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New Interpretations for the Riemann Curvature and the Landsberg Property in Homogeneous Finsler Geometry

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Abstract

In this paper, we provide a new Riemann curvature formula in homogeneous Finsler geometry. Meanwhile, we prove that a homogeneous Finsler metric is Landsberg if and only if its connection operator induces Killing vector fields for a Hessian metric. As an application, we prove that any homogeneous Landsberg sphere with dimension bigger than 1 and constant flag curvature must be Riemannian.

Keywords: homogeneous Finsler manifold; connection operator; constant flag curvature; Landsberg curvature; Riemann curvature; spray vector field

MSC: 22E46; 53C30

1. Introduction

In Finsler geometry, the flag curvature and the Landsberg curvature have been extensively studied. They are related to two open problems, classifying Finsler space forms (i.e., Finsler metrics of constant flag curvature; see [1–3]), and finding unicorns (i.e., Finsler metrics satisfying the Landsberg property but not the Berwald property; see [4–6] and the references therein), respectively.

In Homogeneous Finsler geometry, flag curvature and Landsberg curvature can be algebraically presented, using the spray vector field η and connection operator N introduced by L. Huang (see [7] or Theorem 4 below). In [8], the author provides new proofs, and generalizes some homogeneous curvature formulas to homogeneous spray geometry.

The geometric meaning of the homogeneous curvature formulas in Finsler geometry needs to be explored. In this paper, we explore the geometric meaning of the flag curvature formula and the Landsberg curvature formula in homogeneous Finsler geometry. Before introducing our main theorems, some conventions and notations need to be explained.

For a homogeneous Finsler manifold $(G/H, F)$ with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, we use the same F to denote the Minkowski norm $F(\cdot) = F(o, \cdot)$ on the tangent space $\mathfrak{m} = T_o(G/H)$ at $o = eH \in G/H$, and use g_y to denote the Hessian metric on $\mathfrak{m}\backslash\{0\}$ induced by the Minkowski norm F. For any $v \in \mathfrak{m}$, the connection operator and the curvature operator defines two tangent fields on $\mathfrak{m}\backslash\{0\}$, $N^v(y) = N(y, v)$ and $R^v(y) = R_y(v)$, respectively. In the discussion, we denote by $[\cdot,\cdot]_{\mathfrak{g}}$ the bracket in $\mathfrak{g}$, by $[\cdot,\cdot]_{\mathfrak{h}}$ and $[\cdot,\cdot]_{\mathfrak{m}}$ the projection of $[\cdot,\cdot]_{\mathfrak{g}}$ to its $\mathfrak{h}$- and $\mathfrak{m}$-summand in the given reductive decomposition, and by $[\cdot,\cdot]$ without any subscription the canonical bracket between two smooth tangent fields on $\mathfrak{m}\backslash\{0\}$.

First, we prove the following theorem.

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Theorem 1. *Let $(G/H, F)$ be a homogeneous Finsler manifold with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, and η and N be the corresponding spray vector field and connection operator, respectively. For any $v \in \mathfrak{m}$, $N^v(y) = N(y, v)$ and $R^v(y) = R_y(v)$ are viewed as smooth tangent fields on $\mathfrak{m}\backslash\{0\}$. Then, we have*

$$[N^u, N^v] + N^{[u,v]_\mathfrak{m}} = [y, [u, v]_\mathfrak{h}]_\mathfrak{m} + \tfrac{1}{3}(D_u R^v - D_v R^u),$$

where $D_u R^v : \mathfrak{m}\backslash\{0\} \to \mathfrak{m}$ is determined by $D_u R^v(y) = \frac{\mathrm{d}}{\mathrm{d}t} R_{y+tu}(v)|_{t=0}$.

Theorem 1 provides a new explanation for the homogeneous flag curvature formula. In fact, Theorem 1 is valid in the context of homogeneous spray geometry, which generalizes Theorem 1.6 in [9].

Next, we prove the following theorem.

Theorem 2. *Let $(G/H, F)$ be a homogeneous Finsler manifold with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, and N be the corresponding connection operator. For any $v \in \mathfrak{m}$, $N^v(y) = N(y, v)$ is viewed as a smooth tangent field on $\mathfrak{m}\backslash\{0\}$. Then, F is a Landsberg metric if and only if for any $v \in \mathfrak{m}$, N^v is a Killing vector field for the Hessian metric g_y on $\mathfrak{m}\backslash\{0\}$.*

Theorem 2 provides a new explanation for the Landsberg property in homogeneous Finsler geometry.

