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A New Generalization of Two-Variable q-Gould–Hopper–Hahn–Appell Polynomials via Quantum q-Calculus

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Abstract

In this paper, we introduce a class of two-variable q-Gould–Hopper–Hahn–Appell polynomials obtained by combining a Hahn-type q-Gould–Hopper kernel with a general q-Appell factor. Using a formal generating function approach, we derive explicit series representations, connection and convolution formulas, higher and mixed q-derivative identities, inverse operational formulas, two-direction shift relations, and a determinant representation in the non-singular case $\mathcal{A}_{0,q} \neq 0$. We then develop an operational description of the family and establish its quasi-monomial structure, which yields the corresponding lowering operator, an induced raising operator, and the associated q-differential equation. We also discuss reductions to previously known subclasses, distinguish clearly between non-singular and singular q-Appell specializations, and include representative examples illustrating how an Appell factor affects the low-degree zero pattern.

Keywords: q-calculus; two-variable q-Gould–Hopper–Hahn–Appell polynomials; Appell polynomials; quasi-monomiality; operational identities; fractional derivatives; q-difference equations; zero distribution

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1. Introduction

Appell-type polynomial sequences, Gould–Hopper polynomials, and their q-extensions play an important role in the theory of special functions, generating functions, and operational methods. Their significance lies in the fact that they connect algebraic structures with analytic tools such as recurrence relations, operational identities, and differential or q-difference equations.

From the viewpoint of mathematical physics and computational modelling, the value of such polynomial systems is not limited to formal algebra: generating kernels, deformed shift parameters, and operator factorizations provide reusable analytic templates for future numerical schemes. Related computational studies, including quantum particle swarm optimization for mobile-edge task offloading [1] and active learning multiclass support vector machine methods for network traffic classification [2], illustrate application scenarios in which algebraic kernels and parameterized bases can support optimization, classification, and model reduction analogies.

The classical background for these constructions is broad and each source contributes a specific component to the present framework. Andrews, Askey, and Roy [3] give the standard special function setting in which many hypergeometric and q-hypergeometric polynomial families are placed. Srivastava and Manocha [4] provide the generating function techniques that underlie Appell and Gould–Hopper-type constructions. Ernst [5] gives a systematic treatment of q-calculus and its associated polynomial structures. Kac and Cheung [6] present the elementary quantum-calculus tools used for q-difference operators. Floreanini and Vinet [7] illustrate the connection between quantum algebras and q-special functions. Choi [8] surveys recent applications of special functions, while Zayed et al. [9] show how q-Hermite–Appell-type polynomials can be studied through multidimensional graphical exploration.

In recent years, considerable attention has been devoted to hybrid polynomial families obtained by combining an Appell kernel with an additional generating factor. Such constructions often preserve the Appell character while enriching the underlying algebraic and operational structure.

Ahmad and Khan [10] developed a q-Laguerre-based Appell generalization and studied its quasi-monomiality. Fadel et al. [11] considered bivariate q-Laguerre–Appell polynomials together with applications. Costabile, Khan, and Ali [12] treated q-truncated exponential–Appell polynomials, whereas Khan, Yasmin, and Ahmad [13] investigated truncated exponential-based Appell polynomials in the non-q setting. Khan et al. [14] studied q-Appell convolutions within quantum calculus, and Özat et al. [15] developed truncated-exponential-based general Appell polynomials. In addition, Askey's limit results for q-Laguerre polynomials [16] remain relevant because they clarify how q-families behave under limiting processes and how classical polynomial families can be recovered.

A horizontal comparison with closely related mixed q-Appell models is useful for identifying the precise contribution of the present construction. Hahn–Appell polynomials [17] contain the Hahn translation parameter w and a one-variable Appell factor, but they do not include the Gould–Hopper higher-order variable yt^m. The q-Gould–Hopper–Appell family studied in [18] contains the higher-order Gould–Hopper deformation, but it is not built on the Hahn basis $(x)^n_{q,w}$ and therefore does not describe the additional Hahn shift geometry. The q-Hermite–Appell and q-Laguerre–Appell families [10,11,19] specialize the auxiliary kernel in different directions and are well adapted to their own recurrence and application settings; however, their kernels do not simultaneously encode an arbitrary Appell factor, a Hahn-type deformation, and a Gould–Hopper order parameter m.

Related Model	**Main Structure**	**Difference from the Present Model**
Hahn–Appell polynomials	Hahn shifted basis with one Appell factor	No Gould–Hopper higher-order variable yt^m
q-Gould–Hopper–Appell polynomials	Appell factor plus higher-order Gould–Hopper term	No Hahn basis $(x)^n_{q,w}$ and no w-shift geometry
q-Hermite–Appell and q-Laguerre–Appell families	special auxiliary kernels adapted to Hermite or Laguerre settings	No simultaneous arbitrary Appell factor, Hahn deformation, and Gould–Hopper order m
Present q-Gould–Hopper–Hahn–Appell model	$\mathcal{A}_q(t)e_q^{\widetilde{w}t}(xt)e_q^{wt}(yt^m)$	Combines all three layers and yields determinant, operational, shift, reduction, and zero-geometry consequences

The present model fills the gap by keeping the Appell coefficients, the Hahn shift parameter w, and the Gould–Hopper order m simultaneously active. Thus the Appell coefficients control the Sheffer–Appell component, the Hahn kernel controls the w-shifted basis, and the Gould–Hopper term controls the higher-order m-direction. This separation

of roles is the main theoretical advantage of the proposed model and explains why the family can recover existing subclasses while still producing new determinant, operational, shift, and zero-distribution formulas.

In this context, q-calculus provides a natural setting for extending classical Appell-type constructions. Throughout this paper, we assume $0 < q < 1$ and use the standard notation

$$[n]_q = \frac{1-q^n}{1-q}, \qquad [n]_q! = \prod_{k=1}^{n} [k]_q,$$

together with the q-exponential function

$$e_q(\tau) = \sum_{n=0}^{\infty} \frac{\tau^n}{[n]_q!}.$$

We also use the usual q-derivative operator

$$\widehat{D}_{q,\tau} h(\tau) = \frac{h(\tau) - h(q\tau)}{\tau(1-q)},$$

which plays a central role in the operational description of the polynomial families considered here.

Throughout the paper, the elementary q-derivative identities are written consistently with respect to the differentiation variable τ; the symbols such as α are parameters independent of τ. Thus,

$$\widehat{D}_{q,\tau} \tau^r = [r]_q \tau^{r-1}, \qquad r \in \mathbb{N}, \tag{1}$$

$$\widehat{D}_{q,\tau} e_q(\alpha\tau) = \alpha e_q(\alpha\tau), \qquad \alpha \in \mathbb{C}, \tag{2}$$

and, more generally,

$$\widehat{D}^k_{q,\tau} e_q(\alpha\tau) = \alpha^k e_q(\alpha\tau), \qquad k \in \mathbb{N}, \quad \alpha \in \mathbb{C}. \tag{3}$$

For two functions $f(\tau)$ and $g(\tau)$, the q-Leibniz rule is [20]

$$\widehat{D}_{q,\tau}(f(\tau)g(\tau)) = f(\tau)\widehat{D}_{q,\tau} g(\tau) + g(q\tau)\widehat{D}_{q,\tau} f(\tau). \tag{4}$$

The essential ingredient is the q-Appell family, characterized by the generating function [21–23]

$$\mathcal{A}_q(t) e_q(\tau t) = \sum_{n=0}^{\infty} \mathcal{A}_{n,q}(\tau) \frac{t^n}{[n]_q!}, \tag{5}$$

where

$$\mathcal{A}_q(t) = \sum_{n=0}^{\infty} \mathcal{A}_{n,q} \frac{t^n}{[n]_q!}, \qquad \mathcal{A}_{0,q} \neq 0. \tag{6}$$

This non-singularity condition ensures that the reciprocal series $1/\mathcal{A}_q(t)$ exists whenever determinant and convolution-type representations are required.

The two types of Hahn-type q-exponential functions are defined coefficientwise by [17]

$$e_q^{wt}(zt) := \sum_{n=0}^{\infty} (z)^n_{q,w} \frac{t^n}{[n]_q!}, \tag{7}$$

and

$$E_q^{wt}(zt) := \sum_{n=0}^{\infty} [z]_{q,w}^n \frac{t^n}{[n]_q!}. \tag{8}$$

In this paper, these functions are used through their coefficient expansions and the formal Hahn product rules stated below. In particular, no z-independent infinite product representation is used for either side of (7) or (8). The formal inverse relation is written as

$$e_q^{wt}((\ominus_{q,w} z)t) e_q^{wt}(zt) = 1, \tag{9}$$

where $\ominus_{q,w} z$ denotes the formal Hahn inverse of z. In the limiting case $w = 0$, one obtains $e_q^0(zt) = e_q(zt)$, while $E_q^0(zt)$ is the standard companion q-exponential generated by the factors $[z]_{q,0}^n = q^{n(n-1)/2} z^n$; both tend to the ordinary exponential as $q \to 1^-$. The Hahn polynomial bases in the variable z are $\{(z)_{q,w}^n\}_{n\geq 0}$ and $\{[z]_{q,w}^n\}_{n\geq 0}$, given by

$$(z)_{q,w}^n = z(z-w)(z-[2]_q w)(z-[3]_q w)\cdots(z-[n-1]_q w), \tag{10}$$

$$[z]_{q,w}^n = z(qz+w)(q^2 z+[2]_q w)\cdots(q^{n-1} z+[n-1]_q w). \tag{11}$$

Applying the Hahn difference operator with respect to z to (10) and (11), we obtain

$$\widehat{\mathcal{D}}_{q,w,z}(z)_{q,w}^n = [n]_q [z]_{q,w}^{n-1},$$

$$\widehat{\mathcal{D}}_{q,w,z}[z]_{q,w}^n = [n]_q [qz+w]_{q,w}^{n-1},$$

where $w > 0$, $q \in (0,1)$, and $n \geq 1$. Following [17], the associated one-variable Hahn–Appell kernel is defined by

$$\mathcal{A}_q(t) e_q^{wt}(xt) = \sum_{n=0}^{\infty} \mathcal{P}_n(x; w, q) \frac{t^n}{[n]_q!}. \tag{12}$$

The q-dilatation operator T_τ, acting on a function $f(\tau)$ of one complex variable, is defined by [20]

$$T_\tau^k f(\tau) = f(q^k \tau), \quad k \in \mathbb{R}, \quad 0 < q < 1, \tag{13}$$

where

$$T_\tau^{-1} T_\tau^1 f(\tau) = f(\tau). \tag{14}$$

For the composite q-exponential $e_q(\lambda \tau^m)$, we use the identity [20]

$$\widehat{D}_{q,\tau} e_q(\lambda \tau^m) = \lambda \tau^{m-1} T_{(\lambda;m)} e_q(\lambda \tau^m), \tag{15}$$

where

$$T_{(\lambda;m)} = \frac{1 - q^m T_\lambda^m}{1 - q T_\lambda} = 1 + q T_\lambda + \cdots + q^{m-1} T_\lambda^{m-1}.$$

In recent developments, Cao et al. [20] extended the monomiality principle to q-polynomial sequences, thereby providing an operational framework for quasi-monomial behavior. This framework is useful for q-special polynomials, including the two-variable q-Laguerre polynomials studied in [24]. In this setting, the operators $\hat{\mathcal{M}}_q$ and $\hat{\mathcal{D}}_q$ act as multiplicative and derivative operators on a polynomial sequence $\{\Phi_{n,q}(x)\}_{n\geq 0}$. To avoid confusion with the fixed Gould–Hopper order m used later, the sequence index is denoted here by n. The quasi-monomial relations are

$$\Phi_{n+1,q}(x) = \hat{\mathcal{M}}_q \{\Phi_{n,q}(x)\} \tag{16}$$

and

$$[n]_q \, \Phi_{n-1,q}(x) = \hat{\mathcal{D}}_q\{\Phi_{n,q}(x)\}, \qquad n \geq 1. \tag{17}$$

The commutator of these operators is written as

$$[\hat{\mathcal{D}}_q, \hat{\mathcal{M}}_q] = \hat{\mathcal{D}}_q\hat{\mathcal{M}}_q - \hat{\mathcal{M}}_q\hat{\mathcal{D}}_q. \tag{18}$$

The properties of the quasi-monomial set $\{\Phi_{n,q}(x)\}_{n\geq 0}$ are governed by the interaction of $\hat{\mathcal{M}}_q$ and $\hat{\mathcal{D}}_q$. In particular, the following standard consequences are used later:

(i) The function $\Phi_{n,q}(x)$ satisfies

$$\hat{\mathcal{M}}_q\hat{\mathcal{D}}_q\{\Phi_{n,q}(x)\} = [n]_q \, \Phi_{n,q}(x) \tag{19}$$

and

$$\hat{\mathcal{D}}_q\hat{\mathcal{M}}_q\{\Phi_{n,q}(x)\} = [n+1]_q \, \Phi_{n,q}(x), \tag{20}$$

whenever the involved operators are well defined on the polynomial sequence.

