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A Unified Probabilistic Framework for the Reliability and Robustness Assessment of Series Structural Systems: Axiomatic Foundations and Computational Approximations

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Abstract

This paper develops a unified probabilistic framework for the simultaneous assessment of reliability and robustness of series structural systems. Part I formulates an axiomatic theory of structural robustness: the robustness factor F_{rob} is defined as the negative decadic logarithm of the ratio between the damaged-state system failure probability and a normalised target failure probability, and five formal properties are proven—normalisation, monotonicity in damage severity, exact combination bounds under independent damage scenarios, invariance under reliability-preserving transformations, and boundedness. Part II describes the computational core, the evaluation of series system failure probability: the principal approximation methods (simple bounds, the complement-product reformulation, Ditlevsen's narrow bounds) are formally derived, and a geometric mean-bound approximation over a dominant-element subset is introduced, with four propositions establishing consistency with bounds, monotonicity, convergence, and a complete characterisation of the error direction under positive equicorrelation, including a provably conservative regime above a threshold correlation. Part III synthesises both parts into a computational framework that evaluates F_{rob} from first-order reliability method (FORM) outputs alone, avoiding repeated multi-dimensional numerical integration in the inner loop of design optimisation, with an explicit, computable error bound guaranteeing invariance of the robustness classification; a two-sided Slepian envelope further extends this guarantee to systems with non-equicorrelated, non-negative inter-element correlation. The framework is illustrated through a numerical example and validated against a published case study of a glued–laminated timber truss.

Keywords: series system reliability; structural robustness; axiomatic properties; reliability bounds; first-order reliability method; Ditlevsen bounds; probabilistic framework; failure probability approximation; structural safety

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1. Introduction

Structural reliability theory and structural robustness theory are two pillars of modern probabilistic structural analysis that have despite their close conceptual relationship evolved largely in parallel. Reliability theory, formalised in the 1970s through the contributions of Cornell, Hasofer, Lind, Rackwitz, Ditlevsen, and others, provides a quantitative framework

for evaluating the probability of failure of structural systems under uncertain loading and resistance [1,2]. Robustness theory, motivated by spectacular collapses such as Ronan Point (1968) and the World Trade Center (2001), seeks to characterise the ability of a structure to tolerate local damage without disproportionate global collapse [3,4]. The two theories share probabilistic foundations but employ different mathematical objects: reliability is expressed through failure probabilities and reliability indices, while robustness is typically quantified through scalar measures comparing intact and damaged configurations.

The relationship between the two has been recognised since at least Baker, Schubert, and Faber [3], who proposed a generalised risk-based robustness index combining direct and indirect consequences. Subsequent work by the Joint Committee on Structural Safety [5] and the COST Action TU0601 [6] developed reliability-based robustness frameworks specifically for timber and concrete structures. The robustness factor F_{rob}, introduced for timber trusses in [7] and formalised in [8], expresses robustness as the negative base-10 logarithm of the ratio between damaged-state failure probability and a normalised target failure probability: $F_{rob} = -\log_{10}(p_{Sf,dmg}/p_{f,norm})$. The normalised failure probability is set by code convention to $p_{f,norm} = 8.0 \times 10^{-5}$, corresponding to a target system reliability index $\beta_{S,norm} = 3.775$, yielding F_{rob} values bounded below by -4.097 and above by the intact-state value $F_{rob}(\emptyset)$, which for structures designed to code targets is typically of order +1. Despite its operational simplicity, the formal mathematical properties of this measure have not been systematically established.

On the reliability side, the computation of system failure probability for series systems with more than ten to fifteen elements by direct numerical integration of the multivariate normal distribution function becomes increasingly demanding. Although modern quasi-Monte Carlo schemes keep a single evaluation feasible well beyond this range, the cost compounds when the integration must be repeated across the many damage scenarios and design iterations of a robustness assessment. Approximation methods have been developed over five decades, beginning with the simple Bonferroni bounds [9], advancing through Ditlevsen's narrow bounds [10], and reaching modern compounding and simulation-based approaches [11–14]. The accuracy and error direction of these methods have been studied empirically but rarely characterised through formal propositions.

This paper has three objectives. First, in Part I (Sections 2 and 3), we develop an axiomatic theory of structural robustness based on the F_{rob} measure, with explicit propositions characterising its formal properties. Second, in Part II (Sections 4–6), we provide a unified formal treatment of series system reliability approximations, including a new geometric mean-bound approximation (restricted to a dominant-element subset) with proven formal properties—most importantly a complete characterisation of the error direction under positive inter-element correlation, together with an explicit, computable error bound on the robustness factor. Third, in Part III (Sections 7 and 8), we synthesise the two parts into a unified computational framework allowing robustness assessment with FORM-level computational cost and illustrate the framework through a worked example.

The contribution is purely theoretical. Experimental validation of probabilistic material parameters and applied engineering case studies on timber truss systems are reported in companion publications [8,15,16] and are cited here only as illustrations of the framework's applicability. The intent of this paper is to provide a self-contained mathematical foundation that can support subsequent applied work in any structural material and any series system topology.

The motivation for unifying reliability and robustness theory comes from several practical observations. First, modern structural codes increasingly require probabilistic justification of robustness—Eurocode EN 1991-1-7 [17] explicitly calls for risk-based assessment of disproportionate collapse for Consequence Class 3 structures. Second, performance-

based design frameworks evaluate structures against multiple performance objectives that combine reliability and robustness criteria. Third, the increasing use of computational design tools requires algorithms that can evaluate reliability and robustness in tight design-iteration loops with bounded computational cost. The unified framework developed here addresses all three needs.

The lack of a unified mathematical framework has practical consequences. Engineers compute reliability indices for design verification using one set of tools and assumptions, then attempt to translate these into robustness assessments using a different set of definitions and methods. The translation is often informal, with no guarantee that the resulting robustness measure inherits the desirable properties of the reliability calculation. Codes increasingly require both reliability and robustness justification (EN 1990, EN 1991-1-7), but the connection between them is left implicit. The framework developed here aims to make this connection explicit and mathematically rigorous.

A deeper motivation comes from the recognition that reliability and robustness are not independent properties but two aspects of a single underlying probabilistic structure. A structure's behaviour under uncertain loading and material variability (reliability) determines how it responds to local damage (robustness). The same FORM analyses that produce component reliability indices for the intact configuration can be reused, with modifications for the damaged configurations, to produce robustness factors. The unified framework exploits this overlap to provide robustness assessment at essentially the cost of reliability assessment.

Several recent contributions have advanced this unification. Faber and Sørensen [18] developed indicators that span reliability and inspection planning. Frangopol and Curley [19] established early formulations linking redundancy and reliability. Kim and Frangopol [20] extended these ideas to cost-based optimisation of inspection scheduling under uncertainty. The present paper continues this line of work but focuses specifically on the formal mathematical properties required for the framework to function as a classification tool, with explicit propositions and proofs that have been absent from much of the engineering literature.

The paper is organised as follows. Section 2 develops the axiomatic theory of structural robustness. Section 3 proves five axiomatic properties of the F_{rob} measure. Section 4 introduces the exact formulation of series system reliability and its computational limitations. Section 5 reviews the principal approximation methods with formal derivations. Section 6 presents the proposed mean-bound approximation and its four formal propositions. Section 7 unifies the two parts into a single computational framework. Section 8 provides a worked example. Section 9 concludes and identifies future research directions.

Throughout this paper, we use the following notation. Lower-case bold letters denote vectors; upper-case bold letters denote matrices. Φ (capital Phi) and φ (lower-case phi) denote the standard normal CDF and PDF, respectively. $\Phi_m(\cdot; R)$ denotes the m-variate standard normal CDF with correlation matrix R. The set of natural numbers is $\mathbb{N}$, and 2^S (the power set of S) denotes the set of all subsets of S. P(A) denotes the probability of event A, and E[X] denotes the expectation of random variable X. The foundational works in structural reliability theory by Cornell [21], Hasofer and Lind [22], and Rackwitz and Fiessler [23] establish the FORM framework on which the present analysis relies.

2. Probabilistic Framework for Robustness Assessment

2.1. Notation and Basic Definitions

Let S denote a structural system consisting of m discrete components (elements) indexed by i = 1, 2, . . ., m. Let X = (X1, . . ., Xn) be the vector of basic random variables representing material strengths, geometric quantities, and loads, with joint cumulative

distribution function FX. The state of component i is governed by a limit-state function gi(X), with failure defined by gi(X) ≤ 0. The reliability index of component i is defined via FORM as $\beta_i = \|u^*_i\|$ where u^*_i is the design point vector in the standard normal space U after transformation X $\rightarrow$ U via the Rosenblatt or Nataf transformation [1].

A damage scenario D is a subset D $\subseteq \{1, \ldots, m\}$ of components considered unavailable (failed or removed). The set of all admissible damage scenarios is $D = 2^{\{1,\ldots,m\}}$ (the power set of all component indices), including the empty scenario $\emptyset$ (intact configuration) and the full scenario {1, . . ., m} (collapsed configuration). For computational tractability, only scenarios of cardinality $|D| \leq k$ are typically considered, where k reflects the assumed number of simultaneous damage-initiating events.

The system reliability index $\beta_S(D)$ of configuration D is defined as $\beta_S(D) = \Phi^{-1}(1 - p_{Sf}(D))$, where $p_{Sf}(D)$ is the system failure probability conditioned on the components in D being unavailable, and Φ (capital Phi) is the standard normal CDF. By convention, $\beta_S(\emptyset)$ denotes the reliability of the intact system.

2.2. Definition of the Robustness Factor

The robustness factor F_{rob}, originally proposed by Čizmar [8] and formalised here, is defined as a function F_{rob}: D $\rightarrow$ R (the "robustness factor for normalised failure probability") given by

$$F_{\text{rob}}(D) = -\log_{10}\left(\frac{p_{Sf}(D)}{p_{f,\text{norm}}}\right) \tag{1}$$

By construction, $F_{rob}(D)$ takes values on a logarithmic scale: $F_{rob}(D) < 0$ indicates that the damaged-state failure probability exceeds the normalised (target) failure probability, while $F_{rob}(D) \geq 0$ indicates that the damaged structure satisfies the target. The normalised failure probability $p_{f,norm}$ is set by engineering convention to $p_{f,norm} = 8.0 \times 10^{-5}$, corresponding to a target reliability index $\beta_{S,norm} = 3.775$ [8]. The classification thresholds established in [8] are: $F_{rob} \in [-4.097, -3.097]$: no robustness; $F_{rob} \in (-3.097, -2.097]$: low robustness; $F_{rob} \in (-2.097, -1.097]$: medium robustness; $F_{rob} \in (-1.097, 0]$: high robustness; and $F_{rob} > 0$: ideal robustness.

