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Weighted Simpson-Type Quantum Integral Inequalities for h-Convex Functions

Tuncay Köroğlu [1,*], Muhammet Yazıcı [1], Bahadır Özgür Güler [1] and Abdul Wakil Baidar [2]

[1] Department of Mathematics, Faculty of Science, Karadeniz Technical University, 61080 Trabzon, Turkey; m.yazici@ktu.edu.tr (M.Y.); boguler@ktu.edu.tr (B.Ö.G.)
[2] Department of Mathematics, Kabul University, Kabul 1006, Afghanistan; baidarmath87@ku.edu.af
* Correspondence: tkor@ktu.edu.tr

Abstract

This paper establishes a weighted Simpson-type identity on the parameter domain associated with quantum integral operators. Using this identity together with Hölder's inequality and the power mean inequality, we derive new estimates for classes of functions whose associated parameter-domain q-derivatives satisfy h-convexity assumptions. Additional bounds are obtained under boundedness and Lipschitz conditions. Applications to the s-moment of a random variable and to several special means are derived in the classical limit $q \to 1^-$.

Keywords: Simpson-type inequality; h-convex function; quantum calculus; q-integral; weighted inequality; special means

MSC: 26D15; 26A51; 33D05; 26D10

1. Introduction

Derivative operators, particularly the q-derivative and the two-scale fractional derivative, have been extensively studied in the literature due to their practical applications. Two-scale theory provides a method for analyzing porous media and problems with non-uniform boundary conditions. Specific applications include fractal models for evaluating pressure drops in porous media under low fractal currents and computational models for two-scale population dynamics [1,2]. The mathematical basis for these analyses is quantum calculus. Introduced by Jackson, q-calculus is applied in quantum computing, number theory, combinatorics, and orthogonal polynomials. It is also utilized in non-commutative geometry and Lie algebras within quantum mechanics. The foundational principles of q-calculus are detailed in [3]. Furthermore, the theory of time scales calculus contains quantum calculus as a special case.

Recent studies investigate quantum integral inequalities. Alp and Sarıkaya [4] defined the q-integral using trapezoidal partitions and analyzed its properties. Kunt et al. [5] obtained quantum midpoint-type inequalities for double integrals via Hermite-Hadamard-type inequalities. Deng et al. [6] proved Simpson-type quantum integral inequalities for strictly preinvex functions. Promsakon et al. [7] generalized parameterized Simpson- and Newton-type inequalities. Siricharuanun et al. [8] established Simpson and Newton formulas for convex functions via quantum calculus operators. Additional quantum integral inequalities involving harmonically convex functions are studied in [9,10].

Classical Simpson-type formulas and their error estimates motivate a large literature on midpoint, trapezoid, and Simpson inequalities. In the classical setting, weighted

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Simpson-type inequalities for h-convex functions were studied by Matłoka [11]. In the quantum setting, several studies [6–8,12–22] investigate analogous estimates for convexity classes such as s-convex, (s,m)-convex, and preinvex functions. Several related contributions have further developed quantum integral inequalities for generalized convexity classes. Noor et al. obtained quantum estimates associated with Hermite–Hadamard inequalities and preinvex functions [23,24]. Zhang et al. studied quantum integral inequalities for (α,m)-convex functions [25]. Further estimates involving quasi-convexity, midpoint-type inequalities, and refinements of quantum Hermite–Hadamard-type inequalities were presented in [26–29]. Dynamic inequalities on arbitrary time scales were also considered in [30].

A function $f:[a,b]\to\mathbb{R}$ is convex [31] if for all $u,v\in[a,b]$ and $t\in[0,1]$,

$$f(tu+(1-t)v)\le tf(u)+(1-t)f(v).$$

Varošanec [32] defined h-convexity. Let $I\subset\mathbb{R}$ be an interval and $h:J\to\mathbb{R}$ where $(0,1)\subseteq J$ and $h\neq 0$. A function $f:I\to\mathbb{R}$ is h-convex if

$$f(tx+(1-t)y)\le h(t)f(x)+h(1-t)f(y)$$

for all $x,y\in I$ and $t\in(0,1)$. This class includes nonnegative convex functions, s-convex functions in the second sense, Godunova–Levin functions, and P-functions. For h-convex functions, Sarikaya et al. [33] proved the Hadamard-type inequality:

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le[f(a)+f(b)]\int_0^1 h(t)\,dt.$$

The aim of this paper is to establish weighted Simpson-type estimates compatible with affine reparametrizations on the parameter domain $[0,1]$. The core identity is first established in this parameter-domain setting and is then combined with h-convexity assumptions, Hölder's inequality, and the power mean inequality to derive new bounds. Further estimates are obtained under boundedness and Lipschitz assumptions. Applications to random variables and special means are stated through the classical limit of the main results.

The paper is organized as follows. Section 2 collects the required preliminaries from quantum calculus. Section 3 develops the parameter-domain weighted Simpson-type identity and establishes the corresponding inequalities for h-convex data. Section 4 derives additional estimates under boundedness and Lipschitz assumptions. Section 5 presents applications obtained from the classical limit $q\to 1^-$, including consequences for moments of random variables and special means. Section 6 contains concluding remarks.

2. Preliminaries

Throughout this section, let $[a,b]\subset\mathbb{R}$ and let $q\in(0,1)$ be fixed.