(ii) The explicit operational form is

$$\Phi_{n,q}(x) = \hat{\mathcal{M}}_q{}^n \, \{1\}, \tag{21}$$

with $\Phi_{0,q}(x) = 1$.

(iii) The corresponding generating relation is

$$\mathfrak{E}_q\left\{t\hat{\mathcal{M}}_q\right\}\{1\} = \sum_{n=0}^{\infty} \Phi_{n,q}(x)\frac{t^n}{[n]_q!}, \qquad |t| < \infty, \tag{22}$$

which follows from (21).

Motivated by the interaction between the q-Appell structure and the Hahn-type q-Gould–Hopper kernel, in this paper, we introduce and study a new class of polynomials, called the q-Gould–Hopper–Hahn–Appell polynomials. This family unifies several previously known polynomial sequences and provides a natural extension of both the q-Appell and the q-Gould–Hopper settings.

The importance of the proposed family is twofold. On the structural side, it supplies a single generating function model in which Appell-type coefficient sequences, Hahn shifts, and Gould–Hopper higher-order terms can be handled simultaneously. On the application side, such a unified basis can be used in operational expansions for linear q-difference models: the lowering operator, raising operator, and convolution relations make it possible to transform formal q-differential equations into algebraic relations among expansion coefficients. The numerical zero-distribution examples included below also show a concrete application to the qualitative analysis of polynomial profiles, where the Appell factor acts as a controllable deformation parameter for the location and symmetry of zeros.

The main contributions of the paper are as follows. First, we derive generating functions, explicit series representations, and convolution-type formulas for the new family. Next, we obtain determinant representations, derivative formulas, addition formulas, and summation identities. We then establish an operational characterization and a quasi-monomial structure, which lead to the lowering operator, an induced raising operator, and the corresponding q-differential equation. Finally, we discuss reductions to known special cases and present graphical illustrations concerning the distribution of zeros and representative polynomial profiles.

2. q-Gould–Hopper–Hahn–Appell Polynomials

In this section, we introduce the q-Gould–Hopper–Hahn polynomials $\mathcal{H}_n^{(m)}(x,y;w,q)$ and the two-variable q-Gould–Hopper–Hahn–Appell polynomials, denoted by ${}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q)$, and derive the basic identities needed later. We begin with the following definitions.

Let $m \in \mathbb{N}, \quad 0 < q < 1, \quad x,y,w \in \mathbb{C}$. The two-variable q-Gould–Hopper–Hahn polynomials $\mathcal{H}_n^{(m)}(x,y;w,q)$ are defined by

$$e_q^{wt}(xt)e_q^{wt}(yt^m) = \sum_{n=0}^{\infty} \mathcal{H}_n^{(m)}(x,y;w,q)\frac{t^n}{[n]_q!}, \tag{23}$$

and hence,

$$\mathcal{H}_n^{(m)}(x,y;w,q) = [n]_q! \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(x)_{q,w}^{n-mk}(y)_{q,w}^{k}}{[k]_q![n-mk]_q!}. \tag{24}$$

The two-variable q-Gould–Hopper–Hahn polynomials $\mathcal{H}_n^{(m)}(x,y;w,q)$ are quasi-monomial in the following formal sense. For notational clarity, the subscript qGHH in the operator symbols below abbreviates the underlying q-Gould–Hopper–Hahn polynomial family; it is not an additional variable, parameter, or polynomial sequence. Their lowering operator is

$$\widehat{P}_{\mathrm{qGHH}} = \widehat{D}_{q,x}, \qquad \widehat{P}_{\mathrm{qGHH}}\mathcal{H}_n^{(m)}(x,y;w,q) = [n]_q\mathcal{H}_{n-1}^{(m)}(x,y;w,q), \tag{25}$$

and the associated raising operator is defined on the polynomial basis by

$$\widehat{M}_{\mathrm{qGHH}}\mathcal{H}_n^{(m)}(x,y;w,q) = \mathcal{H}_{n+1}^{(m)}(x,y;w,q), \qquad n \geq 0. \tag{26}$$

Consequently, the corresponding q-differential equation is

$$\left(\widehat{M}_{\mathrm{qGHH}}\widehat{D}_{q,x} - [n]_q\right)\mathcal{H}_n^{(m)}(x,y;w,q) = 0. \tag{27}$$

No universal closed formula for $\widehat{M}_{\mathrm{qGHH}}$ in terms of x, y, T_x, T_y, and powers of $\widehat{D}_{q,x}$ is asserted here; such a formula requires additional operator commutation assumptions beyond the generating function identities used in this paper.

Motivated by Equations (12) and (23), we define the two-variable q-Gould–Hopper–Hahn–Appell polynomials, denoted by ${}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q)$. The generating functions are interpreted coefficientwise in the formal power-series ring $\mathbb{C}[[t]]$. Whenever an analytic interpretation is needed, we additionally assume that the relevant kernels converge in a common neighborhood of the origin; reciprocal-series arguments are then restricted to the non-singular case $\mathcal{A}_{0,q} \neq 0$. The defining generating function is

$$\mathcal{A}_q(t)e_q^{wt}(xt)e_q^{wt}(yt^m) = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q)\frac{t^n}{[n]_q!}. \tag{28}$$

All identities in the sequel are first understood in the formal power-series ring $\mathbb{C}[[t]]$. Under this interpretation, coefficient comparison, Cauchy products, reciprocal-series arguments in the non-singular case $\mathcal{A}_{0,q} \neq 0$, and the operational identity involving $e_q^{wt}(y\widehat{D}_{q,x}^m)$ are algebraic statements. In particular, when the operator acts on a polynomial, the series terminates after finitely many applications of $\widehat{D}_{q,x}^m$, so no analytic convergence condition is needed.

For an analytic reading, assume that $\mathcal{A}_q(t)$ has a positive radius of convergence $R_{\mathcal{A}}$ and that the two Hahn kernels converge absolutely and locally uniformly for $|t| < R_x$ and $|t| < R_y$, respectively. Then (28) is valid in every disc $|t| < R$, where

$$0 < R < \min\{R_{\mathcal{A}}, R_x, R_y\}.$$

In this analytic setting, operator formulas are used on functions for which the corresponding q-difference operators preserve the chosen analytic function space and the relevant operator series converges locally uniformly. The determinant construction remains restricted to the non-singular case $\mathcal{A}_{0,q} \neq 0$; singular kernels, especially the Genocchi kernel, are treated separately in Section 4 through degree-shift identities.

Example 1. *The zeros of the fully generic family in* (28) *depend on the arbitrary Appell factor* $\mathcal{A}_q(t)$*. Therefore, for a reproducible numerical illustration, take*

$$\mathcal{A}_q(t) = 1, \qquad q = 0.6, \qquad w = 0, \qquad m = 2.$$

Then ${}_{\mathcal{H}}\mathcal{A}_n^{(2)}(x, y; 0, 0.6) = H_{n,0.6}^{(2)}(x, y)$ *and* (24) *gives*

$$\begin{aligned}
H_{2,0.6}^{(2)}(x, y) &= x^2 + 1.6y,\\
H_{3,0.6}^{(2)}(x, y) &= x^3 + 3.136xy,\\
H_{4,0.6}^{(2)}(x, y) &= x^4 + 4.26496x^2y + 4.26496y^2,\\
H_{5,0.6}^{(2)}(x, y) &= x^5 + 5.0169856x^3y + 9.833291776xy^2.
\end{aligned}$$

For the fixed section $y = -0.3$*, the corresponding one-variable zero values are*

$$\begin{aligned}
H_{2,0.6}^{(2)}(x, -0.3) = 0 \quad &\Rightarrow \quad x \approx \pm 0.6928203230,\\
H_{3,0.6}^{(2)}(x, -0.3) = 0 \quad &\Rightarrow \quad x \approx 0,\ \pm 0.9699484522,\\
H_{4,0.6}^{(2)}(x, -0.3) = 0 \quad &\Rightarrow \quad x \approx \pm 0.6930286552,\ \pm 0.8939794585,\\
H_{5,0.6}^{(2)}(x, -0.3) = 0 \quad &\Rightarrow \quad x \approx 0,\ \pm(0.9201332121 \pm 0.3067527974i).
\end{aligned}$$

The two-variable zero set is a real algebraic curve in the (x, y)*-plane; the surfaces in Figure 1 show the full two-variable profiles together with the projected zero contour.*

In the operational formulas below, we work at the level of formal power series and use the Hahn identities

$$\widehat{D}_{q,\xi} e_q^{wt}(\xi t) = t\, e_q^{wt}(\xi t), \qquad \widehat{D}_{q,\xi}^r e_q^{wt}(\xi t) = t^r e_q^{wt}(\xi t), \quad r \in \mathbb{N}_0, \tag{29}$$

and the multiplicative law with respect to the formal Hahn sum $\oplus_{q,w}$

$$e_q^{wt}((\xi \oplus_{q,w} \eta)t) = e_q^{wt}(\xi t) e_q^{wt}(\eta t). \tag{30}$$

We also write $\ominus_{q,w}\delta$ for the formal Hahn inverse of δ, characterized by

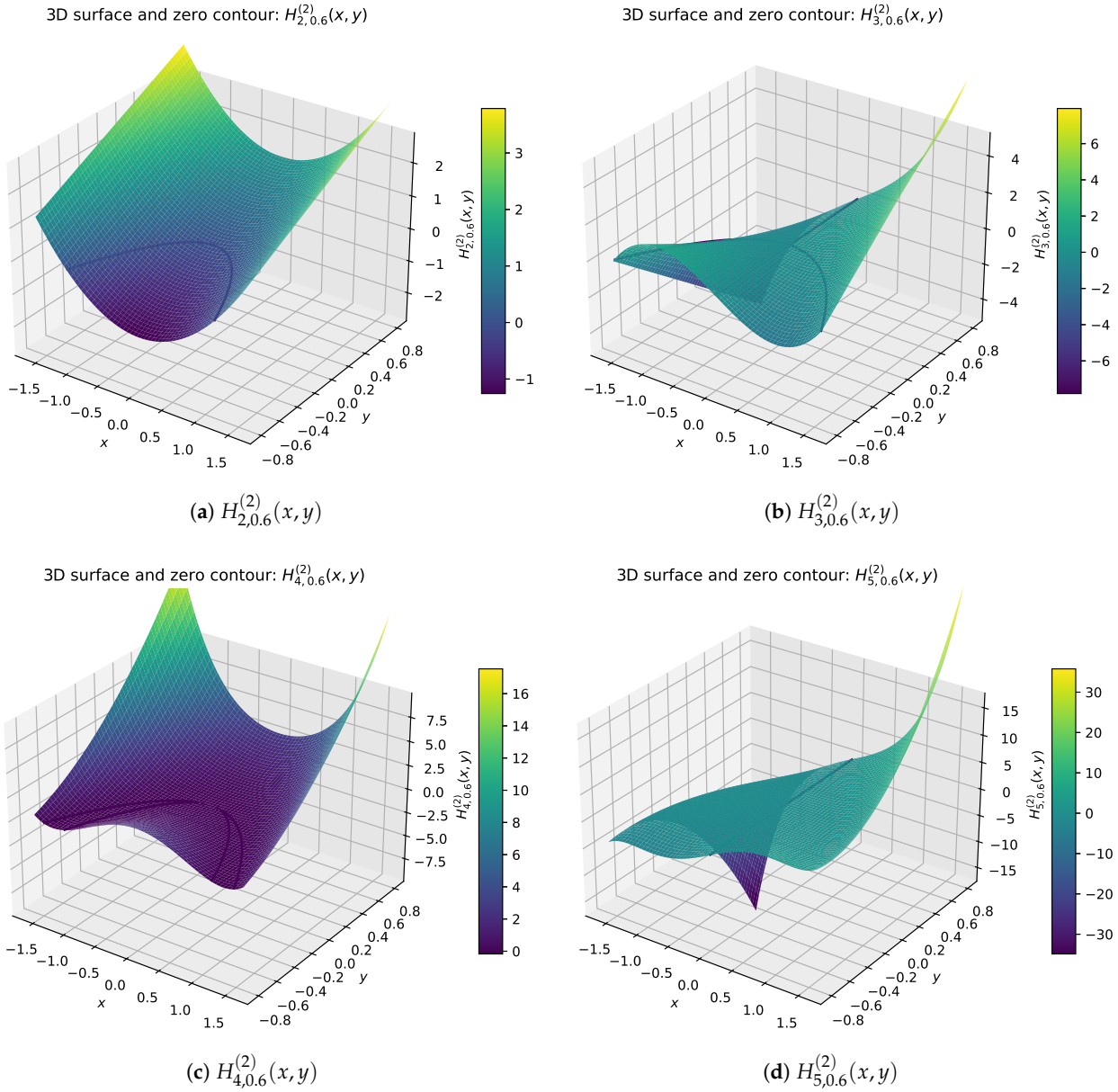
$$e_q^{wt}((\ominus_{q,w}\delta)t) e_q^{wt}(\delta t) = 1.$$

These relations are used as defining formal rules for the Hahn-type kernel throughout the operational part of the paper. Consequently, every identity in which e_q^{wt} is manipulated through (29) and (30) is to be interpreted coefficientwise in the ring of formal power

series. In particular, the quasi-monomial statements below are formal consequences of the generating function together with the assumed Hahn–Appell rules, rather than analytic assertions about a separately constructed Hahn operator model. Ordinary addition is not being asserted in (30); all shift formulas below are written with the Hahn sum $\oplus_{q,w}$.

