

Article

Stochastic Differential Financial Portfolio Game Under CEV Model with CRRA Utility

Shuo Cheng ¹, Ming Cao ² and Hua Zhang ^{2,*}

¹ School of Management, Zhongkai University of Agriculture and Engineering, Guangzhou 510225, China; chengshuo@zhku.edu.cn

² School of Economics & Trade, Guangdong University of Finance, Guangzhou 510521, China; caoming@guf.edu.cn

* Correspondence: 47-173@gduf.edu.cn

Abstract

This paper investigates a stochastic differential portfolio game between two competing investors with relative wealth preferences. The financial market consists of one risk-free asset and one risky asset, whose price dynamics follow the CEV model. We formulate this game as two utility maximization problems, where each investor aims to maximize their relative utility defined as the weighted average of the ratio between their terminal wealth and the competitor's terminal wealth. Firstly, we derive the Hamilton–Jacobi–Bellman (HJB) equations and corresponding value functions through the dynamic programming principle. Next, we obtain the explicit solutions to equilibrium investment strategies and value functions for the non-zero-sum game under the CRRA utility framework. Finally, we conducted numerical simulations to analyze the impacts of model parameters on equilibrium strategies and provide relevant economic explanations.

Keywords: stochastic differential portfolio game; CEV model; CRRA utility; Nash equilibrium

MSC: 91G10; 91A26; 91B06

1. Introduction

Portfolio theory, which focuses on efficient asset allocation by maximizing returns and minimizing risks, has long been a core research topic in modern finance. Theoretical models vary across different types of investors, including retail investors and institutional investors such as funds and insurance companies. Retail investors make decisions independently based on their risk appetite and market conditions, while institutional investors interact closely with one another.

With the development of financial markets, institutional investors have gained growing market influence. To outperform their peers, they must incorporate competitors' behaviors into their investment decisions. In fund management, the ranking effect in behavioral finance drives investors to favor top-tier fund companies. Accordingly, fund managers are motivated to beat their peers to attract capital inflows and boost performance. In addition, from the perspective of investment behavior psychology, investors also pursue higher social status and thus care greatly about their relative performance. Such competitive interactions have attracted extensive academic attention. As Hortaçsu and Syverson (2004) [1] pointed out, the mutual fund literature has generally not addressed strategic interaction among firms. Yet, competitive forces are important determinants of the fortunes of asset management firms.



Academic Editor: Tahir Choulli

Received: 2 April 2026

Revised: 22 June 2026

Accepted: 26 June 2026

Published: 6 July 2026

Copyright: © 2026 by the authors.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and

conditions of the [Creative Commons](#)

[Attribution \(CC BY\)](#) license.

In financial markets, the competition between institutional investors is usually measured by relative performance indicators. In practice, relative performance can better reflect the professional capability of investment manager, and it has been widely adopted as a benchmark for performance evaluation in the financial industry (Dong et al., 2023 [2]). In academic research, two mainstream methods are used to characterize relative performance: the first calculates the weighted average of the performance gap between an investor and their competitors, and the second adopts the weighted average of the performance ratio between peers. The former has received far more research attention.

Espinosa and Touzi (2015) [3] first introduced relative performance to characterize strategic competition among N investment managers. They constructed an N -player investment game within the classical Black–Scholes setting and proved the existence of Nash equilibrium, laying a solid theoretical basis for investment decisions with multi-agent strategic interactions. Following their framework, Li and Qiu (2025) [4] examined a non-zero-sum portfolio game of optimal asset-liability management (ALM) between insurers and reinsurers within the framework of the constant elasticity of variance (CEV) model under common shock dependence. For cases of power utility and mean-variance utility, Luo and Zhou (2025) [5] obtained explicit expressions of the investment strategies and value functions for asset-liability management. Yang et al. (2020) [6] further extended this research stream by incorporating liabilities and adopting the Heston stochastic volatility model for risky asset prices.

In actuarial research, Bensoussan et al. (2014) [7] initially adopted relative performance to describe competition between insurance companies. They explored a two-player non-zero-sum game of investment and reinsurance and analyzed corresponding optimal strategies under competitive setting. Subsequently, a large number of scholars later extended their model to further study similar non-zero-sum games for insurers, such as Siu et al. (2017) [8], Deng et al. (2018) [9], Bin and Zhu (2021) [10], Bin et al. (2024) [11], and Bai et al. (2022) [12], along with their cited works. All of the above literature focuses on continuous-time investment or investment–reinsurance games. Zhou et al. (2019) [13,14] explored multi-stage portfolio and investment games with serially correlated asset returns. Lin et al. (2020) [15] further investigated a discrete-time portfolio game between two competing institutional investors whose decisions interact with each other. Their results show that the findings from discrete-time frameworks are consistent with those obtained in continuous-time models.

In contrast, studies on the second type of relative performance measurement remain relatively scarce. As a seminal work, Basak and Makarov (2014) [16] conducted a comprehensive analysis of strategic interactions in dynamic asset allocation and discussed how portfolio choices and professional specialization interact among fund managers. Lacker and Zariphopoulou (2019) [17] adopted N -player differential games and mean-field game theory to solve multi-agent investment decision problems. They derived explicit analytical expressions for equilibrium strategies under both exponential and power utility assumptions. Building upon this foundation, Lacker and Soret (2020) [18] extended the above framework by incorporating consumption decisions and studied an N -player non-zero-sum investment–consumption game. In an incomplete market, Kraft et al. (2020) [19] discussed a two-player non-zero-sum investment game where risky asset prices follow an extended Heston model. Using the dynamic programming principle, they also obtained closed-form solutions for equilibrium investment strategies. Zhu et al. (2026) [20] has also extended the research on relative performance optimal investment–reinsurance problems to the setting of incomplete financial markets.

The literature review indicates that few studies have explored portfolio games with relative performance under the CEV model. Against this background, this study constructs

a continuous-time two-player stochastic investment game model. We adopt the Constant Relative Risk Aversion (CRRA) utility function to describe investors' risk preferences (Wu and Ma, 2015 [21]), and define relative performance as the weighted average of the wealth ratio between each investor and their competitor. Each investor's objective is to maximize the expected utility of terminal relative wealth. By applying the dynamic programming principle, we further derive the explicit analytical solutions to the Nash equilibrium strategies.

The main contributions of this paper are as follows:

- (1) Specifically, we employ relative performance, defined as the weighted average of an investor's terminal wealth ratio relative to that of the competitor, to characterize the competitive interaction between two investors. On this basis, we build a two-player non-zero-sum investment game where risky asset prices obey the CEV model.
- (2) Based on the dynamic programming principle, we prove the existence conditions of Nash equilibrium strategies and derive the explicit analytical solutions of Nash equilibrium under the CRRA utility framework. These expressions are valuable because they allow us to intuitively observe the direct impact of each parameter on the equilibrium strategy. This analytical tractability further facilitates optimal parameter design and boundary analysis, offering clear portfolio insights that would be obscured in purely numerical approaches.