Definition 1 ([34]). *For a function $f:[a,b]\to\mathbb{R}$ and $x\in[a,b]$, the q-derivative of f at x is defined by*

$$ {}_aD_qf(x)=\frac{f(x)-f(qx+(1-q)a)}{(1-q)(x-a)},\quad x\neq a, \tag{1}$$

and ${}_aD_qf(a)=\lim_{x\to a}{}_aD_qf(x)$.

If $a = 0$ in (1), the operator reduces to ${}_0D_q f(x) = D_q f(x)$, defined as [3]

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}.$$

Lemma 1 ([35]). *For $a \in \mathbb{R}$ and $\alpha \in \mathbb{R}$,*

$${}_aD_q(x-a)^\alpha = \left(\frac{1-q^\alpha}{1-q}\right)(x-a)^{\alpha-1}. \tag{2}$$

Definition 2 ([34]). *The q-integral of a function $f : [a,b] \to \mathbb{R}$ on $[a,b]$ is defined for $x \in [a,b]$ as*

$$\int_a^x f(t)\,{}_ad_qt = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a). \tag{3}$$

If $a = 0$ in (3), then the integral reduces to

$$\int_0^x f(t)\,{}_0d_qt = (1-q)x\sum_{n=0}^{\infty} q^n f(q^n x)$$

for $x \in [0, +\infty)$. Furthermore, for $b \in (a,x)$,

$$\begin{aligned}\int_b^x f(t)\,{}_ad_qt &= \int_a^x f(t)\,{}_ad_qt - \int_a^b f(t)\,{}_ad_qt \\ &= (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) \\ &\quad - (1-q)(b-a)\sum_{n=0}^{\infty} q^n f(q^n b + (1-q^n)a).\end{aligned}$$

Theorem 1 ([36]). *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then,*

(i) ${}_aD_q \int_a^x f(t)\,{}_ad_qt = f(x)$;
(ii) $\int_c^x {}_aD_q f(t)\,{}_ad_qt = f(x) - f(c)$ *for* $c \in (a,x)$.

Theorem 2 ([36]). *Let $f, g : [a,b] \to \mathbb{R}$ be continuous functions and $\alpha \in \mathbb{R}$. Then, for $x \in [a,b]$,*

(i) $\int_a^x [f(t) + g(t)]\,{}_ad_qt = \int_a^x f(t)\,{}_ad_qt + \int_a^x g(t)\,{}_ad_qt$;
(ii) $\int_a^x (\alpha f)(t)\,{}_ad_qt = \alpha \int_a^x f(t)\,{}_ad_qt$;
(iii) $\int_c^x f(t)\,{}_aD_q g(t)\,{}_ad_qt = (fg)(t)\big|_c^x - \int_c^x {}_aD_q f(t)\,g(qt + (1-q)a)\,{}_ad_qt$ *for* $c \in (a,x)$.

Lemma 2 ([36]). *For $\alpha \in \mathbb{R} \setminus \{-1\}$,*

$$\int_a^x (t-a)^\alpha\,{}_ad_qt = \left(\frac{1-q}{1-q^{\alpha+1}}\right)(x-a)^{\alpha+1}. \tag{4}$$

3. Weighted Simpson-Type Inequalities for *h*-Convex Functions

In the sequel, whenever an affine reparametrization of the form

$$\ell_{ab}(t) = \frac{1-t}{2}a + \frac{1+t}{2}b, \qquad \ell_{ba}(t) = \frac{1-t}{2}b + \frac{1+t}{2}a, \qquad t \in [0,1],$$

is used inside a q-integral over t, the symbol ${}_0D_q f(\ell_{ab}(t))$ (and similarly ${}_0D_q f(\ell_{ba}(t))$) is understood as shorthand for the q-derivative of the corresponding composite function with respect to the parameter t. We also denote

$$\|w\|_{[a,b],\infty} := \sup_{x\in[a,b]} |w(x)|.$$

Lemma 3 (Affine substitution rule). *Let $g : [a,b] \to \mathbb{R}$ be quantum integrable and put $A = (a+b)/2$. Then,*

$$\int_0^1 g\Big(\frac{1-t}{2}a + \frac{1+t}{2}b\Big)\,{}_0d_qt = \frac{2}{b-a}\int_A^b g(x)\,{}_Ad_qx, \tag{5}$$

$$\int_0^1 g\Big(\frac{1-t}{2}b + \frac{1+t}{2}a\Big)\,{}_0d_qt = -\frac{2}{b-a}\int_A^a g(x)\,{}_Ad_qx. \tag{6}$$

Here the integrals with base point A are understood in the finite-interval q-integral sense of Definition 2. In particular, the oriented integral in (6) is interpreted as

$$\int_A^a g(x)\,{}_Ad_qx = (1-q)(a-A)\sum_{n=0}^{\infty} q^n g\big(q^n a + (1-q^n)A\big).$$

Proof. By the definition of the ${}_0d_q$-integral,

$$\int_0^1 g(\ell_{ab}(t))\,{}_0d_qt = (1-q)\sum_{n=0}^{\infty} q^n g\big(\ell_{ab}(q^n)\big).$$

On the other hand, by the definition of the ${}_Ad_q$-integral on $[A,b]$,

$$\int_A^b g(x)\,{}_Ad_qx = (1-q)\cdot\frac{b-a}{2}\sum_{n=0}^{\infty} q^n g(q^n b + (1-q^n)A).$$

Since

$$q^n b + (1-q^n)A = \frac{1-q^n}{2}a + \frac{1+q^n}{2}b = \ell_{ab}(q^n),$$

we obtain (5). Similarly,

$$\int_0^1 g(\ell_{ba}(t))\,{}_0d_qt = (1-q)\sum_{n=0}^{\infty} q^n g\big(A - q^n(A-a)\big),$$

whereas

$$\int_A^a g(x)\,{}_Ad_qx = (1-q)(a-A)\sum_{n=0}^{\infty} q^n g(q^n a + (1-q^n)A).$$

