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Structural Properties and Integral Transforms of the k -Kummer Hypergeometric Function

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Abstract

In this paper, we study the k -Kummer hypergeometric function $M_k(a, c; w)$, a generalization of the classical Kummer confluent hypergeometric function, arising as a solution of the k -confluent hypergeometric differential equation. We establish a Kummer-type transformation formula and derive derivative identities, contiguous relations, and addition and multiplication formulas. In addition, we obtain closed-form expressions for the Laplace, Mellin, Stieltjes, Sumudu, and Riemann–Liouville fractional integral transforms involving $M_k(a, c; w)$. The results presented here extend the classical theory of confluent hypergeometric functions to the k -generalized setting and provide analytic tools for further investigations of generalized differential equations and integral transforms.

Keywords: k -Kummer function; k -confluent hypergeometric function; contiguous relations; integral transforms; operational identities; generalized special functions

MSC: 33C15; 33C05; 44A20

1. Introduction

Confluent hypergeometric functions are one of the fundamental classes of special functions appearing in the analytic theory of differential equations. In particular, the Kummer and Tricomi functions arise as solutions of the confluent hypergeometric differential equation

$$w\gamma''(w) + (c - w)\gamma'(w) - a\gamma(w) = 0, \quad (1)$$

which may be regarded as a limiting form of the Gauss hypergeometric equation when two regular singular points coalesce [1–3]. Owing to their rich analytic structure, these functions play a central role in the analytic theory of special functions and frequently arise in mathematical physics, applied analysis, and engineering. Comprehensive investigations concerning their analytic properties, asymptotic behavior, integral representations, and transformation formulas may be found in the classical works of Erdélyi et al. [4,5] and Slater [3].

Confluent hypergeometric functions appear naturally in a broad range of applied problems involving differential equations. One of their classical applications occurs in quantum mechanics, where the radial Schrödinger equation for the hydrogen atom and Coulomb-type potentials can be reduced to the confluent hypergeometric differential



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equation [6,7]. Similar structures arise in heat conduction, wave propagation, electromagnetic theory, and boundary value problems [8,9]. In particular, Morse and Feshbach [10] systematically employed confluent hypergeometric functions in the study of differential equations arising in theoretical physics, especially in problems associated with wave mechanics, scattering theory, and coordinate systems obtained by separation of variables. These applications demonstrate that confluent hypergeometric functions constitute one of the principal analytic tools for constructing explicit solutions of physically significant differential equations.

Another important area in which confluent hypergeometric functions arise is diffusion theory and stochastic analysis. Exact solutions of Fokker–Planck-type equations, drift–diffusion models, and related stochastic differential equations are frequently represented in terms of confluent hypergeometric functions after suitable transformations of the governing equations [11]. Such representations play an essential role in the study of transition probabilities, asymptotic behavior, and first-passage phenomena. Moreover, confluent hypergeometric structures also arise in stochastic models governed by Fokker–Planck equations, including applications in probability theory and population dynamics [12]. These examples further illustrate the importance of confluent hypergeometric functions in diffusion theory and stochastic analysis.

Besides their role as explicit solutions of differential equations, the structural properties of confluent hypergeometric functions themselves are of considerable importance. Transformation formulas, derivative identities, contiguous relations, and addition or multiplication formulas constitute fundamental tools in the analytic investigation of special functions. Such identities establish relations between solutions corresponding to shifted parameters and provide operational representations useful in both theoretical and computational studies [3,13]. In addition, recurrence and contiguous relations play an important role in numerical analysis since they allow the construction of stable recursive algorithms for evaluating special functions with large parameters [14]. Derivative formulas and transformation identities are also closely connected with integral transforms, operational calculus, and asymptotic analysis [5,15,16]. Consequently, the investigation of these structural properties is important not only for the analytic theory of special functions but also for applications involving mathematical modeling and differential equations.

In recent years, generalized versions of classical special functions obtained through deformation parameters have attracted considerable attention. One important approach in this direction is based on the k -gamma function and the corresponding k -Pochhammer symbol introduced and investigated in [17–20]. These generalized structures make it possible to define k -analogues of many classical special functions while preserving substantial parts of their analytic behavior. More recently, several properties of k -generalized hypergeometric functions, including convergence properties, derivative formulas, integral representations, integral transforms, contiguous relations, differential equations, and fractional integral operators, have been investigated in [21,22]. Within this framework, the k -Kummer function arises as a natural extension of the classical confluent hypergeometric function. Since classical confluent hypergeometric functions appear naturally in quantum mechanics, diffusion theory, stochastic processes, and applied differential equations, it is natural to investigate analogous properties of their k -generalized counterparts. Such investigations contribute to the development of analytic tools for the study of generalized differential equations and related mathematical models involving deformation parameters. Although several k -extensions of classical special functions have been introduced in the literature, a unified treatment of the structural properties of the k -Kummer function has not yet been available. In particular, transformation formulas, derivative identities, contiguous relations, addition and multiplication formulas, together with integral transform formulas,

have not previously been developed within a common framework. Consequently, the present work is not merely concerned with constructing individual k -analogues of classical identities, but rather with developing a unified framework for the structural analysis of the k -Kummer function.

Motivated by these observations, the present paper is devoted to the investigation of several structural properties of the k -Kummer function. In particular, we establish transformation formulas, derivative identities, contiguous relations, and various addition and multiplication formulas involving parameter shifts. We also investigate the behavior of the k -Kummer function under several classical integral transforms, including the Laplace, Mellin, Stieltjes, Sumudu, and Riemann–Liouville fractional integral transforms. These results extend several classical identities associated with confluent hypergeometric functions to the k -generalized setting and contribute to the analytic theory of generalized special functions. Furthermore, the obtained identities may be useful in future investigations concerning generalized differential equations, integral transforms, asymptotic analysis, and mathematical models involving k -special functions.

From the viewpoint of differential equations, the obtained identities may also be interpreted as operational properties of solutions of the k -confluent hypergeometric differential equation. In particular, the derivative formulas and contiguous relations describe the behavior of solutions under parameter shifts, while the addition, multiplication, and integral transform formulas provide alternative representations that may be useful in the construction and analysis of exact solutions. Consequently, the results contribute not only to the theory of k -special functions but also to the study of differential equations whose solutions can be expressed in terms of the k -Kummer function.

The paper is organized as follows. Section 2 recalls the basic definitions and preliminary results concerning the k -Gamma function, the k -Pochhammer symbol, the k -Gauss hypergeometric function, and the k -Kummer function. Section 2 also introduces the k -confluent hypergeometric limit function ${}_0F_{1,k}$, because of its close relationship with the k -Kummer function. In Section 3, we establish the main structural properties of $M_k(a, c; w)$, including the fundamental transformation formula, contiguous relations, derivative identities, and addition and multiplication formulas. Section 4 is devoted to the investigation of several integral transforms involving the k -Kummer function, namely the Laplace, Mellin, Stieltjes, Sumudu, and Riemann–Liouville fractional integral transforms.

2. Preliminaries

Throughout this paper, $a, c, w \in \mathbb{C}$ with $c \notin \{0, -k, -2k, \dots\}$, $k > 0$, and $n \in \mathbb{N}$. All series are considered within their domains of convergence.

Definition 1. Let $k > 0$ and $\operatorname{Re}(w) > 0$. The k -Gamma function is defined by [18]

$$\Gamma_k(w) = \int_0^\infty t^{w-1} e^{-t^k/k} dt. \quad (2)$$

Definition 2. Let $k > 0$, $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}_0$. The Pochhammer k -symbol [17] is defined by

$$(\alpha)_{n,k} = \begin{cases} \prod_{j=0}^{n-1} (\alpha + jk), & n \geq 1, \\ 1, & n = 0, \end{cases} \quad (3)$$

which can equivalently be written in terms of the k -Gamma function as

$$(\alpha)_{n,k} = \frac{\Gamma_k(\alpha + nk)}{\Gamma_k(\alpha)}. \quad (4)$$

For $k = 1$, $(\alpha)_{n,k}$ reduces to the classical Pochhammer symbol

$$(\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)}. \quad (5)$$

Proposition 1. Let $k > 0$ and $\alpha \in \mathbb{C}$. Then the following identities hold [17]:

$$(\alpha)_{n+r,k} = (\alpha)_{n,k}(\alpha + nk)_{r,k}, \quad (n, r \in \mathbb{N}_0), \quad (6)$$

$$(\alpha + k)_{n,k} = \frac{(\alpha + nk)}{\alpha} (\alpha)_{n,k}, \quad (7)$$

$$(\alpha)_{n,k} = k^n \left(\frac{\alpha}{k}\right)_n \quad (8)$$

Definition 3. For $k > 0$, the k -Gauss hypergeometric function is defined by

$${}_2F_{1,k}(a, b; c; w) = \sum_{n=0}^{\infty} \frac{(a)_{n,k} (b)_{n,k}}{(c)_{n,k}} \frac{w^n}{n!}. \quad (9)$$

Definition 4. Let $k > 0$ and $a, c, w \in \mathbb{C}$ with $c \notin \{0, -k, -2k, \dots\}$. The k -Kummer hypergeometric function is defined by the series

$$M_k(a, c; w) = \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{w^n}{n!}. \quad (10)$$

This function satisfies the k -confluent hypergeometric differential equation

$$kw\gamma''(w) + (c - kw)\gamma'(w) - a\gamma(w) = 0. \quad (11)$$

In particular, when $k = 1$, this reduces to the classical Kummer confluent hypergeometric function [23].