The numerical values of these thresholds follow directly from the normalisation and require no independent calibration. The scale is anchored at two natural points: the target ($F_{rob} = 0$, where $p_{Sf} = p_{f,norm}$) and certain collapse ($F_{rob} = \log_{10}(p_{f,norm}) \approx -4.097$, where $p_{Sf} = 1$). The interior boundaries -3.097, -2.097, and -1.097 fall one decade of failure probability apart, measured from the collapse floor—each successive class corresponding to a tenfold change in p_{Sf}—so the scale is a direct logarithmic re-expression of the residual failure probability and inherits whatever target the governing code prescribes. The thresholds are therefore sensitive to the code convention only through $p_{f,norm}$: adopting a stricter target—for instance $\beta_{S,norm} = 4.7$ ($p_{f,norm} \approx 1.3 \times 10^{-6}$) for the highest consequence class of EN 1990—shifts the entire scale and moves the collapse floor to $\log_{10}(1.3 \times 10^{-6}) \approx -5.9$, widening the classification range, whereas a less strict target compresses it. The framework itself is invariant to this choice: every proposition is proved for an arbitrary fixed $p_{f,norm}$, and only the reported numerical class boundaries change. Where cross-national or cross-code comparison is required, F_{rob} values should be quoted together with the $p_{f,norm}$ against which they were computed.

Throughout, the structure is modelled as a series system of its dominant failure modes; a component of this series system is a failure mode, which may itself be a parallel mechanism—as in the case study of Section 8.6, modelled as a series system of parallel subsystems—so that structural redundancy is represented at the component level. System failure is identified with the attainment of any one modal limit state, a deliberately conservative definition that does not require the formation of a collapse mechanism, and

the post-damage states are obtained by removing a failed element and recomputing the reliabilities of the remaining modes under load redistribution.

For a robustness classification system, the minimum F_{rob} over the considered damage scenarios D is the relevant quantity:

$$F^*_{rob} = \min_{D \in \mathcal{D},\ |D| \leq k} F_{rob}(D) \tag{2}$$

This is the worst-case robustness factor. Threshold values listed above are established by engineering convention in [8,24] and are not derived axiomatically here; the theory does not depend on specific threshold choices.

Two clarifications about damage-scenario enumeration are needed. First, the set D of admissible damage scenarios depends on the engineering judgement of plausible initiating events. For most structures, single-element damage ($|D| = 1$) is the dominant scenario type, as random failures (manufacturing defects, isolated impacts, localised fires) typically affect one element at a time. For seismic or progressive-collapse scenarios, multi-element damage scenarios ($|D| \geq 2$) may be relevant, with the joint probability of simultaneous damage following from the scenario probability model.

Second, the framework distinguishes between damage to components (elements removed from the system) and partial loss of strength of components (elements with reduced reliability indices). The propositions of Section 3 are stated for full component removal, but the framework extends naturally to partial damage: the component reliability index β_i is reduced by an amount $\delta\beta_i$ reflecting the partial loss of strength, and the system analysis proceeds as before. The propositions remain valid with appropriate redefinition of the damage scenario as a vector of strength reductions rather than a subset of removed components.

The choice of defining the robustness measure directly in probability space through the decadic logarithm of Equation (1) deserves brief justification. The reliability index β remains the standard reporting scale of structural codes [1,2], with interpretable target values (β = 3.8 for moderate consequences, 4.7 for high consequences, per EN 1990 Annex C). The measure $F_{rob}(D) = -\log_{10}(p_{Sf}(D)/p_{f,norm})$ is, however, defined as a logarithmic ratio of failure probabilities, and every derivation in this paper operates on this definition; no step of the framework relies on differences in reliability indices. The decadic logarithm compresses the many orders of magnitude spanned by failure probabilities into a numerically convenient bounded scale, while remaining in one-to-one correspondence with the β-scale through $\beta_S = -\Phi^{-1}(p_{Sf})$, so that thresholds expressed on the F_{rob} scale translate uniquely into β-values whenever code compatibility is required.

Several alternative robustness measures have been proposed in the literature, including the reliability index ratio $\beta_{S,dmg}/\beta_{S,intact}$, the probability ratio $p_{Sf,dmg}/p_{Sf,intact}$, and the consequence-weighted measures of Baker et al. [3]. Each emphasises different aspects of the damage-state behaviour. The logarithmic ratio definition of Equation (1) has three advantages for engineering practice: (i) it is dimensionless and varies over a numerically moderate, bounded range even though the underlying probabilities span many orders of magnitude; (ii) it satisfies the axioms of Section 3 with simple and fully rigorous proofs; (iii) it is anchored to a fixed code-based target $p_{f,norm}$, so that values are directly comparable across structures and damage scenarios without reference to the intact configuration.

It should be made explicit which of these roles F_{rob} plays. Because Equation (1) involves only the damaged-state failure probability $p_{Sf}(D)$ and the fixed code target $p_{f,norm}$—and not the intact-state probability $p_{Sf}(\emptyset)$—F_{rob} is, in the taxonomy of robustness measures, a normalised post-damage residual-reliability measure expressed on a logarithmic scale, rather than a differential measure that compares the damaged and intact configurations. Differential measures—the redundancy and reliability-difference indices of Frangopol

and Curley [19], the probability ratio $p_{Sf,dmg}/p_{Sf,intact}$, and the risk-based index of Baker, Schubert and Faber [3]—quantify the relative degradation caused by damage and therefore tend to zero for a structure that is already weak when intact. F_{rob} deliberately does not: by referencing an absolute, code-calibrated target rather than the intact state, it measures whether the damaged structure retains an acceptable residual safety level in absolute terms. This is appropriate for code-type acceptance, where the operative question is whether a damaged structure remains tolerably safe, not merely how much it has degraded relative to its own intact state; the absolute form also avoids the ill-conditioning of intact-referenced ratios when $p_{Sf}(\emptyset)$ is extremely small, as it is for well-designed structures, where a differential measure can grow arbitrarily large. The intact value $F_{rob}(\emptyset)$ is retained as a separate reference level (Proposition 1) and may be reported alongside $F_{rob}(D)$ whenever an explicitly differential view is required: the difference $F_{rob}(D) - F_{rob}(\emptyset) = -\log_{10}(p_{Sf}(D)/p_{Sf}(\emptyset))$ recovers the logarithmic probability-ratio robustness measure exactly. The framework therefore subsumes both readings, while adopting the absolute residual-reliability form as primary.

The framework can also be extended to incorporate consequences. Define a consequence-weighted robustness factor $F_{rob,C}(D) = F_{rob}(D) \cdot C(D)$, where $C(D)$ is a normalised consequence function reflecting economic, social, or environmental impacts of damage scenario D. This extension preserves the axiomatic properties of Section 3 under mild conditions on $C(D)$ (continuity and monotonicity in damage severity) and provides a natural bridge to risk-based assessment.

3. Axiomatic Properties of the Robustness Factor

This section establishes five formal properties of the robustness factor as defined in Equation (1). These properties characterise F_{rob} mathematically and provide a basis for its use as a structural classification measure.

3.1. Property 1: Normalisation

Proposition 1 (Normalisation). *For any structural system S, the value $F_{rob}(\emptyset) = -log_{10}(p_{Sf}(\emptyset)/p_{f,norm})$ represents the reference (intact) robustness level. For a well-designed structure with $p_{Sf}(\emptyset) \leq p_{f,norm}$, $F_{rob}(\emptyset) \geq 0$. For the fully collapsed configuration, $F_{rob}(\{1,\ldots,m\}) = -log_{10}(1/p_{f,norm}) = log_{10}(p_{f,norm})$, which is a finite negative value (approximately -4.097 for $p_{f,norm} = 8.0 \times 10^{-5}$), representing the lower bound of the F_{rob} scale.*

Proof. For the intact case, $F_{rob}(\emptyset) = -\log_{10}(p_{Sf}(\emptyset)/p_{f,norm})$ is finite and depends only on the design margin between the intact failure probability and the target. For the collapsed case, $p_{Sf}(\{1, \ldots, m\}) = 1$, hence $F_{rob}(\{1, \ldots, m\}) = -\log_{10}(1/p_{f,norm}) = \log_{10}(p_{f,norm})$. With $p_{f,norm} = 8.0 \times 10^{-5}$, this yields $F_{rob}(\{1, \ldots, m\}) \approx -4.097$, the established lower bound of the classification scale [8]. □

3.2. Property 2: Monotonicity in Damage Severity

Proposition 2 (Monotonicity in Damage). *If D_1 is a subset of D_2 ($D_1 \subseteq D_2$), and the structural model satisfies the damage-consistency assumption (A0): the conditional system failure probability is non-decreasing with respect to set inclusion of damage scenarios; $p_{Sf}(D_1) \leq p_{Sf}(D_2)$ whenever $D_1 \subseteq D_2$—physically, additional damage cannot make collapse less likely—then $F_{rob}(D_2) \leq F_{rob}(D_1)$. In other words, adding components to a damage scenario can only worsen (or preserve) the resulting robustness factor.*

Proof. The inequality $p_{Sf}(D_2) \geq p_{Sf}(D_1)$ is exactly assumption (A0); it is not automatic in the surviving-series representation, because removing a component also removes its failure

mode—a counterexample is the removal of a lightly loaded (e.g., near-zero-force) member whose loss does not increase the demand on any surviving component, which strictly reduces p_{Sf}. Assumption (A0) excludes such pathological configurations and is satisfied whenever load redistribution does not relieve any surviving component while the contribution of the removed failure mode is compensated by the increased demand on the survivors. Under (A0), dividing by the fixed constant $p_{f,norm}$ preserves the inequality, and since $-\log_{10}$ is strictly decreasing, $F_{rob}(D_2) = -\log_{10}(p_{Sf}(D_2)/p_{f,norm}) \leq -\log_{10}(p_{Sf}(D_1)/p_{f,norm}) = F_{rob}(D_1)$. □

3.3. Property 3: Combination of Independent Damage Scenarios

Proposition 3 (Exact Bounds under Combination of Independent Damage Scenarios)**.** *Let D_1 and D_2 be damage scenarios satisfying the following assumptions: (A1) $D_1 \cap D_2 = \emptyset$; (A2) the system failure event under the combined scenario decomposes as the union of the failure events under the individual scenarios, $F(D_1 \cup D_2) = F(D_1) \cup F(D_2)$; (A3) the events $F(D_1)$ and $F(D_2)$ are statistically independent. Then, exactly and without any small-probability approximation, $\min(F_{rob}(D_1), F_{rob}(D_2)) - \log_{10} 2 \leq F_{rob}(D_1 \cup D_2) \leq \min(F_{rob}(D_1), F_{rob}(D_2))$. For n pairwise disjoint scenarios satisfying (A2)–(A3), the lower offset $\log_{10} 2$ is replaced by $\log_{10} n$.*