Since $a - A = -(b-a)/2$ and

$$q^n a + (1-q^n)A = A - q^n(A-a) = \frac{1-q^n}{2}b + \frac{1+q^n}{2}a = \ell_{ba}(q^n),$$

identity (6) follows. □

Lemma 4. *Let $f, w : [a,b] \to \mathbb{R}$ and $q \in (0,1)$. Define*

$$F_{ab}(t) := f\big(\ell_{ab}(t)\big), \qquad F_{ba}(t) := f\big(\ell_{ba}(t)\big),$$

$$W_{ab}(t) := w\big(\ell_{ab}(t)\big), \qquad W_{ba}(t) := w\big(\ell_{ba}(t)\big), \qquad t \in [0,1].$$

Suppose that $F_{ab}, F_{ba}, W_{ab}, W_{ba}, {}_0D_qF_{ab}, {}_0D_qF_{ba}$ are quantum integrable on $[0,1]$, and that w is symmetric with respect to $\frac{a+b}{2}$, i.e.,

$$w(a+b-x) = w(x) \qquad (x \in [a,b]).$$

Define

$$U_1(t) := \frac{1}{2}\int_0^t W_{ab}(s)\,{}_0d_qs - \frac{1}{3}\int_0^1 W_{ab}(s)\,{}_0d_qs,$$

$$U_2(t) := \frac{1}{2}\int_0^t W_{ba}(s)\,{}_0d_qs - \frac{1}{3}\int_0^1 W_{ba}(s)\,{}_0d_qs.$$

Then

$$\begin{aligned}&\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]\int_0^1 W_{ab}(s)\,{}_0d_qs\\&\quad-\frac{1}{2}\int_0^1 W_{ab}(t)F_{ab}(qt)\,{}_0d_qt-\frac{1}{2}\int_0^1 W_{ba}(t)F_{ba}(qt)\,{}_0d_qt\\&=\int_0^1 U_1(t)\,{}_0D_qF_{ab}(t)\,{}_0d_qt+\int_0^1 U_2(t)\,{}_0D_qF_{ba}(t)\,{}_0d_qt.\end{aligned}\tag{7}$$

Proof. Set

$$\mathcal{W} := \int_0^1 W_{ab}(s)\,{}_0d_qs.$$

By the fundamental theorem of quantum calculus,

$${}_0D_qU_1(t) = \frac{1}{2}W_{ab}(t), \qquad {}_0D_qU_2(t) = \frac{1}{2}W_{ba}(t).$$

Since $\ell_{ba}(t) = a+b-\ell_{ab}(t)$ and $w(a+b-x) = w(x)$, we have

$$W_{ba}(t) = W_{ab}(t) \qquad (t\in[0,1]).$$

Also,

$$U_1(0) = -\frac{1}{3}\mathcal{W}, \quad U_1(1) = \frac{1}{6}\mathcal{W}, \quad U_2(0) = -\frac{1}{3}\mathcal{W}, \quad U_2(1) = \frac{1}{6}\mathcal{W},$$

and

$$F_{ab}(0) = f\Big(\frac{a+b}{2}\Big), \quad F_{ab}(1) = f(b), \quad F_{ba}(0) = f\Big(\frac{a+b}{2}\Big), \quad F_{ba}(1) = f(a).$$

Applying the ${}_0q$-integration-by-parts formula

$$\int_0^1 g(t)\,{}_0D_qh(t)\,{}_0d_qt = g(1)h(1)-g(0)h(0)-\int_0^1 {}_0D_qg(t)\,h(qt)\,{}_0d_qt$$

with $g = U_1$, $h = F_{ab}$, we obtain

$$\int_0^1 U_1(t)\,{}_0D_qF_{ab}(t)\,{}_0d_qt = \frac{\mathcal{W}}{6}f(b)+\frac{\mathcal{W}}{3}f\Big(\frac{a+b}{2}\Big)-\frac{1}{2}\int_0^1 W_{ab}(t)F_{ab}(qt)\,{}_0d_qt.\tag{8}$$

Similarly, with $g = U_2$, $h = F_{ba}$,

$$\int_0^1 U_2(t)\,{}_0D_qF_{ba}(t)\,{}_0d_qt = \frac{\mathcal{W}}{6}f(a)+\frac{\mathcal{W}}{3}f\Big(\frac{a+b}{2}\Big)-\frac{1}{2}\int_0^1 W_{ba}(t)F_{ba}(qt)\,{}_0d_qt.\tag{9}$$

Adding (8) and (9) yields (7). □

For convenience, define the parameter-domain Simpson functional associated with Lemma 4 by

$$\begin{aligned}\mathcal{S}_q(f,w) := \Bigg|&\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]\int_0^1 W_{ab}(s)\,{}_0d_qs\\&-\frac{1}{2}\int_0^1 W_{ab}(t)F_{ab}(qt)\,{}_0d_qt-\frac{1}{2}\int_0^1 W_{ba}(t)F_{ba}(qt)\,{}_0d_qt\Bigg|.\end{aligned}\tag{10}$$