2. Model Assumptions and Construction

Let $(\Omega, \mathbb{F}, \mathbb{P})$ denote a complete probability space, where $\mathbb{P}$ stands for the probability measure and the filtration $\mathbb{F} \triangleq \{\mathcal{F}_t\}_{t \geq 0}$ satisfies the usual conditions (i.e., it is right-continuous and complete). Here, $\{\mathcal{F}_t\}_{t \geq 0}$ represents all available information up to time t . All stochastic processes and random variables in this paper are defined on this probability space. We further assume continuous trading, permissible short-selling, and the absence of transaction costs and taxes in the financial.

2.1. Model Assumptions

For simplicity and without loss of generality, we consider a financial market with two types of financial assets: one risk-free bond and one risky asset, such as a stock. Assume that the price of the risk-free bond $B(t)$ follows the ordinary differential of Equation (1):

$$dB(t) = rB(t)dt \quad (1)$$

where $r > 0$ stand for the risk-free interest rate. The price of the risky asset $S(t)$ follows the CEV model shown in Equation (2).

$$dS(t) = \mu S(t)dt + \sigma S^{\beta+1}(t)dw(t) \quad (2)$$

In this model, the constant $\mu > 0$ represents the expected return rate of the risky asset, where $\sigma > 0$ is a given constant; β is known as the constant elasticity of variance factor; $\sigma S^{\beta}(t)$ denotes the instantaneous volatility of the risky asset; and $w(t)$ is a one-dimensional standard Brownian motion.

Assume that at any time t , an investor allocates a proportion $\pi(t)$ of wealth to the risky asset, and the rest $1 - \pi(t)$ to the risk-free bond. We use $\pi(t)$ to denote the investor's investment strategy. Then, under this strategy $\pi(t)$, the investor's wealth dynamics satisfy the stochastic differential below (Equation (3)).

$$\begin{aligned} dX(t) &= [1 - \pi(t)]X(t)\frac{dB(t)}{B(t)} + \pi(t)X(t)\frac{dS(t)}{S(t)} \\ &= [r + (\mu - r)\pi(t)]X(t)dt + \pi(t)X(t)\sigma S^{\beta}(t)dw(t). \end{aligned} \quad (3)$$

In financial markets, numerous investors often interact strategically rather than making decisions independently, which essentially forms a stochastic economic game. This paper focuses on two competing investors, namely two insurance companies or two fund management firms, labeled as investor 1 and investor 2, respectively. The investment strategy of investor $k \in \{1, 2\}$ is denoted as $\pi_k(t)$. Accordingly, the wealth process $X_k(t)$ of investor k under strategy $\pi_k(t)$ can be expressed as

$$dX_k(t) = [r + (\mu - r)\pi_k(t)]X_k(t)dt + \pi_k(t)X_k\sigma S^\beta(t)dw(t) \quad (4)$$

Definition 1 (Admissible Strategy). For $k \in \{1, 2\}$, a strategy $\pi_k(t)$ is called admissible for investor k if the following two conditions hold:

- (i) $\pi_k(t)$ is $\mathcal{F}_t$ -progressively measurable and holds for any $\mathbb{E}\left[\int_0^T \|\pi_k(t)\|^2 dt\right] < \infty$;
- (ii) For a given $\forall x_0 \in \mathbb{R}$, the stochastic differential Equation (4) has a unique strong solution $X_k(t)$, satisfying $\mathbb{E}\left[\sup_{0 \leq t \leq T} \|X_k(t)\|^2\right] < \infty$ almost surely for all $T < \infty$.

Denote the set of all admissible strategies for investor k as $k = 1, 2$.

2.2. Construction of the Non-Zero-Sum Game Model

We study a two-player non-zero-sum investment game under a complete information framework. Let $U_k(\cdot)$ denote the utility function of investor k , which is smooth, strictly increasing, and strictly concave ($U'_k(\cdot) > 0$ and $U''_k(\cdot) < 0$). Each investor aims to choose an admissible strategy $\pi_k \in \Pi_k$ to maximize the expected utility of their terminal wealth. In this model, investors pay attention to both the absolute wealth and relative wealth of competitors. Specifically, the optimization problem for each investor can be formulated as follows:

Given the strategy choice $\pi_j \in \Pi_j$ of the competitor $j \in \{1, 2\}$, investor $k \neq j$ seeks an optimal strategy to maximize the objective function (5).

$$\sup_{\pi_k \in \Pi_k} J_k(t, x_k, x_j, \pi_k, \pi_j) = J_k(t, x_k, x_j, \pi_k^*, \pi_j) \quad (5)$$

where

$$\begin{aligned} J_k(t, x_k, x_j, \pi_k, \pi_j) &\triangleq \mathbb{E}\left[U_k\left(X_k^{1-\kappa_k}(T)\left(\frac{X_k(T)}{X_j(T)}\right)^{\kappa_k}\right) \mid X_k(t) = x_k, X_j(t) = x_j\right] \\ &= \mathbb{E}\left[U_k\left(X_k(T)X_j^{-\kappa_k}(T)\right) \mid X_k(t) = x_k, X_j(t) = x_j\right] \end{aligned}$$

The first term in the above expression $X_k^{1-\kappa_k}(T)$ measures the investor's utility derived from absolute wealth, and the second term $\left(\frac{X_k(T)}{X_j(T)}\right)^{\kappa_k}$ measures the utility derived from relative wealth. The parameter $\kappa_k \in [0, 1]$ reflects how sensitive the investor k is to the competitor's ($j \neq k$) wealth, and it also quantifies the degree of market competition. A larger value of this parameter means investor k attaches more importance to relative wealth utility. In extreme cases, when $\kappa_k = 0$, investor k only focuses on absolute wealth utility; conversely, when $\kappa_k = 1$, investor k cares exclusively about relative wealth utility.

Referring to Espinosa and Touzi (2015) [3], the optimization problem (5) can be transformed into the following non-zero-sum game problem.