Equivalently, applying the formal operator $e_q^{wt}\left(y\widehat{D}_{q,x}^m\right)$ to the one-variable kernel yields

$$e_q^{wt}\left(y\widehat{D}_{q,x}^m\right)\mathcal{A}_q(t)e_q^{wt}(xt) = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q)\frac{t^n}{[n]_q!}. \tag{31}$$



(a) $H^{(2)}_{2,0.6}(x,y)$ **(b)** $H^{(2)}_{3,0.6}(x,y)$

(c) $H^{(2)}_{4,0.6}(x,y)$ **(d)** $H^{(2)}_{5,0.6}(x,y)$

Figure 1. Three-dimensional surfaces of the canonical q-Gould–Hopper–Hahn–Appell specialization $\mathcal{A}_q(t) = 1, q = 0.6, w = 0, m = 2$. The zero-level contour is projected onto the surface plot to visualize the real two-variable zero set.

Remark 1. *As $w \to 0^+$, Equation (23) reduces formally to the two-variable q-Gould–Hopper–Appell family [18]*

$$\mathcal{A}_q(t)e_q(xt)e_q(yt^m) = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{n,q}^{(m)}(x,y)\frac{t^n}{[n]_q!}. \tag{32}$$

Remark 2. *If, in addition, $q \to 1^-$, then (23) tends formally to the classical two-variable Gould–Hopper–Appell family ${}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y)$ (see [25–27]).*

Theorem 1. *The two-variable q-Gould–Hopper–Hahn–Appell polynomials admit the series representation*

$$ {}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = [n]_q! \sum_{k=0}^{\lfloor n/m \rfloor} \frac{\mathcal{P}_{n-mk}(x;w,q)(y)_{q,w}^k}{[k]_q![n-mk]_q!}. \tag{33} $$

Proof. Expand the one-variable kernel (12) and the second Hahn-type exponential in (7), then compare coefficients of t^n. □

Theorem 2. *For every $n \geq 0$, the convolution identity*

$$ {}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = \sum_{k=0}^{n} \binom{n}{k}_q \mathcal{A}_{k,q}\mathcal{H}_{n-k}^{(m)}(x,y;w,q) \tag{34} $$

holds, where the coefficients $\mathcal{A}_{k,q}$ are defined by (6).

Proof. By (23),

$$ \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q)\frac{t^n}{[n]_q!} = \mathcal{A}_q(t)\sum_{n=0}^{\infty}\mathcal{H}_n^{(m)}(x,y;w,q)\frac{t^n}{[n]_q!}. \tag{35} $$

Insert the expansion (6) of $\mathcal{A}_q(t)$ and use the Cauchy product. Comparing coefficients of t^n yields (34). □

The determinant form provides a compact and systematic representation of the polynomials ${}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q)$ in terms of the reciprocal coefficients of the q-Appell kernel and the associated q-Gould–Hopper–Hahn polynomials. This representation is valid in the non-singular case $\mathcal{A}_{0,q} \neq 0$, where the reciprocal series of $\mathcal{A}_q(t)$ exists as a formal power series. Its importance lies in the fact that it transforms the coefficient relations obtained from the generating function into a triangular algebraic system, which can be solved explicitly by Cramer's rule. Consequently, the determinant formula not only gives an explicit closed representation of the polynomial sequence, but also reveals how the Appell factor modifies the underlying q-Gould–Hopper–Hahn structure through the reciprocal coefficients $\beta_{n,q}$. Such determinantal representations are particularly useful for deriving recurrence relations, studying special cases, and connecting the present family with the general theory of q-Appell-type polynomial systems.

Determinantal representations are often useful in the study of Appell-type families because they encode the coefficient recursion in a compact algebraic form. Following the determinantal approach developed for q-Appell polynomials in [28,29], we obtain the following representation.

Theorem 3. *Assume in addition that $\mathcal{A}_{0,q} \neq 0$, so that the reciprocal series exists formally, and let*

$$ \frac{1}{\mathcal{A}_q(t)} = \sum_{n=0}^{\infty}\beta_{n,q}\frac{t^n}{[n]_q!}, \qquad \beta_{0,q} = \frac{1}{\mathcal{A}_{0,q}} \neq 0. \tag{36} $$

Then the reciprocal coefficients satisfy, for $n \geq 1$,

$$ \beta_{n,q} = -\frac{1}{\mathcal{A}_{0,q}}\sum_{k=1}^{n}\binom{n}{k}_q \mathcal{A}_{k,q}\beta_{n-k,q}, \tag{37} $$

with the initial value $\beta_{0,q}$ *given in* (36)*. Moreover,*

$$ {}_{\mathcal{H}}\mathcal{A}_0^{(m)}(x,y;w,q) = \mathcal{A}_{0,q}\mathcal{H}_0^{(m)}(x,y;w,q) = \mathcal{A}_{0,q}, $$

and for $n \geq 1$*, the determinant form of* ${}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q)$ *is*

$$ {}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = \frac{1}{(\beta_{0,q})^{n+1}} \begin{vmatrix} \beta_{0,q} & \beta_{1,q} & \cdots & \beta_{n-1,q} & \beta_{n,q} \\ 0 & \beta_{0,q} & \cdots & \binom{n-1}{1}_q \beta_{n-2,q} & \binom{n}{1}_q \beta_{n-1,q} \\ 0 & 0 & \cdots & \binom{n-1}{2}_q \beta_{n-3,q} & \binom{n}{2}_q \beta_{n-2,q} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta_{0,q} & \binom{n}{n-1}_q \beta_{1,q} \\ \mathcal{H}_0^{(m)}(x,y;w,q) & \mathcal{H}_1^{(m)}(x,y;w,q) & \cdots & \mathcal{H}_{n-1}^{(m)}(x,y;w,q) & \mathcal{H}_n^{(m)}(x,y;w,q) \end{vmatrix}. \tag{38} $$

Proof. Since $\mathcal{A}_{0,q} \neq 0$, the series $\mathcal{A}_q(t)$ is a unit of $\mathbb{C}[[t]]$. Multiplying the reciprocal expansion (36) by (6) and comparing the coefficient of $t^n/[n]_q!$ gives

$$ \sum_{k=0}^{n} \binom{n}{k}_q \mathcal{A}_{k,q}\beta_{n-k,q} = \delta_{n,0}, $$

which immediately yields $\beta_{0,q} = 1/\mathcal{A}_{0,q}$ and the recursion (37) for $n \geq 1$.

It remains to derive the determinant. Multiplying the defining generating function (28) by $1/\mathcal{A}_q(t)$ gives the triangular coefficient system

$$ \mathcal{H}_r^{(m)}(x,y;w,q) = \sum_{j=0}^{r} \binom{r}{j}_q \beta_{r-j,q}\,{}_{\mathcal{H}}\mathcal{A}_j^{(m)}(x,y;w,q), \qquad r = 0,1,\ldots,n. \tag{39} $$

The coefficient matrix of this system is lower triangular with diagonal entries $\beta_{0,q}$; hence, its determinant is $(\beta_{0,q})^{n+1}$ and the solution is unique. Applying Cramer's rule to the last unknown ${}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q)$ and then transposing the resulting determinant gives exactly (38). Equivalently, the numerator determinant in (38) is obtained from the coefficient matrix of (39) by replacing the column corresponding to ${}_{\mathcal{H}}\mathcal{A}_n^{(m)}$ with the right-hand side vector $(\mathcal{H}_0^{(m)},\ldots,\mathcal{H}_n^{(m)})^T$. The transposition does not change the determinant, and division by $(\beta_{0,q})^{n+1}$ gives the stated formula. The case $n = 0$ follows from the first equation in (39). □

The following recurrence relations are understood in the same formal Hahn–Appell sense as the identities (29) and (30).

Theorem 4. *Assume that* $\mathcal{A}_{0,q} \neq 0$*, and let*

$$ \frac{1}{\mathcal{A}_q(t)} = \sum_{k=0}^{\infty} \beta_{k,q} \frac{t^k}{[k]_q!}, \qquad \beta_{0,q} = \frac{1}{\mathcal{A}_{0,q}}. $$

Then

$$ {}_{\mathcal{H}}\mathcal{A}_0^{(m)}(x,y;w,q) = \mathcal{A}_{0,q}, $$

and, for every $n \geq 1$*,*

$$ {}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = \frac{1}{\beta_{0,q}} \left[\mathcal{H}_n^{(m)}(x,y;w,q) - \sum_{k=1}^{n} \binom{n}{k}_q \beta_{k,q}\,{}_{\mathcal{H}}\mathcal{A}_{n-k}^{(m)}(x,y;w,q) \right]. \tag{40} $$

Equivalently,

$$ {}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = \mathcal{A}_{0,q}\mathcal{H}_n^{(m)}(x,y;w,q) - \mathcal{A}_{0,q}\sum_{k=1}^{n}\binom{n}{k}_q \beta_{k,q}\, {}_{\mathcal{H}}\mathcal{A}_{n-k}^{(m)}(x,y;w,q). \tag{41}$$

Proof. Multiplying the defining generating function (28) by $1/\mathcal{A}_q(t)$ gives

$$e_q^{wt}(xt)e_q^{wt}(yt^m) = \left(\sum_{k=0}^{\infty}\beta_{k,q}\frac{t^k}{[k]_q!}\right)\left(\sum_{r=0}^{\infty}{}_{\mathcal{H}}\mathcal{A}_r^{(m)}(x,y;w,q)\frac{t^r}{[r]_q!}\right).$$

Since the left-hand side is the generating function of $\mathcal{H}_n^{(m)}(x,y;w,q)$, comparison of the coefficient of $t^n/[n]_q!$ gives

$$\mathcal{H}_n^{(m)}(x,y;w,q) = \sum_{k=0}^{n}\binom{n}{k}_q \beta_{k,q}{}_{\mathcal{H}}\mathcal{A}_{n-k}^{(m)}(x,y;w,q).$$

Separating the term $k=0$ and using $\beta_{0,q} = 1/\mathcal{A}_{0,q}$ yields (40) and (41). □

Theorem 5. *The higher q-derivative formula with respect to x is*

$$\widehat{D}_{q,x}^k {}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = \frac{[n]_q!}{[n-k]_q!}\,{}_{\mathcal{H}}\mathcal{A}_{n-k}^{(m)}(x,y;w,q), \qquad 0\le k\le n. \tag{42}$$

Proof. Apply $\widehat{D}_{q,x}^k$ to both sides of (28). Since $\widehat{D}_{q,x}^k e_q^{wt}(xt) = t^k e_q^{wt}(xt)$, we obtain

$$\sum_{n=0}^{\infty}\widehat{D}_{q,x}^k {}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q)\frac{t^n}{[n]_q!} = t^k\mathcal{A}_q(t)e_q^{wt}(xt)e_q^{wt}(yt^m) = \sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q)\frac{t^{n+k}}{[n]_q!}. \tag{43}$$

Reindex the series on the right-hand side and compare coefficients of t^n. □

Theorem 6. *The family* ${}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q)$ *satisfies the Hahn x-shift formula*

$${}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x\oplus_{q,w}\delta,y;w,q) = \sum_{k=0}^{n}\binom{n}{k}_q (\delta)_{q,w}^k\, {}_{\mathcal{H}}\mathcal{A}_{n-k}^{(m)}(x,y;w,q). \tag{44}$$