Remark 1. *Taking the limit $q \to 1^-$ in Lemma 4, we obtain*

$$\frac{1}{6(b-a)}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]\int_a^b w(x)\,dx - \frac{1}{b-a}\int_a^b w(x)f(x)\,dx$$
$$= \frac{b-a}{2}\int_0^1 \widetilde{U}_1(t)f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt - \frac{b-a}{2}\int_0^1 \widetilde{U}_2(t)f'\left(\frac{1-t}{2}b+\frac{1+t}{2}a\right)dt,$$

where

$$\widetilde{U}_1(t) = \frac{1}{2}\int_0^t w\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)ds - \frac{1}{3}\int_0^1 w\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)ds,$$
$$\widetilde{U}_2(t) = \frac{1}{2}\int_0^t w\left(\frac{1-s}{2}b+\frac{1+s}{2}a\right)ds - \frac{1}{3}\int_0^1 w\left(\frac{1-s}{2}b+\frac{1+s}{2}a\right)ds.$$

Before stating the following results, define the auxiliary functions

$$G_{ab}(t) := \left|{}_0D_qF_{ab}(t)\right|, \qquad G_{ba}(t) := \left|{}_0D_qF_{ba}(t)\right|, \qquad t \in [0,1].$$

In the estimates below, the h-convexity assumptions are imposed directly on these auxiliary functions or, when powers are involved, on their corresponding r-th powers. Thus, these are explicit hypotheses on auxiliary functions defined on the parameter interval $[0,1]$.

Theorem 3. *Let the assumptions of Lemma 4 hold. Assume that G_{ab} and G_{ba} are h-convex on $[0,1]$. Then*

$$\mathcal{S}_q(f,w) \le \frac{5}{6}\|w\|_{[a,b],\infty}\Bigg[G_{ab}(0)\int_0^1 h(1-t)\,{}_0d_qt + G_{ab}(1)\int_0^1 h(t)\,{}_0d_qt$$
$$+ G_{ba}(0)\int_0^1 h(1-t)\,{}_0d_qt + G_{ba}(1)\int_0^1 h(t)\,{}_0d_qt\Bigg]. \tag{11}$$

Proof. By Lemma 4 and the triangle inequality,

$$\mathcal{S}_q(f,w) \le \int_0^1 |U_1(t)|\,G_{ab}(t)\,{}_0d_qt + \int_0^1 |U_2(t)|\,G_{ba}(t)\,{}_0d_qt.$$

Also,

$$|U_1(t)| \le \frac{1}{2}\int_0^t |W_{ab}(s)|\,{}_0d_qs + \frac{1}{3}\int_0^1 |W_{ab}(s)|\,{}_0d_qs$$
$$\le \|w\|_{[a,b],\infty}\left(\frac{1}{2}t+\frac{1}{3}\right) \le \frac{5}{6}\|w\|_{[a,b],\infty},$$

and similarly,

$$|U_2(t)| \le \frac{5}{6}\|w\|_{[a,b],\infty}.$$

Since G_{ab} and G_{ba} are h-convex on $[0,1]$,

$$G_{ab}(t) \le h(1-t)G_{ab}(0) + h(t)G_{ab}(1),$$

$$G_{ba}(t) \le h(1-t)G_{ba}(0) + h(t)G_{ba}(1).$$

Substituting these bounds into the previous estimate gives (11). □

Corollary 1. *Under the assumptions of Theorem 3, if $h(t) = t$, then*

$$\mathcal{S}_q(f,w) \le \frac{5}{6}\|w\|_{[a,b],\infty}\left[\frac{q}{1+q}\left(G_{ab}(0)+G_{ba}(0)\right)+\frac{1}{1+q}\left(G_{ab}(1)+G_{ba}(1)\right)\right]. \tag{12}$$

Proof. If $h(t) = t$, then

$$\int_0^1 h(t)\,{}_0d_qt = \int_0^1 t\,{}_0d_qt = \frac{1}{1+q}, \qquad \int_0^1 h(1-t)\,{}_0d_qt = 1-\frac{1}{1+q} = \frac{q}{1+q}.$$

Now apply Theorem 3. □

Remark 2. *The kernel estimate appearing in the proof of Theorem 3 can be sharpened by computing*

$$\int_0^1 \left(\frac{1}{2}t+\frac{1}{3}\right){}_0d_qt = (1-q)\sum_{n=0}^{\infty} q^n\left(\frac{1}{2}q^n+\frac{1}{3}\right) = \frac{2q+5}{6(q+1)}. \tag{13}$$

For $p > 0$, define

$$\Lambda_p(q) := \int_0^1 \left(\frac{1}{2}t+\frac{1}{3}\right)^p {}_0d_qt = (1-q)\sum_{n=0}^{\infty} q^n\left(\frac{1}{2}q^n+\frac{1}{3}\right)^p. \tag{14}$$

The corresponding classical limit is

$$\lim_{q\to 1^-}\Lambda_p(q) = \int_0^1 \left(\frac{1}{2}t+\frac{1}{3}\right)^p dt = \frac{1}{p+1}\left(\frac{2\cdot 5^{p+1}-2^{p+2}}{6^{p+1}}\right). \tag{15}$$

Theorem 4. *Let the assumptions of Lemma 4 hold, and let $r > 1$ with $\frac{1}{p}+\frac{1}{r} = 1$. Set*

$$G_{ab,r}(t) := G_{ab}(t)^r, \qquad G_{ba,r}(t) := G_{ba}(t)^r, \qquad t \in [0,1].$$