We aim to find the Nash equilibrium strategy pair $(\pi_1^*, \pi_2^*) \in \Pi_1 \times \Pi_2$ such that for any admissible strategies $\pi_1 \in \Pi_1$ and $\pi_2 \in \Pi_2$, the following inequalities hold:

$$\begin{cases} J_1(t, x_1, x_2, \pi_1, \pi_2^*) \leq J_1(t, x_1, x_2, \pi_1^*, \pi_2^*), \\ J_2(t, x_2, x_1, \pi_2, \pi_1^*) \leq J_2(t, x_2, x_1, \pi_2^*, \pi_1^*). \end{cases}$$

3. Solution of the Non-Zero-Sum Game Model

This section adopts the dynamic programming principle to derive the Nash equilibrium strategies for the proposed non-zero-sum game. $\forall t \in [0, T]$ and $j \neq k \in \{1, 2\}$, and given $X_k(t) = x_k$, $X_j(t) = x_j$, and $S(t) = s$, the value function $V_k(t, x_k, x_j, s)$ of investor k is defined as Equation (6):

$$V_k(t, x_k, x_j, s) = \sup_{\pi_k \in \Pi_k} \mathbb{E} \left[U_k \left(X_k(T) X_j^{-\kappa_k}(T) \right) \middle| X_k(t) = x_k, X_j(t) = x_j, S(t) = s \right] \quad (6)$$

According to the stochastic dynamic programming principle (Richard, 1957 [22]; Liberzon, 2012 [23]), the value function $V_k(t, x_k, x_j, s)$ satisfies the Hamilton–Jacobi–Bellman (HJB) (Equation (7))

$$\begin{aligned} \sup_{\pi_k \in \Pi_k} \left\{ \frac{\partial V_k}{\partial t} + [rx_k + (\mu - r)\pi_k x_k] \frac{\partial V_k}{\partial x_k} + \frac{1}{2} \sigma^2 s^{2\beta} \pi_k^2 x_k^2 \frac{\partial^2 V_k}{\partial x_k^2} + \sigma^2 s^{2\beta+1} \pi_k x_k \frac{\partial^2 V_k}{\partial x_k \partial s} \right. \\ \left. + \sigma^2 s^{2\beta} \pi_k \pi_j^* x_k x_j \frac{\partial^2 V_k}{\partial x_k \partial x_j} + [rx_j + (\mu - r)\pi_j^* x_j] \frac{\partial V_k}{\partial x_j} + \frac{1}{2} \sigma^2 s^{2\beta} \pi_j^{*2} x_j^2 \frac{\partial^2 V_k}{\partial x_j^2} \right. \\ \left. + \sigma^2 s^{2\beta+1} \pi_j^* x_j \frac{\partial^2 V_k}{\partial x_j \partial s} + \mu s \frac{\partial V_k}{\partial s} + \frac{1}{2} \sigma^2 s^{2\beta+2} \frac{\partial^2 V_k}{\partial s^2} \right\} = 0, \end{aligned} \quad (7)$$

along with its corresponding terminal condition $V_k(T, x_k, x_j, s) = U_k(x_k x_j^{-\kappa_k})$.

We take the first-order derivative of Equation (7) with respect to π_k and obtain Equation (8).

$$\pi_k^* = - \underbrace{\frac{(\mu - r)x_k \frac{\partial V_k}{\partial x_k}}{\sigma^2 s^{2\beta} x_k^2 \frac{\partial^2 V_k}{\partial x_k^2}}}_{\text{myopic term}} - \underbrace{\frac{\sigma^2 s^{2\beta+1} x_k \frac{\partial^2 V_k}{\partial x_k \partial s}}{\sigma^2 s^{2\beta} x_k^2 \frac{\partial^2 V_k}{\partial x_k^2}}}_{\text{hedging term}} - \underbrace{\frac{\sigma^2 s^{2\beta} \pi_j^* x_k x_j \frac{\partial^2 V_k}{\partial x_k \partial x_j}}{\sigma^2 s^{2\beta} x_k^2 \frac{\partial^2 V_k}{\partial x_k^2}}}_{\text{adjustment term}}. \quad (8)$$

As shown in this formula, the optimal investment strategy π_k^* of investor k consists of three parts. The first term corresponds to the so-called myopic term, the second is the hedging term that accounts for the price volatility of the risky asset, and the third is an adjustment term induced by competition between investors. Therefore, an extra adjustment term emerges in Equation (8) relative to the case without competition.

Substituting Equation (8) into the HJB Equation (7), we simplify the original HJB equation and derive the partial differential Equation (9) below:

$$\begin{aligned} \frac{\partial V_k}{\partial t} + \mu s \frac{\partial V_k}{\partial s} + \frac{1}{2} \sigma^2 s^{2\beta+2} \frac{\partial^2 V_k}{\partial s^2} + rx_k \frac{\partial V_k}{\partial x_k} - \frac{1}{2} \sigma^2 s^{2\beta} \pi_k^{*2} x_k^2 \frac{\partial^2 V_k}{\partial x_k^2} \\ + [rx_j + (\mu - r)\pi_j^* x_j] \frac{\partial V_k}{\partial x_j} + \frac{1}{2} \sigma^2 s^{2\beta} \pi_j^{*2} x_j^2 \frac{\partial^2 V_k}{\partial x_j^2} + \sigma^2 s^{2\beta+1} \pi_j^* x_j \frac{\partial^2 V_k}{\partial x_j \partial s} = 0. \end{aligned} \quad (9)$$

Based on the above analysis, we further derive the sufficient conditions for the existence of Nash equilibrium for the proposed game, which are summarized in Theorem 1 below.

Theorem 1. $\forall (t, x_k, x_j, s) \in [0, T] \times \mathbb{R}^3$, if $\frac{\partial^2 V_k}{\partial x_k^2} \neq 0$, then the Nash equilibrium strategies of the game satisfy the coupled nonlinear Equation (10):

$$\begin{cases} \pi_1^*(t) = -\frac{(\mu-r)x_1 \frac{\partial V_1}{\partial x_1} + \sigma^2 s^{2\beta+1} x_1 \frac{\partial^2 V_1}{\partial x_1 \partial s} + \sigma^2 s^{2\beta} \pi_2^* x_1 x_2 \frac{\partial^2 V_1}{\partial x_1 \partial x_2}}{\sigma^2 s^{2\beta} x_1^2 \frac{\partial^2 V_1}{\partial x_1^2}}, \\ \pi_2^*(t) = -\frac{(\mu-r)x_2 \frac{\partial V_2}{\partial x_2} + \sigma^2 s^{2\beta+1} x_2 \frac{\partial^2 V_2}{\partial x_2 \partial s} + \sigma^2 s^{2\beta} \pi_1^* x_1 x_2 \frac{\partial^2 V_2}{\partial x_1 \partial x_2}}{\sigma^2 s^{2\beta} x_2^2 \frac{\partial^2 V_2}{\partial x_2^2}}, \end{cases} \quad (10)$$

Here, V_1 and V_2 are solutions to the coupled partial differential Equation (11) subject to the terminal conditions (12).