Proof. Replace x by $x\oplus_{q,w}\delta$ in (28). Using the multiplicative property of the Hahn-type exponential,

$$\mathcal{A}_q(t)e_q^{wt}(xt)e_q^{wt}(yt^m)e_q^{wt}(\delta t) = \sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x\oplus_{q,w}\delta,y;w,q)\frac{t^n}{[n]_q!}. \tag{45}$$

Now expand $e_q^{wt}(\delta t)$ and use (28); coefficient comparison gives (44). □

Theorem 7. *The following summation formula holds for the two-variable q-Gould–Hopper–Hahn–Appell polynomials* ${}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q)$*:*

$${}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = \sum_{s=0}^{n}\sum_{k=0}^{n-s}\binom{n}{s}_q\binom{n-s}{k}_q (\ominus_{q,w}\delta)_{q,w}^s \mathcal{H}_{n-s-k}^{(m)}(x,y;w,q)\,\mathcal{P}_k(\delta;w,q). \tag{46}$$

Proof. Starting from (23), insert the identity

$$1 = e_q^{wt}((\ominus_{q,w}\delta)t)e_q^{wt}(\delta t)$$

which follows from the defining relation for the formal Hahn inverse. Then

$$\mathcal{A}_q(t)e_q^{wt}(xt)e_q^{wt}(yt^m) = \big(\mathcal{A}_q(t)e_q^{wt}(\delta t)\big)e_q^{wt}((\ominus_{q,w}\delta)t)e_q^{wt}(xt)e_q^{wt}(yt^m). \tag{47}$$

Using (12), (23), and the series expansion of $e_q^{wt}((\ominus_{q,w}\delta)t)$, we obtain

$$\begin{aligned}\mathcal{A}_q(t)e_q^{wt}(xt)e_q^{wt}(yt^m) &= \left(\sum_{k=0}^{\infty}\mathcal{P}_k(\delta;w,q)\frac{t^k}{[k]_q!}\right)\left(\sum_{s=0}^{\infty}(\ominus_{q,w}\delta)_{q,w}^s\frac{t^s}{[s]_q!}\right)\left(\sum_{r=0}^{\infty}\mathcal{H}_r^{(m)}(x,y;w,q)\frac{t^r}{[r]_q!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{s=0}^{n}\sum_{k=0}^{n-s}\frac{[n]_q!}{[s]_q![k]_q![n-s-k]_q!}(\ominus_{q,w}\delta)_{q,w}^s\mathcal{P}_k(\delta;w,q)\mathcal{H}_{n-s-k}^{(m)}(x,y;w,q)\right)\frac{t^n}{[n]_q!}.\end{aligned} \tag{48}$$

Since

$$\frac{[n]_q!}{[s]_q![k]_q![n-s-k]_q!} = \binom{n}{s}_q\binom{n-s}{k}_q,$$

comparison with the generating function (28) yields (46). □

3. Operational Identities and Quasi-Monomiality Principle

For later use, we record the connection formula (33) in the compact notation

$${}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = [n]_q!\sum_{k=0}^{\lfloor n/m\rfloor}\frac{(y)_{q,w}^k}{[k]_q!\,[n-mk]_q!}\,\mathcal{P}_{n-mk}(x;w,q). \tag{49}$$

Corollary 1. *Let*

$$\mathcal{T} := e_q^{wt}\Big(y\widehat{D}_{q,x}^m\Big). \tag{50}$$

Then the operational identity already implicit in (12) *can be written coefficientwise as*

$${}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = \mathcal{T}\{\mathcal{P}_n(x;w,q)\}, \qquad n = 0,1,2,\ldots. \tag{51}$$

Proof. This is the coefficient form of (31) after expanding the one-variable kernel (12). No additional hypothesis beyond the formal rules (29) and (30) is used. □

Theorem 8. *For every $n \geq 1$,*

$$\widehat{D}_{q,x}\,{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = [n]_q\,{}_{\mathcal{H}}\mathcal{A}_{n-1}^{(m)}(x,y;w,q), \tag{52}$$

and, for $n \geq m$,

$$\widehat{D}_{q,y}\,{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = \frac{[n]_q!}{[n-m]_q!}\,{}_{\mathcal{H}}\mathcal{A}_{n-m}^{(m)}(x,y;w,q), \tag{53}$$

while $\widehat{D}_{q,y}\,{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = 0$ for $0 \leq n < m$.

Proof. The x-lowering identity is the case $k = 1$ of (42); we record it here together with the corresponding y-lowering relation because both operators enter the quasi-monomial discussion.

For the y-derivative, differentiate (28) with respect to y. Since $\widehat{D}_{q,y}e_q^{wt}(yt^m) = t^m e_q^{wt}(yt^m)$, we obtain

$$\sum_{n=0}^{\infty}\widehat{D}_{q,y}\,{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q)\frac{t^n}{[n]_q!} = t^m\,\mathcal{A}_q(t)e_q^{wt}(xt)e_q^{wt}(yt^m) = \sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q)\frac{t^{n+m}}{[n]_q!}.$$

Shift the index on the right-hand side and compare the coefficients of t^n. This yields (53) and the vanishing statement for $0 \le n < m$. □

Corollary 2. *For every $n \ge m$,*

$$\widehat{D}_{q,y}\,{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = \widehat{D}_{q,x}^m\,{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q), \tag{54}$$

and for $0 \le n < m$, both sides vanish.

Proof. Apply (52) repeatedly m times:

$$\widehat{D}_{q,x}^m\,{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = \frac{[n]_q!}{[n-m]_q!}\,{}_{\mathcal{H}}\mathcal{A}_{n-m}^{(m)}(x,y;w,q).$$

This is exactly the right-hand side of (53). □

Theorem 9. *For any auxiliary parameter η,*

$${}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y\oplus_{q,w}\eta;w,q) = [n]_q! \sum_{k=0}^{\lfloor n/m\rfloor} \frac{(\eta)_{q,w}^k}{[k]_q!\,[n-mk]_q!}\,{}_{\mathcal{H}}\mathcal{A}_{n-mk}^{(m)}(x,y;w,q). \tag{55}$$

Proof. Replace y with $y\oplus_{q,w}\eta$ in (28):

$$\mathcal{A}_q(t)e_q^{wt}(xt)e_q^{wt}((y\oplus_{q,w}\eta)t^m) = \sum_{n\ge 0}{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y\oplus_{q,w}\eta;w,q)\frac{t^n}{[n]_q!}.$$

By the multiplicative property of the formal Hahn-type exponential,

$$e_q^{wt}((y\oplus_{q,w}\eta)t^m) = e_q^{wt}(yt^m)e_q^{wt}(\eta t^m) = e_q^{wt}(yt^m)\sum_{k\ge 0}(\eta)_{q,w}^k\frac{t^{mk}}{[k]_q!}.$$

Multiply this by the generating function of ${}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q)$ and compare coefficients of t^n. □

The next identities refine the operational calculus of the family. They are stated in the same formal Hahn–Appell setting used in (29) and (30). Thus all operator products are interpreted coefficientwise in the corresponding formal power-series ring. This point is consistent with the Hahn calculus and Hahn–Appell framework developed in [17,30] and with the operational and quasi-monomial viewpoint used for recent q-polynomial families in [14,18,20,31].

Theorem 10. *Let*

$$\mathcal{T}_y := e_q^{wt}\Big(y\widehat{D}_{q,x}^m\Big).$$

Then, in the formal Hahn sense,

$$\mathcal{T}_y^{-1} = e_q^{wt}\Big((\ominus_{q,w}y)\widehat{D}_{q,x}^m\Big). \tag{56}$$

Consequently, the one-variable Hahn–Appell kernel is recovered from the two-variable family by

$$\mathcal{P}_n(x;w,q) = e_q^{wt}\Big((\ominus_{q,w}y)\widehat{D}_{q,x}^m\Big)\{{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q)\}, \qquad n=0,1,2,\ldots, \tag{57}$$

that is,

$$\mathcal{P}_n(x;w,q) = [n]_q! \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(\ominus_{q,w} y)_{q,w}^k}{[k]_q!\,[n-mk]_q!}\, {}_{\mathcal{H}}\mathcal{A}_{n-mk}^{(m)}(x,y;w,q). \tag{58}$$

Proof. By the defining inverse relation for the formal Hahn inverse,

$$e_q^{wt}((\ominus_{q,w} y)t^m) e_q^{wt}(yt^m) = 1.$$

Multiplying the defining generating function (28) by $e_q^{wt}((\ominus_{q,w} y)t^m)$ gives

$$\mathcal{A}_q(t) e_q^{wt}(xt) = e_q^{wt}((\ominus_{q,w} y)t^m) \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) \frac{t^n}{[n]_q!}.$$

The left-hand side is exactly the generating function (12) of $\mathcal{P}_n(x;w,q)$. Expanding

$$e_q^{wt}((\ominus_{q,w} y)t^m) = \sum_{k=0}^{\infty} (\ominus_{q,w} y)_{q,w}^k \frac{t^{mk}}{[k]_q!}$$

and using the Cauchy product, the coefficient of $t^n/[n]_q!$ is precisely (58). Finally, since (51) says ${}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = \mathcal{T}_y\{\mathcal{P}_n(x;w,q)\}$, the same coefficientwise multiplication proves (56) and (57). □

Theorem 11. *Define the formal q-Appell operator*

$$\mathcal{A}_q\Big(\widehat{D}_{q,x}\Big) := \sum_{k=0}^{\infty} \mathcal{A}_{k,q} \frac{\widehat{D}_{q,x}^k}{[k]_q!}. \tag{59}$$

Then, for every $n \geq 0$,

$${}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = \mathcal{A}_q\Big(\widehat{D}_{q,x}\Big)\{\mathcal{H}_n^{(m)}(x,y;w,q)\}. \tag{60}$$

If $\mathcal{A}_{0,q} \neq 0$ and

$$\mathcal{B}_q(t) := \frac{1}{\mathcal{A}_q(t)} = \sum_{k=0}^{\infty} \beta_{k,q} \frac{t^k}{[k]_q!},$$

then the inverse factorization is

$$\mathcal{H}_n^{(m)}(x,y;w,q) = \mathcal{B}_q\Big(\widehat{D}_{q,x}\Big)\{{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q)\} = \sum_{k=0}^{n} \binom{n}{k}_q \beta_{k,q}\, {}_{\mathcal{H}}\mathcal{A}_{n-k}^{(m)}(x,y;w,q). \tag{61}$$

Proof. Since $\mathcal{H}_n^{(m)}(x,y;w,q)$ is the specialization of ${}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q)$ obtained by taking $\mathcal{A}_q(t) = 1$, its generating function is (23). Applying (59) to this generating function gives

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{A}_q\Big(\widehat{D}_{q,x}\Big)\{\mathcal{H}_n^{(m)}(x,y;w,q)\}\frac{t^n}{[n]_q!} &= \mathcal{A}_q\Big(\widehat{D}_{q,x}\Big)\Big\{e_q^{wt}(xt) e_q^{wt}(yt^m)\Big\} \\ &= \mathcal{A}_q(t) e_q^{wt}(xt) e_q^{wt}(yt^m),\end{aligned}$$

because $\widehat{D}_{q,x}^k e_q^{wt}(xt) = t^k e_q^{wt}(xt)$. The last expression is the defining generating function (28); coefficient comparison proves (60). If $\mathcal{A}_{0,q} \neq 0$, then $\mathcal{B}_q(t)\mathcal{A}_q(t) = 1$. Applying the same argument to $\mathcal{B}_q(\widehat{D}_{q,x})$ and using the Cauchy product yields (61); equivalently, it is the coefficient form of the triangular identity (35). □

Theorem 12. *For $r, s \in \mathbb{N}_0$ and $n \geq 0$, one has*

$$\widehat{D}_{q,x}^{r}\widehat{D}_{q,y}^{s}\,{}_{\mathcal{H}}\mathcal{A}_{n}^{(m)}(x,y;w,q) = \begin{cases} \dfrac{[n]_q!}{[n-r-ms]_q!}\,{}_{\mathcal{H}}\mathcal{A}_{n-r-ms}^{(m)}(x,y;w,q), & n \geq r+ms, \\ 0, & 0 \leq n < r+ms. \end{cases} \tag{62}$$

Proof. Apply $\widehat{D}_{q,x}^{r}\widehat{D}_{q,y}^{s}$ to (28). The formal Hahn rules give

$$\widehat{D}_{q,x}^{r}e_q^{wt}(xt) = t^r e_q^{wt}(xt), \qquad \widehat{D}_{q,y}^{s}e_q^{wt}(yt^m) = t^{ms}e_q^{wt}(yt^m).$$

Hence

$$\sum_{n=0}^{\infty}\widehat{D}_{q,x}^{r}\widehat{D}_{q,y}^{s}\,{}_{\mathcal{H}}\mathcal{A}_{n}^{(m)}(x,y;w,q)\frac{t^n}{[n]_q!} = t^{r+ms}\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{A}_{n}^{(m)}(x,y;w,q)\frac{t^n}{[n]_q!}.$$

Reindexing the right-hand side and comparing coefficients of t^n gives (62). If $n < r+ms$, no term contributes to the coefficient of t^n, so the value is zero. □

Corollary 3. *For every $s \in \mathbb{N}_0$ and $n \geq 0$,*

$$\widehat{D}_{q,y}^{s}\,{}_{\mathcal{H}}\mathcal{A}_{n}^{(m)}(x,y;w,q) = \widehat{D}_{q,x}^{ms}\,{}_{\mathcal{H}}\mathcal{A}_{n}^{(m)}(x,y;w,q). \tag{63}$$

In particular, the case $s = 1$ recovers the first-order relation (54).