Assume that $G_{ab,r}$ and $G_{ba,r}$ are h-convex on $[0,1]$. Then,

$$\begin{aligned}\mathcal{S}_q(f,w) \le{}& \|w\|_{[a,b],\infty}\Lambda_p(q)^{1/p}\\ &\times\left[\left(G_{ab,r}(0)\int_0^1 h(1-t)\,{}_0d_qt + G_{ab,r}(1)\int_0^1 h(t)\,{}_0d_qt\right)^{1/r}\right.\\ &\left.+\left(G_{ba,r}(0)\int_0^1 h(1-t)\,{}_0d_qt + G_{ba,r}(1)\int_0^1 h(t)\,{}_0d_qt\right)^{1/r}\right].\end{aligned} \tag{16}$$

Proof. By Lemma 4, Hölder's inequality, and the estimate

$$|U_1(t)|, |U_2(t)| \le \|w\|_{[a,b],\infty}\left(\frac{1}{2}t+\frac{1}{3}\right),$$

we obtain

$$\begin{aligned}\mathcal{S}_q(f,w) \le{}& \|w\|_{[a,b],\infty}\left(\int_0^1\left(\frac{1}{2}t+\frac{1}{3}\right)^p {}_0d_qt\right)^{1/p}\left(\int_0^1 G_{ab,r}(t)\,{}_0d_qt\right)^{1/r}\\ &+\|w\|_{[a,b],\infty}\left(\int_0^1\left(\frac{1}{2}t+\frac{1}{3}\right)^p {}_0d_qt\right)^{1/p}\left(\int_0^1 G_{ba,r}(t)\,{}_0d_qt\right)^{1/r}.\end{aligned}$$

Since $G_{ab,r}$ and $G_{ba,r}$ are h-convex,

$$G_{ab,r}(t) \le h(1-t)G_{ab,r}(0)+h(t)G_{ab,r}(1),$$

$$G_{ba,r}(t) \le h(1-t)G_{ba,r}(0) + h(t)G_{ba,r}(1).$$

Also, by (14),

$$\int_0^1 \left(\frac{1}{2}t + \frac{1}{3}\right)^p {}_0d_qt = \Lambda_p(q).$$

Substituting these bounds yields (16). □

Corollary 2. *Under the assumptions of Theorem 4, if $h(t) = t$, then*

$$\mathcal{S}_q(f,w) \le \|w\|_{[a,b],\infty}\,\Lambda_p(q)^{1/p} \times \left[\left(\frac{q}{1+q}G_{ab,r}(0) + \frac{1}{1+q}G_{ab,r}(1)\right)^{1/r} + \left(\frac{q}{1+q}G_{ba,r}(0) + \frac{1}{1+q}G_{ba,r}(1)\right)^{1/r}\right]. \tag{17}$$

Theorem 5. *Let the assumptions of Lemma 4 hold and let $r \ge 1$. With $G_{ab,r}(t) := G_{ab}(t)^r$ and $G_{ba,r}(t) := G_{ba}(t)^r$, assume that $G_{ab,r}$ and $G_{ba,r}$ are h-convex on $[0,1]$. Then,*

$$\mathcal{S}_q(f,w) \le \|w\|_{[a,b],\infty}\left(\frac{2q+5}{6(q+1)}\right)^{1-1/r}\Big[(G_{ab,r}(0)J_0 + G_{ab,r}(1)J_1)^{1/r} + (G_{ba,r}(0)J_0 + G_{ba,r}(1)J_1)^{1/r}\Big], \tag{18}$$

where

$$J_0 := \int_0^1 \left(\frac{1}{2}t + \frac{1}{3}\right)h(1-t)\,{}_0d_qt, \qquad J_1 := \int_0^1 \left(\frac{1}{2}t + \frac{1}{3}\right)h(t)\,{}_0d_qt.$$

Proof. By Lemma 4, the weighted power-mean inequality, and the bound

$$|U_1(t)|, |U_2(t)| \le \|w\|_{[a,b],\infty}\left(\frac{1}{2}t + \frac{1}{3}\right),$$

we obtain

$$\mathcal{S}_q(f,w) \le \|w\|_{[a,b],\infty}\left(\int_0^1 \left(\frac{1}{2}t + \frac{1}{3}\right){}_0d_qt\right)^{1-1/r}\left(\int_0^1 \left(\frac{1}{2}t + \frac{1}{3}\right)G_{ab,r}(t){}_0d_qt\right)^{1/r} + \|w\|_{[a,b],\infty}\left(\int_0^1 \left(\frac{1}{2}t + \frac{1}{3}\right){}_0d_qt\right)^{1-1/r}\left(\int_0^1 \left(\frac{1}{2}t + \frac{1}{3}\right)G_{ba,r}(t){}_0d_qt\right)^{1/r}.$$

Since $G_{ab,r}$ and $G_{ba,r}$ are h-convex,

$$\left(\frac{1}{2}t + \frac{1}{3}\right)G_{ab,r}(t) \le \left(\frac{1}{2}t + \frac{1}{3}\right)\big(h(1-t)G_{ab,r}(0) + h(t)G_{ab,r}(1)\big),$$

$$\left(\frac{1}{2}t + \frac{1}{3}\right)G_{ba,r}(t) \le \left(\frac{1}{2}t + \frac{1}{3}\right)\big(h(1-t)G_{ba,r}(0) + h(t)G_{ba,r}(1)\big).$$