Finally, we provide the following application of Theorems 1 and 2.

Theorem 3. *Any homogeneous Landsberg sphere (S^n, F) with dimension $n > 1$ and flag curvature $K \equiv 1$ must be Riemannian.*

This paper is structured as follows. In Section 2, we connect some necessary knowledge in homogeneous Finsler geometry. In Section 3, we prove the three main theorems.

2. Preliminaries in General and Homogeneous Finsler Geometry

2.1. Minkowski Norm, Hessian Metric, and Finsler Metric

A *Minkowski norm* on a finite dimensional real vector space $\mathbf{V}$ is a continuous map $F : \mathbf{V} \to \mathbb{R}$ satisfying the following properties:

1. F is a positive and smooth function when restricted to $\mathbf{V}\backslash\{0\}$;
2. For any $\lambda > 0$ and $y \in \mathbf{V}$, $F(\lambda y) = \lambda F(y)$;
3. For any $y \in \mathbf{V}\backslash\{0\}$, the *fundamental tensor*

$$g_y(w_1, w_2) = \tfrac{\partial^2}{\partial t_1 \partial t_2}|_{t_1=t_2=0} F^2(y + t_1 w_1 + t_2 w_2), \forall w_1, w_2 \in \mathbf{V}$$

is positive definite.

The fundamental tensor g_y defines a Riemannian metric on $\mathbf{V}\backslash\{0\}$, which is called the *Hessian metric* of F.

A Finsler metric on a smooth manifold M is a continuous function $F : TM \to \mathbb{R}$ satisfying the following properties:

1. F is a smooth function when restricted to the slit tangent bundle $TM\backslash 0$;
2. For each $x \in M$, $F(x, \cdot) = F|_{T_x M}$ is Minkowski norm on $T_x M$.

We call the pair (M, F) a *Finsler manifold*.

A Minkowski norm is *Euclidean* when the fundamental tensor g_y is independent of y. A Finsler metric is *Riemannian* when it is Euclidean and when restricted to each tangent space.

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The Cartan tensor for the Minkowski norm F on $\mathbf{V}$ is determined by the symmetric multi-linear functionals $C_y : \mathbf{V} \times \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ for any $y \in \mathbf{V}$, i.e.,

$$C_y(w_1, w_2, w_3) = \frac{1}{4} \frac{\partial^3}{\partial t_1 \partial t_2 \partial t_3} F^2(y + t_1 w_1 + t_2 w_2 + t_3 w_3), \quad \forall w_1, w_2, w_3 \in \mathbf{V}.$$

It is obvious that the Minkowski norm F is Euclidean if and only if its Cartan tensor vanishes everywhere. Since C_y is positive -1-homogeneous, which implies $C_y(y, \cdot, \cdot) \equiv 0$, we have the following obvious fact:

Lemma 1. *A Minkowski norm F on $\mathbf{V}$ is Euclidean if and only if $C_y(v, v, v) = 0$ for any $y \in \mathbf{V}\backslash\{0\}$ and $v \in \mathbf{V}$ satisfying $g_y(y, v) = 0$.*

In Theorem 4 below, we will meet another type of Cartan tensor, which involves more differentiations. For simplicity, we will also call

$$C_y(w_1, w_2, w_3, w_4) = \frac{1}{4} \frac{\partial^4}{\partial t_1 \partial t_2 \partial t_3 \partial t_4} F^2(y + t_1 w_1 + t_2 w_2 + t_3 w_3 + t_4 w_4),$$
$$\forall y \in \mathbf{V}\backslash\{0\}, w_1, w_2, w_3, w_4 \in \mathbf{V}$$

the *Cartan tensor* of the Minkowski norm F.

Both Cartan tensors can be immediately transported to Finsler manifolds, which help us detect if a Finsler manifold is Riemannian or if it can be isometrically embedded in a Minkowski manifold. See [10] for more details.