Proof. Set $(r,s) = (0,s)$ and $(r,s) = (ms,0)$ in (62). Both choices give the same coefficient factor $[n]_q!/[n-ms]_q!$ when $n \geq ms$, and both sides vanish when $n < ms$. □

Theorem 13. *For arbitrary auxiliary parameters δ and η,*

$$\begin{aligned} {}_{\mathcal{H}}\mathcal{A}_{n}^{(m)}(x \oplus_{q,w} \delta, y \oplus_{q,w} \eta; w, q) &= e_q^{wt}\Big(\delta\widehat{D}_{q,x}\Big)e_q^{wt}\Big(\eta\widehat{D}_{q,y}\Big)\{{}_{\mathcal{H}}\mathcal{A}_{n}^{(m)}(x,y;w,q)\} \\ &= [n]_q!\sum_{j=0}^{\lfloor n/m\rfloor}\sum_{i=0}^{n-mj}\frac{(\eta)_{q,w}^{j}(\delta)_{q,w}^{i}}{[j]_q!\,[i]_q!\,[n-i-mj]_q!}\,{}_{\mathcal{H}}\mathcal{A}_{n-i-mj}^{(m)}(x,y;w,q). \end{aligned} \tag{64}$$

Proof. Replacing x by $x \oplus_{q,w} \delta$ and y by $y \oplus_{q,w} \eta$ in (28), and using (30), gives

$$\begin{aligned} &\mathcal{A}_q(t)e_q^{wt}((x \oplus_{q,w} \delta)t)e_q^{wt}((y \oplus_{q,w} \eta)t^m) \\ &\quad = \mathcal{A}_q(t)e_q^{wt}(xt)e_q^{wt}(yt^m)e_q^{wt}(\delta t)e_q^{wt}(\eta t^m). \end{aligned}$$

Expanding the last two factors yields

$$e_q^{wt}(\delta t)e_q^{wt}(\eta t^m) = \sum_{i=0}^{\infty}(\delta)_{q,w}^{i}\frac{t^i}{[i]_q!}\sum_{j=0}^{\infty}(\eta)_{q,w}^{j}\frac{t^{mj}}{[j]_q!}.$$

Multiplication by the generating function of ${}_{\mathcal{H}}\mathcal{A}_{n}^{(m)}(x,y;w,q)$ and coefficient comparison gives the double sum in (64). On the other hand, by the mixed-derivative hierarchy,

$$e_q^{wt}\Big(\delta\widehat{D}_{q,x}\Big)e_q^{wt}\Big(\eta\widehat{D}_{q,y}\Big)\{{}_{\mathcal{H}}\mathcal{A}_{n}^{(m)}(x,y;w,q)\}$$

expands into the same finite double sum. Therefore the operational and coefficient forms coincide. □

The preceding operational identities can be summarized as follows.

Theorem 14. *The two-variable q-Gould–Hopper–Hahn–Appell polynomials satisfy*

$$ {}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = e_q^{wt}\left(y\widehat{D}_{q,x}^m\right)\{\mathcal{P}_n(x;w,q)\}, \tag{65} $$

$$ {}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y\oplus_{q,w} l;w,q) = e_q^{wt}\left(l\widehat{D}_{q,x}^m\right)\left\{{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q)\right\}, \tag{66} $$

and

$$ e_q^{wt}\left((\ominus_{q,w}y)\widehat{D}_{q,x}^m\right){}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = \mathcal{P}_n(x;w,q), \tag{67} $$

where $\ominus_{q,w}y$ *is the formal Hahn inverse of* y.

Proof. Equation (65) is exactly the coefficient form of (31). Equation (66) follows by replacing y with $y\oplus_{q,w} l$ and using the multiplicative rule (30). Finally, (67) is the inverse operational identity (57); it follows from $e_q^{wt}((\ominus_{q,w}y)t^m)e_q^{wt}(yt^m) = 1$ and does not use ordinary negation of y. □

The following quasi-monomial formulation is in the spirit of the monomiality principle, operational methods, and later q-extensions; see, for example, [18–20,25–27,31,32].

Remark 3. *The quasi-monomial structure in this section is conditional on two inputs: first, the formal Hahn identities (29) and (30); second, the assumption that the underlying one-variable family* $\{\mathcal{P}_n(x;w,q)\}_{n\geq 0}$ *admits a raising operator* $\mathcal{M}_0$ *as in (68). Accordingly, the operator* $\mathcal{M}_m$ *is introduced by intertwining on the image of* $\mathcal{T}$*, and the resulting eigen equation is understood in this induced formal-operator sense.*

Definition 1. *Assume that* $\{\mathcal{P}_n(x;w,q)\}_{n\geq 0}$ *is quasi-monomial and satisfies*

$$ \widehat{D}_{q,x}\mathcal{P}_n(x;w,q) = [n]_q\,\mathcal{P}_{n-1}(x;w,q), \qquad \mathcal{M}_0\mathcal{P}_n(x;w,q) = \mathcal{P}_{n+1}(x;w,q). \tag{68} $$

Define $\mathcal{M}_m$ *on the image of* $\mathcal{T}$ *by the intertwining relation*

$$ \mathcal{M}_m\,\mathcal{T} = \mathcal{T}\,\mathcal{M}_0. \tag{69} $$

Example 2. *The assumption (68) is explicit in the basic subclass obtained by taking* $\mathcal{A}_q(t) = 1$ *and passing formally to the ordinary q-exponential limit* $w\to 0^+$*. In that case,*

$$ \mathcal{P}_n(x;0,q) = x^n, \qquad \widehat{D}_{q,x}x^n = [n]_q\,x^{n-1}, \qquad \mathcal{M}_0 = x, $$

So, the underlying one-variable family is quasi-monomial with a concrete raising operator. The corresponding two-variable family becomes the standard q-Gould–Hopper polynomials

$$ H_{n,q}^{(m)}(x,y) = e_q\left(y\widehat{D}_{q,x}^m\right)\{x^n\}, $$

and the induced raising operator on the image of $\mathcal{T} = e_q\left(y\widehat{D}_{q,x}^m\right)$ *may be written formally as*

$$ \mathcal{M}_m = \mathcal{T}\,x\,\mathcal{T}^{-1}. $$

Thus, the present construction recovers the usual monomiality principle in a concrete and non-singular subclass.

Theorem 15. *Under (68) and (69), the family* $\{{}_{\mathcal{H}}\mathcal{A}_n^{(m)}\}_{n\geq 0}$ *is quasi-monomial. More precisely,*

$$ \widehat{D}_{q,x}{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = [n]_q\,{}_{\mathcal{H}}\mathcal{A}_{n-1}^{(m)}(x,y;w,q), \qquad \mathcal{M}_m{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = {}_{\mathcal{H}}\mathcal{A}_{n+1}^{(m)}(x,y;w,q), \tag{70} $$

and hence, it satisfies the q-differential equation

$$\left(\mathcal{M}_m \widehat{D}_{q,x} - [n]_q\right) {}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = 0. \tag{71}$$

Proof. The first identity is already established in (52). Since $\mathcal{T}$ is a series in $\widehat{D}_{q,x}^m$, it commutes with $\widehat{D}_{q,x}$; therefore

$$\begin{aligned}\widehat{D}_{q,x}\,{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) &= \widehat{D}_{q,x}\mathcal{T}\{\mathcal{P}_n(x;w,q)\} = \mathcal{T}\{\widehat{D}_{q,x}\mathcal{P}_n(x;w,q)\}\\ &= [n]_q\,\mathcal{T}\{\mathcal{P}_{n-1}(x;w,q)\} = [n]_q\,{}_{\mathcal{H}}\mathcal{A}_{n-1}^{(m)}(x,y;w,q).\end{aligned}$$

For the raising action, apply (69) and (51):

$$\begin{aligned}\mathcal{M}_m\,{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) &= \mathcal{M}_m\mathcal{T}\{\mathcal{P}_n(x;w,q)\} = \mathcal{T}\mathcal{M}_0\{\mathcal{P}_n(x;w,q)\}\\ &= \mathcal{T}\{\mathcal{P}_{n+1}(x;w,q)\} = {}_{\mathcal{H}}\mathcal{A}_{n+1}^{(m)}(x,y;w,q).\end{aligned}$$

Finally,

$$\mathcal{M}_m\widehat{D}_{q,x}\,{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = \mathcal{M}_m\left([n]_q\,{}_{\mathcal{H}}\mathcal{A}_{n-1}^{(m)}(x,y;w,q)\right) = [n]_q\,{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q),$$

which is equivalent to (71). □

Corollary 4. *Under the quasi-monomiality assumptions* (68) *and* (69)*, one has*

$${}_{\mathcal{H}}\mathcal{A}_{n+1}^{(m)}(x,y;w,q) = \mathcal{M}_m\,{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q), \qquad n = 0,1,2,\ldots. \tag{72}$$

Consequently,

$${}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = \mathcal{M}_m^n\{{}_{\mathcal{H}}\mathcal{A}_0^{(m)}(x,y;w,q)\} = \mathcal{M}_m^n\{\mathcal{A}_{0,q}\}. \tag{73}$$

Moreover,

$$\widehat{D}_{q,x}\,{}_{\mathcal{H}}\mathcal{A}_{n+1}^{(m)}(x,y;w,q) = [n+1]_q\,{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q), \qquad n \geq 0, \tag{74}$$

and

$$\widehat{D}_{q,y}\,{}_{\mathcal{H}}\mathcal{A}_{n+m}^{(m)}(x,y;w,q) = \frac{[n+m]_q!}{[n]_q!}\,{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q), \qquad n \geq 0. \tag{75}$$

Proof. The first recurrence follows directly from the raising relation $\mathcal{M}_m\,{}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = {}_{\mathcal{H}}\mathcal{A}_{n+1}^{(m)}(x,y;w,q)$. Iterating this identity gives (73). The identities (74) and (75) are the lowering formulas with the indices shifted by one and by m, respectively. □

Remark 4. *Equations* (70) *and* (71) *are the quasi-monomial statements used in this paper. In particular, the multiplicative operator for the two-variable family is the induced operator $\mathcal{M}_m$ determined by the intertwining relation* (69)*. We therefore do not assert a universal closed expression involving only x, y, T_x, T_y, $\widehat{D}_{q,x}$, and the logarithmic derivative of $\mathcal{A}_q$; such an expression is not valid without extra commutation hypotheses. With the induced operator, the valid q-differential equation is precisely*

$$\left(\mathcal{M}_m \widehat{D}_{q,x} - [n]_q\right) {}_{\mathcal{H}}\mathcal{A}_n^{(m)}(x,y;w,q) = 0.$$

4. Reductions to Classical Families

This section records the principal reductions obtained from the general two-variable q-Gould–Hopper–Hahn–Appell family by specifying the Appell factor $\mathcal{A}_q(t)$. In particular, the standard q-Bernoulli, q-Euler, and q-Genocchi kernels give the corresponding Bernoulli-, Euler-, and Genocchi-based Gould–Hopper–Hahn–Appell polynomials. For these specializations, we state the generating functions, convolution formulas, operational identities, and determinant consequences. The Bernoulli and Euler kernels are non-singular, namely $\mathcal{A}_{0,q} \neq 0$, and hence the reciprocal-series and determinant results from the preceding sections apply directly. By contrast, the Genocchi kernel has a vanishing constant term, so it is handled separately through its degree-shift relation with the Euler specialization.