Proof. Write $p_i = p_{Sf}(D_i)$, $i \in \{1, 2\}$, and $p_{12} = p_{Sf}(D_1 \cup D_2)$. By (A2), inclusion–exclusion, and independence (A3): $p_{12} = p_1 + p_2 - p_1 \cdot p_2$. Since $0 \leq p_1 \cdot p_2 \leq \min(p_1, p_2)$, it follows that $\max(p_1, p_2) \leq p_{12} \leq p_1 + p_2 \leq 2 \cdot \max(p_1, p_2)$. Dividing by $p_{f,norm}$ and applying the strictly decreasing map $-\log_{10}$ reverses the inequalities: $-\log_{10}(2 \cdot \max(p_1, p_2)/p_{f,norm}) \leq F_{rob}(D_1 \cup D_2) \leq -\log_{10}(\max(p_1, p_2)/p_{f,norm})$. The right-hand side equals $\min(F_{rob}(D_1), F_{rob}(D_2))$ and the left-hand side equals $\min(F_{rob}(D_1), F_{rob}(D_2)) - \log_{10} 2$. The case of n scenarios follows from $\max_i p_i \leq p_{Sf}(D_1 \cup \ldots \cup D_n) \leq n \cdot \max_i p_i$, a consequence of the union bound. □

Remark 1. *Proposition 3 is exact: assumptions (A1)–(A3) are stated explicitly and the proof uses no small-probability approximation and no local linearisation. The asymptotically additive behaviour $p12 \approx p1 + p2$, valid when the product $p1 \cdot p2$ is negligible, is recovered as a special case but is not required. The upper bound expresses sub-additive degradation—a combined scenario can never be classified as more robust than the worse of its constituents—while the lower bound shows that the additional degradation caused by combining two independent scenarios never exceeds $\log10\ 2 \approx 0.301$ on the Frob scale. It should nevertheless be emphasised that (A2)–(A3) constitute an idealisation: in real structures the failure events of distinct damage scenarios share the same load and material variables and are therefore positively dependent, and combined damage generally aggravates load redistribution beyond the union of the individual effects, so that $F(D1 \cup D2) \supseteq F(D1) \cup F(D2)$. Proposition 3 should accordingly be read as an exact result for the idealised model, providing reference bounds rather than guaranteed values for physically coupled scenarios. Because the true combined failure event then contains the union, the resulting $Frob(D1 \cup D2)$ is an optimistic (upper) estimate of the robustness of coupled scenarios; the safety-critical assessment in this work therefore relies on the worst single-element-removal scenario ($|D| = 1$) rather than on scenario combination.*

3.4. Property 4: Invariance Under Reliability-Preserving Transformations

Proposition 4 (Invariance)**.** *Let $T: X \to X'$ be a transformation of basic random variables that preserves the joint distribution of $(g_1, \ldots, g_m)$. Then $F_{rob}(D)$ is invariant under T for all D in D.*

Proof. Since T preserves the joint distribution of the limit-state functions, the system failure probability $p_{Sf}(D)$ is invariant for all D. The normalised value $p_{f,norm}$ is fixed by code convention. Hence $F_{rob}(D) = -\log_{10}(p_{Sf}(D)/p_{f,norm})$ is invariant under T. □

This property is important because it implies that F_{rob} does not depend on the parametrisation of the random variables, only on the joint distribution of the limit-state functions. The choice of transformation (Rosenblatt vs. Nataf, for instance) does not affect F_{rob}.

The invariance property has both theoretical and practical significance. Theoretically, it confirms that F_{rob} is a genuine geometric property of the joint distribution of limit-state functions, not an artefact of any particular parameterisation. Practically, it allows different software implementations to produce identical F_{rob} values regardless of internal coordinate choices, supporting reproducibility and cross-validation across analysis environments.

3.5. Property 5: Boundedness

Proposition 5 (Boundedness)**.** *For any series system S whose structural model satisfies assumption (A0) of Proposition 2, the worst-case robustness factor lies in the bounded interval* $F_{rob}^* \in [\log_{10}(p_{f,norm}), F_{rob}(\emptyset)]$*. With* $p_{f,norm} = 8.0 \times 10^{-5}$*, the lower endpoint is* ≈ -4.097 *and is attained when the most critical damage scenario causes collapse* ($p_{Sf} = 1$)*; the upper endpoint* $F_{rob}(\emptyset)$ *is finite and is set by the design margin of the intact configuration, so that no universal numerical upper bound exists.*

Proof. For a non-redundant system, the most critical damage scenario D* leads to $p_{Sf}(D^*) = 1$, hence $F_{rob}(D^*) = -\log_{10}(1/p_{f,norm}) = \log_{10}(p_{f,norm})$. With the conventional $p_{f,norm} = 8.0 \times 10^{-5}$, this gives $F_{rob}(D^*) \approx -4.097$. For redundant systems, $p_{Sf}(D^*) < 1$ and $F_{rob}(D^*) > \log_{10}(p_{f,norm})$. For the upper endpoint, Proposition 2 (under assumption (A0)) gives $F_{rob}(D) \leq F_{rob}(\emptyset)$ for every admissible D, hence $F_{rob}^* \leq F_{rob}(\emptyset)$, and the interval $[\log_{10}(p_{f,norm}), F_{rob}(\emptyset)]$ follows. This finite lower bound is what makes F_{rob} a useful classification measure; for structures designed to code targets, $F_{rob}(\emptyset)$ is typically of order +1, which motivates the practical range [−4.097, ≈+1] quoted in [8], the classification thresholds spanning this range. □

This proposition establishes that F_{rob} is a meaningful classification measure: the difference between a fully redundant system ($F_{rob} \to 0$) and a brittle non-redundant system ($F_{rob} \to \log_{10}(p_{f,norm}) \approx -4.097$) is large enough to distinguish robustness classes.

These five propositions establish the mathematical foundation for using F_{rob} as a structural classification measure. They guarantee that the measure behaves predictably under variations in the structural configuration and damage scenarios and that classification thresholds can be set based on engineering convention without ambiguity. The framework developed here is distinct from but complementary to the indicator-based approaches of Faber and Sørensen [18] and the consequence-based approaches of Baker et al. [3]. While those frameworks emphasise different aspects of robustness—inspection planning and consequence evaluation, respectively—the F_{rob} measure developed here focuses on the relative reliability degradation, which is the most operationally tractable quantity for routine engineering analysis.

While the propositions and proofs of this paper are framed in general terms applicable to any series structural system, the motivation for this work arose from the analysis of timber truss structures [8]. Timber trusses present several features that make the proposed framework particularly valuable: they are routinely composed of m = 20 to 50 elements (exceeding the limit for exact integration), they typically have positive inter-element correlation due to batch effects in material strength (where Proposition 9 applies), and they require classification into discrete robustness categories for code compliance.

Beyond timber, the framework applies directly to other structural materials where similar features are present: steel lattice girders [25], reinforced concrete frames [19], and

masonry structures. The propositions of Sections 3 and 6 do not depend on the structural material or topology; they hold for any structural material or member type, subject only to the series-system logic and the non-negative inter-element correlation assumed throughout. The axiomatic properties of Section 3 require only the series-system logic (any element failure constitutes system failure) and the damage-consistency assumption (A0), while the reliability-approximation results of Sections 6 and 7 additionally adopt the non-negative-correlation assumption—which, as discussed in Section 5.2, must be verified for the specific modal set rather than presumed, since the FORM modal correlation $\rho_{ij} = \alpha_i^T \alpha_j$ may be negative even when the modes share basic variables. The breadth of applicability of the framework is therefore substantial, subject to that verification.

4. Exact System Reliability and Computational Limitations

This section formalises the computational challenge that motivates the approximation methods of Sections 5 and 6. The exact series system failure probability—exact for jointly Gaussian (i.e., FORM-linearised) safety margins, which is the working assumption throughout—is defined as

$$p_{Sf} = P\left(\bigcup_i \{g_i(\mathbf{X}) \le 0\}\right) = 1 - \Phi_m(\boldsymbol{\beta}; \mathbf{R}) \tag{3}$$

where $\Phi_m(\beta; R)$ is the m-variate standard normal cumulative distribution function with reliability vector $\beta = (\beta_1, \ldots, \beta_m)$ and correlation matrix R (in bold) with elements $R[i,j] = \rho_{ij} = (\alpha_i)^T \alpha_j$, the inner product of the FORM sensitivity vectors α_i and α_j.

The numerical evaluation of Φ_m requires m-dimensional integration. For m = 2, fast algorithms exist [26]. For m = 3 to 10, recursive numerical methods (Gauss–Hermite quadrature, Monte Carlo integration) remain tractable. For larger m, tensor-product quadrature schemes such as Gauss–Hermite become impractical because the number of quadrature points grows exponentially with m; modern quasi-Monte Carlo algorithms (for example the Genz method) nevertheless remain feasible for m values of several tens, although a single evaluation still requires the full correlation matrix and is not available in closed form. The cost becomes a practical concern mainly when this evaluation must be repeated—across the many damage scenarios of a robustness assessment and the many configurations of a design-iteration loop—and in workflows where each limit-state evaluation already requires a finite-element analysis.

This limitation has motivated the development of approximation methods. The remainder of Part II reviews and extends these methods with a focus on formal properties.

5. Existing Approximation Methods for Series System Reliability

5.1. Simple Bonferroni Bounds

The simple bounds, derived from the Bonferroni inequality, are [9]

$$\max_i p(F_i) \le p_{Sf} \le \sum_i p(F_i) \tag{4}$$

where $p(F_i) = \Phi(-\beta_i)$ is the marginal failure probability of component i. The lower bound is the probability of failure of the most critical single component. The upper bound corresponds to fully exclusive failure events. These bounds use no information about pairwise correlations and are typically wide for systems with multiple critical components.

5.2. Sørensen's Complement-Product Reformulation

Sørensen [11] expressed the upper bound in a computationally convenient form:

$$1 - \prod_i (1 - p(F_i)) \geq p_{Sf} \geq \max_i p(F_i) \tag{5}$$

The reformulated upper bound represents the failure probability under the assumption of independent failure events. It is equivalent in Equation (4) to first order in $p(F_i)$ and is numerically more stable when the failure probabilities are small. The upper bound in Equation (5) is valid for non-negatively correlated safety margins ($\rho_{ij} \geq 0$ for all i, j), as guaranteed by the Slepian inequality [27]; under general (possibly negative) dependence, only the Bonferroni sum $\Sigma_i\, p(F_i)$ of Equation (4) is a universal upper bound. Throughout this paper the structural assumption $\rho_{ij} \geq 0$ is adopted, consistent with elements sharing common sources of uncertainty (loads, batch effects in material strength). This assumption is a genuine restriction and not merely a technical convenience. In FORM the modal correlation is the inner product $\rho_{ij} = \alpha_i{}^T\, \alpha_j$ of the unit sensitivity vectors at the two design points, and this inner product is not guaranteed to be non-negative even when the modes share basic variables: opposing failure mechanisms (for example a tension mode and a compression mode driven by the same axial action), modes governed by load effects of opposite sign, or the reorientation of sensitivity vectors after post-damage redistribution can all produce $\rho_{ij} < 0$. The assumption $\rho_{ij} \geq 0$ should therefore be verified for the specific modal set rather than presumed; it holds when the dominant failure modes are driven concordantly by a common load or resistance variable, which is the typical—but not universal—situation for the chord and diagonal bending–axial modes of the trusses that motivate this work. Where some pairs are negatively correlated, the complement-product upper bound of Equation (5), Proposition 9, and the Slepian envelope of Proposition 11 no longer apply; the universal Bonferroni upper bound $\Sigma_i\, p(F_i)$ of Equation (4), which requires no sign assumption, remains valid and may be substituted for pU, at the cost of a wider, but still rigorous, error bound in Proposition 10.