Since several addition and multiplication formulas derived in the sequel rely on operational representations involving derivatives, we recall the following Taylor-type expansions for analytic functions.

Lemma 1 (Taylor Expansion for the Sum (p. 21, [3])). Let f be an analytic function. Then for every $w, z \in \mathbb{C}$ the following identities hold:

$$f(w + z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{d^n}{dw^n} f(w) \quad (12)$$

and

$$\left(\frac{w+z}{w}\right) f(w+z) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{z}{w+z}\right)^n \frac{d^n}{dw^n} \{w^n f(w)\}. \quad (13)$$

Lemma 2 (Taylor Expansion for the Product (p. 22, [3])). Let f be an analytic function of a complex variable. Then for every $w \in \mathbb{C}$ and for suitable $z \in \mathbb{C}$ the following identities hold:

$$f(wz) = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} w^n \frac{d^n}{dw^n} f(w) \quad (14)$$

and

$$zf(wz) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{z-1}{z}\right)^n \frac{d^n}{dw^n} [w^n f(w)]. \quad (15)$$

In order to investigate the behavior of the k -Kummer function under various operational methods, we recall several classical integral transforms and fractional integral operators that will be used throughout the sequel.

$$\mathcal{L}\{f(w); s\} := \int_0^\infty e^{-sw} f(w) dw, \quad (\operatorname{Re}(s) > 0) \quad (16)$$

where $\mathcal{L}$ denotes the Laplace transform.

$$\mathcal{M}\{f(w); s\} = \int_0^\infty w^{s-1} f(w) dw, \quad (\operatorname{Re}(s) > 0) \quad (17)$$

where $\mathcal{M}$ denotes the Mellin transform.

$$\mathcal{S}\{f(w); s\} = \int_0^\infty \frac{f(w)}{w+s} dw, \quad (\operatorname{Re}(s) > 0) \quad (18)$$

where $\mathcal{S}$ denotes the Stieltjes transform.

The Sumudu transform [24] is given by

$$\mathcal{S}\{f(w); u\} = \int_0^\infty e^{-uw} f(uw) dw \quad (u \in \mathbb{C}). \quad (19)$$

Finally, the Riemann–Liouville fractional integral operator [4] is defined as

$$(I_0^\alpha f)(w) = \frac{1}{\Gamma(\alpha)} \int_0^w (w-t)^{\alpha-1} f(t) dt \quad (\operatorname{Re}(\alpha) > 0). \quad (20)$$

Definition 5. The generalized exponential integral function [25] is defined for $\operatorname{Re}(s) > 0$ and $\operatorname{Re}(v) > 0$ by

$$E_v(s) = \int_1^\infty e^{-st} t^{-v} dt. \quad (21)$$

3. Main Results

In this section, we investigate several structural properties of the k -Kummer function $M_k(a, c; w)$. Our aim is to obtain identities that parallel the classical theory of confluent hypergeometric functions while reflecting the effect of the parameter k . In particular, we derive a fundamental transformation formula together with various addition, multiplication, contiguous, and derivative identities. These results describe how the function behaves under shifts of the parameters and changes in the argument, and they provide a useful basis for the subsequent analysis of k -confluent hypergeometric structures.

Theorem 1. Let $k > 0$ and $a, c, w \in \mathbb{C}$ with $c \notin \{0, -k, -2k, \dots\}$. Then the k -Kummer function defined by

$$M_k(a, c; w) = \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{w^n}{n!}$$

is absolutely convergent for every $w \in \mathbb{C}$.

Proof. The convergence of the series at $w = 0$ is evident. Now, let $w \neq 0$ and consider the general term

$$u_n = \frac{(a)_{n,k}}{(c)_{n,k}} \frac{w^n}{n!}.$$

To determine the convergence of the series, we apply the ratio test. By using property (7) of the k -Pochhammer symbol, it follows that

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{a + nk}{c + nk} \cdot \frac{w}{n+1} \right| = |w| \cdot \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0.$$

Since this limit is equal to zero for every fixed $w \in \mathbb{C}$, the series is absolutely convergent for all $w \in \mathbb{C}$ by the ratio test. Consequently, the radius of convergence is $R = \infty$. $\square$

Theorem 2. (The fundamental k -Kummer transformation) For $k > 0$ and $a, c, w \in \mathbb{C}$ with $c \notin \{0, -k, -2k, \dots\}$, the k -Kummer function satisfies

$$e^{-w} M_k(a, c; w) = M_k(c - a, c; -w). \quad (22)$$

Proof. By using the series representation (10) of the k -Kummer function together with the series expansion of the exponential function and applying the Cauchy product formula, we obtain

$$\begin{aligned} e^{-w} M_k(a, c; w) &= \left(\sum_{m=0}^{\infty} \frac{(-w)^m}{m!} \right) \left(\sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{w^n}{n!} \right) \\ &= \sum_{r=0}^{\infty} \frac{w^r}{r!} \sum_{m=0}^r \binom{r}{m} (-1)^m \frac{(a)_{r-m,k}}{(c)_{r-m,k}}. \end{aligned}$$

Using the identity (6) we obtain

$$\frac{(a)_{r-m,k}}{(c)_{r-m,k}} = \frac{(a)_{r,k}}{(c)_{r,k}} \frac{(c + rk - mk)_{m,k}}{(a + rk - mk)_{m,k}}.$$

Hence,

$$e^{-w} M_k(a, c; w) = \sum_{r=0}^{\infty} \frac{w^r}{r!} \frac{(a)_{r,k}}{(c)_{r,k}} \sum_{m=0}^r \binom{r}{m} (-1)^m \frac{(c + rk - mk)_{m,k}}{(a + rk - mk)_{m,k}}.$$

Applying (8), we obtain

$$\frac{(c + rk - mk)_{m,k}}{(a + rk - mk)_{m,k}} = \frac{\left(\frac{c}{k} + r - m\right)_m}{\left(\frac{a}{k} + r - m\right)_m}.$$

Then, using the identities

$$(\alpha - m)_m = (-1)^m (1 - \alpha)_m$$

and

$$\binom{r}{m} (-1)^m = \frac{(-r)_m}{m!},$$

the inner sum can be expressed as the classical hypergeometric series

$$\begin{aligned} \sum_{m=0}^r \binom{r}{m} (-1)^m \frac{(c + rk - mk)_{m,k}}{(a + rk - mk)_{m,k}} &= \sum_{m=0}^r \frac{(-r)_m \left(1 - \frac{c}{k} - r\right)_m}{\left(1 - \frac{a}{k} - r\right)_m} \frac{1}{m!} \\ &= {}_2F_1\left(-r, 1 - \frac{c}{k} - r; 1 - \frac{a}{k} - r; 1\right). \end{aligned}$$

Applying the Chu–Vandermonde summation formula [26] and then using (8), we obtain

$$\begin{aligned} {}_2F_1\left(-r, 1 - \frac{c}{k} - r; 1 - \frac{a}{k} - r; 1\right) &= (-1)^r \frac{\left(\frac{c-a}{k}\right)_r}{\left(\frac{a}{k}\right)_r} \\ &= (-1)^r \frac{(c-a)_{r,k}}{(a)_{r,k}}. \end{aligned}$$

Substituting this expression into the previous series, we find

$$\begin{aligned} e^{-w} M_k(a, c; w) &= \sum_{r=0}^{\infty} \frac{w^r}{r!} \frac{(a)_{r,k}}{(c)_{r,k}} \left((-1)^r \frac{(c-a)_{r,k}}{(a)_{r,k}} \right) \\ &= \sum_{r=0}^{\infty} \frac{(c-a)_{r,k}}{(c)_{r,k}} \frac{(-w)^r}{r!} \\ &= M_k(c-a, c; -w). \end{aligned}$$

This completes the proof. $\square$

Theorem 3. Let $k > 0$ and $c \in \mathbb{C}$ with $c \notin \{0, -k, -2k, \dots\}$. Then the k -hypergeometric function defined by

$${}_0F_{1,k}(-; c; w) = \sum_{n=0}^{\infty} \frac{1}{(c)_{n,k}} \frac{w^n}{n!} \quad (23)$$

is a solution of the differential equation

$$kw y''(w) + cy'(w) - y(w) = 0. \quad (24)$$

Proof. Let

$$a_n = \frac{1}{(c)_{n,k} n!}$$

and consider the function

$$y(w) = \sum_{n=0}^{\infty} a_n w^n.$$

Equation (24) has an ordinary point at every finite point $w \neq 0$. Indeed, after dividing by kw , the equation is written in standard form, and the resulting coefficients are analytic for $w \neq 0$. At $w = 0$, the coefficient of $y'(w)$ has a simple pole, while the coefficient of $y(w)$ also has a simple pole. Therefore, $w = 0$ is a regular singular point. Next, we verify that the power series (23) satisfies Equation (24).