Theorem 16. *Choose $\mathcal{A}_q(t)$ from the usual q-Appell kernels [5,21,33]. For the first two kernels, the non-singularity condition $\mathcal{A}_{0,q} \neq 0$ is satisfied, whereas the third one is singular with vanishing constant term:*

$$\mathcal{A}_q^{(\mathrm{B})}(t) = \frac{t}{e_q(t)-1}, \qquad \mathcal{A}_q^{(\mathrm{E})}(t) = \frac{[2]_q}{e_q(t)+1}, \qquad \mathcal{A}_q^{(\mathrm{G})}(t) = \frac{[2]_q t}{e_q(t)+1}. \tag{76}$$

Then the corresponding q-Bernoulli-, q-Euler-, and q-Genocchi-based Gould–Hopper–Hahn–Appell families are obtained, respectively, by

$$\frac{t}{e_q(t)-1} e_q^{wt}(xt) e_q^{wt}(yt^m) = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{B}_n^{(m)}(x,y;w,q) \frac{t^n}{[n]_q!}, \tag{77}$$

$$\frac{[2]_q}{e_q(t)+1} e_q^{wt}(xt) e_q^{wt}(yt^m) = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{E}_n^{(m)}(x,y;w,q) \frac{t^n}{[n]_q!}, \tag{78}$$

$$\frac{[2]_q t}{e_q(t)+1} e_q^{wt}(xt) e_q^{wt}(yt^m) = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{G}_n^{(m)}(x,y;w,q) \frac{t^n}{[n]_q!}. \tag{79}$$

Moreover,

$${}_{\mathcal{H}}\mathcal{B}_n^{(m)}(x,y;w,q) = \sum_{j=0}^{n} \binom{n}{j}_q \mathcal{B}_{j,q}\, \mathcal{H}_{n-j}^{(m)}(x,y;w,q), \tag{80}$$

$${}_{\mathcal{H}}\mathcal{E}_n^{(m)}(x,y;w,q) = \sum_{j=0}^{n} \binom{n}{j}_q \mathcal{E}_{j,q}\, \mathcal{H}_{n-j}^{(m)}(x,y;w,q), \tag{81}$$

$${}_{\mathcal{H}}\mathcal{G}_n^{(m)}(x,y;w,q) = \sum_{j=0}^{n} \binom{n}{j}_q \mathcal{G}_{j,q}\, \mathcal{H}_{n-j}^{(m)}(x,y;w,q). \tag{82}$$

Proof. The generating functions (77)–(79) are obtained by inserting the corresponding q-Appell kernels into (28). The convolution identities follow directly from the Cauchy product, exactly as in the general decomposition formula (34). For the Bernoulli and Euler specializations, one may also apply the reciprocal-series and determinant results whenever $\mathcal{A}_{0,q} \neq 0$. The Genocchi specialization remains valid at the level of generating functions and convolution identities, but the reciprocal-series construction is not used there because its kernel has a vanishing constant term. □

The preceding reductions can be refined further by recording the operational and determinant consequences for these three canonical Appell kernels. This makes explicit how the general formulation specializes in the Bernoulli–Euler–Genocchi direction and clarifies precisely where the singular character of the Genocchi kernel enters.

Theorem 17. *Define the associated one-variable Hahn–Appell specializations by*

$$\frac{t}{e_q(t)-1}e_q^{wt}(xt)=\sum_{n=0}^{\infty}\mathcal{B}_n(x;w,q)\frac{t^n}{[n]_q!}, \tag{83}$$

$$\frac{[2]_q}{e_q(t)+1}e_q^{wt}(xt)=\sum_{n=0}^{\infty}\mathcal{E}_n(x;w,q)\frac{t^n}{[n]_q!}, \tag{84}$$

$$\frac{[2]_q t}{e_q(t)+1}e_q^{wt}(xt)=\sum_{n=0}^{\infty}\mathcal{G}_n(x;w,q)\frac{t^n}{[n]_q!}. \tag{85}$$

Then, with the operator $\mathcal{T}=e_q^{wt}\left(y\widehat{D}_{q,x}^{m}\right)$ *from* (50)*, the corresponding two-variable families satisfy the operational identities*

$${}_{\mathcal{H}}\mathcal{B}_n^{(m)}(x,y;w,q)=\mathcal{T}\{\mathcal{B}_n(x;w,q)\}, \tag{86}$$

$${}_{\mathcal{H}}\mathcal{E}_n^{(m)}(x,y;w,q)=\mathcal{T}\{\mathcal{E}_n(x;w,q)\}, \tag{87}$$

$${}_{\mathcal{H}}\mathcal{G}_n^{(m)}(x,y;w,q)=\mathcal{T}\{\mathcal{G}_n(x;w,q)\}. \tag{88}$$

Consequently, for $\mathcal{Y}\in\{\mathcal{B},\mathcal{E},\mathcal{G}\}$*,*

$$\widehat{D}_{q,x}\,{}_{\mathcal{H}}\mathcal{Y}_n^{(m)}(x,y;w,q)=[n]_q\,{}_{\mathcal{H}}\mathcal{Y}_{n-1}^{(m)}(x,y;w,q), \tag{89}$$

and

$$\widehat{D}_{q,y}\,{}_{\mathcal{H}}\mathcal{Y}_n^{(m)}(x,y;w,q)=\begin{cases}\dfrac{[n]_q!}{[n-m]_q!}\,{}_{\mathcal{H}}\mathcal{Y}_{n-m}^{(m)}(x,y;w,q), & n\geq m,\\ 0, & 0\leq n<m.\end{cases} \tag{90}$$

Moreover, the Euler and Genocchi specializations are related by

$${}_{\mathcal{H}}\mathcal{G}_0^{(m)}(x,y;w,q)=0,\qquad {}_{\mathcal{H}}\mathcal{G}_n^{(m)}(x,y;w,q)=[n]_q\,{}_{\mathcal{H}}\mathcal{E}_{n-1}^{(m)}(x,y;w,q),\quad n\geq 1. \tag{91}$$

Proof. Multiplying each of the one-variable generating functions (83)–(85) by $e_q^{wt}(yt^m)$ gives precisely the two-variable generating functions (77)–(79). The coefficientwise form of this observation is (86)–(88), and it is exactly the specialization of the general operational identity (51). The lowering relations (89) and (90) are then immediate from (52) and (53). Finally, since

$$\frac{[2]_q t}{e_q(t)+1}=t\,\frac{[2]_q}{e_q(t)+1},$$

we obtain

$$\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{G}_n^{(m)}(x,y;w,q)\frac{t^n}{[n]_q!}=t\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{E}_n^{(m)}(x,y;w,q)\frac{t^n}{[n]_q!}=\sum_{n=1}^{\infty}[n]_q\,{}_{\mathcal{H}}\mathcal{E}_{n-1}^{(m)}(x,y;w,q)\frac{t^n}{[n]_q!},$$

which yields (91). □

Theorem 18. *Let the reciprocal coefficients for the Bernoulli and Euler kernels be defined by*

$$\frac{e_q(t)-1}{t}=\sum_{n=0}^{\infty}\beta_{n,q}^{(\mathrm{B})}\frac{t^n}{[n]_q!},\qquad \frac{e_q(t)+1}{[2]_q}=\sum_{n=0}^{\infty}\beta_{n,q}^{(\mathrm{E})}\frac{t^n}{[n]_q!}, \tag{92}$$

so that $\beta_{0,q}^{(\mathrm{B})}=1$ *and* $\beta_{0,q}^{(\mathrm{E})}=2/[2]_q\neq 0$*. The zero-degree values are*

$${}_{\mathcal{H}}\mathcal{B}_0^{(m)}(x,y;w,q)=1,\qquad {}_{\mathcal{H}}\mathcal{E}_0^{(m)}(x,y;w,q)=\frac{[2]_q}{2}.$$

Then, for $n \geq 1$, the Bernoulli and Euler reductions admit the determinant representations

$$ {}_{\mathcal{H}}\mathcal{B}_n^{(m)}(x,y;w,q) = \frac{1}{\left(\beta_{0,q}^{(\mathrm{B})}\right)^{n+1}} \begin{vmatrix} \beta_{0,q}^{(\mathrm{B})} & \beta_{1,q}^{(\mathrm{B})} & \cdots & \beta_{n-1,q}^{(\mathrm{B})} & \beta_{n,q}^{(\mathrm{B})} \\ 0 & \beta_{0,q}^{(\mathrm{B})} & \cdots & \binom{n-1}{1}_q \beta_{n-2,q}^{(\mathrm{B})} & \binom{n}{1}_q \beta_{n-1,q}^{(\mathrm{B})} \\ 0 & 0 & \cdots & \binom{n-1}{2}_q \beta_{n-3,q}^{(\mathrm{B})} & \binom{n}{2}_q \beta_{n-2,q}^{(\mathrm{B})} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta_{0,q}^{(\mathrm{B})} & \binom{n}{n-1}_q \beta_{1,q}^{(\mathrm{B})} \\ \mathcal{H}_0^{(m)}(x,y;w,q) & \mathcal{H}_1^{(m)}(x,y;w,q) & \cdots & \mathcal{H}_{n-1}^{(m)}(x,y;w,q) & \mathcal{H}_n^{(m)}(x,y;w,q) \end{vmatrix}. \tag{93} $$

and

$$ {}_{\mathcal{H}}\mathcal{E}_n^{(m)}(x,y;w,q) = \frac{1}{\left(\beta_{0,q}^{(\mathrm{E})}\right)^{n+1}} \begin{vmatrix} \beta_{0,q}^{(\mathrm{E})} & \beta_{1,q}^{(\mathrm{E})} & \cdots & \beta_{n-1,q}^{(\mathrm{E})} & \beta_{n,q}^{(\mathrm{E})} \\ 0 & \beta_{0,q}^{(\mathrm{E})} & \cdots & \binom{n-1}{1}_q \beta_{n-2,q}^{(\mathrm{E})} & \binom{n}{1}_q \beta_{n-1,q}^{(\mathrm{E})} \\ 0 & 0 & \cdots & \binom{n-1}{2}_q \beta_{n-3,q}^{(\mathrm{E})} & \binom{n}{2}_q \beta_{n-2,q}^{(\mathrm{E})} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta_{0,q}^{(\mathrm{E})} & \binom{n}{n-1}_q \beta_{1,q}^{(\mathrm{E})} \\ \mathcal{H}_0^{(m)}(x,y;w,q) & \mathcal{H}_1^{(m)}(x,y;w,q) & \cdots & \mathcal{H}_{n-1}^{(m)}(x,y;w,q) & \mathcal{H}_n^{(m)}(x,y;w,q) \end{vmatrix}. \tag{94} $$

For the Genocchi reduction, the vanishing constant term of the kernel prevents a direct reciprocal-series determinant of the same type. Moreover,

$$ {}_{\mathcal{H}}\mathcal{G}_0^{(m)}(x,y;w,q) = 0, \qquad {}_{\mathcal{H}}\mathcal{G}_1^{(m)}(x,y;w,q) = \frac{[2]_q}{2}. $$

For $n \geq 2$, (91) gives the induced determinant form

$$ {}_{\mathcal{H}}\mathcal{G}_n^{(m)}(x,y;w,q) = \frac{[n]_q}{\left(\beta_{0,q}^{(\mathrm{E})}\right)^n} \begin{vmatrix} \beta_{0,q}^{(\mathrm{E})} & \beta_{1,q}^{(\mathrm{E})} & \cdots & \beta_{n-2,q}^{(\mathrm{E})} & \beta_{n-1,q}^{(\mathrm{E})} \\ 0 & \beta_{0,q}^{(\mathrm{E})} & \cdots & \binom{n-2}{1}_q \beta_{n-3,q}^{(\mathrm{E})} & \binom{n-1}{1}_q \beta_{n-2,q}^{(\mathrm{E})} \\ 0 & 0 & \cdots & \binom{n-2}{2}_q \beta_{n-4,q}^{(\mathrm{E})} & \binom{n-1}{2}_q \beta_{n-3,q}^{(\mathrm{E})} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta_{0,q}^{(\mathrm{E})} & \binom{n-1}{n-2}_q \beta_{1,q}^{(\mathrm{E})} \\ \mathcal{H}_0^{(m)}(x,y;w,q) & \mathcal{H}_1^{(m)}(x,y;w,q) & \cdots & \mathcal{H}_{n-2}^{(m)}(x,y;w,q) & \mathcal{H}_{n-1}^{(m)}(x,y;w,q) \end{vmatrix}. \tag{95} $$

Proof. For $n \geq 1$, the formulas (93) and (94) are the direct specializations of the general determinant representation (38) obtained by taking $\mathcal{A}_q(t) = t/(e_q(t) - 1)$ and $\mathcal{A}_q(t) = [2]_q/(e_q(t) + 1)$, respectively. The reciprocal series in (92) are exactly the corresponding inverses. For Genocchi, the kernel has zero constant term, so the reciprocal-series argument from Section 2 does not apply directly in $\mathbb{C}[[t]]$. Nevertheless, the shift relation (91) converts the Genocchi case into the Euler case one degree lower, which gives (95) for $n \geq 2$. The stated $n = 0$ and $n = 1$ Genocchi values follow directly from the generating function and (91). □

The singular Genocchi case is therefore incorporated into the theory without forcing a non-existent reciprocal series. The direct determinant theorem applies only to kernels with a non-zero constant term. For $\mathcal{A}_q(t) = [2]_q t/(e_q(t) + 1)$, the constant term vanishes, so $1/\mathcal{A}_q(t)$ is not a formal power series. The correct treatment is to keep the Genocchi generating function and its convolution and lowering identities, and to transfer determinant information through the Euler relation (91). This explicitly separates the non-singular and singular branches of the theory.

