) $0 \leq \alpha_i^1 < 1$, for $i = 1, 2, \cdots, n-1$;
(2) $0 \leq \alpha_i^j < 1$, for $i = j = 2, \cdots, n-1$.

Corollary 1. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a nonsingular $\mathcal{M}$-tensor. If Condition 2 is satisfied, then $\mathcal{A}^{(\alpha)}$ is also a nonsingular $\mathcal{M}$-tensor.*

Remark 2. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a $\mathcal{Z}$-tensor. It is obvious that the tensor $\mathcal{F}$ and $S_{\alpha\beta}\mathcal{L}$ are nonnegative. By simple computation, Condition 1 can also guarantee $\mathcal{F}^{(\alpha\beta)}$ is nonnegative. Moreover, the entries in $\mathcal{F}^{(\alpha\beta)}$ have the same symbol as the corresponding entries in $\mathcal{F}$ with Condition 1. Under Condition 2, the tensor $\mathcal{F}^{(\alpha)}$ is also nonnegative.*

4. The Comparison Theorem

In this section, we mainly give the comparison results of the spectral radius of iterative tensors $\mathcal{T}$, $\mathcal{T}^{(\alpha\beta)}$, $\mathcal{T}^{(\alpha)}$ and $\check{\mathcal{T}}^{(\alpha\beta)}$. First of all, we provide a lemma as follows.

Condition 3.

(1) $0 \leq -\alpha_i^1 a_{1i+1\cdots i+1} + \beta_i^1 < \frac{-1}{(n-1)\min_{t\in\{2,\cdots,n\}}\{a_{t1\cdots 1}\}}$, for $i = 1, 2, \cdots, n-1$;

(2) $0 \leq -\alpha_i^j a_{ji+1\cdots i+1} + \beta_i^j < \frac{-1}{a_{i+1i\cdots i}}$, for $i = j = 2, \cdots, n-1$.

Theorem 2. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a weakly irreducible $\mathcal{Z}$-tensor with positive diagonal entries. Assume $0 \leq \gamma < \omega \leq 1$ or $0 < \gamma = \omega < 1$. Then the following statements are true:*

(1) $\mathcal{T}$ defined in (4) is weakly irreducible nonnegative tensor;

(2) If Condition 1 is satisfied, then $\breve{\mathcal{T}}^{(\alpha\beta)}$ defined in (7) is weakly irreducible nonnegative tensor;

(3) If Condition 1 is satisfied and one of the following conditions holds:

(a) $\rho(\mathcal{T}) < 1$;

(b) $\rho(\mathcal{T}) \geq 1$, and Condition 3 holds,

where $\rho(\mathcal{T})$is the spectral radius of tensor $\mathcal{T}$, then $\mathcal{T}^{(\alpha\beta)}$ defined in (5) is a weakly irreducible nonnegative tensor.

Proof. Since $\mathcal{A}$ is a $\mathcal{Z}$-tensor with positive diagonal entries, it follows from the splitting $\mathcal{A} = \mathcal{I}_m - \mathcal{L} - \mathcal{F}$ that

$$\begin{aligned}\mathcal{T} &= M(\mathcal{I}_m - \gamma\mathcal{L})^{-1}[(1-\omega)\mathcal{I}_m + (\omega-\gamma)\mathcal{L} + \omega\mathcal{F}] \\ &= (I - \gamma L)^{-1}[(1-\omega)\mathcal{I}_m + (\omega-\gamma)\mathcal{L} + \omega\mathcal{F}] \\ &= [I + \gamma L + (\gamma L)^2 + \cdots][(1-\omega)\mathcal{I}_m + (\omega-\gamma)\mathcal{L} + \omega\mathcal{F}] \\ &\geq (1-\omega)\mathcal{I}_m + (\omega-\gamma)\mathcal{L} + \gamma(1-\omega)\mathcal{L} + \omega\mathcal{F}.\end{aligned} \tag{8}$$

So, $\mathcal{T}$ is a weakly irreducible nonnegative because $\mathcal{L}, \mathcal{F}$ are nonnegative and $\mathcal{L} + \mathcal{F}$ is weakly irreducible. Since $\mathcal{A}^{(\alpha\beta)}$ is a $\mathcal{Z}$-tensor and the entries have the same symbol as the entries in $\mathcal{A}$ under the Condition 1, $\breve{\mathcal{T}}^{(\alpha\beta)}$ is a weakly irreducible nonnegative tensor similarly.

For case (a). From Lemma 1, we know that $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor. Now from Theorem 1, $\mathcal{A}^{(\alpha\beta)}$ is also a nonsingular $\mathcal{M}$-tensor. Therefore, from Lemma 3, we know that $\mathcal{D}^{(\alpha\beta)} > 0$. Following the fact $\mathcal{L}^{(\alpha\beta)}, \mathcal{F}^{(\alpha\beta)} \geq 0$, we have

$$\begin{aligned}\mathcal{T}^{(\alpha\beta)} &= M(\mathcal{D}^{(\alpha\beta)} - \gamma\mathcal{L}^{(\alpha\beta)})^{-1}[(1-\omega)\mathcal{D}^{(\alpha\beta)} + (\omega-\gamma)\mathcal{L}^{(\alpha\beta)} + \omega\mathcal{F}^{(\alpha\beta)}] \\ &= (D^{(\alpha\beta)} - \gamma L^{(\alpha\beta)})^{-1}[(1-\omega)\mathcal{D}^{(\alpha\beta)} + (\omega-\gamma)\mathcal{L}^{(\alpha\beta)} + \omega\mathcal{F}^{(\alpha\beta)}] \\ &= [I + \gamma(D^{(\alpha\beta)})^{-1}L^{(\alpha\beta)} + (\gamma(D^{(\alpha\beta)})^{-1}L^{(\alpha\beta)})^2 + \cdots] \\ &\quad (D^{(\alpha\beta)})^{-1}[(1-\omega)\mathcal{D}^{(\alpha\beta)} + (\omega-\gamma)\mathcal{L}^{(\alpha\beta)} + \omega\mathcal{F}^{(\alpha\beta)}] \\ &\geq (1-\omega)\mathcal{I}_m + \gamma(1-\omega)(D^{(\alpha\beta)})^{-1}\mathcal{L}^{(\alpha\beta)} + (\omega-\gamma)(D^{(\alpha\beta)})^{-1}\mathcal{L}^{(\alpha\beta)} \\ &\quad + \omega(D^{(\alpha\beta)})^{-1}\mathcal{F}^{(\alpha\beta)}\end{aligned}$$

which means $\mathcal{T}^{(\alpha\beta)}$ is nonnegative. Since the entries in $\mathcal{F}^{(\alpha\beta)}$ have the same symbol as the corresponding entries in $\mathcal{F}$ and $\mathcal{L}^{(\alpha\beta)} \geq \mathcal{L}$, we further have that $\mathcal{T}^{(\alpha\beta)}$ is weakly irreducible due to $\mathcal{L} + \mathcal{F}$ being weakly irreducible.