2.2. Homogeneous Finsler Manifold and Reductive Decomposition

A connected Finsler manifold (M, F) is called *homogeneous* if its connected isometry group $I_0(M, F)$ acts transitively on M [11]. In this situation, we can present M as a smooth coset space G/H, in which G is a Lie subgroup of $I_0(M, F)$ which acts transitively on M, and H is the isotropy subgroup at the origin $o = eH \in G/H$.

Since the subgroup H here is a compactly embedded subgroup in G, we can find a *reductive decomposition* $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ for the homogeneous manifold $(G/H, F)$, in which $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{h} = \mathrm{Lie}(H)$, and the reductiveness means that this decomposition is an $\mathrm{Ad}(H)$-invariant. The tangent space $T_o(G/H)$ can be canonically identified with $\mathfrak{m}$ such that the isotropy H-action on $T_o(G/H)$ coincides with the $\mathrm{Ad}(H)$-action on $\mathfrak{m}$. Then, the homogeneous (i.e., G-invariant) Finsler metric F on G/H corresponds one-to-one to an arbitrary $\mathrm{Ad}(H)$-invariant Minkowski norm on $\mathfrak{m}$.

In our discussion below, $[\cdot, \cdot]_\mathfrak{g}$ means the bracket in $\mathfrak{g}$, $[\cdot, \cdot]_\mathfrak{h}$ and $[\cdot, \cdot]_\mathfrak{m}$ mean the projections of $[\cdot, \cdot]_\mathfrak{g}$ to the $\mathfrak{h}$ and $\mathfrak{m}$, respectively, with respect to the given reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. So, the reductiveness of this decomposition implies $[\mathfrak{h}, \mathfrak{m}]_\mathfrak{g} = [\mathfrak{h}, \mathfrak{m}]_\mathfrak{m} \subset \mathfrak{m}$ in the Lie algebra level. Meanwhile, we will use $[\cdot, \cdot]$, without any subscript, to denote the canonical bracket between two smooth tangent fields.

2.3. Spray Vector Field, Connection Operator, and Homogeneous Curvature Formula

For a homogeneous Finsler manifold $(G/H, F)$ with a reductive decomposition, $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, L. Huang defines its *spray vector field* $\eta : \mathfrak{m}\backslash\{0\} \to \mathfrak{m}$ and *connection operator* $N : (\mathfrak{m}\backslash\{0\}) \times \mathfrak{m} \to \mathfrak{m}$ by

$$g_y(\eta(y), u) = g_y(y, [u, y]_\mathfrak{m}), \quad \forall y \in \mathfrak{m}\backslash\{0\}, u \in \mathfrak{m}, \quad \text{and}$$
$$2g_y(N(y, v), u) = g_y([u, v]_\mathfrak{m}, y) + g_y([u, y]_\mathfrak{m}, v) + g_y([v, y]_\mathfrak{m}, u) - 2C_y(u, v, \eta(y)),$$
$$\forall y \in \mathfrak{m}\backslash\{0\}, u, v \in \mathfrak{m},$$

respectively. In [12], an equivalent description for the connection operator is given, i.e.,

$$N(y,v) = \tfrac{1}{2}D_v\eta(y) - \tfrac{1}{2}[y,v]_m, \quad \forall y \in \mathfrak{m}\backslash\{0\}, v \in \mathfrak{m}.$$

in which $D_v\eta(y) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\eta(y+tv)$.

Using η and N, the curvatures of $(G/H, F)$ can be explicitly presented by homogeneous curvature formulas. In later discussion, we will concern the Riemann curvature and Landsberg curvature. The corresponding homogeneous curvature formulas are as follows (see Proposition 4.6, Theorem 4.8 in [7], and Theorem F, Theorem 6.6 in [8]).