$$\begin{cases} \frac{\partial V_1}{\partial t} + \mu s \frac{\partial V_1}{\partial s} + \frac{1}{2} \sigma^2 s^{2\beta+2} \frac{\partial^2 V_1}{\partial s^2} + r x_1 \frac{\partial V_1}{\partial x_1} - \frac{1}{2} \sigma^2 s^{2\beta} \pi_1^{*2} x_1^2 \frac{\partial^2 V_1}{\partial x_1^2} \\ + [r x_2 + (\mu - r) \pi_2^* x_2] \frac{\partial V_1}{\partial x_2} + \frac{1}{2} \sigma^2 s^{2\beta} \pi_2^{*2} x_2^2 \frac{\partial^2 V_1}{\partial x_2^2} + \sigma^2 s^{2\beta+1} \pi_2^* x_2 \frac{\partial^2 V_1}{\partial x_2 \partial s} = 0, \\ \frac{\partial V_2}{\partial t} + \mu s \frac{\partial V_2}{\partial s} + \frac{1}{2} \sigma^2 s^{2\beta+2} \frac{\partial^2 V_2}{\partial s^2} + r x_2 \frac{\partial V_2}{\partial x_2} - \frac{1}{2} \sigma^2 s^{2\beta} \pi_2^{*2} x_2^2 \frac{\partial^2 V_2}{\partial x_2^2} \\ + [r x_1 + (\mu - r) \pi_1^* x_1] \frac{\partial V_2}{\partial x_1} + \frac{1}{2} \sigma^2 s^{2\beta} \pi_1^{*2} x_1^2 \frac{\partial^2 V_2}{\partial x_1^2} + \sigma^2 s^{2\beta+1} \pi_1^* x_1 \frac{\partial^2 V_2}{\partial x_1 \partial s} = 0, \end{cases} \quad (11)$$

$$\begin{cases} V_1(T, x_1, x_2, s) = U_1(x_1 x_2^{-\kappa_1}), \\ V_2(T, x_2, x_1, s) = U_2(x_2 x_1^{-\kappa_2}). \end{cases} \quad (12)$$

As noted by Bensoussan et al. (2014) [7] and Siu et al. (2017) [8], it is difficult to prove the existence and uniqueness of solutions for the coupled partial differential Equation (11) under general cases $T > 0$. Nevertheless, when the relevant parameter T is sufficiently small and investors adopt CRRA utility functions, we can obtain the explicit analytical solutions for optimal value functions and Nash equilibrium strategies. The next section will elaborate on the derivation of equilibrium strategies under the CRRA utility setting.

Theorem 1 only provides the sufficient conditions for the existence of Nash equilibrium. We further present the corresponding necessary conditions in Theorem 2, which verifies that the solution $V_k(t, x_k, x_j, s)$ to the HJB Equation (7) is exactly the value function of our game model.

Theorem 2. For $k \in \{1, 2\}$, assume that V_k is a solution to the HJB Equation (7). Let π_k^* denote the admissible strategy defined in Definition 1 and X_k^* be the associated wealth process under this strategy π_k^* . If the family $\left\{ V_k(\tau_i \wedge T, X_k^*(\tau_i \wedge T), X_j^*(\tau_i \wedge T), S(\tau_i \wedge T)) \right\}_{i \in \mathbb{N}}$ is uniformly integrable for any sequence of stopping times $\{\tau_i\}_{i \in \mathbb{N}}$, then the admissible strategy π_k^* is the optimal strategy of investor k for the optimization problem (5), and the function V_k is the corresponding value function.

The proof of Theorem 2 is similar to the relevant proof in Bensoussan et al. (2014) [7]. Limited by the article length, we omit the detailed proof here. Readers who wish to know more can refer to the above literature.

4. Nash Equilibrium Strategies Under CRRA Utility Functions

This section focuses on deriving the explicit analytical solutions to Nash equilibrium strategies, where both investors adopt the CRRA utility function.

Suppose the utility function of investor k follows the standard CRRA form as shown in Equation (13):

$$U_k(x_k, x_j) = \frac{1}{1 - \gamma_k} \left(x_k x_j^{-\kappa_k} \right)^{1 - \gamma_k} \quad (13)$$

where $k, j \in \{1, 2\}$, $k \neq j$, $\gamma_k > 0$, and $\gamma_k \neq 1$ denotes the coefficient of relative risk aversion.

Under the CRRA utility framework, we summarize the analytical results of Nash equilibrium strategies for the proposed game in Theorem 3.

Theorem 3. For the proposed non-zero-sum game, if $B_1(t)$ and $B_2(t)$ satisfy the coupled ordinary differential Equations (14) and (15),

$$\dot{B}_1(t) = \phi_{1,1} + \phi_{1,2}B_1(t) + \phi_{1,3}B_1^2(t) + \phi_{1,4}B_2(t) + \phi_{1,5}B_2^2(t) + \phi_{1,6}B_1(t)B_2(t) \quad (14)$$

$$\dot{B}_2(t) = \phi_{2,1} + \phi_{2,2}B_2(t) + \phi_{2,3}B_2^2(t) + \phi_{2,4}B_1(t) + \phi_{2,5}B_1^2(t) + \phi_{2,6}B_1(t)B_2(t) \quad (15)$$

with the corresponding terminal conditions $B_1(T) = B_2(T) = 0$, where $\phi_{k,1}, \dots, \phi_{k,6}$ and $k = 1, 2$ is determined by Equation (26), then the coupled Equation (11) has explicit solutions formulated as Equations (16) and (17):

$$V_1(t, x_1, x_2, s) = \frac{1}{1 - \gamma_1} \left(x_1 x_2^{-\kappa_1} \right)^{1 - \gamma_1} e^{A_1(t) - B_1(t)s^{-2\beta}} \quad (16)$$

$$V_2(t, x_2, x_1, s) = \frac{1}{1 - \gamma_2} \left(x_2 x_1^{-\kappa_2} \right)^{1 - \gamma_2} e^{A_2(t) - B_2(t)s^{-2\beta}} \quad (17)$$

where

$$A_1(t) = r(1 - \kappa_1)(1 - \gamma_1)(T - t) - \beta(2\beta + 1)\sigma^2 \int_t^T B_1(s)ds \quad (18)$$

$$A_2(t) = r(1 - \kappa_2)(1 - \gamma_2)(T - t) - \beta(2\beta + 1)\sigma^2 \int_t^T B_2(s)ds \quad (19)$$

Furthermore, the corresponding Nash equilibrium strategies (π_1^*, π_2^*) can be expressed as

$$\begin{cases} \pi_1^*(t) = \frac{\mu - r}{(1 - \nu_1 \nu_2)\sigma^2 S^{2\beta}(t)} \left(\frac{1}{\gamma_1} - \frac{\nu_1}{\gamma_2} \right) + \frac{2\beta}{S^{2\beta}(t)} \left[\frac{B_1(t)}{\gamma_1} - \frac{\nu_1 B_2(t)}{\gamma_2} \right], \\ \pi_2^*(t) = \frac{\mu - r}{(1 - \nu_1 \nu_2)\sigma^2 S^{2\beta}(t)} \left(\frac{1}{\gamma_2} - \frac{\nu_2}{\gamma_1} \right) + \frac{2\beta}{S^{2\beta}(t)} \left[\frac{B_2(t)}{\gamma_2} - \frac{\nu_2 B_1(t)}{\gamma_1} \right], \end{cases} \quad (20)$$

where $\nu_1 \triangleq \frac{\kappa_1(1 - \gamma_1)}{\gamma_1}$, $\nu_2 \triangleq \frac{\kappa_2(1 - \gamma_2)}{\gamma_2}$, and $1 - \nu_1 \nu_2 > 0$.