For case (b). Since the entries of diagonal tensor $\mathcal{D}^{(\alpha\beta)}$ of $\mathcal{A}^{(\alpha\beta)}$ are

$$a_{jj\cdots j}^{(\alpha\beta)} = \begin{cases} 1 + \sum_{t=1}^{n-1}(-\alpha_t^1 a_{1t+1\cdots t+1} + \beta_t^1)a_{t+11\cdots 1}, & \text{for } j = 1, \\ 1 + (-\alpha_i^j a_{ii+1\cdots i+1} + \beta_i^j)a_{i+1i\cdots i}, & \text{for } i = j = 2, \cdots, n-1, \\ a_{nn\cdots n}. \end{cases}$$

Under Condition 3, we have $a^{(\alpha\beta)}_{ii\cdots i} > 0$, so $\mathcal{D}^{(\alpha\beta)} > 0$. By the same discussion as case (a), we obtain that $\mathcal{T}^{(\alpha\beta)}$ is weakly irreducible and nonnegative. □

For the special case, all $\beta^j_i = 0$; we also have same results for $\bar{\mathcal{T}}$ defined in (6).

Condition 4.

(1) $0 \le -\alpha^1_i a_{1i+1\cdots i+1} < \frac{-1}{(n-1)\min_{t\in\{2,\cdots,n\}}\{a_{t1\cdots 1}\}}$, for $i = 1, 2, \cdots, n-1$;

(2) $0 \le -\alpha^j_i a_{ji+1\cdots i+1} < \frac{-1}{a_{i+1i\cdots i}}$, for $i = j = 2, \cdots, n-1$.

Corollary 2. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a weakly irreducible $\mathcal{Z}$-tensor with positive diagonal entries. Assume $0 \le \gamma < \omega \le 1$ or $0 < \gamma = \omega < 1$, if Condition 2 is satisfied, and one of the following conditions holds:*

(1) $\rho(\mathcal{T}) < 1$;

(2) $\rho(\mathcal{T}) \ge 1$, and Condition 4 holds,

where $\rho(\mathcal{T})$is the spectral radius of the tensor $\mathcal{T}$. Then $\mathcal{T}^{(\alpha)}$ is a weakly irreducible nonnegative tensor.

Next, we consider the spectral radius comparison results between the preconditioned AOR iterative method and the original AOR iterative method.

Theorem 3. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a weakly irreducible $\mathcal{Z}$-tensor with positive diagonal entries. If $0 \le \gamma \le \omega \le 1$, and Condition 1 is satisfied, then for the spectral radius of $\mathcal{T}$ defined in (4) and $\mathcal{T}^{(\alpha\beta)}$ defined in (5), the following statements hold:*

(1) $\rho(\mathcal{T}^{(\alpha\beta)}) \le \rho(\mathcal{T}) < 1$, if $\rho(\mathcal{T}) < 1$;

(2) $\rho(\mathcal{T}^{(\alpha\beta)}) \ge \rho(\mathcal{T}) > 1$, if $\rho(\mathcal{T}) > 1$ and Condition 3 holds.

Proof. Following from (8) in Theorem 2, we know that $\mathcal{T}$ is always nonnegative whether $\gamma = \omega$ or not. According to Lemma 5, there exists a nonzero vector $\mathbf{x} \ge 0$ such that

$$\mathcal{T}\mathbf{x}^{m-1} = \rho(\mathcal{T})\mathbf{x}^{[m-1]}.$$

For simplicity, we denote $\rho(\mathcal{T})$ as λ. From Lemma 6, we know that $\lambda > 0$. And it can be deduced that

$$[(1-\omega)\mathcal{I}_m + (\omega-\gamma)\mathcal{L} + \omega\mathcal{F}]\mathbf{x}^{m-1} = \lambda(\mathcal{I}_m - \gamma\mathcal{L})\mathbf{x}^{m-1}. \tag{9}$$

Now, we consider

$$\begin{aligned}
&\mathcal{T}^{(\alpha\beta)}\mathbf{x}^{m-1} - \lambda\mathbf{x}^{[m-1]}\\
=&M(\mathcal{D}^{(\alpha\beta)} - \gamma\mathcal{L}^{(\alpha\beta)})^{-1}[(1-\omega)\mathcal{D}^{(\alpha\beta)} + (\omega-\gamma)\mathcal{L}^{(\alpha\beta)} + \omega\mathcal{F}^{(\alpha\beta)} - \lambda(\mathcal{D}^{(\alpha\beta)} - \gamma\mathcal{L}^{(\alpha\beta)})]\mathbf{x}^{m-1}\\
=&M(\mathcal{D}^{(\alpha\beta)} - \gamma\mathcal{L}^{(\alpha\beta)})^{-1}[(1-\omega-\lambda)\mathcal{D}^{(\alpha\beta)} + (\omega-\gamma+\lambda\gamma)\mathcal{L}^{(\alpha\beta)} + \omega\mathcal{F}^{(\alpha\beta)}]\mathbf{x}^{m-1}\\
=&M(\mathcal{D}^{(\alpha\beta)} - \gamma\mathcal{L}^{(\alpha\beta)})^{-1}[(1-\omega-\lambda)(\mathcal{I}_m - \mathcal{D}_{\alpha\beta}) + (\omega-\gamma+\lambda\gamma)(\mathcal{L} - \mathcal{L}_{\alpha\beta})]\\
&+\omega(\mathcal{F} - S_{\alpha\beta}\mathcal{I}_m - \mathcal{F}_{\alpha\beta} + S_{\alpha\beta}\mathcal{F})\mathbf{x}^{m-1}
\end{aligned}$$

From (9), the above expression can be further written as follows

$$\begin{aligned}
&\mathcal{T}^{(\alpha\beta)}\mathbf{x}^{m-1} - \lambda\mathbf{x}^{[m-1]} \\
&= M(\mathcal{D}^{(\alpha\beta)} - \gamma\mathcal{L}^{(\alpha\beta)})^{-1}[-(1-\omega-\lambda)\mathcal{D}_{\alpha\beta} - (\omega-\gamma+\lambda\gamma)\mathcal{L}_{\alpha\beta} \\
&- \omega(S_{\alpha\beta}\mathcal{I}_m + \mathcal{F}_{\alpha\beta} - S_{\alpha\beta}\mathcal{F})]\mathbf{x}^{m-1} \\
=&M(\mathcal{D}^{(\alpha\beta)} - \gamma\mathcal{L}^{(\alpha\beta)})^{-1}[(\lambda-1)\mathcal{D}_{\alpha\beta} + \omega(\mathcal{D}_{\alpha\beta} - \mathcal{L}_{\alpha\beta} - \mathcal{F}_{\alpha\beta}) - (\lambda\gamma-\gamma)\mathcal{L}_{\alpha\beta} \\
&- \omega(S_{\alpha\beta}\mathcal{I}_m - S_{\alpha\beta}\mathcal{F})]\mathbf{x}^{m-1} \\
=&M(\mathcal{D}^{(\alpha\beta)} - \gamma\mathcal{L}^{(\alpha\beta)})^{-1}[(\lambda-1)\mathcal{D}_{\alpha\beta} - (\lambda-1)\gamma\mathcal{L}_{\alpha\beta} - \omega S_{\alpha\beta}(\mathcal{I}_m - \mathcal{L} - \mathcal{F})]\mathbf{x}^{m-1} \\
=&M(\mathcal{D}^{(\alpha\beta)} - \gamma\mathcal{L}^{(\alpha\beta)})^{-1}\{(\lambda-1)\mathcal{D}_{\alpha\beta} - (\lambda-1)\gamma\mathcal{L}_{\alpha\beta} + S_{\alpha\beta}[(1-\omega)\mathcal{I}_m + (\omega-\gamma)\mathcal{L} \\
&+ \omega\mathcal{F} - (\mathcal{I}_m - \gamma\mathcal{L})]\}\mathbf{x}^{m-1} \\
=&M(\mathcal{D}^{(\alpha\beta)} - \gamma\mathcal{L}^{(\alpha\beta)})^{-1}\{(\lambda-1)\mathcal{D}_{\alpha\beta} - (\lambda-1)\gamma\mathcal{L}_{\alpha\beta} + S_{\alpha\beta}[\lambda(\mathcal{I}_m - \gamma\mathcal{L}) \\
&- (\mathcal{I}_m - \gamma\mathcal{L})]\}\mathbf{x}^{m-1} \\
=&M(\mathcal{D}^{(\alpha\beta)} - \gamma\mathcal{L}^{(\alpha\beta)})^{-1}[(\lambda-1)\mathcal{D}_{\alpha\beta} - (\lambda-1)\gamma\mathcal{L}_{\alpha\beta} + (\lambda-1)S_{\alpha\beta}(\mathcal{I}_m - \gamma\mathcal{L})]\mathbf{x}^{m-1}
\end{aligned}$$