Theorem 4. *Let $(G/H, F)$ be a homogeneous Finsler manifold endowed with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, and denote by $\eta : \mathfrak{m}\backslash\{0\} \to \mathfrak{m}$ and $N : (\mathfrak{m}\backslash\{0\}) \times \mathfrak{m} \to \mathfrak{m}$ the corresponding spray vector field and the connection operator, respectively. Then, for any $y \in \mathfrak{m}\backslash\{0\} = T_o(G/H)$, the Riemann curvature operator $R_y : \mathfrak{m} \to \mathfrak{m}$ satisfies*

$$R_y(v) = [y,[v,y]_\mathfrak{h}]_\mathfrak{m} + D_\eta N(y,v) - N(y,N(y,v)) + N(y,[y,v]_\mathfrak{m}) - [y,N(y,v)]_\mathfrak{m},$$

in which $D_\eta N(y,v) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}N(y+t\eta(y),v)$, and the Landsberg curvature $L_y : \mathfrak{m}\times\mathfrak{m}\times\mathfrak{m} \to \mathfrak{m}$ satisfies

$$L_y(v,v,v) = 3C_y(v,v,[v,y]_\mathfrak{m} - N(y,v)) - C_y(v,v,v,\eta(y)).$$

Any smooth map $V : \mathfrak{m}\backslash\{0\} \to \mathfrak{m}$ can be naturally viewed as a smooth tangent field $y \in \mathfrak{m}\backslash\{0\} \mapsto \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}y + tV(y)$. More explicitly, using a real linear basis $\{e_1, \cdots, e_n\}$ and the corresponding coordinate $(y^1, \cdots, y^n)$ for $y = y^i e_i$, $V = V^i(y)e_i$ represents the tangent field $V^i(y)\partial_{y^i}$ on $\mathfrak{m}\backslash\{0\}$. So, for the smooth tangent fields represented by $U, V : \mathfrak{m}\backslash\{0\} \to \mathfrak{m}$, their bracket $[U,V]$ is given by $[U,V] = D_UV - D_VU$, in which $(D_UV)(y) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}V(y+tU(y))$, $\forall y \in \mathfrak{m}\backslash\{0\}$.

For example, the spray vector field η can also be viewed as a tangent field on $\mathfrak{m}\backslash\{0\}$. Any vector $v \in \mathfrak{m}$ can be viewed as a parallel tangent field on $\mathfrak{m}\backslash\{0\}$. For each vector $v \in \mathfrak{m}$, the connection operator N defines the tangent field $N^v(y) = N(y,v)$ on $\mathfrak{m}\backslash\{0\}$, and the Riemann curvature operator R defines the tangent field $R^v(y) = R_y(v)$.

The Minkowski space is a trivial example, which can be viewed as a connected Abelian group endowed with a left invariant metric. Its flag curvature, as well as its spray vector field and connection operator, vanish everywhere.

3. Proof of the Main Theorems

3.1. Proof of Theorem 1

Let $(G/H, F)$ be a homogeneous Finsler manifold with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Denote by η and N the corresponding spray vector field and connection operator, respectively. Theorem 4 indicates that R^v can be decomposed as the sum $R^v = R_1^v + R_2^v$ for each $v \in \mathfrak{m}$, where

$$\begin{aligned} &R_1^v(y) = [y,[v,y]_\mathfrak{h}]_m, \quad \text{and} \\ &R_2^v(y) = (D_\eta N^v)(y) - N(y,N(y,v)) + N(y,[y,v]_\mathfrak{m}) - [y,N(y,v)]_\mathfrak{m}, \\ &\forall y \in \mathfrak{m}\backslash\{0\}. \end{aligned}$$

Lemma 2. *For any $y \in \mathfrak{m}\backslash\{0\}$ and $v \in \mathfrak{m}$, we have*

$$\begin{aligned} R_2^v(y) \quad = \quad & \tfrac{1}{2}(D_\eta D_v\eta)(y) - \tfrac{1}{2}[\eta(y),v]_\mathfrak{m} - \tfrac{1}{4}(D_{D_v\eta}\eta)(y) + \tfrac{3}{4}(D_{[y,v]_\mathfrak{m}}\eta)(y) \\ & -\tfrac{1}{4}[y,(D_v\eta)(y)]_\mathfrak{m} - \tfrac{1}{4}[y,[y,v]_\mathfrak{m}]_\mathfrak{m}. \end{aligned}$$