5.3. Ditlevsen's Narrow Bounds

Ditlevsen [10] derived bounds that incorporate pairwise correlation information through bivariate normal probabilities. With elements sorted in descending order $p(F_1) \geq p(F_2) \geq \ldots \geq p(F_m)$:

$$p(F_1) + \sum_{i=2}^{m} \max \left[0,\, p(F_i) - \sum_{j=1}^{i-1} p\left(F_i \cap F_j\right) \right] \leq p_{Sf} \tag{6}$$

$$p_{Sf} \leq p(F_1) + \sum_{i=2}^{m} \left[p(F_i) - \max_{j<i} p\left(F_i \cap F_j\right) \right] \tag{7}$$

where $p(F_i \cap F_j) = \Phi_2(-\beta_i, -\beta_j; \rho_{ij})$ is the bivariate normal joint failure probability. The bounds are narrower than Equation (5) but require $m(m-1)/2$ bivariate integrations and depend on element ordering.

6. Mean-Bound Approximation

6.1. Definition

Sort the elements in descending order of failure probability: $p_1 \geq p_2 \geq \ldots \geq p_m > 0$, with corresponding reliability indices $\beta_i = \Phi^{-1}(1 - p_i)$. Define the dominant-element set

$$D_j = \{1, 2, \ldots, j\}, \quad \text{where } j = \max\{\, k : p_k \geq \varepsilon \cdot p_1 \,\} \tag{8}$$

for a threshold epsilon in the open interval (0, 1). The recommended value is $\varepsilon = 10^{-2}$, selecting elements whose failure probability is at least 1% of the most critical element. The mean-bound approximation is defined as the geometric mean of the simple bounds restricted to Dj. The geometric mean is the natural central estimator on the logarithmic probability scale on which F_{rob} is defined: it places the estimate at the exact midpoint between the bounds in log-space, so that the error of F_{rob} is symmetric and attains the tight half-width bound of Proposition 10. Among all estimators formed from the two bounds, the geometric mean is the unique one that equalises the worst-case over- and underestimation of F_{rob} on the logarithmic scale—a minimax property on that scale that the arithmetic mean does not possess; the choice is therefore derived from the objective of controlling the F_{rob} error rather than adopted heuristically. The approximation is:

$$\widetilde{p}_{Sf} = \left[\max_{i \in D_j} p_i \cdot \left(1 - \prod_{i \in D_j} (1 - p_i)\right)\right]^{1/2} = (p_L \cdot p_U)^{1/2} \tag{9}$$

$$\widetilde{\beta}_S = \Phi^{-1}\left(1 - \widetilde{p}_{Sf}\right) \tag{10}$$

6.2. Formal Properties

Proposition 6 (Consistency with Bounds)**.** *Let pL = max_{i∈Dj} p_i and pU = 1 − ∏_{i ∈ Dj}(1 − p_i). Then pL ≤ $\widetilde{p}_{Sf}$ ≤ pU.*

Proof. $\widetilde{p}_{Sf}$ = (pL · pU)$^{1/2}$ is the geometric mean of pL and pU. Since 0 < pL ≤ pU, the geometric mean lies between them: pL = (pL · pL)$^{1/2}$ ≤ (pL · pU)$^{1/2}$ ≤ (pU · pU)$^{1/2}$ = pU. □

Proposition 7 (Monotonicity)**.** *The approximation $\widetilde{p}_{Sf}$ is monotonically non-decreasing in each component failure probability p_i for i ∈ Dj.*

Proof. Both pL = max p_i (a maximum) and pU = 1 − ∏(1 − p_i) (a complement product) are non-negative and non-decreasing in each p_i. The geometric mean (pL · pU)$^{1/2}$ of two non-negative non-decreasing functions is non-decreasing, since the product of non-negative non-decreasing functions is non-decreasing and the square root is monotone. □

Proposition 8 (Convergence)**.** *As j → m, the approximation evaluated over Dj converges to the geometric mean of the simple bounds over the full system: lim (j → m) $\widetilde{p}_{Sf}$ = (max$_i$ p_i · [1 − ∏$_i$(1 − p_i)])$^{1/2}$.*

Proof. As j → m, the dominant set Dj approaches the full element set {1, . . ., m}, and both pL and pU evaluated over Dj approach their values over the full system. The mean follows by continuity. □

Proposition 9 (Error Direction under Positive Equicorrelation)**.** *Consider a series system whose safety margins are standard normal with uniform inter-element correlation $\rho_{ij} = \rho \in [0, 1]$ for all i ≠ j and fixed component failure probabilities $p_1, \ldots, p_m$. Let $p_{Sf}(\rho)$ denote the exact system failure probability. Then: (i) $p_{Sf}(\rho)$ is continuous and non-increasing in ρ, with $p_{Sf}(0) = pU = 1 - \prod_i(1 - p_i)$ and lim_(ρ→1) $p_{Sf}(\rho) = pL = \max_i p_i$; (ii) if pL < pU, there exists a threshold correlation $\rho^* \in (0, 1)$ such that the mean-bound approximation $\widetilde{p}_{Sf} = (pL \cdot pU)^{1/2}$ satisfies $\widetilde{p}_{Sf} \geq p_{Sf}(\rho)$ for all $\rho \geq \rho^*$ and $\widetilde{p}_{Sf} \leq p_{Sf}(\rho)$ for all $\rho \leq \rho^*$. In particular, for $\rho \geq \rho^*$ the approximation overestimates*

the system failure probability and underestimates β_S—a conservative (safe-side) error with respect to reliability verification—whereas for $\rho < \rho^*$ the error is anti-conservative.

Proof. (i) By the Slepian inequality [27], if two zero-mean multivariate normal vectors share marginal variances and the correlation matrix of the first dominates that of the second componentwise, then the joint CDF of the first dominates pointwise: $\Phi_m(\beta; R(\rho_2)) \geq \Phi_m(\beta; R(\rho_1))$ for $\rho_2 \geq \rho_1$. Hence $p_{Sf}(\rho) = 1 - \Phi_m(\beta; R(\rho))$ is non-increasing in ρ, and it is continuous because the multivariate normal CDF is continuous in its correlation parameter. At $\rho = 0$ the safety margins are independent, so $p_{Sf}(0) = 1 - \prod_i(1 - p_i) = pU$ exactly. As $\rho \to 1$ the margins become comonotonic; the system then fails if and only if its most critical component fails, and $p_{Sf}(\rho) \to \max_i p_i = pL$. (ii) By Proposition 6, $pL < \widetilde{p}_{Sf} < pU$ whenever $pL < pU$. Since $p_{Sf}(\rho)$ decreases continuously from pU to pL, the intermediate value theorem yields $\rho^* \in (0, 1)$ with $p_{Sf}(\rho^*) = \widetilde{p}_{Sf}$, and monotonicity gives the directional inequalities on either side of ρ^*. The statements for $\widetilde{\beta}_S$ follow because $\beta = -\Phi^{-1}(p)$ is strictly decreasing in p. □

Proposition 9 is the key result of Part II. It replaces an empirical estimate by a complete characterisation of the error direction. Two distinct guarantees follow. First, the upper simple bound pU is universally conservative under positive correlation: by the Slepian inequality, $p_{Sf}(\rho) \leq pU$ for all $\rho \geq 0$, so an analysis based on pU always overestimates the failure probability and underestimates β_S. Second, the mean-bound approximation $\widetilde{p}_{Sf}$ is conservative precisely for correlations above the structural threshold ρ^* and anti-conservative below it; no universally guaranteed error direction is claimed for $\widetilde{p}_{Sf}$, since the exact value may lie on either side of the geometric mean of the bounds depending on ρ.

The proof of Proposition 9 relies on the Slepian inequality [27], a fundamental result in the theory of multivariate normal distributions. The Slepian inequality states that for two zero-mean multivariate normal vectors X and Y with the same marginal variances, if the correlation matrix of X is componentwise greater than or equal to that of Y, then the CDF of X is greater than or equal to that of Y at every quantile. Applied to the failure event of a series system, this means that positive correlation reduces the joint failure probability relative to the independent case.

The practical significance of Proposition 9 is twofold. When a guaranteed conservative estimate is required, the upper bound pU should be used: overestimating the failure probability and underestimating the reliability index is conservative (safe-side) with respect to reliability verification. When the mean-bound approximation is preferred for its superior average accuracy, Proposition 9 identifies the correlation regime $\rho \geq \rho^*$ in which it inherits the same conservative character. For strongly correlated systems—monolithic structures, single-batch materials, strongly coupled elements—the approximation is therefore both accurate and safe-sided; for weakly correlated systems the user should be aware that the estimate may be anti-conservative, with the magnitude of the error nevertheless bounded by Proposition 10, and with pU available as a strictly conservative fallback. Two caveats apply. First, Proposition 9 is proven for the equicorrelated case; for non-uniform correlation matrices the threshold behaviour is indicative rather than guaranteed. Second, ρ^* is defined implicitly through the exact $p_{Sf}(\rho)$ and is therefore not known a priori; in practice, a conservative estimate of ρ^* should be obtained (e.g., from the two most critical elements via the bivariate normal probability), or pU adopted whenever a guaranteed safe-side estimate is required.

An analogous result for negative correlation does not hold in general. For systems with negative inter-element correlation, the mean-bound approximation may overestimate or underestimate β_S depending on the structure of the correlation matrix and the failure

probability distribution. As discussed in Section 5.2, negative modal correlations are not excluded a priori—the FORM modal correlation $\rho_{ij} = \alpha_i{}^T \alpha_j$ may be negative for opposing or oppositely loaded mechanisms—so this case must be checked rather than assumed away; where it arises, the directional guarantees of Proposition 9 do not apply and the universal Bonferroni bound of Equation (4) should be used instead.

7. Synthesis of the Two Parts

7.1. Algorithmic Implementation

The unified computational framework combines Part I (robustness measure F_{rob}) with Part II (mean-bound approximation of β_S) into a single algorithm that requires only FORM-level computational cost. The algorithm proceeds as follows:

Step 1. For the intact system S, compute the component reliability indices $\beta_1, \ldots, \beta_m$ via FORM.