Proof. Suppose the value function of investor k adopts the following separable form (21):

$$V_k(t, x_k, x_j, s) = \frac{1}{1 - \gamma_k} \left(x_k x_j^{-\kappa_k} \right)^{1 - \gamma_k} e^{A_k(t) - B_k(t)s^{-2\beta}} \quad (21)$$

where $A_k, B_k : [0, T] \rightarrow \mathbb{R}$ are undetermined functions defined on Equation (21), and they satisfy the terminal conditions $A_k(T) = B_k(T) = 0$.

We calculate the partial derivatives of Equation (21) with respect to relevant variables, and the results are shown below.

$$\begin{aligned} \frac{\partial V_k}{\partial t} &= (\dot{A}_k - \dot{B}_k s^{-2\beta}) V_k, \quad \frac{\partial V_k}{\partial x_k} = \frac{1 - \gamma_k}{x_k} V_k, \quad \frac{\partial V_k}{\partial x_j} = \frac{-\kappa_k(1 - \gamma_k)}{x_j} V_k, \\ \frac{\partial V_k}{\partial s} &= 2\beta B_k s^{-2\beta-1} V_k, \quad \frac{\partial^2 V_k}{\partial x_k^2} = \frac{-\gamma_k(1 - \gamma_k)}{x_k^2} V_k, \quad \frac{\partial^2 V_k}{\partial x_k \partial x_j} = \frac{-\kappa_k(1 - \gamma_k)^2}{x_k x_j} V_k, \\ \frac{\partial^2 V_k}{\partial x_k \partial s} &= \frac{2(1 - \gamma_k)\beta B_k s^{-2\beta-1}}{x_k} V_k, \quad \frac{\partial^2 V_k}{\partial s^2} = 4\beta^2 B_k^2 s^{-4\beta-2} V_k - 2\beta(2\beta + 1)B_k s^{-2\beta-2} V_k, \\ \frac{\partial^2 V_k}{\partial x_j^2} &= \frac{\kappa_k(1 - \gamma_k)[1 + \kappa_k(1 - \gamma_k)]}{x_j^2} V_k, \quad \frac{\partial^2 V_k}{\partial x_j \partial s} = \frac{-\kappa_k(1 - \gamma_k)2\beta B_k s^{-2\beta-1}}{x_j} V_k \end{aligned}$$

Substituting the above expressions into Equation (10) and performing algebraic simplification, we derive Equation (22):

$$\begin{cases} \pi_1^* = \frac{\mu-r}{\gamma_1 \sigma^2 s^{2\beta}} + \frac{2\beta B_1}{\gamma_1 s^{2\beta}} - \nu_1 \pi_2^*, \\ \pi_2^* = \frac{\mu-r}{\gamma_2 \sigma^2 s^{2\beta}} + \frac{2\beta B_2}{\gamma_2 s^{2\beta}} - \nu_2 \pi_1^*, \end{cases} \quad (22)$$

By solving Equation (22), we acquire the solutions of π_1^* and π_2^* , which are presented in Equation (20).

We further substitute the expression for V_k from Equation (22) and the relevant partial derivatives into Equation (9) and simplify the formula to obtain Equation (23).

$$\begin{aligned} & \dot{A}_k - \dot{B}_k s^{-2\beta} + 2\beta B_k \left(r + \frac{\mu-r}{\gamma_k} \right) s^{-2\beta} - \beta(2\beta+1)\sigma^2 B_k \\ & + r(1-\kappa_k)(1-\gamma_k) + \frac{(\mu-r)^2(1-\gamma_k)}{2\gamma_k \sigma^2} s^{-2\beta} + \frac{2\beta^2 \sigma^2 B_k^2}{\gamma_k} s^{-2\beta} \\ & - [(\mu-r) + 2\beta \sigma^2 B_k] \nu_k \left[\frac{\mu-r}{(1-\nu_k \nu_j) \sigma^2} \left(\frac{1}{\gamma_j} - \frac{\nu_j}{\gamma_k} \right) + \frac{2\beta}{(1-\nu_k \nu_j)} \left(\frac{B_j}{\gamma_j} - \frac{\nu_j B_k}{\gamma_k} \right) \right] s^{-2\beta} \\ & + \frac{1}{2} \sigma^2 \nu_k \gamma_k (1+\nu_k) \left[\frac{\mu-r}{(1-\nu_k \nu_j) \sigma^2} \left(\frac{1}{\gamma_j} - \frac{\nu_j}{\gamma_k} \right) + \frac{2\beta}{(1-\nu_k \nu_j)} \left(\frac{B_j}{\gamma_j} - \frac{\nu_j B_k}{\gamma_k} \right) \right]^2 s^{-2\beta} = 0. \end{aligned} \quad (23)$$

This formula above can be decomposed into two independent ordinary differential equations, as shown in (24) and (25).

$$\begin{aligned} & -\dot{B}_k + \frac{(\mu-r)^2(1-\gamma_k)}{2\gamma_k \sigma^2} + 2\beta \left(r + \frac{\mu-r}{\gamma_k} \right) B_k + \frac{2\beta^2 \sigma^2 B_k^2}{\gamma_k} \\ & - [(\mu-r) + 2\beta \sigma^2 B_k] \nu_k \left[\frac{\mu-r}{(1-\nu_k \nu_j) \sigma^2} \left(\frac{1}{\gamma_j} - \frac{\nu_j}{\gamma_k} \right) + \frac{2\beta}{(1-\nu_k \nu_j)} \left(\frac{B_j}{\gamma_j} - \frac{\nu_j B_k}{\gamma_k} \right) \right] \\ & + \frac{1}{2} \sigma^2 \gamma_k \nu_k (1+\nu_k) \left[\frac{\mu-r}{(1-\nu_k \nu_j) \sigma^2} \left(\frac{1}{\gamma_j} - \frac{\nu_j}{\gamma_k} \right) + \frac{2\beta}{(1-\nu_k \nu_j)} \left(\frac{B_j}{\gamma_j} - \frac{\nu_j B_k}{\gamma_k} \right) \right]^2 = 0, \end{aligned} \quad (24)$$

$$\dot{A}_k - \beta(2\beta+1)\sigma^2 B_k + r(1-\kappa_k)(1-\gamma_k) = 0. \quad (25)$$