Proof. Direct calculation shows that

$$\begin{aligned}
R_2^v(y) &= R_y(v)\\
&= (D_\eta N^v)(y) - N(y, N(y,v)) + N(y,[y,v]_{\mathfrak{m}}) - [y, N(y,v)]_m\\
&= (D_\eta(\tfrac{1}{2}D_v\eta - \tfrac{1}{2}[y,v]_{\mathfrak{m}}))(y) - (\tfrac{1}{2}(D_{\frac{1}{2}D_v\eta - \frac{1}{2}[y,v]_{\mathfrak{m}}}\eta)(y) - \tfrac{1}{2}[y, \tfrac{1}{2}(D_v\eta)(y)\\
&\quad -\tfrac{1}{2}[y,v]_m]_m) + (D_{[y,v]_{\mathfrak{m}}}\eta)(y) - \tfrac{1}{2}[y,[y,v]_{\mathfrak{m}}]_{\mathfrak{m}} - \tfrac{1}{2}[y,(D_v\eta)(y)]_m + \tfrac{1}{2}[y,[y,v]_{\mathfrak{m}}]_{\mathfrak{m}}\\
&= \tfrac{1}{2}(D_\eta D_v\eta)(y) - \tfrac{1}{2}[\eta(y),v]_{\mathfrak{m}} - \tfrac{1}{4}(D_{D_v\eta}\eta)(y) + \tfrac{1}{4}(D_{[y,v]_{\mathfrak{m}}}\eta)(y) + \frac{1}{4}[y,(D_v\eta)(y)]_{\mathfrak{m}}\\
&\quad -\tfrac{1}{4}[y,[y,v]_{\mathfrak{m}}]_{\mathfrak{m}} + \tfrac{1}{2}(D_{[y,v]_{\mathfrak{m}}}\eta)(y) - \tfrac{1}{2}[y,[y,v]_{\mathfrak{m}}]_{\mathfrak{m}} - \tfrac{1}{2}[y,(D_v\eta)(y)]_{\mathfrak{m}} + \tfrac{1}{2}[y,[y,v]_{\mathfrak{m}}]_{\mathfrak{m}}\\
&= \tfrac{1}{2}(D_\eta D_v\eta)(y) - \tfrac{1}{2}[\eta(y),v]_{\mathfrak{m}} - \tfrac{1}{4}(D_{D_v\eta}\eta)(y) + \tfrac{3}{4}(D_{[y,v]_{\mathfrak{m}}}\eta)(y)\\
&\quad -\tfrac{1}{4}[y,(D_v\eta)(y)]_{\mathfrak{m}} - \tfrac{1}{4}[y,[y,v]_{\mathfrak{m}}]_{\mathfrak{m}},
\end{aligned}$$

which ends the proof of Lemma 2. □

Proof. Now, we start to prove Theorem 1. Lemma 2 implies that for any $y \in \mathfrak{m}\backslash\{0\}$,

$$\begin{aligned}
(D_{D_u\eta}\eta)(y) &= 2(D_\eta D_u\eta)(y) + 3(D_{[y,u]_{\mathfrak{m}}}\eta)(y) - [y,[y,u]_{\mathfrak{m}}]_{\mathfrak{m}} - 2[\eta(y),u]_{\mathfrak{m}}\\
&\quad -[y,(D_u\eta)(y)]_{\mathfrak{m}} - 4R_2^u(y).
\end{aligned}$$

So, at y, we have

$$\begin{aligned}
D_{D_u\eta}D_v\eta &= D_vD_{D_u\eta}\eta + D_{[D_u\eta,v]}\eta\\
&= D_vD_{D_u\eta}\eta - D_{D_vD_u\eta}\eta\\
&= D_v(2D_\eta D_u\eta + 3D_{[y,u]_{\mathfrak{m}}}\eta - [y,[y,u]_{\mathfrak{m}}]_{\mathfrak{m}} - 2[\eta(y),u]_{\mathfrak{m}} - [y,(D_u\eta)(y)]_{\mathfrak{m}} - 4R_2^u)\\
&\quad -D_{D_vD_u\eta}\eta\\
&= 2D_vD_\eta D_u\eta + 3D_vD_{[y,u]_{\mathfrak{m}}}\eta - [y,[v,u]_{\mathfrak{m}}]_{\mathfrak{m}} - [v,[y,u]_{\mathfrak{m}}]_m - 2[(D_v\eta)(y),u]_{\mathfrak{m}}\\
&\quad -[v,(D_u\eta)(y)]_{\mathfrak{m}} - 4D_vR_2^u - [y,(D_vD_u\eta)(y)]_m - D_{D_vD_u\eta}\eta. \qquad (1)
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
D_{D_v\eta}D_u\eta &= 2D_{D_v\eta}D_u\eta + 3D_uD_{[y,v]_{\mathfrak{m}}}\eta - [y,[u,v]_{\mathfrak{m}}]_{\mathfrak{m}} - [u,[y,v]_{\mathfrak{m}}]_{\mathfrak{m}} - 2[(D_u\eta)(y),v]_{\mathfrak{m}}\\
&\quad -[u,(D_v\eta)(y)]_{\mathfrak{m}} - 4D_uR_2^v - [y,(D_uD_v\eta)(y)]_{\mathfrak{m}} - D_{D_uD_v\eta}\eta + 2D_\eta D_uD_v\eta. \qquad (2)
\end{aligned}$$