Using (13), we obtain (18). □

Corollary 3. *Under the assumptions of Theorem 5, if $h(t) = t$, then*

$$J_0 = \int_0^1 \left(\frac{1}{2}t + \frac{1}{3}\right)(1-t)\,{}_0d_qt = \frac{q(2q^2+5q+2)}{6(1+q)(1+q+q^2)},$$

$$J_1 = \int_0^1 \left(\frac{1}{2}t + \frac{1}{3}\right)t\,{}_0d_qt = \frac{2q^2+5q+5}{6(1+q)(1+q+q^2)},$$

and

$$\mathcal{S}_q(f,w) \le \|w\|_{[a,b],\infty}\left(\frac{2q+5}{6(q+1)}\right)^{1-1/r}\Big[(G_{ab,r}(0)J_0 + G_{ab,r}(1)J_1)^{1/r} + (G_{ba,r}(0)J_0 + G_{ba,r}(1)J_1)^{1/r}\Big].$$

4. Additional Estimates for Bounded and Lipschitz Derivatives

Throughout this section, let the assumptions of Lemma 4 hold. Since

$$W_{ba}(t) = W_{ab}(t) \qquad (t \in [0,1]),$$

we have

$$U_2(t) = U_1(t) \qquad (t \in [0,1]).$$

For convenience, define

$$H(t) := U_1(t) = \frac{1}{2}\int_0^t W_{ab}(s)\,{}_0d_qs - \frac{1}{3}\int_0^1 W_{ab}(s)\,{}_0d_qs.$$

Then Lemma 4 can be rewritten in the form

$$\mathcal{S}_q(f,w) = \left|\int_0^1 H(t)\Big({}_0D_qF_{ab}(t) + {}_0D_qF_{ba}(t)\Big)\,{}_0d_qt\right|. \tag{19}$$

Moreover,

$$|H(t)| \le \|w\|_{[a,b],\infty}\left(\frac{1}{2}t + \frac{1}{3}\right) \qquad (t \in [0,1]). \tag{20}$$

Consequently,

$$\int_0^1 |H(t)|\,{}_0d_qt \le \|w\|_{[a,b],\infty}\frac{2q+5}{6(q+1)}, \tag{21}$$

and

$$\int_0^1 t|H(t)|\,{}_0d_qt \le \|w\|_{[a,b],\infty}\frac{5+5q+2q^2}{6(q+1)(1+q+q^2)}. \tag{22}$$

Theorem 6. *Let the assumptions of Lemma 4 hold. Assume that there exists a constant $M > 0$ such that*

$$\left|{}_0D_qF_{ab}(t) + {}_0D_qF_{ba}(t)\right| \le M \qquad (t \in [0,1]).$$

Then

$$\mathcal{S}_q(f,w) \le M\|w\|_{[a,b],\infty}\frac{2q+5}{6(q+1)}. \tag{23}$$

Proof. By (19),

$$\mathcal{S}_q(f,w) \le \int_0^1 |H(t)|\left|{}_0D_qF_{ab}(t) + {}_0D_qF_{ba}(t)\right|\,{}_0d_qt.$$

Using the assumed bound, we obtain

$$\mathcal{S}_q(f,w) \le M\int_0^1 |H(t)|\,{}_0d_qt.$$

Now (21) yields (23). □

The following theorem is compatible with affine-type q-derivatives arising from polynomial test functions and allows us to derive explicit Lipschitz-type estimates.

Theorem 7. *Let the assumptions of Lemma 4 hold. Assume that there exist a constant $c \in \mathbb{R}$ and $L > 0$ such that*

$$\left|{}_0D_qF_{ab}(t) - c\right| \le Lt, \qquad \left|{}_0D_qF_{ba}(t) + c\right| \le Lt \qquad (t \in [0,1]).$$

Then

$$\mathcal{S}_q(f,w) \le L\|w\|_{[a,b],\infty}\frac{5+5q+2q^2}{3(q+1)(1+q+q^2)}. \tag{24}$$

Proof. By (19),

$$\mathcal{S}_q(f,w) \le \int_0^1 |H(t)|\left|{}_0D_qF_{ab}(t) + {}_0D_qF_{ba}(t)\right|{}_0d_qt.$$

For each $t \in [0,1]$, we write

$${}_0D_qF_{ab}(t) + {}_0D_qF_{ba}(t) = \left({}_0D_qF_{ab}(t) - c\right) + \left({}_0D_qF_{ba}(t) + c\right).$$

Hence, by the triangle inequality and by the hypotheses,

$$\begin{aligned}\left|{}_0D_qF_{ab}(t) + {}_0D_qF_{ba}(t)\right| &\le \left|{}_0D_qF_{ab}(t) - c\right| + \left|{}_0D_qF_{ba}(t) + c\right| \\ &\le 2Lt.\end{aligned}$$

Therefore,

$$\mathcal{S}_q(f,w) \le 2L\int_0^1 t|H(t)|\,{}_0d_qt.$$

Using (22), we obtain (24). □

Example 1. *Let $a < b$, set*

$$A := \frac{a+b}{2}, \qquad d := \frac{b-a}{2},$$

and choose

$$f(x) = x^2, \qquad w_\lambda(x) = 1 + \lambda(x-A)^2, \qquad \lambda \ge 0.$$

Then w_λ is symmetric with respect to A, and for $\lambda > 0$ it is not identically one. Moreover,

$$\|w_\lambda\|_{[a,b],\infty} = 1 + \lambda d^2.$$

Since

$$F_{ab}(t) = (A+dt)^2, \qquad F_{ba}(t) = (A-dt)^2,$$

we have

$${}_0D_qF_{ab}(t) = 2Ad + d^2(1+q)t, \qquad {}_0D_qF_{ba}(t) = -2Ad + d^2(1+q)t.$$