For this purpose, the derivatives of $y(w)$ are obtained as

$$y'(w) = \sum_{n=0}^{\infty} (n+1) a_{n+1} w^n \quad \text{and} \quad y''(w) = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} w^n.$$

Substituting these expressions into the differential equation and shifting the indices appropriately, we obtain

$$kw y'' + cy' - y = \sum_{n=0}^{\infty} w^n \left[(n+1)(kn+c) a_{n+1} - a_n \right].$$

Using property (7) of the k -Pochhammer symbol, the recurrence relation between consecutive coefficients becomes

$$(n+1)(kn+c)a_{n+1} = (n+1)(kn+c) \cdot \frac{1}{(c)_{n+1,k}(n+1)!} = \frac{1}{(c)_{n,k}n!} = a_n.$$

Hence, for every $n \geq 0$,

$$(n+1)(kn+c)a_{n+1} - a_n = 0.$$

Therefore,

$$kwy''(w) + cy'(w) - y(w) = 0,$$

which completes the proof. $\square$

The connection between (24) and (11) is established in the next theorem.

Theorem 4. Let $k > 0$ and $c \in \mathbb{C}$ with $c \notin \{0, -k, -2k, \dots\}$ and $2c - k \notin \{0, -k, -2k, \dots\}$. Then

$${}_0F_{1,k}\left(-; c; \frac{kw^2}{16}\right) = e^{-w/2} M_k\left(c - \frac{k}{2}, 2c - k; w\right). \quad (25)$$

Proof. As shown in Theorem 3, the function ${}_0F_{1,k}(-; c; w)$ satisfies the differential equation

$$kwy''(w) + cy'(w) - y(w) = 0. \quad (26)$$

We apply the change in variable

$$w = \frac{kt^2}{16} \quad (27)$$

and define

$$u(t) = y\left(\frac{kt^2}{16}\right).$$

By the chain rule, we obtain

$$u'(t) = \frac{kt}{8} y'\left(\frac{kt^2}{16}\right), \quad u''(t) = \frac{k}{8} y'\left(\frac{kt^2}{16}\right) + \frac{k^2 t^2}{64} y''\left(\frac{kt^2}{16}\right).$$

Hence,

$$y'\left(\frac{kt^2}{16}\right) = \frac{8}{kt} u'(t), \quad y''\left(\frac{kt^2}{16}\right) = \frac{64}{k^2 t^2} \left(u''(t) - \frac{1}{t} u'(t)\right).$$

Substituting these expressions into (26) and using $w = \frac{kt^2}{16}$, we arrive at

$$ktu''(t) + (2c - k)u'(t) - \frac{kt}{4}u(t) = 0. \quad (28)$$

Next, we apply the transformation

$$u(t) = e^{-t/2} v(t). \quad (29)$$

Then

$$u'(t) = e^{-t/2} \left(v'(t) - \frac{1}{2}v(t) \right),$$

$$u''(t) = e^{-t/2} \left(v''(t) - v'(t) + \frac{1}{4}v(t) \right).$$

Substituting these expressions into (28) and simplifying, we obtain

$$ktv''(t) + (2c - k - kt)v'(t) - \left(c - \frac{k}{2} \right)v(t) = 0. \quad (30)$$

This equation coincides with the k -confluent hypergeometric differential equation (11) with

$$\alpha = c - \frac{k}{2}, \quad \beta = 2c - k.$$

Therefore, a solution of (30) is given by

$$v(t) = M_k \left(c - \frac{k}{2}, 2c - k; t \right).$$

Combining the transformations (27) and (29), we obtain

$${}_0F_{1,k} \left(-; c; \frac{kt^2}{16} \right) = e^{-t/2} M_k \left(c - \frac{k}{2}, 2c - k; t \right).$$

Replacing t by w completes the proof of (25). $\square$

3.1. Contiguous Relations for the k -Kummer Function

We now present contiguous relations for the k -Kummer function, connecting functions whose parameters differ by $\pm k$.

Theorem 5. For $k > 0$, the following relation holds:

$$M_k(a + k, c; w) = M_k(a, c; w) + \frac{k w}{c} M_k(a + k, c + k; w). \quad (31)$$

Proof. Applying $a \mapsto a + k$ in the series representation (10), we obtain

$$M_k(a + k, c; w) = \sum_{n=0}^{\infty} \frac{(a + k)_{n,k}}{(c)_{n,k}} \frac{w^n}{n!}.$$

Using the identity (7) for $\alpha = a$, we get

$$\begin{aligned} M_k(a + k, c; w) &= \sum_{n=0}^{\infty} \frac{a + nk}{a} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{w^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{w^n}{n!} + \frac{k}{a} \sum_{n=0}^{\infty} \frac{n \cdot (a)_{n,k}}{(c)_{n,k}} \frac{w^n}{n!}. \end{aligned}$$

By reindexing the second sum, we obtain

$$\begin{aligned} M_k(a + k, c; w) &= \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{w^n}{n!} + \frac{k w}{c} \sum_{n=0}^{\infty} \frac{(a + k)_{n,k}}{(c + k)_{n,k}} \frac{w^n}{n!} \\ &= M_k(a, c; w) + \frac{k w}{c} M_k(a + k, c + k; w). \end{aligned}$$

$\square$

Theorem 6. For $k > 0$, the following relation holds:

$$M_k(a, c; w) = \frac{a}{kw + a} M_k(a + k, c; w) + \frac{kw(c - a)}{c(kw + a)} M_k(a, c + k; w). \quad (32)$$

Proof. Applying the transformation $c \mapsto c + k$ to the series representation (10) and using the identity (7) for $\alpha = c$, we obtain

$$M_k(a, c + k; w) = \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c + k)_{n,k}} \frac{w^n}{n!} = \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \left(\frac{c}{c + nk} \right) \frac{w^n}{n!}. \quad (33)$$

After rearranging the above expression, we have

$$M_k(a, c + k; w) = c \cdot \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \left(\frac{1}{a + (n - 1)k} - \frac{c - a + k}{(c + nk)(a + (n - 1)k)} \right) \frac{w^n}{n!}.$$

Using the identity $\frac{(a)_{n,k}}{a + (n - 1)k} = \frac{(a - k)_{n,k}}{a - k}$, we obtain

$$M_k(a, c + k; w) = \frac{c}{a - k} M_k(a - k, c; w) - \frac{c - a + k}{a - k} M_k(a - k, c + k; w).$$

Applying the substitution $a \mapsto a + k$ in the identity, we get

$$M_k(a + k, c + k; w) = \frac{c}{a} M_k(a, c; w) - \frac{c - a}{a} M_k(a, c + k; w). \quad (34)$$

Substituting this expression and rearranging, we obtain the desired result. $\square$

Theorem 7. For $k > 0$, we have the following relation:

$$M_k(a, c; w) = \frac{a}{a - c + k} M_k(a + k, c; w) + \frac{c - k}{c - a - k} M_k(a, c - k; w). \quad (35)$$

Proof. If the transformation $c \mapsto c - k$ is applied in the series definition (10), we obtain

$$M_k(a, c - k; w) = \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c - k)_{n,k}} \frac{w^n}{n!} = \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{c + (n - 1)k}{c - k} \frac{w^n}{n!}. \quad (36)$$

Taking $c + (n - 1)k = (c - a - k) + (a + nk)$ in this series, we get

$$M_k(a, c - k; w) = \frac{c - a - k}{c - k} \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{w^n}{n!} + \frac{1}{c - k} \sum_{n=0}^{\infty} \frac{(a)_{n,k}(a + nk)}{(c)_{n,k}} \frac{w^n}{n!}. \quad (37)$$

Using the identity (7) for $\alpha = a$, we obtain

$$M_k(a, c - k; w) = \frac{c - a - k}{c - k} M_k(a, c; w) + \frac{a}{c - k} M_k(a + k, c; w). \quad (38)$$

Rearranging the expression for $M_k(a, c; w)$ yields the desired relation. $\square$

Theorem 8. The following identity holds for $k > 0$:

$$M_k(a, c; w) = \frac{c - k}{c + k(w - 1)} M_k(a, c - k; w) + \frac{kw(c - a)}{c(c + k(w - 1))} M_k(a, c + k; w)$$

Proof. From the relation (35), we have

$$M_k(a+k, c; w) = \frac{a-c+k}{a} M_k(a, c; w) + \frac{c-k}{a} M_k(a, c-k; w).$$

Substituting this expression into relation (32), we obtain

$$\begin{aligned} M_k(a, c; w) &= \frac{a}{kw+a} \frac{a-c+k}{a} M_k(a, c; w) + \frac{a}{kw+a} \frac{c-k}{a} M_k(a, c-k; w) \\ &\quad + \frac{(c-a)kw}{c(kw+a)} M_k(a, c+k; w). \end{aligned}$$