Since D_u and D_v commute, (1) minus (2) provides

$$\begin{aligned}
[D_u\eta, D_v\eta] &= D_{D_u\eta}D_v\eta - D_{D_v\eta}D_u\eta\\
&= D_vD_{[y,u]_{\mathfrak{m}}}\eta - D_uD_{[y,v]_{\mathfrak{m}}}\eta + \tfrac{2}{3}[y,[u,v]_{\mathfrak{m}}]_{\mathfrak{m}} + \tfrac{1}{3}[u,[y,v]_{\mathfrak{m}}]_{\mathfrak{m}} - \tfrac{1}{3}[v,[y,u]_{\mathfrak{m}}]_{\mathfrak{m}}\\
&\quad +[(D_u\eta)(y),v]_{\mathfrak{m}} - [(D_v\eta)(y),u]_{\mathfrak{m}} + \tfrac{4}{3}(D_uR_2^v - D_vR_2^u). \qquad (3)
\end{aligned}$$

Meanwhile, it is not hard to see that

$$\begin{aligned}
[D_u\eta,[y,v]_{\mathfrak{m}}] &= D_{D_u\eta}[y,v]_{\mathfrak{m}} - D_{[y,v]_{\mathfrak{m}}}D_u\eta\\
&= [(D_u\eta)(y),v]_m - D_{[y,v]_{\mathfrak{m}}}D_u\eta, \qquad (4)\\
[D_v\eta,[y,u]_{\mathfrak{m}}] &= D_{D_v\eta}[y,u]_{\mathfrak{m}} - D_{[y,u]_{\mathfrak{m}}}D_v\eta\\
&= [(D_v\eta)(y),u]_{\mathfrak{m}} - D_{[y,u]_{\mathfrak{m}}}D_v\eta, \qquad (5)\\
[[y,u]_{\mathfrak{m}},[y,v]_{\mathfrak{m}}] &= D_{[y,u]_{\mathfrak{m}}}[y,v]_{\mathfrak{m}} - D_{[y,v]_{\mathfrak{m}}}[y,u]_m\\
&= [[y,u]_{\mathfrak{m}},v] - [[y,v]_m,u]_m. \qquad (6)
\end{aligned}$$

Then, summarizing (3)–(6), we get

$$\begin{aligned} 4[N^u, N^v] &= [D_u\eta - [y,u]_m, D_v\eta - [y,v]_{\mathfrak{m}}] \\ &= [D_u\eta, D_v\eta] + [[y,u]_{\mathfrak{m}}, [y,v]_{\mathfrak{m}}] + [D_v\eta, [y,u]_{\mathfrak{m}}] - [D_u\eta, [y,v]_{\mathfrak{m}}] \\ &= (D_vD_{[y,u]_{\mathfrak{m}}} - D_{[y,u]_{\mathfrak{m}}}D_v)\eta - (D_uD_{[y,v]_{\mathfrak{m}}} - D_{[y,v]_{\mathfrak{m}}}D_u)\eta + \tfrac{2}{3}[y,[u,v]_{\mathfrak{m}}]_{\mathfrak{m}} \\ &\quad + \tfrac{4}{3}[u,[y,v]_{\mathfrak{m}}]_{\mathfrak{m}} - \tfrac{4}{3}[v,[y,u]_{\mathfrak{m}}]_{\mathfrak{m}} + \tfrac{4}{3}(D_uR_2^v - D_vR_2^u). \end{aligned} \tag{7}$$