Step 2. Apply the geometric mean-bound approximation (Equations (8)–(10)) to obtain $\widetilde{\beta}_S(\emptyset)$, the system reliability of the intact configuration.

Step 3. Enumerate the relevant damage scenarios $D \in \mathcal{D}$ (typically single-element removals, $|D| = 1$) or as restricted by the assumed damage cardinality k.

Step 4. For each damage scenario D, recompute the component reliability indices for the remaining elements (those not in D), accounting for any change in load distribution. Apply the mean-bound approximation to obtain $\widetilde{\beta}_S(D)$.

Step 5. Compute the robustness factors $F_{rob}(D) = -\log_{10}(\widetilde{p}_{Sf}(D)/p_{f,norm})$ for each damage scenario, where $\widetilde{p}_{Sf}$ is the mean-bound approximation from Section 6 and $p_{f,norm} = 8.0 \times 10^{-5}$.

Step 6. Identify the worst-case robustness $F_{rob}{}^* = \min_D F_{rob}(D)$ and classify the system using the established threshold conventions.

7.2. Computational Cost

The total computational cost of the framework is dominated by Step 4, which requires $|\mathcal{D}|$ damaged-configuration analyses, each comprising up to $m - 1$ component-level FORM analyses. For typical systems with $m = 30$ components and $k = 1$ (single-element damage, $|\mathcal{D}| = 30$), this amounts to approximately $30 \times 29 \approx 870$ component-level FORM analyses per configuration. Each such analysis is inexpensive when the limit state is available in closed form (of the order of milliseconds), but may take seconds to minutes when the limit state requires a finite-element evaluation. In the latter case, the roughly 870 analyses dominate the runtime, and the additional m-dimensional system integration that direct evaluation would require is—as quantified in the next paragraph—a minor cost beside them. In this regime, the contribution of the present framework is to supply that aggregation in closed form and with an explicit error bound, rather than to reduce a cost that is already small.

The framework can be further accelerated through several techniques without compromising the formal guarantees of Propositions 6 to 11. First, FORM analyses for nearby damage scenarios share design point information, allowing warm-starting that can reduce iteration counts (the reduction is problem-dependent and is reported in the literature to be of the order of a factor of two for smooth limit states). Second, sensitivity analysis can identify damage scenarios that are unlikely to govern $F_{rob}{}^*$, allowing them to be excluded from explicit evaluation. Third, parallel computation across damage scenarios is embarrassingly parallel, scaling linearly with the number of available processors. It should be emphasised that the cost of the system-probability evaluation is, in absolute terms, very small and is not where the runtime of a full analysis is spent. On a single core of a standard workstation (Python 3.11/SciPy), one mean-bound evaluation of Equation (9) takes of the

order of 40 μs, one one-factor 1D integral (Proposition 11(iii)) of the order of 11 ms, and one full m-dimensional Genz evaluation of the order of 14 ms for m = 10 and 90 ms for m = 30. All of these are negligible beside a single finite-element limit-state evaluation, which may take seconds to minutes. The runtime of a robustness analysis is therefore dominated by the $|D| \cdot (m - 1)$ component-level FORM/FE calls, not by the system aggregation; the contribution of the present framework is to remove the additional m-dimensional system integration that direct evaluation of Equation (3) would require, while the closed-form estimate of Equation (9) avoids even the millisecond-scale 1D integral in the inner loop of a design optimisation.

For applications requiring assessment of many candidate designs in a design optimisation loop, the framework can be embedded in surrogate model frameworks following Echard et al. [28]. The robustness factor F_{rob}^* serves as a constraint in the optimisation, and the surrogate is trained on a sparse set of full evaluations. This combination of formal error bounds from the mean-bound approximation and surrogate modelling for design exploration provides a practical pathway from theory to design practice.

This contrasts with direct multi-dimensional integration of Φ_m in Equation (3): a single such evaluation remains feasible for m values of several tens with modern quasi-Monte Carlo methods, but it requires the full correlation matrix, is not available in closed form, and must be repeated in full for every configuration—which is less suited to the repeated design-iteration loops, large systems, and partial-correlation settings that the present framework targets. The framework's polynomial cost in m, combined with the formal error guarantees of Proposition 9, makes it suitable for routine engineering analysis.

7.3. Error Propagation

Because Equation (1) normalises the damaged-state failure probability by the fixed code constant $p_{f,norm}$, the error in $F_{rob}(D)$ is governed solely by the approximation error of the damaged configuration: $\widetilde{F}_{rob}(D) - F_{rob}(D) = -\log_{10}(\widetilde{p}_{Sf}(D)/p_{Sf}(D))$. No intact-state quantity enters the definition of F_{rob}, and consequently no cancellation between intact and damaged configurations can occur in F_{rob} itself. The relevant question is therefore how large the logarithmic estimation error can be; Proposition 10 answers it with an explicit and computable bound.

Beyond this error characterisation, the framework benefits from additional robustness in its computational pipeline. The mean-bound approximation requires only the component reliability indices $\beta_1, \ldots, \beta_m$ and is insensitive to errors in the FORM design points u^*_i provided that $\|u^*_i\|$ is accurately computed. A distinction must, however, be drawn between two different notions of FORM error. The first is the numerical convergence of the FORM iteration itself—the convergence of the HLRF (or equivalent) algorithm to the design point—which typically reaches a tolerance of 10^{-4} on $\|u^*\|$ within 10–15 iterations [23]; this iteration error is indeed negligible. The second, and entirely separate error, is the approximation error of FORM as a method: even when the iteration has fully converged, the reliability index $\beta_i = \|u^*_i\|$ is only a first-order (linearised) estimate of the true component reliability and carries an error that the convergence of the iteration does not bound. This FORM approximation error arises from the curvature of nonlinear limit-state functions (which second-order reliability methods address), from non-Gaussian basic variables after the Rosenblatt/Nataf transformation, from the possible existence of multiple design points, from the change in the limit-state surface produced by post-damage load redistribution, and from finite-element model uncertainty in the limit-state evaluation. The present framework takes the component indices β_i as exact inputs and bounds only the error incurred in aggregating them into the system probability p_{Sf}; it does not bound the error in the β_i indices themselves, which must be controlled by the choice of component reliability method

(FORM, SORM, importance sampling, or subset simulation). The scope of Proposition 10 is, accordingly, the series-system aggregation error: its bound $|\widetilde{F}_{rob}(D) - F_{rob}(D)| \leq$ ½·$\log_{10}(pU(D)/pL(D))$ governs the discrepancy between the mean-bound estimate and the exact series probability given the component indices and is independent of—and additional to—any error in computing those indices by FORM. The total error of an F_{rob} estimate can accordingly be decomposed additively on the logarithmic scale as in Equation (11)

$$\Delta F_{rob,total} = \Delta F_{agg} + \Delta F_{FORM} + \Delta F_{FE} + \Delta F_{redist} + \Delta F_{param} \quad (11)$$

where ΔF_{agg} is the series-aggregation error bounded by Proposition 10; ΔF_{FORM} is the first-order (linearisation) error of FORM, controllable by SORM or importance sampling; ΔF_{FE} is the finite-element discretisation and model error in the limit-state evaluation; ΔF_{redist} is the error in the assumed post-damage load redistribution that sets the damaged-state indices $\beta_i(D)$; and ΔF_{param} is the error in the statistical parameters of the basic variables. Only the first term is bounded by the present framework; the remaining terms are inherited from the component-level reliability analysis and the structural model and must be controlled by the corresponding established techniques. This decomposition delimits precisely what Proposition 10 does and does not certify.

The matrix-based system reliability method of Kang, Song, and Gardoni [29] provides an alternative computational pathway when the user wishes higher accuracy at the cost of additional bivariate normal integrations. The MSR method computes the system failure probability exactly to second order through products of structured probability matrices and can be combined with the F_{rob} definition of Equation (1) without changing the axiomatic properties of Section 3. The framework presented here is therefore compatible with multiple computational backends: simple bounds, mean-bound approximation, Ditlevsen narrow bounds, MSR, or subset simulation [30] can all be used to compute $p_{Sf}(D)$, with F_{rob} then evaluated directly from Equation (1). The user can choose the computational backend appropriate for the application without modifying the conceptual framework.

Hohenbichler and Rackwitz [31] established that for general system topologies (combinations of series and parallel subsystems), the failure probability can be expressed as a function of an extended vector of component reliability indices and correlation matrix. The framework presented here for pure series systems can be extended to such general topologies by following the recursive decomposition of [31], with the mean-bound approximation applied at each level of the decomposition. This extension is left for future work.

Proposition 10 (Computable Error Bound for F_{rob})**.** *Let D be a damage scenario and let $\widetilde{p}_{Sf}(D)$ be any estimate satisfying $pL(D) \leq \widetilde{p}_{Sf}(D) \leq pU(D)$, where $pL(D)$ and $pU(D)$ are the simple bounds (Equation (5)) evaluated for the damaged configuration over the full surviving element set and under non-negative inter-element correlation ($\rho_{ij} \geq 0$, Section 5.2); by the same bounds, the exact value satisfies $pL(D) \leq p_{Sf}(D) \leq pU(D)$. Then the error of the estimated robustness factor obeys $|\widetilde{F}_{rob}(D) - F_{rob}(D)| = |\log_{10}(\widetilde{p}_{Sf}(D)/p_{Sf}(D))| \leq \log_{10}(pU(D)/pL(D))$. Since the estimator of Equation (9) is the geometric mean $\widetilde{p}_G(D) = (pL(D)\cdot pU(D))^{1/2}$, the bound sharpens to the symmetric half-width $|\widetilde{F}_{rob}(D) - F_{rob}(D)| \leq$ ½·$\log_{10}(pU(D)/pL(D))$. If $pU(D)$ is evaluated only over the dominant subset Dj of Equation (8), it must be augmented by the truncation correction $(m - j)\cdot\varepsilon\cdot p_1$, since the exact probability may marginally exceed the truncated bound.*

Proof. From Equation (1), $\widetilde{F}_{rob}(D) - F_{rob}(D) = -\log_{10}(\widetilde{p}_{Sf}(D)/p_{Sf}(D))$. Both $\widetilde{p}_{Sf}(D)$ and $p_{Sf}(D)$ lie in $[pL(D), pU(D)]$, hence their ratio lies in $[pL(D)/pU(D), pU(D)/pL(D)]$ and the absolute value of its decadic logarithm is at most $\log_{10}(pU(D)/pL(D))$. For the geometric-mean estimator, $\widetilde{p}_G(D)/p_{Sf}(D) \in [(pL/pU)^{1/2}, (pU/pL)^{1/2}]$, because $\widetilde{p}_G(D)/pU(D) =$