5. Distribution of Zeros and Graphical Representation

Because the Appell factor $\mathcal{A}_q(t)$ is arbitrary, the fully generic family does not have a single universal zero distribution. Therefore, the zero discussion is organized into two complementary parts. First, we give a non-singular Bernoulli–Appell specialization, which remains within the genuinely Appell-modified class and shows how the additional Appell factor changes the low-degree zero geometry. Second, we use the canonical baseline choice $\mathcal{A}_q(t) = 1$ only for transparent graphical comparison, since its parity symmetry makes the movement and symmetry of the zeros easier to visualize.

Example 3. *Consider the non-singular q-Bernoulli specialization*

$$\mathcal{A}_q(t) = \frac{t}{e_q(t) - 1}, \qquad w = 0, \qquad m = 2. \tag{96}$$

Then

$$\frac{t}{e_q(t) - 1} e_q(xt) e_q(yt^2) = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{B}_n^{(2)}(x, y; 0, q) \frac{t^n}{[n]_q!}, \tag{97}$$

and, by the general convolution formula (34),

$${}_{\mathcal{H}}\mathcal{B}_n^{(2)}(x, y; 0, q) = \sum_{j=0}^{n} \binom{n}{j}_q \mathcal{B}_{j,q} H_{n-j,q}^{(2)}(x, y), \tag{98}$$

where $\mathcal{B}_{j,q}$ denotes the jth q-Bernoulli number. Since $\mathcal{A}_{0,q} = 1$, the reciprocal-series and determinant results of Section 2 apply to this specialization without modification.

For the concrete choice $q = 0.6$ and $y = -0.3$, the first few members are

$${}_{\mathcal{H}}\mathcal{B}_0^{(2)}(x, -0.3; 0, 0.6) = 1, \tag{99}$$

$${}_{\mathcal{H}}\mathcal{B}_1^{(2)}(x, -0.3; 0, 0.6) = x - 0.625, \tag{100}$$

$${}_{\mathcal{H}}\mathcal{B}_2^{(2)}(x, -0.3; 0, 0.6) = x^2 - x - 0.3652040816, \tag{101}$$

$${}_{\mathcal{H}}\mathcal{B}_3^{(2)}(x, -0.3; 0, 0.6) = x^3 - 1.225x^2 - 0.7158x + 0.6128161765, \tag{102}$$

$${}_{\mathcal{H}}\mathcal{B}_4^{(2)}(x, -0.3; 0, 0.6) = x^4 - 1.36x^3 - 0.973488x^2 + 1.333488x + 0.2383673723. \tag{103}$$

Their zeros are approximately

$$\begin{aligned} {}_{\mathcal{H}}\mathcal{B}_2^{(2)} &: \quad x \approx -0.28435,\ 1.28435, \\ {}_{\mathcal{H}}\mathcal{B}_3^{(2)} &: \quad x \approx -0.76356,\ 0.56299,\ 1.42557, \\ {}_{\mathcal{H}}\mathcal{B}_4^{(2)} &: \quad x \approx -0.93049,\ -0.16414,\ 1.22732 \pm 0.23322\, i. \end{aligned}$$

This example concretely shows how the Appell factor breaks the pure parity pattern visible in the canonical Gould–Hopper subclass and produces a genuinely shifted zero geometry already at low degrees.

The preceding numerical data lead to the following qualitative conclusions rather than merely visual observations. First, the canonical subclass $\mathcal{A}_q(t) = 1$ has the parity symmetry stated below in Proposition 1; hence, its real zeros are symmetric with respect to the origin and its non-real zeros occur in conjugate pairs when the coefficients are real. Second, the Bernoulli–Appell specialization contains non-zero Appell coefficients of different parity; therefore, the convolution formula (98) mixes even and odd canonical terms and generally destroys origin symmetry. Third, the appearance of a non-real conjugate pair in ${}_{\mathcal{H}}\mathcal{B}_4^{(2)}$

shows that the Appell factor can change not only the location but also the real/complex type of low-degree zeros. These conclusions explain why the graphs are included: they illustrate specific algebraic consequences of the Appell deformation.

The Bernoulli–Appell modification in the preceding example is also visible graphically. Figure 2 displays the shifted real-axis profiles, while Figure 3 records the corresponding zero distribution in the complex plane. These two plots complement the numerical zero list and make explicit that the Bernoulli–Appell factor destroys the even/odd parity pattern of the canonical subclass.

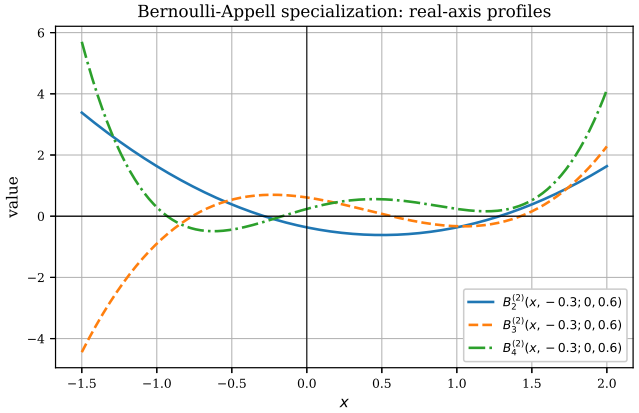


Figure 2. Real-axis profiles of the Bernoulli–Appell specialization ${}_{\mathcal{H}}\mathcal{B}_n^{(2)}(x,-0.3;0,0.6)$ for $n=2,3,4$. The shifted intersections with the real axis visualize the loss of the origin-symmetric parity pattern seen in the canonical case.

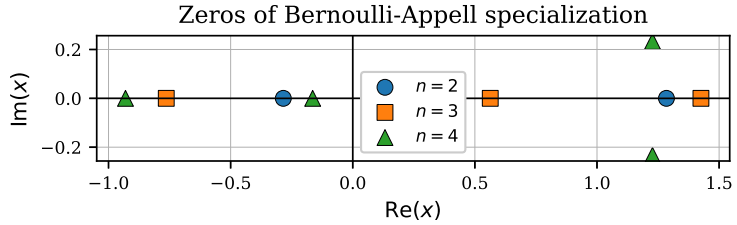


Figure 3. Zeros of ${}_{\mathcal{H}}\mathcal{B}_n^{(2)}(x,-0.3;0,0.6)$ for $n=2,3,4$ in the complex plane. The $n=4$ member already has a non-real conjugate pair, while the root pattern is shifted by the Bernoulli–Appell factor.

For the graphical part, the most transparent baseline choice is

$$\mathcal{A}_q(t)=1, \qquad w=0, \qquad m=2, \tag{104}$$

which reduces the family to the standard two-variable q-Gould–Hopper polynomials from [34,35]

$$H_{n,q}^{(2)}(x,y)=[n]_q!\sum_{k=0}^{\lfloor n/2\rfloor}\frac{y^k x^{n-2k}}{[k]_q!\,[n-2k]_q!}. \tag{105}$$

We emphasize that this graphical baseline is included for visual transparency only; it should be read together with the non-singular Bernoulli specialization above, which already belongs to the genuinely Appell-modified class.

Proposition 1. *For the specialization* (104),

$$H_{n,q}^{(2)}(-x,y)=(-1)^n H_{n,q}^{(2)}(x,y). \tag{106}$$

Hence, if x_0 is a zero, then $-x_0$ is also a zero. If $q, y \in \mathbb{R}$, complex zeros appear in conjugate-symmetric pairs as well.

Proof. In (105), the exponent of x is $n - 2k$. Replacing x by $-x$ therefore produces the factor $(-1)^{n-2k} = (-1)^n$ in each summand, proving (106). For real q and y, all coefficients are real, so non-real zeros come in conjugate pairs. □

For the concrete numerical illustration shown in Figures 4–8, take

$$q = 0.6, \qquad m = 2, \qquad y = -0.3. \tag{107}$$

Then the first few polynomials are

$$H^{(2)}_{2,0.6}(x, -0.3) = x^2 - 0.48, \tag{108}$$
$$H^{(2)}_{3,0.6}(x, -0.3) = x^3 - 0.9408x, \tag{109}$$
$$H^{(2)}_{4,0.6}(x, -0.3) = x^4 - 1.279488x^2 + 0.3838464, \tag{110}$$
$$H^{(2)}_{5,0.6}(x, -0.3) = x^5 - 1.50509568x^3 + 0.88499625984x. \tag{111}$$

Their zeros are approximately

$$\begin{aligned} H^{(2)}_{2,0.6} &: \quad x = \pm 0.6928, \\ H^{(2)}_{3,0.6} &: \quad x = 0, \ \pm 0.9699, \\ H^{(2)}_{4,0.6} &: \quad x = \pm 0.6930, \ \pm 0.8940, \\ H^{(2)}_{5,0.6} &: \quad x = 0, \ \pm(0.9201 \pm 0.3068i). \end{aligned}$$

To complement the separate real-axis profiles and the complex zero plot, we add four further graphical views of the same canonical specialization. The normalized comparison in Figure 9 places the degrees $n = 2, 3, 4, 5$ on the same scale. Figure 10 gives the zero-level sets in the full (x, y)-plane, so it displays the two-variable zero geometry directly. Figure 11 shows a density representation of $\log(1 + |H^{(2)}_{5,0.6}(x, y)|)$ with the zero contour superimposed. Finally, Figure 12 illustrates how the non-real zeros of $H^{(2)}_{5,q}(x, -0.3)$ move as q varies.

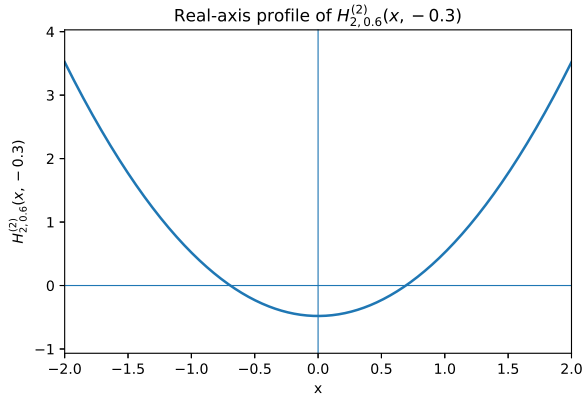


Figure 4. Real-axis profile of $H^{(2)}_{2,0.6}(x, -0.3) = x^2 - 0.48$. The graph is even, opens upward, and intersects the real axis exactly at $x = \pm 0.6928$.

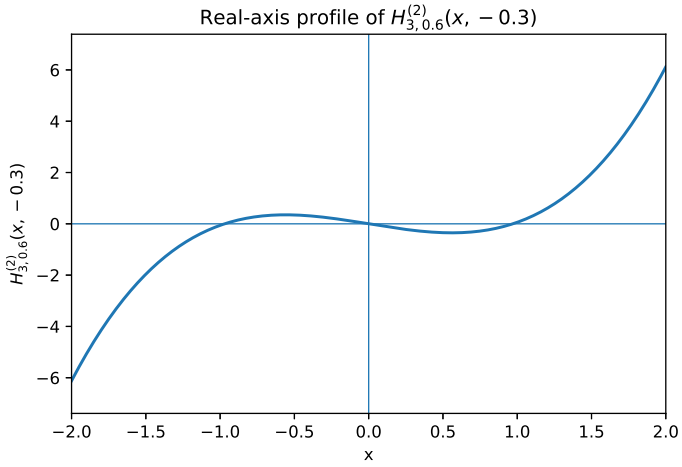


Figure 5. Real-axis profile of $H^{(2)}_{3,0.6}(x,-0.3) = x^3 - 0.9408x$. The graph is odd, hence origin-symmetric, and its three real zeros are $x = 0$ and $x = \pm 0.9699$.