To study the effect of $\rho(\mathcal{T})$ on the preconditioned iteration tensor, we extract the common factor $\frac{\lambda-1}{\lambda}$, which leads to the following result

$$\begin{aligned}
&\mathcal{T}^{(\alpha\beta)}\mathbf{x}^{m-1} - \lambda\mathbf{x}^{[m-1]} \\
=&M(\mathcal{D}^{(\alpha\beta)} - \gamma\mathcal{L}^{(\alpha\beta)})^{-1}\{(\lambda-1)\mathcal{D}_{\alpha\beta} - (\lambda-1)\gamma\mathcal{L}_{\alpha\beta} \\
&+ \frac{\lambda-1}{\lambda}S_{\alpha\beta}[(1-\omega)\mathcal{I}_m + (\omega-\gamma)\mathcal{L} + \omega\mathcal{F}]\}\mathbf{x}^{m-1} \\
=&\frac{\lambda-1}{\lambda}M(\mathcal{D}^{(\alpha\beta)} - \gamma\mathcal{L}^{(\alpha\beta)})^{-1}[\lambda\mathcal{D}_{\alpha\beta} - \lambda\gamma\mathcal{L}_{\alpha\beta} + S_{\alpha\beta}[(1-\omega)\mathcal{I}_m + (\omega-\gamma)\mathcal{L} + \omega\mathcal{F}]\mathbf{x}^{m-1}.
\end{aligned}$$

By (9), due to the corresponding definitions, we have $\lambda\mathcal{D}_{\alpha\beta} - \lambda\gamma\mathcal{L}_{\alpha\beta} + S_{\alpha\beta}[(1-\omega)\mathcal{I}_m + (\omega-\gamma)\mathcal{L} + \omega\mathcal{F}] \geq 0$, then we consider the following:

When $0 \leq \gamma < \omega \leq 1$ or $0 < \gamma = \omega < 1$. It follows from Theorem 2 that $\mathcal{T}$ and $\tilde{\mathcal{T}}$ are weakly irreducible and nonnegative in both cases, then the vector $\mathbf{x}$ can be further positive.

Case (1): $\lambda < 1$.

Following on from the process of (8), we know that $\mathcal{A} = \mathcal{U} - \mathcal{V}$ is a convergent regular splitting, then $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor. According to Theorem 1, $\mathcal{A}^{(\alpha\beta)}$ is also a nonsingular $\mathcal{M}$-tensor, so $\mathcal{D}^{(\alpha\beta)} - \gamma\mathcal{L}^{(\alpha\beta)}$ is a nonsingular $\mathcal{M}$-tensor. By Lemma 2, $M(\mathcal{D}^{(\alpha\beta)}) - \gamma(\mathcal{L}^{(\alpha\beta)})$ is a nonsingular M-matrix, then $M(\mathcal{D}^{(\alpha\beta)}) - \gamma(\mathcal{L}^{(\alpha\beta)})^{-1} \geq 0$. Therefore, we have $\mathcal{T}^{(\alpha\beta)}\mathbf{x}^{m-1} - \lambda\mathbf{x}^{[m-1]} \leq 0$. By Lemma 7, we get $\rho(\mathcal{T}^{(\alpha\beta)}) \leq \rho(\mathcal{T}) < 1$.

Case (2): $\lambda > 1$.

As discussed in Theorem 2, Condition 3 leads to $\mathcal{D}^{(\alpha\beta)}) > 0$, then $M(\mathcal{D}^{(\alpha\beta)}) - \gamma(\mathcal{L}^{(\alpha\beta)})^{-1} = M(\mathcal{I}_m - \gamma(\mathcal{D}^{(\alpha\beta)})^{-1}\tilde{\mathcal{L}})^{-1}M(\mathcal{D}^{(\alpha\beta)})^{-1} \geq 0$. Therefore, we have $\mathcal{T}^{(\alpha\beta)}\mathbf{x}^{m-1} - \lambda\mathbf{x}^{[m-1]} \geq 0$. By Lemma 7, we get $\rho(\mathcal{T}^{(\alpha\beta)}) \geq \rho(\mathcal{T}) > 1$.

When $\gamma = \omega = 1$. $\mathbf{x}$ is just a nonnegative and nonzero vector. Here we denote $\tilde{\mathcal{T}}$ as $\tilde{\mathcal{T}}_{\gamma,\omega}$, and denote $\mathcal{T}$ as $\mathcal{T}_{\gamma,\omega}$.

Case (1): $\lambda < 1$.

We have $\rho(\mathcal{T}_{1,1}^{(\alpha\beta)}) = \lim_{\gamma\to 1}\rho(\mathcal{T}_{\gamma,1}^{(\alpha\beta)}) \leq \lim_{\gamma\to 1}\rho(\mathcal{T}_{\gamma,1}) = \rho(\mathcal{T}_{1,1})$

Case (2): $\lambda > 1$. Even though $\mathbf{x}$ is just a nonnegative and nonzero vector, we still have the same result from the fact $M(\mathcal{D}^{(\alpha\beta)} - \gamma\mathcal{L}^{(\alpha\beta)})^{-1} \geq 0$ and the Lemma 7. □

Using a similar argument, we have the spectral radius comparison results between the iterative tensor $\check{\mathcal{T}}^{(\alpha\beta)}$, $\mathcal{T}^{(\alpha)}$ and the original iterative tensor $\mathcal{T}$.

Theorem 4. *Let $A \in \mathbb{R}^{[m,n]}$ be a weakly irreducible $\mathcal{Z}$-tensor with positive diagonal entries. If $0 \le \gamma \le \omega \le 1$, and Condition 2 is satisfied, then for the spectral radius of $\mathcal{T}$ defined in (4) and $\mathcal{T}^{(\alpha)}$ defined in (6), the following statements hold:*

(1) $\rho(\mathcal{T}^{(\alpha)}) \le \rho(\mathcal{T}) < 1$, *if* $\rho(\mathcal{T}) < 1$;
(2) $\rho(\mathcal{T}^{(\alpha)}) \ge \rho(\mathcal{T}) > 1$, *if* $\rho(\mathcal{T}) > 1$ *and Condition 4 holds.*

Theorem 5. *Let $A \in \mathbb{R}^{[m,n]}$ be a weakly irreducible $\mathcal{Z}$-tensor with positive diagonal entries. If $0 \le \gamma \le \omega \le 1$, and Condition 1 is satisfied, then for the spectral radius of $\mathcal{T}$ defined in (4) and $\check{\mathcal{T}}^{(\alpha\beta)}$ defined in (7), one of the following statements hold:*

(1) $\rho(\check{\mathcal{T}}^{(\alpha\beta)}) \le \rho(\mathcal{T}) < 1$, *if* $\rho(\mathcal{T}) < 1$;
(2) $\rho(\check{\mathcal{T}}^{(\alpha\beta)}) \ge \rho(\mathcal{T}) > 1$, *if* $\rho(\mathcal{T}) > 1$.