$(pL/pU)^{1/2}$ and $\widetilde{p}_G(D)/pL(D) = (pU/pL)^{1/2}$; taking logarithms gives the symmetric half-width bound, which is attained at the endpoints and is therefore tight. □

Corollary 1 (Invariance of the Robustness Classification)**.** *If the error bound of Proposition 10 is smaller than the distance from $\widetilde{F}_{rob}(D)$ to the nearest classification threshold, then the estimated and the exact robustness factor are assigned to the same robustness class.*

Proof. Under the stated condition, the interval $\widetilde{F}_{rob}(D) \pm$ bound lies entirely within a single classification interval and $F_{rob}(D)$ belongs to that interval by Proposition 10. □

Remark 2. *Proposition 10 supersedes a heuristic error-cancellation argument used in an earlier formulation of the framework. Cancellation between intact-state and damaged-state errors cannot operate in Equation (1), which contains only the damaged-state probability and a fixed normalisation constant. A cancellation property does hold for differences in robustness factors: if $\widetilde{p}_{Sf}(D) = pSf(D)\cdot(1 + \varepsilon)$ with a common relative error ε across scenarios, then $\widetilde{F}_{rob}(D) - \widetilde{F}_{rob}(D') = Frob(D) - Frob(D')$ exactly. This weaker statement is retained as an observation only; it is not required by the framework.*

Proposition 11 (Two-Sided Slepian Envelope for Bounded Non-Negative Correlation)**.** *Let a series system have standard-normal safety margins with any valid (positive semidefinite) correlation matrix R whose off-diagonal entries satisfy $\rho_{min} \le R_{ij} \le \rho_{max}$ for all $i \neq j$, with $0 \le \rho_{min} \le \rho_{max} \le 1$, and fixed component failure probabilities $p_1, \ldots, p_m$ (thresholds β_i). Let $p_{Sf}(R) = 1 - \Phi_m(\beta; R)$ be the exact system failure probability and let $p_{Sf}^{eq}(\rho)$ denote the equicorrelated failure probability with all off-diagonals equal to ρ. Then: (i) $p_{Sf}^{eq}(\rho_{max}) \le p_{Sf}(R) \le p_{Sf}^{eq}(\rho_{min})$; (ii) consequently $F_{rob}^{eq}(\rho_{min}) \le F_{rob}(R) \le F_{rob}^{eq}(\rho_{max})$, where $F_{rob}^{eq}(\rho) = -log_{10}(p_{Sf}^{eq}(\rho)/p_{f,norm})$; (iii) each bounding quantity is evaluated by a single one-dimensional integral via the one-factor representation $p_{Sf}^{eq}(\rho) = 1 - \int \varphi(z) \prod_i \Phi((\beta_i - \sqrt{\rho}\cdot z)/\sqrt{(1-\rho)})\, dz$, at $O(m)$ cost per evaluation; (iv) if $F_{rob}^{eq}(\rho_{min})$ and $F_{rob}^{eq}(\rho_{max})$ fall in the same robustness class, the class of the system with any valid R having off-diagonal entries in $[\rho_{min}, \rho_{max}]$ is determined exactly, without evaluating $\Phi_m(\beta; R)$.*

Proof. Let $R(\rho)$ be the equicorrelation matrix with off-diagonals ρ. Since $\rho_{min} \le R_{ij} \le \rho_{max}$ for every $i \neq j$, the off-diagonals satisfy $R(\rho_{min})_{ij} \le R_{ij} \le R(\rho_{max})_{ij}$ componentwise, while all diagonals equal one. Both $R(\rho_{min})$ and $R(\rho_{max})$ are themselves valid correlation matrices, since an equicorrelation matrix with common off-diagonal ρ is positive semidefinite for every $\rho \in [-1/(m-1), 1]$ and hence for $\rho \in [0, 1]$; together with the hypothesis that R is a valid (positive semidefinite) correlation matrix, this makes the comparison admissible. By the Slepian inequality [27], the multivariate normal CDF is non-decreasing under componentwise increase in the correlation entries, hence $\Phi_m(\beta; R(\rho_{min})) \le \Phi_m(\beta; R) \le \Phi_m(\beta; R(\rho_{max}))$. Taking complements reverses the inequalities and yields (i). Statement (ii) follows because $-\log_{10}(\cdot/p_{f,norm})$ is strictly decreasing. Statement (iii) is the standard single-factor decomposition $U_i = \sqrt{\rho}\cdot Z + \sqrt{(1-\rho)}\cdot\xi_i$ with Z, ξ_i independent standard normal, under which the survival events become conditionally independent given Z. Statement (iv) is immediate from (ii) and the definition of the classification intervals. □

Remark 3. *Proposition 11 extends the equicorrelation result of Proposition 9 to non-equicorrelated correlation structures with non-negative pairwise entries: knowledge of only the smallest and largest pairwise correlations suffices to bracket both the system failure probability and the robustness factor, at the cost of two one-factor integrals. Together with Corollary 1 this gives a computable guarantee on the robustness classification that does not require the exact correlation matrix—information that is rarely available in practice. The result is not claimed to hold for arbitrary correlation structures:*

the envelope requires the off-diagonal entries to lie in a non-negative interval [ρmin, ρmax] ⊆ [0, 1], consistent with the structural assumption ρij ≥ 0 adopted throughout (Section 5.2). For correlation matrices with negative entries the Slepian comparison against an equicorrelation envelope is no longer available and the bracketing of Proposition 11 does not apply.

8. Illustrative Numerical Example

8.1. System Definition

To illustrate the framework, consider a series system of m = 10 components with element reliability indices

$$\boldsymbol{\beta} = (3.29,\ 3.50,\ 3.92,\ 4.00,\ 5.00,\ 5.30,\ 5.50,\ 6.00,\ 6.50,\ 7.00) \tag{12}$$

This distribution is representative of structural systems in which a few critical elements govern the system response, where two to three components govern the system reliability and the remaining components have substantial reserve. Assume positive uniform inter-element correlation ρ = 0.6, consistent with values reported for materials sharing common sources of variability. Note that the intact configuration itself has $\widetilde{\beta}_S(\emptyset) = 3.22 < \beta_{S,norm} = 3.775$, i.e., $F_{rob}(\emptyset) = -\log_{10}(6.3 \times 10^{-4}/8.0 \times 10^{-5}) \approx -0.90$; the example is deliberately chosen below the intact target so that the damage scenarios exercise the interior of the classification scale rather than its idealised upper end.

8.2. Application of the Framework

Step 1 (component reliability). The β indices are taken from Equation (12).

Step 2 (intact system reliability). Failure probabilities: $p_1 = 5.0 \times 10^{-4}$, $p_2 = 2.3 \times 10^{-4}$, $p_3 = 4.4 \times 10^{-5}$, etc. The threshold $\varepsilon = 10^{-2}$ selects Dj = {1, 2, 3, 4}, i.e., j = 4 dominant elements. Applying Equation (9), $pL = 5.0 \times 10^{-4}$, $pU = 8.0 \times 10^{-4}$, $\widetilde{p}_{Sf} = (pL \cdot pU)^{1/2} = 6.3 \times 10^{-4}$, $\widetilde{\beta}_S(\emptyset) = 3.22$.

Step 3 (damage scenarios). Consider four damage scenarios D_1, D_2, D_3, D_4, corresponding to single-element removal of components 1, 2, 3, and 4, respectively. For each, recompute the remaining component reliabilities (assumed redistributed by static analysis).

Step 4 (damaged-state reliability). For damage scenario D_1 (removal of the most critical component), the redistribution causes several surviving components to become more critical. The hypothetical new reliability indices of the nine surviving components after redistribution are $\beta' = (\beta'_2, \ldots, \beta'_{10}) = (3.50, 2.8, 3.7, 4.0, 5.3, 5.5, 6.0, 6.5, 7.0)$, where components 3, 4, and 5 have decreased due to load increase (β_3: 3.92 → 2.8; β_4: 4.00 → 3.7; β_5: 5.00 → 4.0), while component 2 and components 6–10 are unaffected. Applying the framework to the damaged configuration: the threshold $\varepsilon = 10^{-2}$ selects the four most critical remaining components, with $pL(D_1) = \Phi(-2.8) = 2.56 \times 10^{-3}$ and $pU(D_1) = 2.93 \times 10^{-3}$; hence $\widetilde{p}_{Sf}(D_1) = (pL \cdot pU)^{1/2} = 2.74 \times 10^{-3}$ and $\widetilde{\beta}_S(D_1) = 2.78$.

Step 5 (robustness factors). For damage scenario D_1 with $\widetilde{p}_{Sf}(D_1) = 2.74 \times 10^{-3}$, the robustness factor follows directly from Equation (1): $F_{rob}(D_1) = -\log_{10}(\widetilde{p}_{Sf}(D_1)/p_{f,norm}) = -\log_{10}(2.74 \times 10^{-3}/8.0 \times 10^{-5}) = -\log_{10}(34.3) \approx -1.53$. By Proposition 10, the associated symmetric error bound for the geometric-mean estimator is $\frac{1}{2} \cdot \log_{10}(pU(D_1)/pL(D_1)) = \frac{1}{2} \cdot \log_{10}(1.145) \approx 0.03$ (the full two-sided width being 0.06). The resulting robustness factors for all four damage scenarios are presented in Table 1.

Table 1. Worked example: Robustness factors for four single-element damage scenarios applied to the system defined in Equation (12). The worst-case $F_{rob}^* = -1.53$ places the system in the "medium-robustness" class per the classification thresholds of [8]: $F_{rob} \in (-2.097, -1.097]$ = medium robustness.

Damage Scenario D	Removed Component	$\tilde{\beta}_S$ (D)	F_{rob}(D)
D_1	Component 1 (most critical)	2.78	−1.53
D_2	Component 2	3.10	−1.08
D_3	Component 3	3.18	−0.96
D_4	Component 4	3.20	−0.93
Worst case	—	—	−1.53

8.3. Discussion

The example illustrates the framework in operation. The intact-state $\tilde{\beta}_S(\emptyset) = 3.22$ is computed using only the four most critical elements; the inclusion of elements 5 through 10 (with $\beta \geq 5.0$) would change the result by less than 0.01, justifying the dominant-element reduction. The worst-case robustness $F_{rob}^* = -1.53$ corresponds to the removal of component 1, which causes the largest reliability drop. By the classification scheme of [8] ($F_{rob} \in (-2.097, -1.097]$ = medium robustness), this system is in the medium-robustness class. The classification thresholds, based on logarithmic ratios of damaged-to-target failure probabilities, span from a lower limit of −4.097 (attained at collapse) upward, with values $F_{rob} > 0$ denoting ideal robustness.