Notice that when $v \in \mathfrak{g}$ is viewed as a parallel tangent field on $\mathfrak{g}\backslash\{0\}$, its bracket with the field $y \mapsto [y,u]$ coincides with the parallel field $y \mapsto [v,u]$, i.e, $[v,u] \in \mathfrak{g}$. So, we have $(D_vD_{[y,u]_{\mathfrak{m}}} - D_{[y,u]_{\mathfrak{m}}}D_v)\eta = D_{[v,u]}\eta$ and $(D_uD_{[y,v]_{\mathfrak{m}}} - D_{[y,v]_{\mathfrak{m}}}D_u)\eta = D_{[u,v]}\eta$, and then (7) can be continued as

$$\begin{aligned} 4[N^u, N^v] &= -2D_{[u,v]_{\mathfrak{m}}}\eta + \tfrac{2}{3}[y,[u,v]_{\mathfrak{m}}]_{\mathfrak{m}} + \tfrac{4}{3}[u,[y,v]_{\mathfrak{m}}]_{\mathfrak{m}} - \tfrac{4}{3}[v,[y,u]_{\mathfrak{m}}]_{\mathfrak{m}} \\ &\quad + \tfrac{4}{3}(D_uR_2^v - D_vR_2^u) \\ &= -4N^{[u,v]_{\mathfrak{m}}} + \tfrac{4}{3}([u,[y,v]_{\mathfrak{m}}]_{\mathfrak{m}} + [[y,u]_m,v]_m - [y,[u,v]_{\mathfrak{m}}]_{\mathfrak{m}}) \\ &\quad + \tfrac{4}{3}(D_uR_2^v - D_vR_2^u). \end{aligned} \tag{8}$$

Finally, we have

$$\begin{aligned} D_uR_1^v - D_vR_1^u &= [u,[v,y]_{\mathfrak{h}}]_{\mathfrak{m}} + [y,[v,u]_{\mathfrak{h}}]_{\mathfrak{m}} - [v,[u,y]_{\mathfrak{h}}]_{\mathfrak{m}} - [y,[u,v]_{\mathfrak{h}}]_{\mathfrak{m}} \\ &= [u,[v,y]_{\mathfrak{h}}]_{\mathfrak{m}} + [[u,y]_{\mathfrak{h}},v]_{\mathfrak{m}} - 2[y,[u,v]_{\mathfrak{h}}]_{\mathfrak{m}}, \end{aligned}$$

and by Jacobi equality,

$$\begin{aligned} [u,[y,v]_{\mathfrak{m}}]_{\mathfrak{m}} + [[y,u]_m,v]_m - [y,[u,v]_{\mathfrak{m}}]_{\mathfrak{m}} &= ([u,[y,v]] + [[y,u],v] - [y,[u,v]])_{\mathfrak{m}} \\ &\quad -([u,[y,v]_{\mathfrak{h}}]_{\mathfrak{m}} + [[y,u]_h,v]_m - [y,[u,v]_{\mathfrak{h}}]_{\mathfrak{m}}) \\ &= [u,[v,y]_{\mathfrak{h}}]_{\mathfrak{m}} + [[u,y]_h,v]_{\mathfrak{m}} + [y,[u,v]_{\mathfrak{h}}]_{\mathfrak{m}}. \end{aligned}$$

So, (8) provides

$$\begin{aligned} 4[N^u, N^v] &= -4N^{[u,v]_{\mathfrak{m}}} + 4[y,[u,v]_{\mathfrak{h}}]_{\mathfrak{m}} + \tfrac{4}{3}(D_uR_1^v - D_vR_1^u) + \tfrac{4}{3}(D_uR_2^v - D_vR_2^u) \\ &= -4N^{[u,v]_{\mathfrak{m}}} + 4[y,[u,v]_{\mathfrak{h}}]_{\mathfrak{m}} + \tfrac{4}{3}(D_uR^v - D_vR^u), \end{aligned}$$

which ends the proof of Theorem 1. □

3.2. Proof of Theorem 2

Proof. Assume that for any $v \in \mathfrak{m}$, N^v is a Killing vector field for the Hessian metric g_y on $\mathfrak{m}$. This property can be equivalently described as follows: for any $u, v \in \mathfrak{m}$, we have

$$C_y(N(y,v),u,u) = \tfrac{\mathrm{d}}{\mathrm{d}t}|_{t=0} g_{y+tN(y,v)}(u,u) = g_y([N^v,u](y),u) = -g_y((D_uN^v)(y),u). \tag{9}$$