Now, we consider the comparison results of two different iterative tensors $\check{\mathcal{T}}^{(\alpha\beta)}$ and $\check{\mathcal{T}}^{(\alpha)}$.

Theorem 6. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a weakly irreducible $\mathcal{Z}$-tensor with positive diagonal entries. If $0 \le \gamma \le \omega \le 1$, and Condition 1 is satisfied, then for the spectral radius of $\mathcal{T}$ defined in (4) and $\check{\mathcal{T}}$ defined in (7), the following statements hold:*

(1) $\rho(\mathcal{T}^{(\alpha\beta)}) \le \rho(\check{\mathcal{T}}^{(\alpha\beta)}) < 1$, *if* $\rho(\mathcal{T}) < 1$;
(2) $\rho(\mathcal{T}^{(\alpha\beta)}) \ge \rho(\check{\mathcal{T}}^{(\alpha\beta)}) > 1$, *if* $\rho(\mathcal{T}) > 1$ *and Condition 3 holds.*

Proof. Firstly, the tensor $\check{\mathcal{T}}^{(\alpha\beta)}$ is nonnegative from Theorem 2. Thus from Lemma 5, there exists a nonnegative vector $\mathbf{x}$ such that

$$\check{\mathcal{T}}^{(\alpha\beta)}\mathbf{x}^{m-1} = \rho(\check{\mathcal{T}}^{(\alpha\beta)})\mathbf{x}^{[m-1]}.$$

Denote $\rho(\check{\mathcal{T}}^{(\alpha\beta)})$ as $\check{\lambda}^{(\alpha\beta)}$. Note that $\check{\mathcal{V}}^{(\alpha\beta)} = \mathcal{V}^{(\alpha\beta)} + \mathcal{D}_{\alpha\beta}$ and $\mathcal{D}^{(\alpha\beta)} = \mathcal{I}_m - \mathcal{D}_{\alpha\beta}$, then by a similar argument in Theorem 3, we get

$$\mathcal{T}^{(\alpha\beta)}\mathbf{x}^{m-1} - \check{\lambda}^{(\alpha\beta)}\mathbf{x}^{[m-1]} = (\check{\lambda}^{(\alpha\beta)} - 1)M(\mathcal{D}^{(\alpha\beta)} - \gamma\mathcal{L}^{(\alpha\beta)})^{-1}\mathcal{D}_{\alpha\beta}\mathbf{x}^{m-1}.$$

The rest of the proof is similar to Theorem 3; we omitted it here. □

Note that, when all $\beta_i^j = 0$, the preconditioner $S_{\alpha\beta}$ reduced some known preconditioners. By a similar argument in Theorem 3, we give the comparison results between $\tilde{\mathcal{T}}^{(\alpha\beta)}$ defined in (5) and $\mathcal{T}^{(\alpha)}$ defined in (6).

Theorem 7. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a weakly irreducible $\mathcal{Z}$-tensor with positive diagonal entries. Assume all $\beta_i^1 \ge 0$, $i = 1, \ldots, n-1$ and $\beta_i^j \ge 0$, $i = j = 2, \ldots, n-1$. If $0 \le \gamma \le \omega \le 1$, and Condition 1 is satisfied, then for the spectral radius of $\tilde{\mathcal{T}}$ and $\bar{\mathcal{T}}$, the following statements hold:*

(1) $\rho(\mathcal{T}^{(\alpha\beta)}) \le \rho(\mathcal{T}^{(\alpha)}) < 1$, *if* $\rho(\mathcal{T}) < 1$;
(2) $\rho(\mathcal{T}^{(\alpha\beta)}) \ge \rho(\mathcal{T}^{(\alpha)}) > 1$, *if* $\rho(\mathcal{T}) > 1$ *and Condition 3 holds.*

Proof. Firstly, when all $\beta_i^j \ge 0$, it is obvious that Conditions 2 and 4 are satisfied due to Conditions 1 and 3 being satisfied. Then the tensor $\mathcal{T}^{(\alpha)}$ is nonnegative from Corollary 2. Thus from Lemma 5, there exists a nonnegative vector $\mathbf{x}$ such that

$$\mathcal{T}^{(\alpha)}\mathbf{x}^{m-1} = \rho(\mathcal{T}^{(\alpha)})\mathbf{x}^{[m-1]}.$$

Denote $\rho(\mathcal{T}^{(\alpha)})$ as $\lambda^{(\alpha)}$; we have $\lambda^{(\alpha)} > 0$ according to Corollaries 1 and 2. By a similar argument in Theorem 3, we get

$$\mathcal{T}^{(\alpha\beta)}\mathbf{x}^{m-1} - \lambda^{(\alpha)}\mathbf{x}^{[m-1]}$$
$$= \frac{\lambda^{(\alpha)}-1}{\lambda^{(\alpha)}} M(\mathcal{D}^{(\alpha\beta)} - \gamma\mathcal{L}^{(\alpha\beta)})^{-1}[\lambda^{(\alpha)}\mathcal{D}_\beta - \lambda^{(\alpha)}\gamma\mathcal{L}_\beta + S_\beta[(1-\omega)\mathcal{I}_m + (\omega-\gamma)\mathcal{L} + \omega\mathcal{F}]\mathbf{x}^{m-1}.$$

Note that we need to use Theorem 4 for the spectral radius of $\rho(\mathcal{T}^{(\alpha)})$. The rest of the proof is similar to Theorem 3; we omitted it here. □

Corollary 3. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is a nonsingular $\mathcal{M}$-tensor. Assume all $\beta_i^1 \geq 0$, $i = 1, \ldots, n-1$ and $\beta_i^j \geq 0$, $i = j = 2, \ldots, n-1$. If $0 \leq \gamma \leq \omega \leq 1$, and Condition 1 is satisfied, then $\rho(\mathcal{T}^{(\alpha\beta)}) \leq \rho(\mathcal{T}^{(\alpha)}) < 1$.*

Proof. Since $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor and $\mathcal{A} = \mathcal{U} - \mathcal{V}$ is a regular splitting, we have $\rho(\mathcal{T}) < 1$ from Lemma 1. Consequently, the results are directly from Theorem 6. □

Remark 3. *Corollary 3 shows that the presented preconditioner $I + S_{\alpha\beta}$ can improve the convergence compared with some known preconditioners when $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor.*

Remark 4. *Throughout this paper, similar results about Jacobi type, Gauss–Seidel type, and SOR type method can be obtained directly from the above results by selecting appropriate parameters for γ and ω.*

Finally, we give a conclusion about the influence of parameters γ and ω on the spectral radius of the preconditioned iterative tensor.