Taking

$$c = 2Ad, \qquad L = d^2(1+q),$$

gives

$$\left|{}_0D_qF_{ab}(t) - c\right| = Lt, \qquad \left|{}_0D_qF_{ba}(t) + c\right| = Lt.$$

Indeed,

$$\begin{aligned}{}_0D_qF_{ab}(t) + {}_0D_qF_{ba}(t) &= \left({}_0D_qF_{ab}(t) - c\right) + \left({}_0D_qF_{ba}(t) + c\right) \\ &= 2Lt.\end{aligned}$$

Hence the assumptions of Theorem 7 are satisfied, and

$$\mathcal{S}_q(f, w_\lambda) \le d^2(1+q)(1+\lambda d^2)\frac{5+5q+2q^2}{3(q+1)(1+q+q^2)}.$$

Equivalently,

$$\mathcal{S}_q(f, w_\lambda) \le d^2(1+\lambda d^2)\frac{5+5q+2q^2}{3(1+q+q^2)}.$$

Remark 3. *If $w(x) \equiv 1$ on $[a, b]$, then*

$$H(t) = \frac{1}{2}t - \frac{1}{3}.$$

In this case, the general estimates (21) and (22) reduce to

$$\int_0^1 |H(t)|\,{}_0d_q t \le \frac{2q+5}{6(q+1)}, \qquad \int_0^1 t|H(t)|\,{}_0d_q t \le \frac{5+5q+2q^2}{6(q+1)(1+q+q^2)}.$$

Hence Theorems 6 and 7 yield explicit one-dimensional bounds for the parameter-domain Simpson functional associated with f.

5. Applications

In the present parameter-domain formulation, closed-form q-applications are less convenient to state than in the original x-space formulation. For this reason, we record below several representative applications obtained from the classical limit $q \to 1^-$ described in Remark 1.

5.1. Random Variables

Let $0 < a < b$ and let $w : [a, b] \to [0, +\infty)$ be a continuous probability density function related to a continuous random variable X, assumed to be symmetric about $\frac{a+b}{2}$. Denote

$$E_s(X) := \int_a^b x^s w(x)\,dx, \qquad E(X) := \int_a^b x w(x)\,dx.$$

Since w is symmetric and $\int_a^b w(x)\,dx = 1$, we have

$$E(X) = \frac{a+b}{2}.$$

Proposition 1. *Let $s > 1$, $r > 1$, $\frac{1}{p} + \frac{1}{r} = 1$, and assume that $r(s-1) \ge 1$. Then,*

$$\begin{aligned}
&\left|\frac{1}{6}\left[a^s + b^s + 4E(X)^s\right] - E_s(X)\right| \\
&\le \frac{(b-a)^2}{2}\, s\, \|w\|_{[a,b],\infty}\left(\frac{2\cdot 5^{p+1} - 2^{p+2}}{6^{p+1}(p+1)}\right)^{1/p}\left(\frac{1}{2}\right)^{1/r} \\
&\quad \times \left[\left(E(X)^{r(s-1)} + b^{r(s-1)}\right)^{1/r} + \left(a^{r(s-1)} + E(X)^{r(s-1)}\right)^{1/r}\right].
\end{aligned} \tag{25}$$

Proof. Apply the classical limit of Corollary 2 with $f(x) = x^s$. Since

$$\frac{1}{b-a}\int_a^b x^s w(x)\,dx = \frac{E_s(X)}{b-a}, \qquad E(X) = \frac{a+b}{2},$$

the claim follows after multiplying both sides of the classical limiting inequality by $b - a$. □

5.2. Special Means

For arbitrary positive real numbers a and b with $a \neq b$, we denote by

$$\mathcal{A}(a,b) := \frac{a+b}{2}$$

the arithmetic mean,

$$\mathcal{G}(a,b) := \sqrt{ab}$$

the geometric mean,

$$\mathcal{L}(a,b) := \frac{b-a}{\ln b - \ln a}$$

the logarithmic mean, and

$$\mathcal{L}_\gamma(a,b) := \left(\frac{b^{\gamma+1}-a^{\gamma+1}}{(\gamma+1)(b-a)}\right)^{1/\gamma}, \qquad \gamma \in \mathbb{R}\setminus\{0,-1\},$$

the generalized logarithmic mean.

Proposition 2. *Let $0 < a < b$ and $n \in \mathbb{N}$. Then*

$$\begin{aligned}&\left|\frac{1}{3}\mathcal{A}(a^n,b^n) + \frac{2}{3}\mathcal{A}(a,b)^n - \mathcal{L}_n^n(a,b)\right| \\ &\leq \frac{5(b-a)}{24}\, n\left[a^{n-1} + b^{n-1} + 2\mathcal{A}(a,b)^{n-1}\right].\end{aligned} \tag{26}$$

Proof. Take $w(x) \equiv 1$ and $f(x) = x^n$ in the classical limit of Corollary 1. Since

$$\frac{1}{b-a}\int_a^b x^n\,dx = \mathcal{L}_n^n(a,b), \qquad \frac{1}{6}\left[a^n + b^n + 4\mathcal{A}(a,b)^n\right] = \frac{1}{3}\mathcal{A}(a^n,b^n) + \frac{2}{3}\mathcal{A}(a,b)^n,$$

we obtain (26). □

The obtained estimate provides a weighted Simpson-type approximation for generalized logarithmic means and illustrates how the parameter-domain formulation recovers classical inequalities in the limit $q \to 1^-$.