Rearranging the expression for $M_k(a, c; w)$ gives the required result. $\square$

Theorem 9. For $k > 0$, the following relation holds:

$$M_k(a, c; w) = \frac{c-a}{c-2a-kw} M_k(a-k, c; w) + \frac{a}{2a-c+kw} M_k(a+k, c; w). \quad (39)$$

Proof. Applying the transformation $a \mapsto a-k$ in the series definition (10), we have

$$\begin{aligned} M_k(a-k, c; w) &= \sum_{n=0}^{\infty} \frac{(a-k)_{n,k}}{(c)_{n,k}} \frac{w^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(a-k)(a)_{n,k}}{(a+(n-1)k)(c)_{n,k}} \frac{w^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \left(1 - \frac{nk}{a+(n-1)k} \right) \frac{w^n}{n!}. \end{aligned}$$

Multiplying both sides by $(c-a)$, we obtain

$$(c-a)M_k(a-k, c; w) = (c-a)M_k(a, c; w) - \frac{kw(c-a)}{c} M_k(a, c+k; w). \quad (40)$$

Substituting the relation (32) for $M_k(a, c+k; w)$ into this equality and simplifying, we obtain the desired result. $\square$

Theorem 10. The following relation holds for $k > 0$:

$$M_k(a, c; w) = \frac{c-k}{a+k(w-1)} M_k(a, c-k; w) + \frac{a-c}{a+k(w-1)} M_k(a-k, c; w).$$

Proof. Substituting the relation (39) for $M_k(a+k, c; w)$ into the relation (38) and simplifying, we obtain the desired relation. $\square$

Theorem 11. For $k > 0$, the following identity holds:

$$M_k(a, c; w) = \frac{kw}{c} M_k(a, c+k; w) + M_k(a-k, c; w). \quad (41)$$

Proof. Using the identity $\frac{c}{c+nk} = \frac{c}{c-a} \left(1 - \frac{a+nk}{c+nk}\right)$ in the series representation (33), we obtain

$$\begin{aligned} M_k(a, c+k; w) &= \frac{c}{c-a} \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \left(1 - \frac{a+nk}{c+nk}\right) \frac{w^n}{n!} \\ &= \frac{c}{c-a} \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{w^n}{n!} - \frac{c}{c-a} \sum_{n=0}^{\infty} \frac{(a)_{n,k}(a+nk)}{(c)_{n,k}(c+nk)} \frac{w^n}{n!} \\ &= \frac{c}{c-a} \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{w^n}{n!} - \frac{c}{c-a} \cdot \frac{a}{c} \sum_{n=0}^{\infty} \frac{(a+k)_{n,k}}{(c+k)_{n,k}} \frac{w^n}{n!} \\ &= \frac{c}{c-a} M_k(a, c; w) - \frac{a}{c-a} M_k(a+k, c+k; w). \end{aligned}$$

Substituting the relations (31) for $M_k(a+k, c+k; w)$ and (39) for $M_k(a+k, c; w)$ into the above expression, we get

$$\begin{aligned} M_k(a, c+k; w) &= \frac{c}{c-a} M_k(a, c; w) - \frac{a}{c-a} \cdot \frac{c}{kw} [M_k(a+k, c; w) - M_k(a, c; w)] \\ &= \frac{ckw + ac}{(c-a)kw} M_k(a, c; w) - \frac{ac}{(c-a)kw} M_k(a+k, c; w) \\ &= \frac{c}{kw} [M_k(a, c; w) - M_k(a-k, c; w)]. \end{aligned}$$

Hence, the desired result follows. $\square$

3.2. Derivative Identities of the k -Kummer Function

In this subsection, we will consider derivative identities for the k -Kummer function, including both ordinary and weighted derivative formulas involving parameter shifts. These results are k -generalizations of the corresponding classical derivative formulas for the classical Kummer function [3].

Corollary 1. Let $k > 0$. Then the k -Kummer function satisfies the following first-order derivative formula:

$$\frac{d}{dw} \{M_k(a, c; w)\} = \frac{a}{c} M_k(a+k, c+k; w). \quad (42)$$

Proof. Taking the first derivative of the k -Kummer function (10), we have

$$\begin{aligned} \frac{d}{dw} \{M_k(a, c; w)\} &= \sum_{n=1}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{nw^{n-1}}{n!} \\ &= \sum_{n=1}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{w^{n-1}}{(n-1)!}. \end{aligned}$$

Now, replacing n by $n+1$, we get

$$\frac{d}{dw} \{M_k(a, c; w)\} = \sum_{n=0}^{\infty} \frac{(a)_{n+1,k}}{(c)_{n+1,k}} \frac{w^n}{n!}.$$

Since $(a)_{n+1,k} = a(a+k)_{n,k}$ and $(c)_{n+1,k} = c(c+k)_{n,k}$, it follows that

$$\begin{aligned} \frac{d}{dw} \{M_k(a, c; w)\} &= \frac{a}{c} \sum_{n=0}^{\infty} \frac{(a+k)_{n,k}}{(c+k)_{n,k}} \frac{w^n}{n!} \\ &= \frac{a}{c} M_k(a+k, c+k; w). \end{aligned}$$

This completes the proof. $\square$

Corollary 2. Let $k > 0$. Then the k -Kummer function satisfies the following n th-order derivative formula:

$$\frac{d^n}{dw^n} \{M_k(a, c; w)\} = \frac{(a)_{n,k}}{(c)_{n,k}} M_k(a + nk, c + nk; w). \quad (43)$$

Proof. The result follows by induction on $n \in \mathbb{N}$ using (42). $\square$

Corollary 3. Let $k > 0$. Then the k -Kummer function satisfies the following derivative formula:

$$\frac{d}{dw} \left\{ w^{\frac{a}{k}} M_k(a, c; w) \right\} = \frac{a}{k} w^{\frac{a}{k}-1} M_k(a + k, c; w) \quad (44)$$

Proof. Taking the first derivative of the function $w^{\frac{a}{k}} M_k(a, c; w)$ and using the derivative formula (42), we obtain

$$\begin{aligned} \frac{d}{dw} \left\{ w^{\frac{a}{k}} M_k(a, c; w) \right\} &= \frac{d}{dw} \left\{ w^{\frac{a}{k}} \right\} M_k(a, c; w) + w^{\frac{a}{k}} \frac{d}{dw} M_k(a, c; w) \\ &= \frac{a}{k} w^{\frac{a}{k}-1} M_k(a, c; w) + w^{\frac{a}{k}} \left(\frac{a}{c} M_k(a + k, c + k; w) \right). \end{aligned}$$

If the relation (31) for $M_k(a + k, c + k; w)$ is applied to the expression on the right-hand side, we arrive at (44). $\square$

Corollary 4. Let $n \in \mathbb{N}$ and $k > 0$. Then the following n th-order derivative formula holds for the k -Kummer function:

$$\frac{d^n}{dw^n} \left\{ w^{\frac{a}{k}+n-1} M_k(a, c; w) \right\} = \frac{(a)_{n,k}}{k^n} w^{\frac{a}{k}-1} M_k(a + nk, c; w) \quad (45)$$

Proof. The result follows by induction on n . The case $n = 1$ is (44). Assume that the formula holds for some $n \in \mathbb{N}$. Then, applying the Leibniz rule to

$$\frac{d^{n+1}}{dw^{n+1}} \left\{ w^{\frac{a}{k}+n} M_k(a, c; w) \right\} = \frac{d^{n+1}}{dw^{n+1}} \left\{ w \cdot w^{\frac{a}{k}+n-1} M_k(a, c; w) \right\}$$

and using the induction hypothesis together with (44) with $a \mapsto a + nk$ and the contiguous relation (41) with $a \mapsto a + (n + 1)k$, we obtain (45). $\square$

Corollary 5. Let $k > 0$. Then the following relation holds:

$$\frac{d}{dw} \left\{ w^{\frac{c}{k}-1} M_k(a, c; w) \right\} = \frac{(c - k)}{k} w^{\frac{c}{k}-2} M_k(a, c - k; w) \quad (46)$$

Proof. We first compute

$$\begin{aligned} \frac{d}{dw} \left\{ w^{\frac{c}{k}-1} M_k(a, c; w) \right\} &= \frac{d}{dw} \left\{ w^{\frac{c}{k}-1} \right\} M_k(a, c; w) + w^{\frac{c}{k}-1} \frac{d}{dw} M_k(a, c; w) \\ &= \left(\frac{c}{k} - 1 \right) w^{\frac{c}{k}-2} M_k(a, c; w) + w^{\frac{c}{k}-1} \frac{a}{c} M_k(a + k, c + k; w). \end{aligned} \quad (47)$$

Multiplying (31) by $\frac{a}{a-c+k}$, adding the result to (35), and simplifying, we obtain

$$M_k(a + k, c + k; w) = \frac{c(c - k)}{(akw)} (M_k(a, c - k; w) - M_k(a, c; w)). \quad (48)$$