On the other hand, we apply D_u to the defining equation for $N(y,v)$, i.e.,

$$2g_y(N(y,v),u) = g_y([u,v]_{\mathfrak{m}},y) + g_y([u,y]_{\mathfrak{m}},v) + g_y([v,y]_{\mathfrak{m}},u) - 2C_y(u,v,\eta(y));$$

then, we get

$$\begin{aligned} & 2g_y(D_uN^v(y),u) + 4C_y(N(y,v),u,u) \\ = \; & g_y([u,v]_{\mathfrak{m}},u) + 2C_y([u,y]_m,u,v) + 2C_y([v,y]_m,u,u) \\ & + g_y([v,u]_{\mathfrak{m}},u) - 2C_y(u,u,v,\eta(y)) - 2C_y(u,v,D_u\eta(y)). \end{aligned} \tag{10}$$

Inputting (9) and $N(y,u) = \frac{1}{2}D_u\eta(y) - \frac{1}{2}[y,u]_{\mathfrak{m}}$ into (10) and simplifying it, we get

$$\begin{aligned} L_y(u,u,v) &= C_y(u,u,[v,y]_{\mathfrak{m}} - N(y,v)) + 2C_y(u,v,[u,y]_{\mathfrak{m}} - N(y,u)) \\ &\quad -C_y(u,u,v,\eta(y)) = 0. \end{aligned} \tag{11}$$

Since L_y is symmetric, (11) is enough for us to argue that F is a Landsberg metric.

To summarize, the above argument proves Lemma Theorem 2 in one direction. Reversing the above argument, the other direction can be proved similarly. □

3.3. Proof of Theorem 3

Proof. Let $(G/H, F)$ be the homogeneous Finsler sphere indicated in the corollary, which is equipped with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Without loss of generality, we assume its flag curvature is constantly 1. Then, for any $y \in \mathfrak{m}\backslash\{0\}$ and $u, v \in \mathfrak{m}$, we have $R_y(u) = F(y)^2u - g_y(u,y)y$ and then,

$$\tfrac{1}{3}(D_uR^v - D_vR^u)(y) = g_y(u,y)v - g_y(v,y)u.$$

Since F is a Landsberg metric, for any $u, v \in \mathfrak{m}$, N^u, N^v, and $N^{[u,v]_{\mathfrak{m}}}$ are Killing vector fields on $(\mathfrak{m}\backslash\{0\})$. Meanwhile, $\mathrm{Ad}(H)$-actions preserve the Minkowski norm F on $\mathfrak{m}$, so they also preserve the corresponding Hessian metric g_y. This implies that $y \in \mathfrak{m}\backslash\{0\} \mapsto [[u,v]_{\mathfrak{h}}, y]_{\mathfrak{m}}$ is also a Killing vector field on $(\mathfrak{m}\backslash\{0\}, g_y)$. Using Theorem 1, we see that

$$y \in \mathfrak{m}\backslash\{0\} \mapsto \tfrac{1}{3}(D_uR^v - D_vR^u)(y) = g_y(u,y)v - g_y(v,y)u$$

is a Killing vector field on $(\mathfrak{m}\backslash\{0\}, g_y)$.

Taking any $w \in \mathfrak{m}$, the Killing property then provides

$$\begin{aligned} 2C_y(g_y(u,y)v - g_y(v,y)u, w, w) &= 2g_y([g_y(u,y)v - g_y(v,y)u, w](y), w) \\ &= -2g_y(g_y(u,w)v - g_y(v,w)u, w) \\ &= -2g_y(u,w)g_y(v,w) + 2g_y(v,w)g_y(u,w) \\ &= 0. \end{aligned} \tag{12}$$

In particular, when $u = y$ and $v = w$ satisfies $g_y(v,y) = 0$, we get $C_y(v,v,v) = 0$. By the symmetric and homogeneous properties of C_y, we see the Cartan tensor of F vanishes everywhere. So, F is Riemannian, which ends the proof. □

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