Define the auxiliary variable as Equation (26):

$$\begin{aligned} \phi_{k,1} &= \frac{(\mu-r)^2(1-\gamma_k)}{2\gamma_k \sigma^2} + \frac{1}{2} \sigma^2 \gamma_k \nu_k (1+\nu_k) M_k^2 - M_k(\mu-r)\nu_k, \\ \phi_{k,2} &= 2\beta \left(r + \frac{\mu-r}{\gamma_k} \right) - 2\beta \sigma^2 \nu_k M_k + (\mu-r) N \frac{\nu_j}{\gamma_k} - \sigma^2 \nu_k \nu_j M_k (1+\nu_k), \\ \phi_{k,3} &= \frac{2\beta^2 \sigma^2}{\gamma_k} + 2\beta \sigma^2 N \frac{\nu_j}{\gamma_k} + \frac{1}{2} \sigma^2 \nu_k (1+\nu_k) N^2 \frac{\nu_j^2}{\gamma_k}, \\ \phi_{k,4} &= -(\mu-r) \frac{\nu_k}{\gamma_j} N + \sigma^2 \gamma_k (1+\nu_k) M_k N \frac{\nu_k}{\gamma_j}, \\ \phi_{k,5} &= \frac{1}{2} \sigma^2 \gamma_k \nu_k (1+\nu_k) N^2 \frac{1}{\gamma_j^2}, \\ \phi_{k,6} &= -2\beta \sigma^2 \frac{\nu_k}{\gamma_j} N - \sigma^2 \frac{\nu_k}{\gamma_j} \nu_j (1+\nu_k) N. \end{aligned} \quad (26)$$

where $M_k = \frac{\mu-r}{(1-\nu_k \nu_j) \sigma^2} \left(\frac{1}{\gamma_j} - \frac{\nu_j}{\gamma_k} \right)$, $N = \frac{2\beta}{(1-\nu_k \nu_j)}$.

Accordingly, Equation (24) can be rearranged into the simplified form (27).

$$\dot{B}_k(t) = \phi_{k,1} + \phi_{k,2} B_k(t) + \phi_{k,3} B_k^2(t) + \phi_{k,4} B_j(t) + \phi_{k,5} B_j^2(t) + \phi_{k,6} B_k(t) B_j(t). \quad (27)$$

When k and j are set to 1 and 2, respectively, we can derive Equations (14) and (15). Thus, The proof is provided in Theorem 2 and is therefore omitted here. $\square$

Theorem 3 also covers a special case: when the parameter $\beta = 0$, the risky asset price process degenerates into the classic Black–Scholes model. The relevant results for this case are presented in Corollary 1 below.

Corollary 1. For the proposed non-zero-sum game, if $B_1(t)$ and $B_2(t)$ satisfy the coupled ordinary differential Equation (28)

$$\dot{B}_1(t) = \phi_{1,1} + \phi_{1,2}B_1(t), \quad \dot{B}_2(t) = \phi_{2,1} + \phi_{2,2}B_2(t) \quad (28)$$

with terminal conditions $B_1(T) = B_2(T) = 0$, where $\phi_{k,1}$ and $\phi_{k,2}$, $k = 1, 2$, are determined by Equation (29),

$$\begin{aligned} \phi_{k,1} &= \frac{(\mu-r)^2(1-\gamma_k)}{2\gamma_k\sigma^2} + \frac{1}{2}\sigma^2\gamma_kv_k(1+v_k)M_k^2 - M_k(\mu-r)v_k, \\ \phi_{k,2} &= -\sigma^2v_kv_jM_k(1+v_k), \\ M_k &= \frac{\mu-r}{(1-v_kv_j)\sigma^2} \left(\frac{1}{\gamma_j} - \frac{v_j}{\gamma_k} \right). \end{aligned} \quad (29)$$

then the optimal value functions for investor 1 and investor 2 are given by Equations (30) and (31):

$$V_1(t, x_1, x_2) = \frac{1}{1-\gamma_1} \left(x_1 x_2^{-\kappa_1} \right)^{1-\gamma_1} e^{A_1(t)-B_1(t)} \quad (30)$$

$$V_2(t, x_2, x_1) = \frac{1}{1-\gamma_2} \left(x_2 x_1^{-\kappa_2} \right)^{1-\gamma_2} e^{A_2(t)-B_2(t)} \quad (31)$$

where

$$A_1(t) = r(1-\kappa_1)(1-\gamma_1)(T-t), \quad A_2(t) = r(1-\kappa_2)(1-\gamma_2)(T-t) \quad (32)$$

Moreover, the corresponding Nash equilibrium strategies (π_1^*, π_2^*) take the form shown in Equation (33):

$$\begin{cases} \pi_1^* = \frac{\mu-r}{(1-v_1v_2)\sigma^2} \left(\frac{1}{\gamma_1} - \frac{v_1}{\gamma_2} \right), \\ \pi_2^* = \frac{\mu-r}{(1-v_1v_2)\sigma^2} \left(\frac{1}{\gamma_2} - \frac{v_2}{\gamma_1} \right), \end{cases} \quad (33)$$

where $v_1 \triangleq \frac{\kappa_1(1-\gamma_1)}{\gamma_1}$, $v_2 \triangleq \frac{\kappa_2(1-\gamma_2)}{\gamma_2}$, and $1 - v_1v_2 > 0$.

It can be observed that the Nash equilibrium strategy (π_1^*, π_2^*) in Equation (33) is consistent with the conclusion of Corollary 3.2 of Lacker and Zariphopoulou (2019) [17] when the number of investors $n = 2$ in the market. To a certain degree, this study extends the research of the above literature.

5. Numerical Simulation Analysis

To visually reveal how parameter changes affect equilibrium investment strategies, this section presents sensitivity analysis results for Equation (20) based on numerical simulations. Unless otherwise stated, we referred to Basak and Makarov (2014) [16] and Kraft et al. (2020) [19] to set the basic model parameters, as listed in Tables 1 and 2.

Table 1. Basic parameters of the financial market.

r	0.05	μ	0.1
σ	0.15	s	5
β	1	T	10

Table 2. Basic parameters of the investors.

Investor 1		Investor 2	
γ_1	4	γ_2	5
κ_1	0.6	κ_2	0.8

We adopted MATLAB 2022 to carry out numerical calculations, and the results reflecting parameter effects on equilibrium strategies are presented in Figures 1–4.

Figure 1 illustrates the impact of the risk aversion coefficient γ_k on the equilibrium investment strategy $\pi_k^*(0)$. The graph shows a negative correlation between $\pi_k^*(0)$ and γ_k , which is consistent with economic intuition. As γ_k measures investor's risk preference, a higher γ_k indicates stronger risk aversion, thereby reducing the proportion of wealth allocated to risky assets.