Substituting this relation into (47), we arrive at (46). $\square$

Corollary 6. Let $n \in \mathbb{N}$ and $k > 0$. Then the following identity holds:

$$\frac{d^n}{dw^n} \left\{ w^{\frac{c}{k}-1} M_k(a, c; w) \right\} = \frac{(-1)^n}{k^n} (k-c)_{n,k} w^{\frac{c}{k}-1-n} M_k(a, c-nk; w). \quad (49)$$

Proof. The proof follows by induction on n . For $n = 1$, the result follows from (46). Assume that the formula holds for $n \in \mathbb{N}$. Differentiating the induction hypothesis and using (46) with c replaced by $c - nk$, together with the contiguous relation (48) with c replaced by $c - nk$, gives the result for $n + 1$. $\square$

Corollary 7. Let $k > 0$. Then the following derivative formula holds:

$$\frac{d}{dw} \left\{ e^{-w} M_k(a, c; w) \right\} = -\frac{c-a}{c} e^{-w} M_k(a, c+k; w). \quad (50)$$

Proof. Applying the product rule and using (42), we obtain

$$\frac{d}{dw} \left\{ e^{-w} M_k(a, c; w) \right\} = -e^{-w} M_k(a, c; w) + e^{-w} \left(\frac{a}{c} M_k(a+k, c+k; w) \right).$$

Using the relation (34) for $M_k(a+k, c+k; w)$, we find

$$\frac{d}{dw} \left\{ e^{-w} M_k(a, c; w) \right\} = -e^{-w} \frac{c-a}{c} M_k(a, c+k; w).$$

$\square$

Corollary 8. Let $n \in \mathbb{N}$ and $k > 0$. Then the following n th-order derivative formula holds:

$$\frac{d^n}{dw^n} \left\{ e^{-w} M_k(a, c; w) \right\} = (-1)^n \frac{(c-a)_{n,k}}{(c)_{n,k}} M_k(c-a+nk, c+nk; -w). \quad (51)$$

Proof. The statement is obtained by induction on n . The result for $n = 1$ is given by (50). Assuming the formula holds for $n \in \mathbb{N}$, differentiation of the induction hypothesis together with (42) applied to $M_k(c-a+nk, c+nk; -w)$ yields the corresponding formula for $n + 1$. $\square$

Corollary 9. Let $k > 0$. Then the following derivative formula holds:

$$\frac{d}{dw} \left\{ e^{-w} w^{\frac{c-a}{k}} M_k(a, c; w) \right\} = \frac{c-a}{k} e^{-w} w^{\frac{c-a}{k}-1} M_k(a-k, c; w). \quad (52)$$

Proof. Applying the product rule yields

$$\begin{aligned} \frac{d}{dw} \left\{ e^{-w} w^{\frac{c-a}{k}} M_k(a, c; w) \right\} &= e^{-w} \left(\frac{c-a}{k} \right) w^{\frac{c-a}{k}-1} M_k(a, c; w) \\ &\quad - e^{-w} w^{\frac{c-a}{k}} M_k(a, c; w) + e^{-w} w^{\frac{c-a}{k}} \frac{d}{dw} M_k(a, c; w). \end{aligned}$$

Using the derivative formula (42) and the relation (31), we get

$$\begin{aligned}
\frac{d}{dw} \left\{ e^{-w} w^{\frac{c-a}{k}} M_k(a, c; w) \right\} &= e^{-w} w^{\frac{c-a}{k}-1} \left(\frac{c-a}{k} - \frac{c-a}{k} \right) M_k(a, c; w) \\
&\quad + e^{-w} w^{\frac{c-a}{k}-1} \frac{c-a}{k} M_k(a-k, c; w) \\
&= \left(\frac{c-a}{k} \right) e^{-w} w^{\frac{c-a}{k}-1} M_k(a-k, c; w) \\
&= \left(\frac{c-a}{k} \right) w^{\frac{c-a}{k}-1} M_k(c-a+k, c; -w).
\end{aligned}$$

This proves the formula. $\square$

Corollary 10. Let $n \in \mathbb{N}$ and $k > 0$. Then the following n th-order derivative formula holds:

$$\frac{d^n}{dw^n} \left\{ e^{-w} w^{\frac{c-a}{k}+n-1} M_k(a, c; w) \right\} = \frac{(c-a)_{n,k}}{k^n} e^{-w} w^{\frac{c-a}{k}-1} M_k(a-nk, c; w). \quad (53)$$

Proof. The result follows by induction on n . For $n = 1$, the formula follows from (52). Assume that it holds for $n \in \mathbb{N}$. Applying the Leibniz rule to

$$\frac{d^{n+1}}{dw^{n+1}} \left\{ w e^{-w} w^{\frac{c-a}{k}+n-1} M_k(a, c; w) \right\},$$

and using the induction hypothesis together with (52) with a replaced by $a - nk$ and (31) with $a \mapsto a - (n+1)k$, shows that the identity holds for $n+1$. $\square$

Corollary 11. Let $k > 0$. Then the following derivative formula holds:

$$\frac{d}{dw} \left\{ e^{-w} w^{\frac{c}{k}-1} M_k(a, c; w) \right\} = \left(\frac{c-k}{k} \right) e^{-w} w^{\frac{c}{k}-2} M_k(a-k, c-k; w). \quad (54)$$

Proof. We compute

$$\begin{aligned}
\frac{d}{dw} \left\{ e^{-w} w^{\frac{c}{k}-1} M_k(a, c; w) \right\} &= \left(\frac{c}{k} - 1 \right) w^{\frac{c}{k}-2} e^{-w} M_k(a, c; w) \\
&\quad + w^{\frac{c}{k}-1} \left(-e^{-w} \frac{c-a}{c} M_k(a, c+k; w) \right) \\
&= \left(\frac{c}{k} - 1 \right) w^{\frac{c}{k}-2} e^{-w} M_k(a, c; w) \\
&\quad - w^{\frac{c}{k}-1} e^{-w} \frac{c-a}{c} M_k(a, c+k; w).
\end{aligned}$$

For the function $M_k(a, c+k; w)$, applying the transformation $a \mapsto a-k$ in (35) and adding this relation to (41), we obtain

$$M_k(a, c+k; w) = \frac{c(c-k)}{kw(c-a)} \{ M_k(a, c; w) - M_k(a-k, c-k; w) \}.$$

Substituting this into the above expression gives

$$\begin{aligned}
\frac{d}{dw} \left\{ e^{-w} w^{\frac{c}{k}-1} M_k(a, c; w) \right\} &= w^{\frac{c}{k}-2} e^{-w} M_k(a, c; w) \left[\frac{c}{k} - 1 - \frac{c-k}{k} \right] \\
&\quad + w^{\frac{c}{k}-2} e^{-w} \left(\frac{c-k}{k} \right) M_k(a-k, c-k; w) \\
&= \frac{c-k}{k} e^{-w} w^{\frac{c}{k}-2} M_k(a-k, c-k; w).
\end{aligned}$$

Therefore, we arrive at (54). $\square$

Corollary 12. Let $n \in \mathbb{N}$ and $k > 0$. Then the following n th-order derivative formula holds:

$$\frac{d^n}{dw^n} \left\{ e^{-w} w^{\frac{c}{k}-1} M_k(a, c; w) \right\} = \frac{(-1)^n}{k^n} (k-c)_{n,k} e^{-w} w^{\frac{c}{k}-n-1} M_k(a-nk, c-nk; w). \quad (55)$$

Proof. The result follows by induction on n . For $n = 1$, the formula follows from (54). Assume that it holds for some $n \in \mathbb{N}$. Differentiating the induction hypothesis and applying (54) with a replaced by $a - nk$ and c replaced by $c - nk$ shows that the identity holds for $n + 1$. $\square$

3.3. Addition Formulas for the k -Kummer Function

In this part, we derive addition formulas for the k -Kummer function and express $M_k(a, c; w + z)$ in terms of series involving shifted parameters. These results are k -generalizations of the corresponding classical addition formulas [3].