It is instructive to perform a sensitivity analysis on the worked example to illustrate the dependence of the approximation error on inter-element correlation, in line with Proposition 9. (Exact values are computed by one-dimensional conditional integration of the equicorrelated multivariate normal CDF.) For $\rho = 0.0$ (full independence), the exact $\beta_S(\emptyset) = 3.15$ while the approximation gives $\tilde{\beta}_S(\emptyset) = 3.22$: the approximation overestimates β_S by 0.07, an anti-conservative error, consistent with the regime $\rho < \rho^*$. For $\rho = 0.6$ the exact value is $\beta_S(\emptyset) = 3.17$ and the error is +0.05, and for $\rho = 0.8$ it is +0.02—still anti-conservative, but shrinking. The error changes sign at the threshold correlation, here $\rho^* \approx 0.87$ by numerical solution of $p_{Sf}(\rho^*) = \tilde{p}_{Sf}$, beyond which the approximation becomes conservative; in the comonotonic limit $\rho \to 1$ the exact value approaches $\beta_S = 3.29$ (corresponding to pL $= 5.0 \times 10^{-4}$) and the error reaches −0.07, on the conservative side. The worked example thus illustrates both regimes identified by Proposition 9.

This behaviour has practical consequences. For systems with high inter-element correlation (typical of monolithic structures, single-batch materials, or strongly coupled elements), the mean-bound approximation is accurate and conservative. For systems with low correlation, the approximation carries a larger, anti-conservative—but bounded and quantifiable (Proposition 10)—error, and the upper bound pU remains available as a strictly conservative alternative. In both regimes the user knows the error direction in advance.

The error analysis can be extended to multi-component damage scenarios. For damage scenarios involving $k > 1$ simultaneous component removals, the post-damage system has $m - k$ components. The same propositions apply unchanged to the damaged configuration, and the error propagation analysis of Proposition 10 continues to hold. This consistency across damage cardinalities is what makes the framework applicable to general robustness assessment.

The error bound of Proposition 10 is operative here. For the damaged configuration D_1, the simple bounds of the dominant set give $\log_{10}(pU(D_1)/pL(D_1)) = \log_{10}(2.93 \times 10^{-3}/2.56 \times 10^{-3}) \approx 0.06$. With the geometric-mean estimator adopted in Equation (9), the robustness factor $F_{rob}(D_1)$ is therefore determined to within the sym-

metric half-width ±0.03 (the full two-sided width being 0.06), while the distance from $\tilde{F}_{rob}(D_1) = -1.53$ to the nearest classification threshold (−1.097) is 0.43—more than fourteen times the half-width bound. By Corollary 1, the medium-robustness classification is guaranteed irrespective of the approximation error.

The worked example also reveals an important feature of the framework: the dominant-element set Dj is dynamic. In the intact configuration, the four most critical components dominate. After damage to component 1, the redistribution can elevate previously secondary components into the dominant set. The threshold criterion (Equation (8)) automatically tracks this change, ensuring that the approximation remains accurate as configurations change. This property is particularly valuable in design exploration, where many candidate configurations must be evaluated quickly without manual tuning of the approximation.

The selection of the threshold ε deserves brief discussion. The recommended value $\varepsilon = 10^{-2}$ is based on the consideration that elements with failure probability below 1% of the most critical element contribute negligibly to system failure under any reasonable correlation structure. For systems where multiple components have comparable failure probabilities, the dominant set may include most or all components. For systems where one or two components clearly dominate, the dominant set may include only those one or two. The framework adapts to both extremes through the same criterion.

It is instructive to compare the mean-bound approximation with the Ditlevsen narrow bounds (Equations (6) and (7)) applied to the same system. For ρ = 0.6 and the β vector of Equation (12), Ditlevsen's narrow bounds give $\beta_{S,L} = 3.166$ and $\beta_{S,U} = 3.169$, with midpoint 3.17, enclosing the exact value $\beta_S(\emptyset) = 3.17$ reported above; the narrow bounds are here practically exact. The mean-bound approximation gives $\tilde{\beta}_S(\emptyset) = 3.22$. The two approximations differ by 0.05 in β, below the engineering accuracy of 0.1. Ditlevsen's narrow bounds are more accurate but require bivariate normal integrations; the mean-bound approximation trades accuracy for computational simplicity while retaining the formal error characterisation of Proposition 9 and the computable error bound of Proposition 10.

To address the request for a more convincing demonstration of the computational benefit, Table 2 benchmarks the mean-bound approximation against the principal alternatives for the intact system of Equation (12) at ρ = 0.6: the Ditlevsen narrow bounds, the one-dimensional one-factor integration applicable to equicorrelated systems (Proposition 11(iii)), and a full m-dimensional multivariate-normal evaluation by the Genz quasi-Monte Carlo algorithm; the matrix-based system reliability (MSR) method of Kang, Song and Gardoni [29] provides a further exact pathway at the cost of structured bivariate integrations. The comparison must be read carefully. For an equicorrelated system the one-factor 1D integral is itself essentially exact at negligible cost and the Genz and Ditlevsen evaluations agree with it to three significant figures ($\beta_S = 3.17$); the mean-bound estimate ($\beta_S = 3.22$) is therefore not more accurate than these methods and its error of 0.05 in β—within the 0.1 engineering tolerance—is the price of its simplicity. The computational benefit of the mean-bound approximation is consequently not raw speed against a single multivariate evaluation, but three properties that the integration-based methods do not provide. First, the estimate is a closed-form, integration-free expression evaluated from the component reliability indices alone, hence differentiable and suited to design-iteration and optimisation loops. Second, it carries the explicit, computable error bound of Proposition 10 and the classification-invariance guarantee of Corollary 1, so the result can be reported as a certified robustness class rather than a point estimate of unknown accuracy. Third, together with the two-sided envelope of Proposition 11, it requires no knowledge of the full correlation matrix—only of the smallest and largest pairwise correlations—whereas the Genz, MSR and Ditlevsen methods all require the complete matrix R. For non-equicorrelated systems,

where the one-factor 1D integral no longer applies, this last property is decisive: the envelope brackets the answer from β and $[\rho_{min}, \rho_{max}]$ alone, at the cost of two one-factor integrals, without the m-dimensional integration that the exact methods demand.

Table 2. Computational benchmark for the intact system of Equation (12) at $\rho = 0.6$. All exact methods agree at $\beta_S = 3.17$ ($p_{Sf} = 7.7 \times 10^{-4}$); the mean-bound geometric estimate gives $\beta_S = 3.22$ at closed-form cost. The benefit of the mean-bound method is its closed form, its computable error bound (Proposition 10), and its independence from the full correlation matrix (Proposition 11), not a reduction in the cost of a single equicorrelated evaluation.

Method	β_S	p_{Sf}	Computational Cost
Mean-bound (geometric, dominant set)	3.22	6.3×10^{-4}	O(j), j = 4; closed-form, no integration
Ditlevsen narrow bounds (midpoint)	3.17	7.7×10^{-4}	$m(m-1)/2 = 45$ bivariate integrals
One-factor 1D integration (exact, equicorr.)	3.17	7.7×10^{-4}	one univariate integral
Full MVN—Genz quasi-MC/MSR [29]	3.17	7.7×10^{-4}	m-dimensional integration

8.4. Sensitivity Analysis and Numerical Validation

To address the dependence of the robustness factor on the assumed correlation, a parametric study was performed for the worst-case damage scenario D_1 over the full surviving element set. Exact system failure probabilities were obtained by one-dimensional integration of the one-factor representation (Proposition 11(iii)); the resulting robustness factors are reported in Table 3 and plotted in Figure 1. The robustness factor increases monotonically with correlation—from $F_{rob}(D_1) = -1.563$ at $\rho = 0$ to -1.504 in the comonotonic limit—consistent with the monotonicity established in Proposition 9, and remains within the medium-robustness band across the entire range, so the classification is insensitive to the correlation assumption.

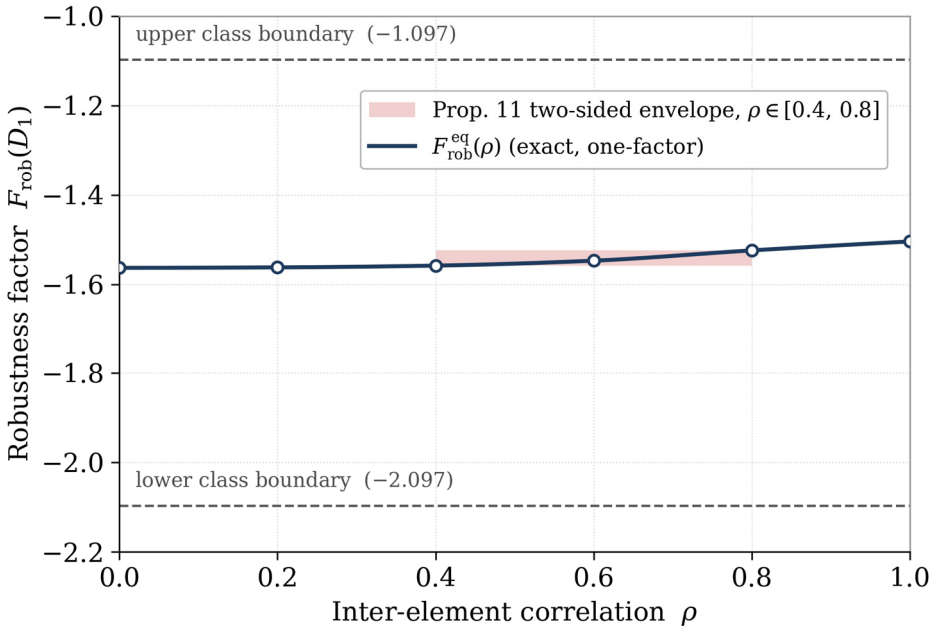


Figure 1. Robustness factor $F_{rob}(D_1)$ versus inter-element correlation ρ (solid curve, exact one-factor integration). The shaded band is the two-sided Slepian envelope of Proposition 11 for a system with off-diagonal correlations in [0.4, 0.8]. Dashed lines mark the medium-robustness class boundaries.

Table 3. Exact robustness factor for damage scenario D_1 as a function of inter-element correlation ρ, computed by one-factor integration (Proposition 11(iii)). The value remains within the medium-robustness class (−2.097, −1.097] throughout. These are exact values over the full nine-element surviving set; the headline $F_{rob}{}^{*} = -1.53$ of Table 3 is the mean-bound estimate over the dominant subset, the two differing by less than the Proposition 10 bound.