Theorem 8. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a weakly irreducible $\mathcal{Z}$-tensor with positive diagonal entries. Suppose $\mathcal{A}^{(\alpha\beta)} = \mathcal{U}_1^{(\alpha\beta)} - \mathcal{V}_1^{(\alpha\beta)} = \mathcal{U}_2^{(\alpha\beta)} - \mathcal{V}_2^{(\alpha\beta)}$, where $\mathcal{U}_i^{(\alpha\beta)} = \frac{1}{\omega_i}(\mathcal{D}^{(\alpha\beta)} - \gamma_i\mathcal{L}^{(\alpha\beta)})$, $\mathcal{V}_i^{(\alpha\beta)} = \frac{1}{\omega_i}[(1-\omega_i)\mathcal{D}^{(\alpha\beta)} + (\omega_i - \gamma_i)\mathcal{L}^{(\alpha\beta)} + \omega_i\mathcal{F}^{(\alpha\beta)}]$, $i = 1, 2$. If $\omega_1 \geq \omega_2 > 0$, $\gamma_1 \geq \gamma_2$, and Condition 1 is satisfied, then for the spectral radius of $\mathcal{T}_1^{(\alpha\beta)}$ and $\mathcal{T}_2^{(\alpha\beta)}$, the following statements hold:*

(1) $\rho(\mathcal{T}_1^{(\alpha\beta)}) \leq \rho(\mathcal{T}_2^{(\alpha\beta)}) < 1$, if $\rho(\mathcal{T}) < 1$;

(2) $\rho(\mathcal{T}_1^{(\alpha\beta)}) \geq \rho(\mathcal{T}_2^{(\alpha\beta)}) > 1$, if $\rho(\mathcal{T}) > 1$ and Condition 3 holds,

where $\mathcal{T}_1^{(\alpha\beta)}$ and $\mathcal{T}_2^{(\alpha\beta)}$ respectively are the iterative tensors with respect to the parameter ω_i, γ_i.

Proof. As $\mathcal{T}_2^{(\alpha\beta)}$ is a nonnegative tensor from Theorem 2, there exists a nonnegative nonzero vector $\mathbf{x}$ such that

$$\mathcal{T}_2^{(\alpha\beta)}\mathbf{x}^{m-1} = \rho(\mathcal{T}_2^{(\alpha\beta)})\mathbf{x}^{[m-1]}.$$

Denote $\rho(\mathcal{T}_2^{(\alpha\beta)})$ as λ_2. Similar to the proof in Theorem 3, we have

$$\mathcal{T}_1^{(\alpha\beta)}\mathbf{x}^{m-1} - \lambda_2\mathbf{x}^{[m-1]}$$
$$= \frac{\lambda_2 - 1}{\lambda_2} M(\mathcal{D}^{(\alpha\beta)} - \gamma_1\mathcal{L}^{(\alpha\beta)})^{-1}$$
$$\cdot \left\{ \frac{\omega_1 - \omega_2}{\omega_2} \left[(1-\omega_2)\mathcal{D}^{(\alpha\beta)} + (\omega_2 - \gamma_2)\mathcal{L}^{(\alpha\beta)} + \omega_2\mathcal{F}^{(\alpha\beta)} \right] + \lambda_2(\gamma_1 - \gamma_2)\mathcal{L}^{(\alpha\beta)} \right\} \mathbf{x}^{m-1}.$$

The rest of the argument is similar to Theorem 3, which we omitted here. □

Theorem 9. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a weakly irreducible $\mathcal{Z}$-tensor with positive diagonal entries. Suppose $\mathcal{A}^{(\alpha\beta)} = \breve{\mathcal{U}}_1^{(\alpha\beta)} - \breve{\mathcal{V}}_1^{(\alpha\beta)} = \breve{\mathcal{U}}_2^{(\alpha\beta)} - \breve{\mathcal{V}}_2^{(\alpha\beta)}$, where $\breve{\mathcal{U}}_i^{(\alpha\beta)} = \frac{1}{\omega_i}(\breve{\mathcal{D}}^{(\alpha\beta)} - \gamma_i\breve{\mathcal{L}}^{(\alpha\beta)})$, $\breve{\mathcal{V}}_i^{(\alpha\beta)} = \frac{1}{\omega_i}[(1-\omega_i)\breve{\mathcal{D}}^{(\alpha\beta)} + (\omega_i - \gamma_i)\breve{\mathcal{L}}^{(\alpha\beta)} + \omega_i\breve{\mathcal{F}}^{(\alpha\beta)}]$, $i = 1, 2$. If $\omega_1 \geq \omega_2 > 0$, $\gamma_1 \geq \gamma_2$,*

and Condition 1 is satisfied, then for the spectral radius of $\check{\mathcal{T}}_1^{(\alpha\beta)}$ *and* $\check{\mathcal{T}}_2^{(\alpha\beta)}$*, the following statements hold:*

(1) $\rho(\check{\mathcal{T}}_1^{(\alpha\beta)}) \leq \rho(\check{\mathcal{T}}_2^{(\alpha\beta)}) < 1$*, if* $\rho(\mathcal{T}) < 1$*;*

(2) $\rho(\check{\mathcal{T}}_1^{(\alpha\beta)}) \geq \rho(\check{\mathcal{T}}_2^{(\alpha\beta)}) > 1$*, if* $\rho(\mathcal{T}) > 1$*,*

where $\check{\mathcal{T}}_1^{(\alpha\beta)}$ *and* $\check{\mathcal{T}}_2^{(\alpha\beta)}$ *respectively are the iterative tensors with respect to the parameter* ω_i*,* γ_i*.*

When $\beta_i^j = 0$, we also have the following results.

Theorem 10. *Let* $\mathcal{A} \in \mathbb{R}^{[m,n]}$ *be a weakly irreducible* $\mathcal{Z}$*-tensor with positive diagonal entries. Suppose* $\mathcal{A}^{(\alpha)} = \mathcal{U}_1^{(\alpha)} - \mathcal{V}_1^{(\alpha)} = \mathcal{U}_2^{(\alpha)} - \mathcal{V}_2^{(\alpha)}$*, where* $\mathcal{U}_i^{(\alpha)} = \frac{1}{\omega_i}(\mathcal{D}^{(\alpha)} - \gamma_i \mathcal{L}^{(\alpha)})$*,* $\mathcal{V}_i^{(\alpha)} = \frac{1}{\omega_i}[(1 - \omega_i)\mathcal{D}^{(\alpha)} + (\omega_i - \gamma_i)\mathcal{L}^{(\alpha)} + \omega_i \mathcal{F}^{(\alpha)}]$*,* $i = 1, 2$*. If* $\omega_1 \geq \omega_2 > 0$*,* $\gamma_1 \geq \gamma_2$*, and Condition 2 is satisfied, then for the spectral radius of* $\mathcal{T}_1^{(\alpha)}$ *and* $\mathcal{T}_2^{(\alpha)}$*, the following statements hold:*

(1) $\rho(\mathcal{T}_1^{(\alpha)}) \leq \rho(\mathcal{T}_2^{(\alpha)}) < 1$*, if* $\rho(\mathcal{T}) < 1$*;*

(2) $\rho(\mathcal{T}_1^{(\alpha)}) \geq \rho(\mathcal{T}_2^{(\alpha)}) > 1$*, if* $\rho(\mathcal{T}) > 1$ *and Condition 4 holds,*

where $\mathcal{T}_1^{(\alpha)}$ *and* $\mathcal{T}_2^{(\alpha)}$ *respectively are the iterative tensors with respect to the parameter* ω_i*,* γ_i*.*