Corollary 13. Let $k > 0$ and $a, c \in \mathbb{C}$. Then the following identity holds

$$M_k(a, c; w + z) = \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} M_k(a + nk, c + nk; w) \frac{z^n}{n!}. \quad (56)$$

Proof. If we take $f(w) = M_k(a, c; w)$ in (12), we obtain

$$M_k(a, c; w + z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{d^n}{dw^n} M_k(a, c; w). \quad (57)$$

Substituting the n -th derivative of f given in (43) into the above expression and simplifying, we arrive at (56). $\square$

Corollary 14. Let $k > 0$ and $a, c \in \mathbb{C}$. Then the following relation holds:

$$M_k(a, c; w + z) = \left(\frac{w}{w + z} \right)^{\frac{a}{k}} \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{k^n n!} M_k(a + nk, c; w) \left(\frac{z}{w + z} \right)^n. \quad (58)$$

Proof. If we set $f(w) = w^{\frac{a}{k}-1} M_k(a, c; w)$ in (13), we have

$$\left(\frac{w + z}{w} \right) (w + z)^{\frac{a}{k}-1} M_k(a, c; w + z) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{z}{w + z} \right)^n \frac{d^n}{dw^n} \left[w^{n+\frac{a}{k}-1} M_k(a, c; w) \right].$$

Using the formula (45) for the n -th derivative of f and substituting it into the above series, we obtain (58). $\square$

Corollary 15. Let $k > 0$ and $a, c \in \mathbb{C}$. Then the following addition formula holds:

$$M_k(a, c; w + z) = \left(\frac{w}{w + z} \right)^{\frac{c}{k}-1} \sum_{n=0}^{\infty} \frac{(k-c)_{n,k}}{k^n n!} M_k(a, c - nk; w) \left(-\frac{z}{w} \right)^n. \quad (59)$$

Proof. If we choose $f(w) = w^{\frac{c}{k}-1} M_k(a, c; w)$ in (12), we find

$$(w + z)^{\frac{c}{k}-1} M_k(a, c; w + z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{d^n}{dw^n} \left\{ w^{\frac{c}{k}-1} M_k(a, c; w) \right\}.$$

Using the identity (49) for the n -th derivative and substituting it into the above series, we arrive at (59). $\square$

Corollary 16. Let $k > 0$ and $a, c \in \mathbb{C}$. Then the following identity holds:

$$M_k(a, c; w + z) = e^z \sum_{n=0}^{\infty} \frac{(c-a)_{n,k}}{(c)_{n,k}} M_k(a, c + nk; w) \frac{(-z)^n}{n!}. \quad (60)$$

Proof. Applying (12) with $f(w) = e^{-w} M_k(a, c; w)$, we have

$$e^{-(w+z)} M_k(a, c; w + z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{d^n}{dw^n} \{e^{-w} M_k(a, c; w)\}. \quad (61)$$

Using the formula (51) and substituting it into the series, we obtain the desired result. $\square$

Corollary 17. Let $k > 0$ and $a, c \in \mathbb{C}$. Then the following identity holds:

$$M_k(a, c; w + z) = e^z \left(\frac{w}{w+z} \right)^{\frac{c}{k}-1} \sum_{n=0}^{\infty} \frac{(k-c)_{n,k}}{k^n n!} M_k(a - nk, c - nk; w) \left(-\frac{z}{w} \right)^n. \quad (62)$$

Proof. Setting $f(w) = e^{-w} w^{\frac{c}{k}-1} M_k(a, c; w)$ in (12), we get

$$e^{-(w+z)} (w+z)^{\frac{c}{k}-1} M_k(a, c; w + z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{d^n}{dw^n} \{e^{-w} w^{\frac{c}{k}-1} M_k(a, c; w)\}.$$

Applying the derivative formula (55) and substituting it into the series, we obtain the desired result (62). $\square$

Corollary 18. Let $k > 0$ and $a, c \in \mathbb{C}$. Then the following identity holds:

$$M_k(a, c; w + z) = \left(\frac{w}{w+z} \right)^{\frac{c-a}{k}} e^z \sum_{n=0}^{\infty} \frac{(c-a)_{n,k}}{k^n n!} M_k(a - nk, c; w) \left(\frac{z}{w+z} \right)^n. \quad (63)$$

Proof. If we take $f(w) = e^{-w} w^{\frac{c-a}{k}-1} M_k(a, c; w)$ in (13), we have

$$\begin{aligned} & \left(\frac{w+z}{w} \right) e^{-(w+z)} (w+z)^{\frac{c-a}{k}-1} M_k(a, c; w + z) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{z}{w+z} \right)^n \frac{d^n}{dw^n} \left[e^{-w} w^{\frac{c-a}{k}+n-1} M_k(a, c; w) \right]. \end{aligned}$$

Applying the derivative formula (53) to the n th derivative term and substituting into the series, we obtain (63). $\square$

3.4. Multiplication Formulas for the k -Kummer Function

We now establish multiplication formulas for the k -Kummer function, describing the behavior of $M_k(a, c; wz)$ through suitable series expansions. These formulas are k -generalizations of the corresponding classical multiplication formulas [3].

Corollary 19. Let $k > 0$ and $a, c \in \mathbb{C}$. Then the following identity holds

$$M_k(a, c; wz) = \sum_{n=0}^{\infty} \frac{((z-1)w)^n}{n!} \frac{(a)_{n,k}}{(c)_{n,k}} M_k(a + nk, c + nk; w). \quad (64)$$

Proof. If we set $f(w) = M_k(a, c; w)$ in (14), we obtain

$$M_k(a, c; wz) = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} w^n \frac{d^n}{dw^n} M_k(a, c; w). \quad (65)$$

Substituting the n -th derivative relation (43) into the above series, we have (65). $\square$

Corollary 20. Let $k > 0$ and $a, c \in \mathbb{C}$. Then the following relation holds:

$$M_k(a, c; wz) = z^{-\frac{a}{k}} \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{n!} \left(\frac{z-1}{kz} \right)^n M_k(a + nk, c; w). \quad (66)$$

Proof. Setting $f(w) = w^{\frac{a}{k}-1} M_k(a, c; w)$ in (15), we get

$$z(wz)^{\frac{a}{k}-1} M_k(a, c; wz) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{z-1}{z} \right)^n \frac{d^n}{dw^n} \left[w^{n+\frac{a}{k}-1} M_k(a, c; w) \right].$$

Using the derivative identity (45) and simplifying, we obtain (66). $\square$

Corollary 21. Let $k > 0$ and $a, c \in \mathbb{C}$. Then the following multiplication formula holds:

$$M_k(a, c; wz) = z^{1-\frac{c}{k}} \sum_{n=0}^{\infty} \frac{(k-c)_{n,k}}{n!} \left(\frac{1-z}{k} \right)^n M_k(a, c - nk; w). \quad (67)$$

Proof. If we set $f(w) = w^{\frac{c}{k}-1} M_k(a, c; w)$ in (14), we find

$$(wz)^{\frac{c}{k}-1} M_k(a, c; wz) = \sum_{n=0}^{\infty} \frac{(z-1)^n w^n}{n!} \frac{d^n}{dw^n} \left[w^{\frac{c}{k}-1} M_k(a, c; w) \right].$$

Substituting the derivative identity (49) into the above series, we obtain (67). $\square$

Corollary 22. Let $k > 0$ and $a, c \in \mathbb{C}$. Then the following identity holds:

$$M_k(a, c; wz) = e^{w(z-1)} \sum_{n=0}^{\infty} \frac{(c-a)_{n,k}}{(c)_{n,k}} \frac{(1-z)^n w^n}{n!} M_k(a, c + nk; w). \quad (68)$$

Proof. If we choose $f(w) = e^{-w} M_k(a, c; w)$ in (14), we have

$$e^{-wz} M_k(a, c; wz) = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} w^n \frac{d^n}{dw^n} \left[e^{-w} M_k(a, c; w) \right].$$

Substituting the identity (51) into the above series and multiplying both sides by e^{wz} , we obtain (68). $\square$

Corollary 23. Let $k > 0$ and $a, c \in \mathbb{C}$. Then the following relation holds:

$$M_k(a, c; wz) = e^{w(z-1)} z^{\frac{a-c}{k}} \sum_{n=0}^{\infty} \frac{(c-a)_{n,k}}{n!} \left(\frac{z-1}{z} \right)^n M_k(a - nk, c; w). \quad (69)$$

Proof. Applying (15) with $f(w) = e^{-w} w^{\frac{c-a}{k}-1} M_k(a, c; w)$, we obtain

$$ze^{-wz} (wz)^{\frac{c-a}{k}-1} M_k(a, c; wz) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{z-1}{z} \right)^n \frac{d^n}{dw^n} \left[e^{-w} w^{n+\frac{c-a}{k}-1} M_k(a, c; w) \right].$$

Using the identity (53) and simplifying yields to (69). $\square$

Corollary 24. Let $k > 0$ and $a, c \in \mathbb{C}$. Then the following multiplication identity holds:

$$M_k(a, c; wz) = e^{w(z-1)} z^{1-\frac{c}{k}} \sum_{n=0}^{\infty} \frac{(k-c)_{n,k}}{n!} \left(\frac{(1-z)w}{kz} \right)^n M_k(a - nk, c - nk; w). \quad (70)$$

Proof. If we take $f(w) = e^{-w} w^{\frac{c}{k}-1} M_k(a, c; w)$ in (14), we obtain

$$e^{-wz} (wz)^{\frac{c}{k}-1} M_k(a, c; wz) = \sum_{n=0}^{\infty} \frac{(z-1)^n w^n}{n!} \frac{d^n}{dw^n} \left[e^{-w} w^{\frac{c}{k}-1} M_k(a, c; w) \right].$$

Using the derivative identity (55) in the above series, we find (70). $\square$

4. Integral Transforms Involving the k -Kummer Function

In this section, we examine the behavior of the k -Kummer function under several integral transforms. By employing the series representation of $M_k(a, c; w)$ and interchanging the order of summation and integration under suitable convergence conditions, explicit transform formulas are derived. These results can be regarded as natural extensions of the corresponding classical formulas for the confluent hypergeometric function, and they provide a unified framework for analyzing the transform-domain representations of k -generalized special functions.