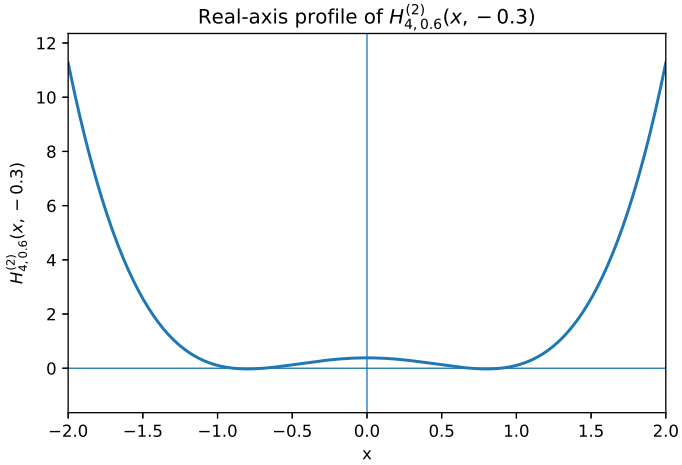


Figure 6. Real-axis profile of $H^{(2)}_{4,0.6}(x,-0.3) = x^4 - 1.279488x^2 + 0.3838464$. The even symmetry is preserved, and the quartic has four real zeros, grouped as two symmetric pairs.

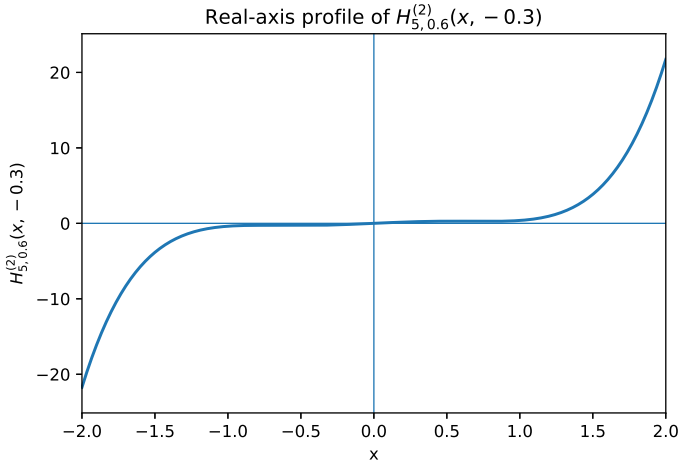


Figure 7. Real-axis profile of $H^{(2)}_{5,0.6}(x,-0.3) = x^5 - 1.50509568x^3 + 0.88499625984x$. Although the polynomial is odd, only one zero remains on the real axis; the other four form a complex quartet.

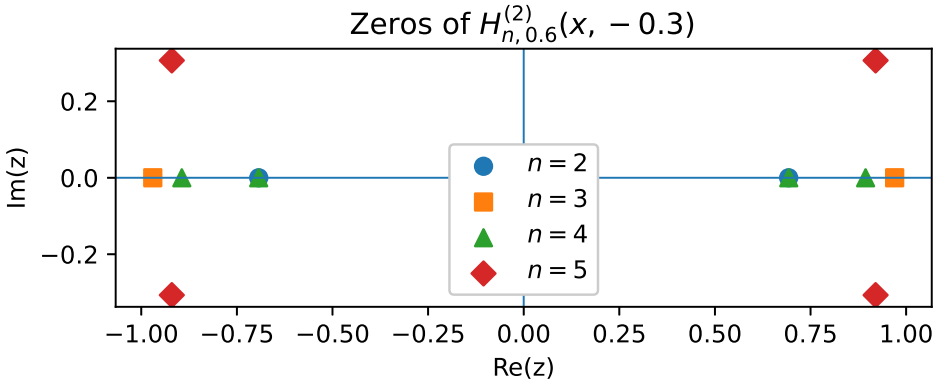


Figure 8. Zeros of $H^{(2)}_{n,0.6}(x, -0.3)$ for $n = 2, 3, 4, 5$ in the complex plane. The configuration is symmetric with respect to the real axis, and parity adds symmetry with respect to the origin.

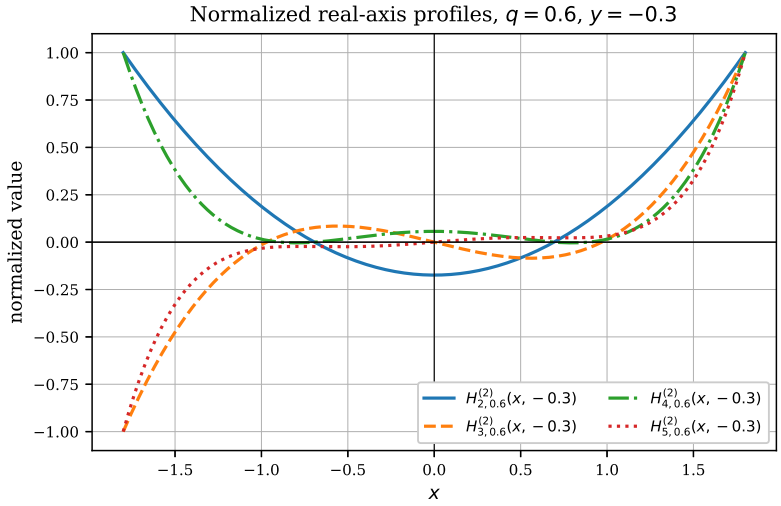


Figure 9. Normalized real-axis comparison of $H^{(2)}_{n,0.6}(x, -0.3)$ for $n = 2, 3, 4, 5$. Normalization by the maximum absolute value over the plotted interval allows oscillatory behavior and zero crossings across different degrees to be compared on a single panel.

Figures 4–7 show how the real-axis profiles change as n increases. The even degrees preserve even symmetry, the odd degrees preserve odd symmetry, and the number of real-axis intersections agrees with the list of real zeros given above. In particular, Figure 7 shows that $H^{(2)}_{5,0.6}$ has only one real zero, namely $x = 0$, while the remaining four zeros form a complex quartet.

Figures 8–12 highlight several persistent features of this baseline subclass:

1. Real zeros are symmetric with respect to the origin.
2. Non-real zeros are symmetric with respect to the real axis because the coefficients are real; together with odd/even parity, the full zero pattern becomes origin-symmetric as well.
3. As the degree increases, some zeros leave the real axis and bifurcate into conjugate pairs; this marks the onset of genuinely two-dimensional zero geometry.

These baseline features should be interpreted cautiously: they are representative of the canonical subclass (104), whereas the Bernoulli specialization (96), together with Figures 2 and 3, illustrates how the additional Appell factor modifies the low-degree coefficient pattern and the associated zero geometry.

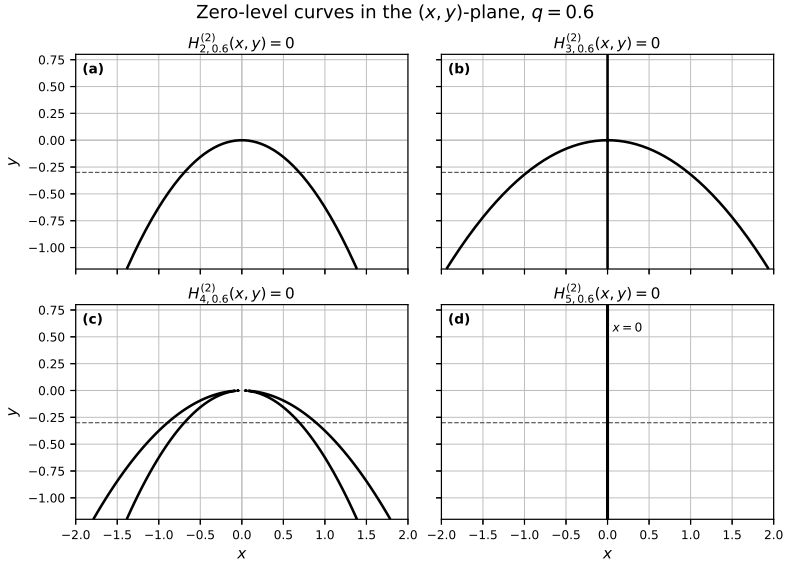


Figure 10. Zero-level curves $H^{(2)}_{n,0.6}(x, y) = 0$ in the (x, y)-plane. (**a**) The zero-level curve for $n = 2$; (**b**) the zero-level curve for $n = 3$; (**c**) the zero-level curve for $n = 4$; (**d**) the zero-level curve for $n = 5$. The dashed horizontal line marks the section $y = -0.3$ used in the one-variable profiles and zero computations. In panel (**d**), the real zero-level set is only the vertical curve $x = 0$, so this curve is drawn explicitly for visibility.

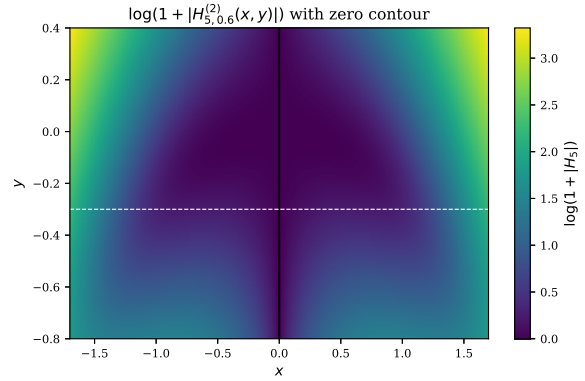


Figure 11. Heatmapof $\log(1 + |H^{(2)}_{5,0.6}(x, y)|)$ with the zero-level contour superimposed. The dashed horizontal line marks the section $y = -0.3$ used in the corresponding one-variable profile and zero computations. This density representation highlights the valley structure around the real zero set and complements the three-dimensional surface plot.

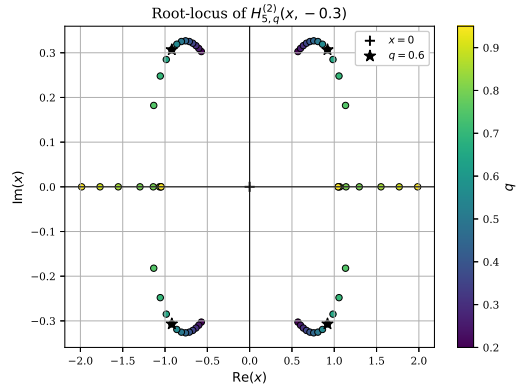


Figure 12. Root locus of $H^{(2)}_{5,q}(x, -0.3)$ for $0.2 \leq q \leq 0.95$. The origin remains a zero for all q, while the remaining four zeros move symmetrically in the complex plane; the markers at $q = 0.6$ correspond to the numerical values listed above.

6. Conclusions

We introduced the two-variable q-Gould–Hopper–Hahn–Appell polynomials by combining a general q-Appell kernel with a Hahn-type q-Gould–Hopper factor at the level of the generating function. This leads to a unified family that connects Appell theory, Hahn-type deformations, and Gould–Hopper-type constructions within the same formal setting.

Using generating function and operational methods, we obtained explicit series expansions, convolution and shift formulas, summation identities, higher and mixed q-derivative relations, inverse Gould–Hopper operational formulas, and Appell-factor operator factorizations. In the non-singular case $\mathcal{A}_{0,q} \neq 0$, we also derived the reciprocal-series coefficients and a determinant representation. The operational formulation further yields the quasi-monomial structure of the family together with its lowering operator, induced raising operator, and the associated q-differential equation.

The treatment of singular Appell kernels is separate but explicit. When $\mathcal{A}_{0,q} = 0$, the reciprocal series $1/\mathcal{A}_q(t)$ does not exist in $\mathbb{C}[[t]]$, so the determinant construction cannot be applied directly. The Genocchi specialization is handled by retaining the generating function, convolution, and operational identities and then using the Euler–Genocchi degree-shift relation to obtain the induced determinant formula stated in Theorem 18. Thus the singular case is not excluded; it is incorporated through a shifted non-singular Euler representation rather than through a direct reciprocal-series inverse.

We also clarified how the new class reduces to previously known families, including the two-variable q-Gould–Hopper–Hahn and q-Gould–Hopper–Appell polynomials, as well as Bernoulli-, Euler-, and Genocchi-type specializations. In particular, these specializations inherit explicit operational identities, while the determinant structure extends directly to the non-singular Bernoulli and Euler cases and, through a degree-shift relation, also yields an induced determinant description for the Genocchi family. Finally, the selected numerical examples show that the additional Appell factor can modify the low-degree zero pattern in a substantial way. These results indicate that the proposed family provides a flexible setting for further work on q-special functions, operational identities, and related q-difference models.

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