5. Numerical Examples

In this section, we will compare the performance of our preconditioner $P = I + S_{\alpha\beta}$ with the preconditioner in Chen and Li [11] and Cui et al. [13]. Our numerical experiment includes the tensor equation and the higher-order Markov chain. All tests will be performed in MATLAB R2016b. We set $\alpha = [\alpha_1^1, \cdots, \alpha_{n-1}^1, \alpha_2^2, \cdots, \alpha_{n-1}^{n-1}]$, $\beta = [\beta_1^1, \cdots, \beta_{n-1}^1, \beta_2^2, \cdots, \beta_{n-1}^{n-1}]$ in $S_{\alpha\beta}$. We denote the number of iteration steps by "IT" and the CPU time in seconds by "CPU (s)". The maximum number of iterations is set as 1000, the stopping criterion is set as $\epsilon = 10^{-11}$. The test will be stopped whenever

$$\|\mathcal{A}\mathbf{x}^{m-1} - \mathbf{b}\|_2 < \epsilon.$$

Example 1. *Let* $\mathcal{B}$ *be the following 3-order 4-dimensional tensor*

$$\mathcal{B}(:,:,1) = \begin{bmatrix} 0.270 & 0.417 & 0.105 & 0.573 \\ 0.208 & 0.206 & 0.142 & 0.052 \\ 0.565 & 0.947 & 0.166 & 0.931 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{B}(:,:,2) = \begin{bmatrix} 0.737 & 0.984 & 0.177 & 0.939 \\ 0.063 & 0.858 & 0.398 & 0.301 \\ 0.860 & 0.785 & 0.133 & 0.295 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{B}(:,:,3) = \begin{bmatrix} 0.467 & 0.559 & 0.054 & 0.898 \\ 0.648 & 0.854 & 0.177 & 0.118 \\ 0.025 & 0.347 & 0.662 & 0.988 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{B}(:,:,4) = \begin{bmatrix} 0.706 & 0.464 & 0.178 & 0.335 \\ 0.999 & 0.764 & 0.359 & 0.175 \\ 0.287 & 0.818 & 0.056 & 0.208 \\ 0 & 0 & 0 & 0.896 \end{bmatrix}.$$

According to Cui et al. [29], this tensor is weakly irreducible nonnegative. We compute the spectral radius of tensors by the power method proposed by Ng et al. [41]. The spectral radius of $\mathcal{B}$ is 3.8505. Let $\mathcal{A}_1 = \mathcal{I}_m - \mathcal{B}$, then $\mathcal{A}_1$ is a weakly irreducible $\mathcal{Z}$-tensor. Taking

the value of $\alpha = 0.1$, $\beta = 0.05$ in the preconditioner which satisfied Conditions 1 and 2. Then we set $\omega = 0.1$, $\gamma = 0.05$. By calculation, we have $\rho(\mathcal{T}) = 2.1327$, $\rho(\mathcal{T}^{(\alpha\beta)}) = 3.0546$, and $\rho(\check{\mathcal{T}}^{(\alpha\beta)}) = 2.3804$, that is

$$\rho(\mathcal{T}^{(\alpha\beta)}) \geq \rho(\check{\mathcal{T}}^{(\alpha\beta)}) \geq \rho(\mathcal{T}) > 1.$$

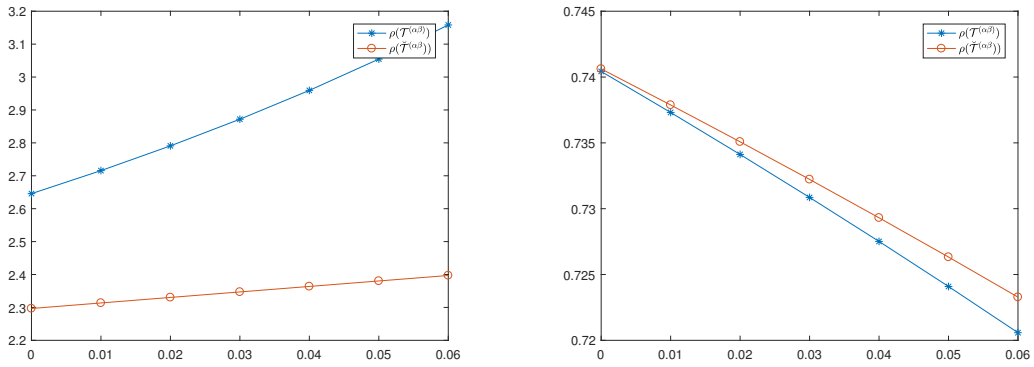
When $\beta = 0$, the preconditioner $I + S_{\alpha\beta}$ reduced to the preconditioner in Cui et al. [13]. We also take $\alpha = 0.1$, then we have $\rho(\mathcal{T}^{(\alpha)}) = 2.6457$, that is $\rho(\mathcal{T}^{(\alpha\beta)}) \geq \rho(\mathcal{T}^{(\alpha)}) \geq \rho(\mathcal{T}) > 1$.

Let $\mathcal{A}_2 = 5\mathcal{I}_m - \mathcal{B}$; then $\mathcal{A}_2$ is a weakly irreducible nonsingular $\mathcal{M}$-tensor. We take $\omega = 0.9$, $\gamma = 0.8$, then $\rho(\mathcal{T}) = 0.7428$, $\rho(\mathcal{T}^{(\alpha\beta)}) = 0.7268$, $\rho(\check{\mathcal{T}}^{(\alpha\beta)}) = 0.7290$, that is

$$\rho(\mathcal{T}^{(\alpha\beta)}) \leq \rho(\check{\mathcal{T}}^{(\alpha\beta)}) \leq \rho(\mathcal{T}) < 1.$$

We also consider the case $\beta = 0$, then we have $\rho(\mathcal{T}^{(\alpha)}) = 0.7404$, that is $\rho(\mathcal{T}^{(\alpha\beta)}) \leq \rho(\mathcal{T}^{(\alpha)}) \leq \rho(\mathcal{T}) < 1$. We verify the theoretical results of Theorems 3–7.

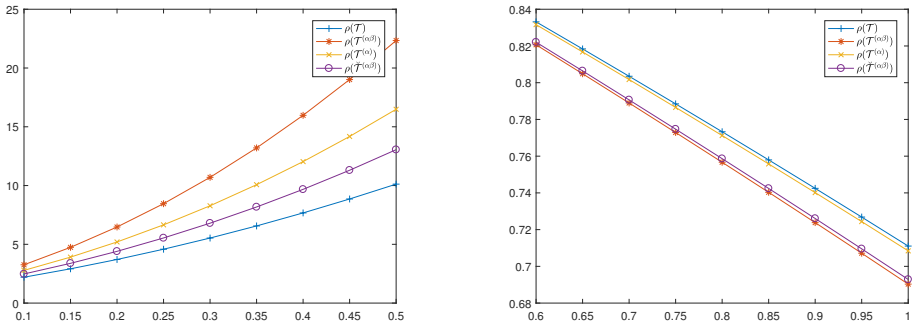
In this example, we also analyze the influence of parameter β on the spectral radius of tensor $\mathcal{T}^{(\alpha\beta)}$ and $\check{\mathcal{T}}^{(\alpha\beta)}$. With a fixed $\alpha = 0.1$, we take $\beta = 0{:}0.01{:}0.06$ which satisfied Conditions 1 and 3. The change in spectral radius tensor $\mathcal{T}^{(\alpha\beta)}$ and $\check{\mathcal{T}}^{(\alpha\beta)}$ are shown in Figure 1 respectively for tensor $\mathcal{A}_1 = \mathcal{I}_m - \mathcal{B}$ and $\mathcal{A}_2 = 5\mathcal{I}_m - \mathcal{B}$.