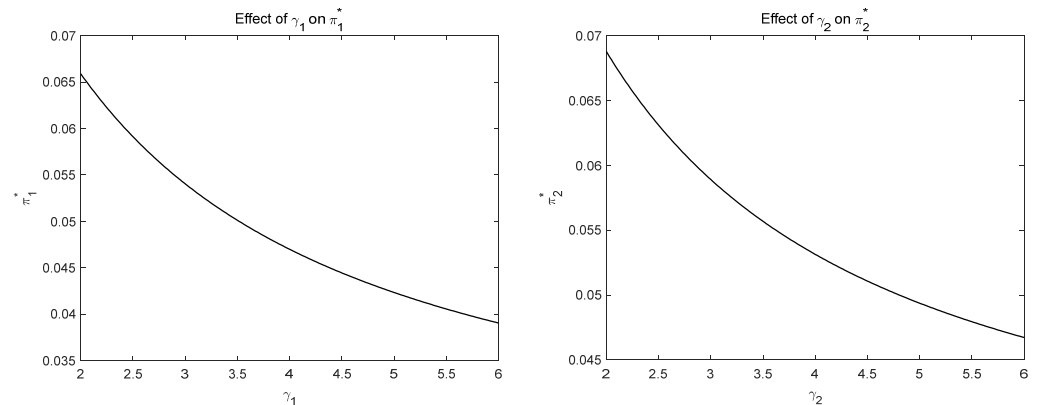


Figure 1. Impact of γ_k on equilibrium investment strategies $\pi_k^*(0)$.

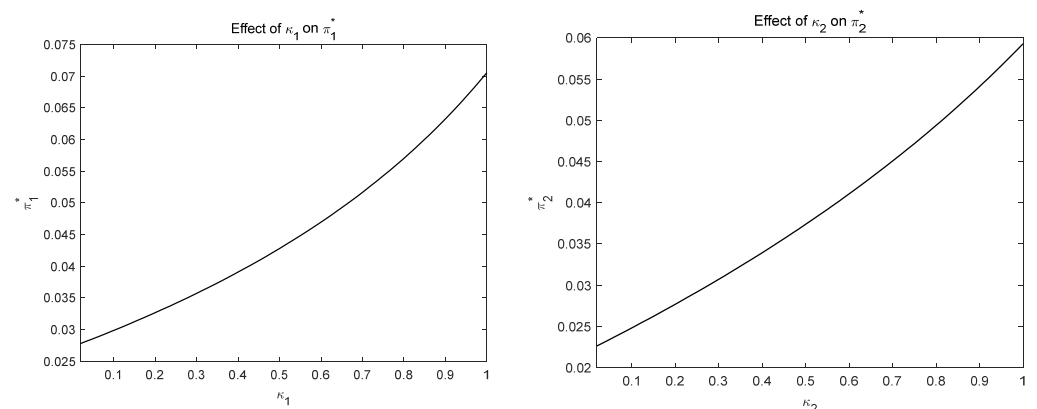


Figure 2. Impact of κ_k on equilibrium investment strategies $\pi_k^*(0)$.

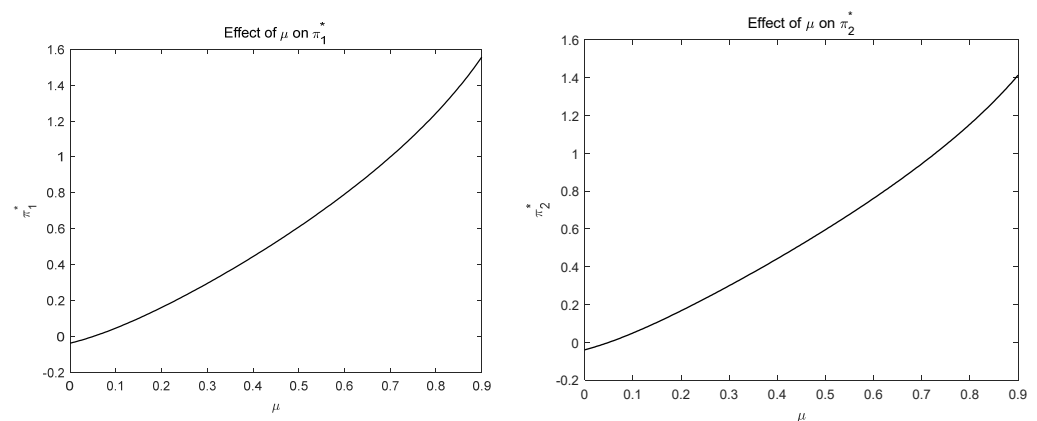


Figure 3. Impact of μ on equilibrium investment strategies $\pi_k^*(0)$.

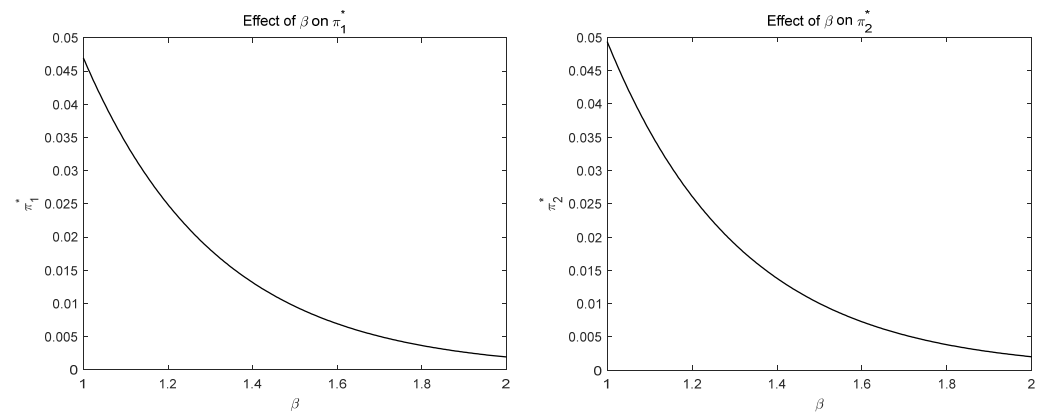


Figure 4. Impact of β on equilibrium investment strategies $\pi_k^*(0)$.

Figure 2 reveals the impact of the market competition coefficient κ_k on the equilibrium investment strategy $\pi_k^*(0)$. The graph indicates that $\pi_k^*(0)$ rises as κ_k increases, a monotonic relationship consistent with the findings of Espinosa and Touzi (2015) [3] and Kraft et al. (2020) [19]. The former adopted the CARA utility function to depict investors' risk attitudes, while the latter applied the CRRA utility function. No matter which utility form is used, a larger κ_k implies that investors pay more attention to relative wealth. In such cases, to obtain competitive advantages, investors will adopt more aggressive strategies and raise the share of risky assets in their portfolios.

Figure 3 depicts the impact of the expected return μ of the risky asset on the equilibrium investment strategy $\pi_k^*(0)$. There exists a significant positive correlation: $\pi_k^*(0)$ grows along with the rise in expected return μ . Intuitively, a higher expected return brings higher potential profits from risky assets, so rational investors will increase their holding of risky assets. This outcome aligns with the basic assumption of rational investors in economics and finance.

Figure 4 illustrates the impact of the parameter β on the equilibrium investment strategy $\pi_k^*(0)$. The figure shows a negative correlation between the two variables. When β increases, the price volatility and investment risk of the risky asset will increase. For risk-averse investors, they will accordingly cut down the proportion invested in risky assets, which conforms to the general investment logic of rational market participants.