Furthermore, the obtained expressions reveal that the transformed forms preserve the hypergeometric-type structure, which is particularly useful in applications involving fractional calculus, integral equations, and diffusion-type models. Throughout this section, all derivations are carried out under conditions ensuring the absolute convergence of the involved series and integrals.

Theorem 12. Under the conditions $k > 0$, $\operatorname{Re}(\mu) > 0$ and $\operatorname{Re}(s) > |\beta|$, the following identity holds:

$$\mathcal{L}\{w^{\mu-1} M_k(a, c; \beta w); s\} = \frac{\Gamma(\mu)}{s^\mu} {}_2F_{1,k}\left(a, k\mu; c; \frac{\beta}{sk}\right). \quad (71)$$

Proof. By choosing $f(w) = w^{\mu-1} M_k(a, c; \beta w)$ in the definition of the Laplace transform given in (16), we obtain

$$\mathcal{L}\{w^{\mu-1} M_k(a, c; \beta w); s\} = \int_0^\infty e^{-sw} w^{\mu-1} M_k(a, c; \beta w) dw.$$

Using the series representation (10) of the k -Kummer function, we get

$$\mathcal{L}\{w^{\mu-1} M_k(a, c; \beta w); s\} = \int_0^\infty e^{-sw} \left(\sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{\beta^n}{n!} w^{n+\mu-1} \right) dw.$$

Since $\operatorname{Re}(\mu) > 0$ and $\operatorname{Re}(s) > |\beta|$, the series of the integrals of the absolute values of the terms is convergent. Therefore, Fubini's theorem justifies the interchange of summation and integration. Hence, we obtain

$$\mathcal{L}\{w^{\mu-1} M_k(a, c; \beta w); s\} = \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{\beta^n}{n!} \int_0^\infty e^{-sw} w^{n+\mu-1} dw.$$

Using the formula ([5], p. 137, 4.3. (1)) to the integral on the right-hand side under the conditions $\operatorname{Re}(s) > 0$ and $\operatorname{Re}(\mu) > 0$, we obtain

$$\mathcal{L}\{w^{\mu-1} M_k(a, c; \beta w); s\} = \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{\beta^n}{n!} \frac{\Gamma(n+\mu)}{s^{n+\mu}}.$$

Applying (5) and then using (8) with $\alpha = k\mu$, we obtain

$$\begin{aligned}\mathcal{L}\{w^{\mu-1}M_k(a, c; \beta w); s\} &= \frac{\Gamma(\mu)}{s^\mu} \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{(\mu)_n}{n!} \left(\frac{\beta}{s}\right)^n \\ &= \frac{\Gamma(\mu)}{s^\mu} \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{(k\mu)_{n,k}}{n!} \left(\frac{\beta}{sk}\right)^n\end{aligned}$$

Finally, using the definition (9), we arrive at (71). $\square$

Theorem 13. Under the conditions $k > 0$, $\operatorname{Re}(s) > 0$ and $\operatorname{Re}(\rho) > |\beta|$, the following identity holds:

$$\mathcal{M}\{e^{-\rho w}M_k(a, c; \beta w); s\} = \frac{\Gamma(s)}{\rho^s} {}_2F_{1,k}\left(a, ks; c; \frac{\beta}{k\rho}\right). \quad (72)$$

Proof. By choosing $f(w) = e^{-\rho w}M_k(a, c; \beta w)$ in the definition of the Mellin transform given in (17), we have

$$\mathcal{M}\{e^{-\rho w}M_k(a, c; \beta w); s\} = \int_0^\infty w^{s-1}e^{-\rho w}M_k(a, c; \beta w)dw.$$

Applying the series representation (10) of the k -Kummer function in the integrand and since $\operatorname{Re}(s) > 0$ and $\operatorname{Re}(\rho) > |\beta|$ ensure the convergence of the series of the integrals of the absolute values of the terms, Fubini's theorem justifies the interchange of summation and integration. Hence,

$$\mathcal{M}\{e^{-\rho w}M_k(a, c; \beta w); s\} = \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{\beta^n}{n!} \int_0^\infty w^{n+s-1}e^{-\rho w}dw.$$

Using ([5], p. 312, 6.3. (1)) under the conditions $\operatorname{Re}(s) > 0$ and $\operatorname{Re}(\rho) > 0$, we get

$$\mathcal{M}\{e^{-\rho w}M_k(a, c; \beta w); s\} = \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{\beta^n}{n!} \frac{\Gamma(n+s)}{\rho^{n+s}}.$$

In the last series obtained, applying (5) and (8) successively and then using the definition (9) of the k -Gauss hypergeometric function, we arrive at (72). This proves the assertion. $\square$

Theorem 14. Under the conditions $k > 0$, $\rho > |\beta|$ and $\operatorname{Re}(s) > 0$, the following identity holds:

$$\mathcal{S}\{e^{-\rho w}M_k(a, c; \beta w); s\} = e^{\rho s} \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \left(\frac{\beta}{\rho}\right)^n E_{n+1}(\rho s). \quad (73)$$

Proof. By taking $f(w) = e^{-\rho w}M_k(a, c; \beta w)$ in the definition of the Stieltjes transform given in (18), we obtain

$$\mathcal{S}\{e^{-\rho w}M_k(a, c; \beta w); s\} = \int_0^\infty \frac{e^{-\rho w}M_k(a, c; \beta w)}{w+s}dw.$$

Using the series representation (10) of the k -Kummer function, and observing that $\rho > |\beta|$ and $\operatorname{Re}(s) > 0$ ensure the convergence of the series of the integrals of the absolute values of the terms, we may apply Fubini's theorem to interchange the order of summation and integration. Therefore,

$$\mathcal{S}\{e^{-\rho w}M_k(a, c; \beta w); s\} = \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{\beta^n}{n!} \int_0^\infty \frac{w^n e^{-\rho w}}{w+s}dw.$$

Using the representation

$$\frac{1}{w+s} = \int_0^\infty e^{-(w+s)t} dt.$$

Since $\rho > 0$ and $\operatorname{Re}(s) > 0$, the resulting double integral is absolutely convergent. Therefore, by Fubini's theorem, the order of integration may be interchanged. Thus,

$$\mathcal{S}\{e^{-\rho w} M_k(a, c; \beta w); s\} = \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{\beta^n}{n!} \int_0^\infty e^{-st} \left(\int_0^\infty w^n e^{-(\rho+t)w} dw \right) dt.$$

Applying ([5], p. 133, 4.2. (3)) to the inner integral under the condition $\operatorname{Re}(\rho + t) > 0$, we obtain

$$\mathcal{S}\{e^{-\rho w} M_k(a, c; \beta w); s\} = \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{\beta^n}{n!} \int_0^\infty e^{-st} \frac{n!}{(\rho + t)^{n+1}} dt.$$

Applying the change in variables $t = \rho(u - 1)$, which is valid since $\rho > 0$, gives

$$\mathcal{S}\{e^{-\rho w} M_k(a, c; \beta w); s\} = \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \beta^n \rho^{-n} e^{\rho s} \int_1^\infty e^{-\rho s u} u^{-(n+1)} du.$$

By the definition of the generalized exponential integral function (21), it follows that

$$\mathcal{S}\{e^{-\rho w} M_k(a, c; \beta w); s\} = e^{\rho s} \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \left(\frac{\beta}{\rho} \right)^n E_{n+1}(\rho s).$$

The proof is complete. $\square$

Theorem 15. Under the condition $k > 0$, $\operatorname{Re}(\mu) > 0$ and $|\beta u| < 1$, the following identity holds:

$$\mathcal{S}\{w^{\mu-1} M_k(a, c; \beta w); u\} = u^{\mu-1} \Gamma(\mu) {}_2F_{1,k} \left(a, k\mu; c; \frac{\beta u}{k} \right). \quad (74)$$

Proof. By taking $f(w) = w^{\mu-1} M_k(a, c; \beta w)$ in the definition of the Sumudu transform given in (19), we obtain

$$\begin{aligned} \mathcal{S}\{w^{\mu-1} M_k(a, c; \beta w); u\} &= \int_0^\infty e^{-w} (uw)^{\mu-1} M_k(a, c; \beta uw) dw \\ &= u^{\mu-1} \int_0^\infty e^{-w} w^{\mu-1} M_k(a, c; \beta uw) dw. \end{aligned}$$

Applying the series representation (10) of the k -Kummer function, and observing that $\operatorname{Re}(\mu) > 0$ and $|\beta u| < 1$ ensure the convergence of the series of the integrals of the absolute values of the terms, Fubini's theorem permits the interchange of summation and integration. Hence,

$$\mathcal{S}\{w^{\mu-1} M_k(a, c; \beta w); u\} = u^{\mu-1} \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{(\beta u)^n}{n!} \int_0^\infty e^{-w} w^{n+\mu-1} dw.$$