(**a**) case $\rho(\mathcal{T}) > 1$ (**b**) case $\rho(\mathcal{T}) < 1$

Figure 1. Spectral radius $\rho(\mathcal{T}^{(\alpha\beta)})$ and $\rho(\check{\mathcal{T}}^{(\alpha\beta)})$ as functions of β with fixed $\alpha = 0.1$. The left and right figures correspond to the AOR parameters $(\omega, \gamma) = (0.1, 0.05)$ and $(\omega, \gamma) = (0.9, 0.8)$, respectively.

In Figure 2, we present a comparison of the iteration tensors $\mathcal{T}$, $\mathcal{T}^{(\alpha\beta)}$, $\check{\mathcal{T}}^{(\alpha\beta)}$, and $\bar{\mathcal{T}}^{(\alpha)}$ with respect to the parameters ω and γ, where $\gamma = 0.9\omega$. Specifically, ω ranges from 0.1 to 0.5 in steps of 0.05 for the case $\rho(\mathcal{T}) > 1$, and from 0.6 to 1 in steps of 0.05 when $\rho(\mathcal{T}) < 1$. This figure shows the results given in Theorems 8–10.



(**a**) case $\rho(\mathcal{T}) > 1$ (**b**) case $\rho(\mathcal{T}) < 1$

Figure 2. Spectral radius $\rho(\mathcal{T})$, $\rho(\mathcal{T}^{(\alpha\beta)})$, $\rho(\mathcal{T}^{(\alpha)})$and $\rho(\check{\mathcal{T}}^{(\alpha\beta)})$ with respect to ω, with $\gamma = 0.9\omega$ and $(\alpha, \beta) = (0.1, 0.05)$.

Next, we compare the experimental results of our original preconditioner with the different preconditioners in Cui et al. [13], Chen and Li [11] in Examples 2 and 3. We denote them as AOR, $P^{(\alpha\beta)}_{AOR}$, P^{1}_{AOR} and P^{2}_{AOR} respectively in the following context.

Example 2. *Let*

$$\mathcal{A} = \left[\begin{array}{ccc|ccc|ccc} 1 & -0.12 & -0.13 & -0.04 & -0.02 & -0.03 & -0.03 & -0.02 & -0.04 \\ -0.12 & -0.03 & -0.06 & -0.01 & 1 & -0.02 & -0.02 & -0.06 & -0.03 \\ -0.13 & -0.02 & -0.10 & -0.03 & -0.04 & -0.02 & -0.02 & -0.10 & 1 \end{array}\right].$$

by Chen and Li. [11]. Taking $\mathbf{b} = \mathbf{1}$*, we consider solving* $\mathcal{A}\mathbf{x}^2 = \mathbf{b}$ *by the proposed preconditioned splitting method.*

In Example 2, we choose initial value $\mathbf{x_0} = \frac{1}{n}\mathbf{1}$. We search for the optimal parameters α and β over the ranges $\alpha = 0.5{:}0.1{:}3$ and $\beta = 0.05{:}0.1{:}1$. The numerical results of Example 2 are shown in Table 1 under different values with respect to parameters ω and r.

Table 1. The numerical results of preconditioned AOR methods for Example 2 under different ω and γ.

(ω, γ)	AOR		$P^{(\alpha\beta)}_{AOR}$			P^{1}_{AOR}			P^{2}_{AOR}		
	IT	**CPU (s)**	(α, β)	**IT**	**CPU (s)**	(α, β)	**IT**	**CPU (s)**	(α, β)	**IT**	**CPU (s)**
(0.6, 0.4)	59	0.0049	(0.80, 0.65)	37	0.0015	(0.90, 0.15)	55	0.0021	(1.00, 0.45)	42	0.0016
(0.7, 0.5)	48	0.0044	(0.70, 0.85)	29	0.0011	(0.80, 0.35)	45	0.0017	(0.90, 0.45)	34	0.0013
(0.7, 0.7)	47	0.0026	(0.70, 0.85)	29	0.0011	(0.80, 0.25)	44	0.0018	(0.90, 0.35)	33	0.0013
(0.8, 0.4)	41	0.0041	(0.70, 0.25)	23	0.0009	(0.70, 0.65)	38	0.0014	(0.80, 0.75)	27	0.0011
(0.8, 0.6)	40	0.0038	(0.70, 0.25)	23	0.0009	(0.70, 0.65)	37	0.0014	(0.80, 0.65)	27	0.0010
(0.8, 0.8)	39	0.0022	(0.70, 0.25)	23	0.0008	(0.70, 0.65)	36	0.0013	(0.80, 0.55)	27	0.0010
(0.9, 0.5)	34	0.0035	(0.60, 0.85)	19	0.0007	(0.70, 0.15)	31	0.0013	(0.80, 0.05)	22	0.0008
(0.9, 0.7)	33	0.0038	(0.60, 0.85)	19	0.0007	(0.70, 0.15)	30	0.0011	(0.70, 0.95)	22	0.0015
(0.9, 0.9)	32	0.0037	(0.60, 0.85)	19	0.0007	(0.70, 0.15)	30	0.0012	(0.70, 0.95)	22	0.0008
(1.0, 0.0)	31	0.0034	(0.60, 0.55)	16	0.0006	(0.60, 0.85)	28	0.0011	(0.70, 0.75)	19	0.0007
(1.0, 0.5)	29	0.0035	(0.60, 0.55)	16	0.0006	(0.60, 0.85)	27	0.0010	(0.70, 0.65)	19	0.0007
(1.0, 1.0)	27	0.0032	(0.60, 0.45)	15	0.0006	(0.70, 0.35)	24	0.0009	(0.70, 0.45)	25	0.0009

From Table 1, we know that no matter what the value of ω and γ, the new preconditioner presented in this article has a better performance in terms of iterations and CPU time than the other preconditioner. We also find that the larger the ω and γ, the better the numerical effect. Except in some special cases, such as $(\omega, \gamma) = (1, 1)$, the iteration step is 25 for P^{2}_{AOR} which is larger than the case $(\omega, \gamma) = (1.0, 0.5)$. We think this is due to the method being a G-S-type method when $(\omega, \gamma) = (1, 1)$ for which the numerical results are generally worse than the AOR-type method.

Example 3. *Let* $\mathcal{A} \in \mathbb{R}^{[3,n]}$ *and* $\mathbf{b} \in \mathbb{R}^{\mathbf{n}\times\mathbf{n}}$ *with*

$$\begin{cases} a_{111} = a_{nnn} = 1, & \\ a_{iii} = 2, & i = 2, 3, \cdots, n-1 \\ a_{ii-1i} = -1/2, & i = 2, 3, \cdots, n-1 \\ a_{ii-1i-1} = -1/2, & i = 2, 3, \cdots, n-1 \\ a_{ii+1i+1} = -1/2, & i = 2, 3, \cdots, n-1, \end{cases}$$

$$\begin{cases} b_1 = c_0^2, & \\ b_i = \dfrac{a}{(n-1)^2}, & i = 2, 3, \cdots, n-1 \\ b_n = c_1^2. & \end{cases}$$

by Li et al. [36]. We take $c_0 = 1/2$, $c_1 = 1/3$ *and* $a = 2$ *in this example.*

Using the same parameter settings as in Example 2, Table 2 presents the numerical results for Example 3 with different dimensions n.

In Example 3, we solve the multi-linear system using the AOR method and the preconditioned AOR method when $(\omega, \gamma) = (1, 0.5)$. For $n = 5, 10, 20, 30, 50, 100, 150, 200, 300$, the numerical results in Table 2 show that our preconditioned method is better than other preconditioned methods when suitable parameters are chosen. Notably, the preferred parameters for our preconditioned method are found to be $(\alpha, \beta) = (0.8, 0.25)$, which are adopted in the following example.