6. Conclusions

Financial markets are complex and dynamic. To better characterize the volatility features of risky asset prices, this study adopted the CEV model to describe the price dynamics of risky assets, constructing a financial market consisting of one risk-free bond and one risky stock modeled under the CEV framework. Considering the stochastic game interactions between market participants, we establish a two-player non-zero-sum stochastic differential investment game, where investors' relative performance is defined as the weighted average of terminal wealth ratios. With the dynamic programming principle, we derive the sufficient conditions for the existence of Nash equilibrium and obtain the explicit analytical solutions to equilibrium strategies under the CRRA utility functions. Numerical simulations were carried out to discuss how model parameter affect equilibrium strategies. The simulation results reveal that the expected return of the risky asset, the constant elasticity of variance factor, the risk aversion coefficient, and the market competition parameter all exert significant influence on the equilibrium investment strategies. Specifically, investors tend to increase their allocation to risky assets when the expected return of the risky asset is higher, the constant elasticity of variance factor is smaller, the risk aversion coefficient is lower, or the market competition parameter is larger—and vice

versa. To some extent, this indicates that intensified competition in financial markets leads to heightened rivalry among investors, prompting each to increase investments in risky assets and thereby contribute to investment exuberance in financial markets.

Author Contributions: Methodology, M.C.; software, H.Z.; formal analysis, S.C.; data curation, H.Z.; writing—original draft, S.C.; writing—review and editing, H.Z.; supervision, M.C.; project administration, M.C. and H.Z.; funding acquisition, S.C. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Young Scientists Fund of the National Natural Science Foundation of China (No. 72103049), Basic and Applied Basic Research Foundation of Guangdong Province (No. 2021A1515110139), Philosophy and Social Sciences “14th Five-Year Plan” Project of Guangdong Province (No. GD23XGL128), General Program of Humanities and Social Sciences Research, Ministry of Education of China (No. 25YJC630009), and the Guangzhou “14th Five-Year Plan” Philosophy and Social Science Project (No. 2025GZGJ120).

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflict of interest.

References

- Hortaçsu, A.; Syverson, C. Product differentiation, search costs, and competition in the mutual fund industry: A case study of S&P 500 index funds. *Q. J. Econ.* **2004**, *119*, 403–456. [\[CrossRef\]](#)
- Dong, Y.H.; Wei, S.Y.; Yin, Z.; Wang, L. Optimal investment for asset managers based on relative performance with inflation risk. *Chin. J. Eng. Math.* **2023**, *40*, 20–40.
- Espinosa, G.E.; Touzi, N. Optimal investment under relative performance concerns. *Math. Financ.* **2015**, *25*, 221–257.
- Li, L.; Qiu, Z.J. The investment and reinsurance game on asset-liability management with common shock dependence under CEV model. *RAIRO-Oper. Res.* **2025**, *59*, 3913–3944. [\[CrossRef\]](#)
- Luo, X.; Zhou, Q. Stochastic differential games on robust optimal asset-liability management with delay under the CEV model. *J. Ind. Manag. Optim.* **2025**, *21*, 1771–1796. [\[CrossRef\]](#)
- Yang, L.; Zhu, H.N.; Zhang, C.K. Portfolio game with liability based on the Heston stochastic volatility model. *Oper. Res. Manag. Sci.* **2020**, *29*, 27–34.
- Bensoussan, A.; Siu, C.C.; Yam, S.C.P.; Yang, H. A class of non-zero-sum stochastic differential investment and reinsurance games. *Automatica* **2014**, *50*, 2025–2037. [\[CrossRef\]](#)
- Siu, C.C.; Yam, S.C.P.; Yang, H.; Zhao, H. A class of nonzero-sum investment and reinsurance games subject to systematic risks. *Scand. Actuar. J.* **2017**, *2017*, 670–707.
- Deng, C.; Zeng, X.; Zhu, H. Non-zero-sum stochastic differential reinsurance and investment games with default risk. *Eur. J. Oper. Res.* **2018**, *264*, 1144–1158.
- Bin, N.; Zhu, H.N. Stochastic differential investment and reinsurance strategy with ambiguity aversion and delay. *Syst. Eng.-Theory Pract.* **2021**, *41*, 1439–1453.
- Bin, N.; Zhu, H.; Liu, Q. Non-zero-sum stochastic differential games on investment, consumption and proportional reinsurance. *J. Ind. Manag. Optim.* **2024**, *20*, 2032–2049. [\[CrossRef\]](#)
- Bai, Y.F.; Zhou, Z.B.; Xiao, H.L.; Gao, R.; Zhong, F. A hybrid stochastic differential reinsurance and investment game with bounded memory. *Eur. J. Oper. Res.* **2022**, *296*, 717–737. [\[CrossRef\]](#)
- Zhou, Z.B.; Ren, T.T.; Xiao, H.L.; Wu, S.; Liu, W. Multi-period portfolio game model based on relative wealth utility. *Chin. J. Manag. Sci.* **2019**, *27*, 34–43.
- Zhou, Z.B.; Ren, T.T.; Xiao, H.L.; Jin, Q.; Wu, S. Multi-period portfolio game model with serially correlated returns. *J. Manag. Sci. China* **2019**, *22*, 66–88.
- Lin, X.; Wu, X.P.; Qian, Y.P. Optimal portfolio selection game problem with interacting agents under relative performance concerns. *J. Quant. Econ.* **2020**, *37*, 53–62.
- Basak, S.; Makarov, D. Strategic asset allocation in money management. *J. Financ.* **2014**, *69*, 179–217. [\[CrossRef\]](#)
- Lacker, D.; Zariphopoulou, T. Mean field and n-agent games for optimal investment under relative performance criteria. *Math. Financ.* **2019**, *29*, 1003–1038. [\[CrossRef\]](#)
- Lacker, D.; Soret, A. Many-player games of optimal consumption and investment under relative performance criteria. *Math. Financ. Econ.* **2020**, *14*, 263–281. [\[CrossRef\]](#)

19. Kraft, H.; Meyer-Wehmann, A.; Seifried, F.T. Dynamic asset allocation with relative wealth concerns in incomplete markets. *J. Econ. Dyn. Control* **2020**, *113*, 103857. [[CrossRef](#)]
20. Zhu, H.N.; Bin, N.; Liu, Q.Y. Stochastic differential games for optimal investment and risk control problems in an incomplete market. *J. Dyn. Games* **2026**, *13*, 104–129. [[CrossRef](#)]
21. Wu, H.; Ma, C.Q. Non-zero-sum stochastic differential portfolio game under a constant elasticity of variance model. *Syst. Eng.* **2015**, *33*, 1–7.
22. Richard, E.B. *Dynamic Programming*; Princeton University Press: Princeton, NJ, USA, 1957.
23. Liberzon, D. *Calculus of Variations and Optimal Control Theory: A Concise Introduction*; Princeton University Press: Princeton, NJ, USA, 2012.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.