Proposition 3. *Let $0 < a < b$, $n \in \mathbb{N}$, $r > 1$, and $\frac{1}{p} + \frac{1}{r} = 1$. Then*

$$\begin{aligned}&\left|\frac{1}{3}\mathcal{A}(a^n,b^n) + \frac{2}{3}\mathcal{A}(a,b)^n - \mathcal{L}_n^n(a,b)\right| \\ &\leq \frac{b-a}{2}\, n\left(\frac{2\cdot 5^{p+1} - 2^{p+2}}{6^{p+1}(p+1)}\right)^{1/p}\left(\frac{1}{2}\right)^{1/r} \\ &\quad\times\left[\left(\mathcal{A}(a,b)^{r(n-1)} + b^{r(n-1)}\right)^{1/r} + \left(a^{r(n-1)} + \mathcal{A}(a,b)^{r(n-1)}\right)^{1/r}\right].\end{aligned} \tag{27}$$

Proof. Take $w(x) \equiv 1$ and $f(x) = x^n$ in the classical limit of Corollary 2. The identities used in the proof of Proposition 2 yield (27). □

Example 2 (A nonconstant symmetric weight)**.** *Let $0 < a < b$, $n \in \mathbb{N}$, $r > 1$, and $\frac{1}{p} + \frac{1}{r} = 1$. Set*

$$A_0 := \mathcal{A}(a,b) = \frac{a+b}{2}, \qquad d := \frac{b-a}{2},$$

and consider the nonconstant symmetric weight

$$w_\lambda(x) := 1 + \lambda(x - A_0)^2, \qquad \lambda > 0.$$

Then,

$$w_\lambda(a + b - x) = w_\lambda(x), \qquad \|w_\lambda\|_{[a,b],\infty} = 1 + \lambda d^2,$$

and

$$\frac{1}{b-a}\int_a^b w_\lambda(x)\,dx = 1 + \frac{\lambda d^2}{3}.$$

For $f(x) = x^n$, define

$$M_{n,\lambda}(a,b) := \frac{1}{b-a}\int_a^b x^n w_\lambda(x)\,dx.$$

Since

$$x^n w_\lambda(x) = x^n + \lambda x^n (x - A_0)^2,$$

we obtain

$$M_{n,\lambda}(a,b) = \mathcal{L}_n^n(a,b) + \lambda\Big[\mathcal{L}_{n+2}^{n+2}(a,b) - 2A_0\,\mathcal{L}_{n+1}^{n+1}(a,b) + A_0^2\mathcal{L}_n^n(a,b)\Big].$$

Applying the classical limit of Corollary 2 with $f(x) = x^n$ and $w = w_\lambda$, we get

$$\begin{aligned}
&\left|\left(1 + \frac{\lambda d^2}{3}\right)\left[\frac{1}{3}\mathcal{A}(a^n,b^n) + \frac{2}{3}\mathcal{A}(a,b)^n\right] - M_{n,\lambda}(a,b)\right| \\
&\le \frac{b-a}{2}\,n(1+\lambda d^2)\left(\frac{2\cdot 5^{p+1} - 2^{p+2}}{(p+1)6^{p+1}}\right)^{1/p}\left(\frac{1}{2}\right)^{1/r} \\
&\quad\times\left[\left(A_0^{r(n-1)} + b^{r(n-1)}\right)^{1/r} + \left(a^{r(n-1)} + A_0^{r(n-1)}\right)^{1/r}\right].
\end{aligned} \tag{28}$$

Proposition 4. *Let $0 < a < b$, $n \in \mathbb{N}$, and $r \ge 1$. Then,*

$$\begin{aligned}
&\left|\frac{1}{3}\mathcal{A}(a^n,b^n) + \frac{2}{3}\mathcal{A}(a,b)^n - \mathcal{L}_n^n(a,b)\right| \\
&\le \frac{b-a}{2}\,n\left(\frac{7}{12}\right)^{1-1/r}\left[\left(\frac{1}{4}\mathcal{A}(a,b)^{r(n-1)} + \frac{1}{3}b^{r(n-1)}\right)^{1/r}\right. \\
&\quad\left. + \left(\frac{1}{3}a^{r(n-1)} + \frac{1}{4}\mathcal{A}(a,b)^{r(n-1)}\right)^{1/r}\right].
\end{aligned} \tag{29}$$

Proof. Take $w(x) \equiv 1$ and $f(x) = x^n$ in the classical limit of Corollary 3. Since for $h(t) = t$ we have

$$J_0 \to \int_0^1 \left(\frac{1}{2}t + \frac{1}{3}\right)(1-t)\,dt = \frac{1}{4}, \qquad J_1 \to \int_0^1 \left(\frac{1}{2}t + \frac{1}{3}\right)t\,dt = \frac{1}{3},$$

Formula (29) follows. □

Remark 4. *By taking $f(x) = e^x$ and $w(x) \equiv 1$ in the classical limit of Corollary 1, we obtain*

$$\left|\frac{1}{3}\mathcal{A}(e^a,e^b) + \frac{2}{3}\mathcal{G}(e^a,e^b) - \mathcal{L}(e^a,e^b)\right| \le \frac{5(b-a)}{24}\left[e^a + e^b + 2e^{\frac{a+b}{2}}\right].$$

6. Conclusions

In this work, a weighted Simpson-type identity was established on the parameter domain associated with affine reparametrizations in quantum calculus. Based on this identity, several bounds were derived under h-convexity, boundedness, and Lipschitz assumptions. Applications to moments of random variables and to special means were then obtained through the classical limit $q \to 1^-$. This approach may serve as a basis for further studies of weighted quadrature-type inequalities in quantum calculus and related operator settings.

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