Applying ([5], p. 137, 4.3. (1)) to the inner integral under the condition $\operatorname{Re}(n + \mu) > 0$, we get

$$\mathcal{S}\{w^{\mu-1} M_k(a, c; \beta w); u\} = u^{\mu-1} \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{(\beta u)^n}{n!} \Gamma(n + \mu).$$

After applying (5) and (8) to the resulting series, the right-hand side can be expressed in terms of the k -Gauss hypergeometric function by means of (9), we obtain (74). $\square$

Theorem 16. Under the conditions $k > 0$, $\operatorname{Re}(\alpha) > 0$ and $\operatorname{Re}(\mu) > 0$, the following identity holds:

$$I_0^\alpha \{w^{\mu-1} M_k(a, c; \beta w); s\} = \frac{\Gamma(\mu)}{\Gamma(\mu + \alpha)} s^{\mu+\alpha-1} \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{(\mu)_n}{(\mu + \alpha)_n} \frac{(\beta s)^n}{n!}. \quad (75)$$

Proof. By taking $f(w) = w^{\mu-1} M_k(a, c; \beta w)$ in the definition of the Riemann–Liouville fractional integral given in (20), we obtain

$$I_0^\alpha \{w^{\mu-1} M_k(a, c; \beta w); s\} = \frac{1}{\Gamma(\alpha)} \int_0^s (s-w)^{\alpha-1} w^{\mu-1} M_k(a, c; \beta w) dw.$$

Applying the series expansion (10) of the k -Kummer function, and observing that the series converges absolutely and uniformly on the compact interval $[0, s]$, term-by-term integration is justified. Hence,

$$I_0^\alpha \{w^{\mu-1} M_k(a, c; \beta w); s\} = \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} \frac{\beta^n}{n!} \int_0^s (s-w)^{\alpha-1} w^{n+\mu-1} dw.$$

Applying the change in variables $w = su$, we find

$$\begin{aligned} \int_0^s (s-w)^{\alpha-1} w^{n+\mu-1} dw &= s^{n+\mu+\alpha-1} \int_0^1 (1-u)^{\alpha-1} u^{n+\mu-1} du \\ &= s^{n+\mu+\alpha-1} B(n+\mu, \alpha). \end{aligned}$$

Using the relation between the beta and gamma functions and subsequently the relation (5), we arrive at (75). $\square$

5. Conclusions

Several structural properties of the k -Kummer hypergeometric function $M_k(a, c; w)$ were investigated. A Kummer-type transformation formula, derivative identities, contiguous relations, and various addition and multiplication formulas were established by using Taylor-type expansions. The behavior of $M_k(a, c; w)$ under several classical integral transforms, including the Laplace, Mellin, Stieltjes, Sumudu, and Riemann–Liouville fractional integral transforms, was also investigated. Compared with the existing literature on the classical Kummer function and its k -generalization [1,17,18], the present work provides a unified collection of structural identities together with new integral transform formulas, thereby extending several known results to the k -generalized setting. The formulas derived here may be useful in future studies concerning asymptotic analysis, generalized differential equations, fractional calculus, and applications of k -special functions in mathematical physics and diffusion-type models. For the convenience of the reader, the main identities are summarized in Appendix A.

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Appendix A. Tables of Identities for the k -Kummer Function

Table A1. Contiguous relations for the k -Kummer function.

$M_k(a + k, c; w)$	$M_k(a - k, c; w)$	$M_k(a, c + k; w)$	$M_k(a, c - k; w)$
$\frac{a}{a + kw}$		$\frac{kw(c - a)}{c(a + kw)}$	
$\frac{a}{a - c + k}$			$\frac{c - k}{c - a - k}$
		$\frac{kw(c - a)}{c[c + k(w - 1)]}$	$\frac{c - k}{c + k(w - 1)}$
$\frac{a}{2a - c + kw}$	$\frac{c - a}{c - 2a - kw}$		
$\frac{a - c}{a + k(w - 1)}$			$\frac{c - k}{a + k(w - 1)}$
	1	$\frac{kw}{c}$	

Table A2. Derivative identities associated with the k -Kummer function.

$f(w)$	$\frac{d^n}{dw^n} \{f(w)\}$
$M_k(a, c; w)$	$\frac{(a)_{n,k}}{(c)_{n,k}} M_k(a + nk, c + nk; w)$
$w^{\frac{a}{k} + n - 1} M_k(a, c; w)$	$\frac{(a)_{n,k}}{k^n} w^{\frac{a}{k} - 1} M_k(a + nk, c; w)$
$w^{\frac{c}{k} - 1} M_k(a, c; w)$	$\frac{(-1)^n}{k^n} (k - c)_{n,k} w^{\frac{c}{k} - n - 1} M_k(a, c - nk; w)$
$e^{-w} M_k(a, c; w)$	$(-1)^n \frac{(c - a)_{n,k}}{(c)_{n,k}} M_k(c - a + nk, c + nk; -w)$
$e^{-w} w^{\frac{c - a}{k} + n - 1} M_k(a, c; w)$	$\frac{(c - a)_{n,k}}{k^n} e^{-w} w^{\frac{c - a}{k} - 1} M_k(a - nk, c; w)$
$e^{-w} w^{\frac{c}{k} - 1} M_k(a, c; w)$	$\frac{(-1)^n}{k^n} (k - c)_{n,k} e^{-w} w^{\frac{c}{k} - n - 1} M_k(a - nk, c - nk; w)$

Table A3. Addition formulas for k -Kummer function.

$f(w)$	$M_k(a, c; w + z)$
$M_k(a, c; w)$	$\sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(c)_{n,k}} M_k(a + nk, c + nk; w) \frac{z^n}{n!}$
$w^{\frac{a}{k} - 1} M_k(a, c; w)$	$\left(\frac{w}{w + z}\right)^{\frac{a}{k}} \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{k^n n!} M_k(a + nk, c; w) \left(\frac{z}{w + z}\right)^n$
$w^{\frac{c}{k} - 1} M_k(a, c; w)$	$\left(\frac{w}{w + z}\right)^{\frac{c}{k} - 1} \sum_{n=0}^{\infty} \frac{(k - c)_{n,k}}{k^n n!} M_k(a, c - nk; w) \left(-\frac{z}{w}\right)^n$
$e^{-w} M_k(a, c; w)$	$e^z \sum_{n=0}^{\infty} \frac{(c - a)_{n,k}}{(c)_{n,k}} M_k(a, c + nk; w) \frac{(-z)^n}{n!}$
$e^{-w} w^{\frac{c}{k} - 1} M_k(a, c; w)$	$e^z \left(\frac{w}{w + z}\right)^{\frac{c}{k} - 1} \sum_{n=0}^{\infty} \frac{(k - c)_{n,k}}{k^n n!} M_k(a - nk, c - nk; w) \left(-\frac{z}{w}\right)^n$
$e^{-w} w^{\frac{c - a}{k} - 1} M_k(a, c; w)$	$\left(\frac{w}{w + z}\right)^{\frac{c - a}{k}} e^z \sum_{n=0}^{\infty} \frac{(c - a)_{n,k}}{k^n n!} M_k(a - nk, c; w) \left(\frac{z}{w + z}\right)^n$

Table A4. Multiplication formulas for k -Kummer function.

$f(w)$	$M_k(a, c; wz)$
$M_k(a, c; w)$	$\sum_{n=0}^{\infty} \frac{((z-1)w)^n}{n!} \frac{(a)_{n,k}}{(c)_{n,k}} M_k(a + nk, c + nk; w)$
$w^{\frac{a}{k}-1} M_k(a, c; w)$	$z^{-\frac{a}{k}} \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{n!} \left(\frac{z-1}{kz}\right)^n M_k(a + nk, c; w)$
$w^{\frac{c}{k}-1} M_k(a, c; w)$	$z^{1-\frac{c}{k}} \sum_{n=0}^{\infty} \frac{(k-c)_{n,k}}{n!} \left(\frac{1-z}{k}\right)^n M_k(a, c - nk; w)$
$e^{-w} M_k(a, c; w)$	$e^{w(z-1)} \sum_{n=0}^{\infty} \frac{(c-a)_{n,k}}{(c)_{n,k}} \frac{(1-z)^n w^n}{n!} M_k(a, c + nk; w)$
$e^{-w} w^{\frac{c-a}{k}-1} M_k(a, c; w)$	$e^{w(z-1)} z^{\frac{a-c}{k}} \sum_{n=0}^{\infty} \frac{(c-a)_{n,k}}{n!} \left(\frac{z-1}{z}\right)^n M_k(a - nk, c; w)$
$e^{-w} w^{\frac{c}{k}-1} M_k(a, c; w)$	$e^{w(z-1)} z^{1-\frac{c}{k}} \sum_{n=0}^{\infty} \frac{(k-c)_{n,k}}{n!} \left(\frac{(1-z)w}{kz}\right)^n M_k(a - nk, c - nk; w)$

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