Table 2. The numerical results of preconditioned AOR methods for Example 3 with different problem sizes n.

n	*AOR*		$P_{AOR}^{(\alpha\beta)}$			P_{AOR}^{1}			P_{AOR}^{2}		
	IT	**CPU (s)**	(α, β)	**IT**	**CPU (s)**	(α, β)	**IT**	**CPU (s)**	(α, β)	**IT**	**CPU (s)**
5	38	0.0040	(0.60, 0.85)	19	0.0007	(0.70, 0.15)	22	0.0008	(0.70, 0.95)	30	0.0012
10	56	0.0079	(0.70, 0.75)	28	0.0012	(0.70, 0.85)	29	0.0025	(1.00, 0.05)	51	0.0022
20	70	0.0085	(0.80, 0.15)	32	0.0015	(0.80, 0.45)	35	0.0017	(1.10, 0.35)	64	0.0043
30	71	0.0338	(0.80, 0.25)	33	0.0020	(0.80, 0.35)	34	0.0026	(1.10, 0.55)	66	0.0065
50	73	0.0270	(0.80, 0.25)	33	0.0034	(0.80, 0.45)	35	0.0030	(1.10, 0.65)	67	0.0116
100	75	0.1538	(0.80, 0.25)	33	0.0275	(0.80, 0.45)	35	0.0485	(1.10, 0.75)	68	0.0579
150	76	0.1831	(0.80, 0.25)	33	0.0694	(0.90, 0.15)	42	0.1069	(1.10, 0.85)	69	0.1638
200	76	0.6035	(0.80, 0.25)	37	0.1749	(0.90, 0.45)	51	0.2305	(1.00, 0.75)	77	0.3972
300	77	1.0459	(0.80, 0.25)	33	0.4157	(0.80, 0.35)	51	0.6519	(1.00, 0.85)	78	1.0685

Example 4. *An approximated higher order Markov chain model was given by Li and Ng [3]:*

$$\mathbf{x} = \mathcal{P}\mathbf{x}^{m-1}, \quad \|\mathbf{x}\|_1 = 1, \tag{10}$$

where $\mathcal{P}$ *is a transition probability tensor.*

Li and Ng [3] proposed a fixed-point iterative method by computing $\mathbf{x}_k = \mathcal{P}\mathbf{x}_{k-1}$. Let $\mathcal{A} = \mathcal{I}_m - \theta\mathcal{P}$ with a positive real number θ, then (10) equals to

$$\mathcal{A}\mathbf{x}^{m-1} = \mathbf{x}^{[m-1]} - \theta\mathbf{x}, \quad \|\mathbf{x}\|_1 = 1.$$

The tensor splitting method for solving this model as follows:

$$\mathbf{x}_k = \left[M(\mathcal{E})^{-1}\mathcal{F}\mathbf{x}_{k-1}^{m-1} + M(\mathcal{E})^{-1}(\mathbf{x}_{k-1}^{[m-1]} - \theta\mathbf{x}_{\mathbf{k-1}})\right]^{[\frac{1}{m-1}]}, \tag{11}$$

where $\mathcal{A} = \mathcal{E} - \mathcal{F}$.

In this example, we consider the following transition probability tensor:

$$\mathcal{P} = \left[\begin{array}{ccc|ccc|ccc} 0.9000 & 0.3340 & 0.3106 & 0.6700 & 0.1040 & 0.0805 & 0.6604 & 0.0945 & 0.0710 \\ 0.0690 & 0.6108 & 0.0754 & 0.2892 & 0.8310 & 0.2956 & 0.0716 & 0.6133 & 0.0780 \\ 0.0310 & 0.0552 & 0.6140 & 0.0408 & 0.0650 & 0.6239 & 0.2680 & 0.2922 & 0.8510 \end{array}\right].$$

We compare the performance of the preconditioned method $P_{AOR}^{(\alpha\beta)}$ and P_{AOR}^{2} with the original method stated in (11) and the fixed-point method in Li and Ng [3]. In numerical experiments, we set initial value $\mathbf{x_0} = \frac{1}{n}\mathbf{1}$, and we search the parameter θ from 0.3 to 0.9. We search for the optimal parameters ω and r for the experiment from the ranges ω = 0.2:0.1:1 and r = 0.2:0.1:ω.

From Table 3, we can see that as θ increases, the preconditioning method proposed in this paper outperforms other preconditioning methods and the fixed-point method. For example, when $\theta = 0.9$, we find that the proposed preconditioning method has a significant advantage over the fixed-point method in terms of both iteration count and computation time.

Table 3. The numerical results of preconditioned AOR methods for Example 4 with different parameter θ.

θ	*AOR*			$P_{AOR}^{(\alpha\beta)}$			P_{AOR}^{2}			Fixed-Point Method	
	(ω, r)	IT	CPU (s)	(ω, r)	IT	CPU (s)	(ω, r)	IT	CPU (s)	IT	CPU (s)
0.3	(1.00, 0.20)	191	0.0077	(1.00, 0.20)	195	0.0079	(1.00, 1.00)	192	0.0078	115	0.0029
0.4	(1.00, 0.20)	123	0.0049	(1.00, 0.20)	125	0.0051	(1.00, 1.00)	124	0.0051	115	0.0028
0.5	(1.00, 0.20)	83	0.0034	(1.00, 0.20)	83	0.0034	(1.00, 1.00)	83	0.0034	115	0.0029
0.6	(1.00, 0.20)	55	0.0023	(1.00, 0.20)	55	0.0023	(1.00, 1.00)	56	0.0023	115	0.0021
0.7	(1.00, 0.20)	36	0.0014	(1.00, 0.20)	34	0.0014	(1.00, 1.00)	36	0.0015	115	0.0029
0.8	(1.00, 0.20)	20	0.0009	(1.00, 0.20)	16	0.0007	(1.00, 1.00)	20	0.0009	115	0.0028
0.9	(0.80, 0.40)	15	0.0006	(0.80, 0.40)	11	0.0004	(0.90, 0.90)	13	0.0008	115	0.0028

6. Conclusions and Outlook

In this paper, a new preconditioned AOR method is proposed for solving the multilinear system with a more general $\mathcal{Z}$-tensor. We consider two different AOR-type iterative methods (5) and (7), and give the comparison results for spectral radius. IN particular, we study the effect of the parameter β_i^j on the spectral radius; theoretical and numerical experiments confirm its effectiveness compared with the preconditioner without β_i^j when $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor. As an application, we use the method to solve higher order Markov chain problems. Numerical examples are also given to show the proposed method is better than some known methods.

While the proposed preconditioned AOR method demonstrates a satisfactory convergence performance for the tested problems, we acknowledge that the convergence rate of the AOR-type iteration is not always guaranteed to be fast. As suggested in the literature, Anderson acceleration has proven effective in accelerating fixed-point iterations and has been successfully applied to various splitting methods [23,24]. A promising direction for future research is to incorporate this acceleration technique into the proposed preconditioned framework to further improve computational efficiency. Additionally, other extrapolation approaches, such as the Shanks transformation, may also be explored in this context. We believe this topic is of interest and deserves further investigation.

Another promising direction for future research is the implementation of the proposed algorithm on GPU architectures, since the tensor–vector products involved in the iteration are inherently parallelizable and well suited for such computing platforms. This would enable the study of larger real-world datasets, such as multi-layer and temporal tensors from the SNAP open collection, and further validate the practical applicability of our method.

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