ρ	$p_{Sf}(D_1)$	$\beta_S(D_1)$	$F_{rob}(D_1)$
0.0	2.93×10^{-3}	2.756	−1.563
0.2	2.92×10^{-3}	2.757	−1.562
0.4	2.89×10^{-3}	2.760	−1.558
0.6	2.82×10^{-3}	2.768	−1.547
0.8	2.68×10^{-3}	2.785	−1.524
1.0	2.56×10^{-3}	2.800	−1.504

The two-sided envelope of Proposition 11 was verified numerically on a non-equicorrelated configuration. A correlation matrix R with off-diagonal entries drawn uniformly from [0.4, 0.8] (smallest eigenvalue 0.12, hence admissible) was generated for the D_1 element set. The exact failure probability, estimated by 4×10^6 Monte Carlo samples, was $p_{Sf}(R) = 2.76 \times 10^{-3}$ ($\pm 2.6 \times 10^{-5}$), bracketed as predicted by $p_{Sf}{}^{eq}(0.8) = 2.68 \times 10^{-3} \leq p_{Sf}(R) \leq p_{Sf}{}^{eq}(0.4) = 2.89 \times 10^{-3}$. The corresponding robustness factor $F_{rob}(R) = -1.538$ lies within the envelope [−1.558, −1.524], and since both envelope endpoints fall in the medium-robustness class, the classification is determined exactly without computing the nine-dimensional normal integral, illustrating Proposition 11(iv).

As an independent check on the one-factor integration used throughout the parametric study, the equicorrelated case ρ = 0.6 was also evaluated by direct Monte Carlo simulation (4×10^6 samples). The analytic value $p_{Sf} = 2.820 \times 10^{-3}$ ($\beta_S = 2.768$) agreed with the simulated $p_{Sf} = 2.843 \times 10^{-3}$ ($\pm 2.7 \times 10^{-5}$; $\beta_S = 2.765$) to within the Monte Carlo standard error, confirming the numerical accuracy of the analytic evaluations. The exact $F_{rob}(D_1) = -1.547$ obtained here is consistent with the mean-bound estimate $\tilde{F}_{rob}(D_1) = -1.53$ of Table 1, the difference (0.017) lying well within the Proposition 10 half-width bound of 0.03.

To address the request for validation over a broad range of reliability levels, system dimensions, and correlation structures, the geometric mean-bound estimate was compared against exact equicorrelated values (one-factor integration) for system sizes m ∈ {5, 10, 20, 30, 50} and correlations ρ ∈ {0.0, 0.3, 0.6, 0.9}, using a representative β-profile with four critical elements (β = 3.29, 3.50, 3.92, 4.00) and the remainder distributed in [4.5, 7.0]. Table 4 reports, for each case, the exact system reliability β_S, the mean-bound estimate $\tilde{\beta}_S$, the error in β, the absolute error in F_{rob}, and the Proposition 10 half-width bound $\frac{1}{2} \cdot \log_{10}(pU/pL)$. The estimated F_{rob} error remains within the Proposition 10 bound in every one of the twenty cases—tightly so at ρ = 0, where the exact probability coincides with pU and the bound is attained, and with substantial margin for ρ ≥ 0.3. The error in β is at most +0.072 (anti-conservative, at low correlation) and reverses to −0.015 (conservative) at ρ = 0.9, exactly as Proposition 9 predicts, and is essentially independent of m because the dominant-set reduction selects the same four critical elements throughout. An independent Monte Carlo check at m = 20, ρ = 0.6 (4×10^6 samples) gave $p_{Sf} = 7.70 \times 10^{-4}$ ($\beta_S = 3.167$), matching the one-factor value 7.71×10^{-4} ($\beta_S = 3.167$) to within the sampling error.

Table 4. Systematic validation of the geometric mean-bound approximation against exact equicorrelated values over system sizes m and correlations ρ. The estimated F_{rob} error $|\Delta F_{rob}|$ stays within the Proposition 10 half-width bound in all cases (attained with equality at ρ = 0). The error in β is anti-conservative at low ρ, reverses sign by ρ = 0.9 (Proposition 9), and is insensitive to m owing to the dominant-set reduction.

m	ρ	β_S (Exact)	$\tilde{\beta}_S$ (MB)	Δβ	$\lvert \Delta F_{rob} \rvert$	½·$\log_{10}$(pU/pL)
5	0.0	3.151	3.221	+0.070	0.105	0.105
5	0.3	3.153	3.221	+0.068	0.103	0.105
5	0.6	3.167	3.221	+0.054	0.081	0.105
5	0.9	3.234	3.221	−0.013	0.020	0.105
10	0.6	3.167	3.221	+0.054	0.081	0.105
10	0.9	3.234	3.221	−0.013	0.020	0.105
20	0.0	3.150	3.221	+0.070	0.106	0.106
20	0.6	3.167	3.221	+0.054	0.081	0.106
30	0.0	3.149	3.220	+0.071	0.107	0.107
30	0.6	3.166	3.220	+0.054	0.081	0.107
30	0.9	3.234	3.220	−0.014	0.021	0.107
50	0.0	3.147	3.219	+0.072	0.108	0.108
50	0.6	3.165	3.219	+0.054	0.081	0.108
50	0.9	3.234	3.219	−0.015	0.022	0.108

For reproducibility, the numerical procedures are as follows. Component failure probabilities are obtained from the reliability indices as $p_i = \Phi(-\beta_i)$; the dominant set is selected with the threshold $\varepsilon = 10^{-2}$, i.e., retaining elements with $p_i \geq \varepsilon \cdot p_1$, which in the present systems selects the four critical elements and changes $\tilde{\beta}_S$ by less than 0.01 relative to the full set (a smaller ε may be used when several elements have comparable failure probabilities, at negligible cost). Equicorrelated reference probabilities are computed by the one-factor representation of Proposition 11(iii) using adaptive Gauss–Kronrod quadrature (SciPy quad) over $z \in [-8, 8]$; general correlation matrices are evaluated by the Genz algorithm (SciPy multivariate_normal); Monte Carlo checks use a Cholesky factor of R applied to standard-normal samples. No proprietary software is required, and the complete computational pipeline is available from the corresponding author.

8.5. Practical Application: Timber Trusses and General Series Structural Systems

The framework was developed for, and is directly applicable to, the robustness assessment of timber truss roof structures—the structural class that motivated this work [7,8,24]. Such trusses are routinely composed of m = 20–50 members, a range over which repeated exact multi-dimensional integration of Equation (3)—required afresh for each damage scenario and design iteration—becomes the costly step, and they exhibit positive inter-element correlation arising from shared timber batches and common moisture and load history, placing them squarely in the regime addressed by Propositions 9 and 11. In a typical design check the engineer obtains the component reliability indices β_i from FORM analyses of the dominant failure modes (bending–compression in the chords, tension and compression in the diagonals), removes each critical member in turn to form the damage scenarios D, and evaluates $F_{rob}(D)$ from the FORM outputs alone using the mean-bound approximation.

Two features make the framework practical in this setting. First, because the exact pairwise correlations of timber members are seldom known, Proposition 11 allows the engineer to bracket F_{rob} using only conservative estimates of the smallest and largest correlations, and Corollary 1 then certifies the robustness class whenever the resulting interval does not cross a class boundary—as is the case for the worked example throughout $0 \leq \rho \leq 1$ (Figure 1). Second, the explicit error bound of Proposition 10 lets the result be

reported as a classified robustness category suitable for code-type acceptance under EN 1991-1-7, rather than as a point estimate of unknown accuracy. The procedure thus converts a sequence of single-member FORM analyses into a defensible robustness classification at negligible additional computational cost, which is the principal requirement for routine application to timber truss systems built on spans of 15–20 m.

It should be emphasised, finally, that the framework is in no way specific to timber: timber trusses are merely the motivating application. All results of this paper—the axiomatic properties of F_{rob} (Propositions 1–5), the series-system reliability propositions (Propositions 6–10), and the general-correlation bound (Proposition 11)—are established in general terms, subject only to the series-system logic and the non-negative-correlation assumption stated above, and invoke no assumption regarding material, member type, or structural typology. The method therefore applies unchanged to steel, reinforced and prestressed concrete, composite, masonry, and mixed structures: any structural system whose failure can be represented as a series system of dominant failure modes characterised by component reliability indices can be assessed by the same procedure, at the same FORM-level computational cost. The choice of a timber truss for the worked example reflects only the origin of the study and not a restriction of its scope.

8.6. Validation Against a Published Case Study

The framework is validated against the published reliability-based robustness analysis of a real glued–laminated timber truss: the main hall of the Samobor sports centre [24], a 36.5 m GL32c plane-frame truss spaced at 5 m. In [24] four dominant FORM failure modes were identified—combined bending and axial force (M + N) in the bottom and top chords and tension and compression in the diagonals—and the structure was modelled as a series system of two parallel subsystems, {bottom chord, top chord} and {tension diagonal, compression diagonal}, the series probability being obtained by the mean-bound combination of the dominant modes (Equation (9)); because in every state one parallel subsystem dominates the series probability by several orders of magnitude; the geometric and arithmetic mean bounds coincide to the reported precision, so the published values are reproduced unchanged. Applying the present F_{rob} measure to the published system failure probabilities, converted from the one-year reference period of [24] to the 50-year reference of the normalisation $p_{f,norm}$ by $p_{f,50} = 1 - (1 - p_{f,1})^{50}$, gives the values in Table 5. This is the EN 1990 convention for reference-period transformation under independent annual maxima of the governing (snow) load; since the time-invariant resistance is correlated across years, the convention overestimates the multi-year failure probability and is therefore conservative—it can only lower the reported F_{rob}, never overstate robustness.

Three observations follow. First, the intact structure is classified as ideal ($F_{rob}(\emptyset) = +1.51$), whereas the most severe damage scenario—failure of the top chord—yields $F_{rob} = -3.43$, falling in the no-robustness class. This is in full agreement with the original analysis [24], which identified top-chord failure as the scenario with the lowest robustness index and an insufficient residual reliability. Second, the worst-case classification of the real structure (no-to-medium robustness for the chord-failure scenarios) brackets that of the synthetic example of Section 8.2 (medium robustness), confirming that the reliability indices used in the synthetic example are representative of an actual design rather than arbitrary.

Third, because in every state one parallel subsystem dominates the series probability by several orders of magnitude, the mean-bound estimate of Equation (9) reproduces the published system reliability indices of [24] essentially exactly—for instance $\beta_S = 5.33$ for the intact system and $\beta_S = 2.59$ for top-chord failure—which illustrates the sharpness of the bound established in Proposition 6 when a single term governs. The case study therefore

demonstrates, on real published data, both the practical applicability of the framework and the accuracy of its central approximation.

Table 5. Robustness factor for the Samobor sports-hall timber truss [24], obtained by applying the F_{rob} measure to the published system failure probabilities. One-year FORM results are converted to the 50-year reference period of $p_{f,norm} = 8.0 \times 10^{-5}$ ($\beta_{S,norm} = 3.775$) assuming independent annual load maxima.

Damage Scenario	p_{Sf} (1-yr) [24]	p_{Sf} (50-yr)	β_S (50-yr)	F_{rob}	Class
Intact	5.00×10^{-8}	2.50×10^{-6}	4.57	+1.51	ideal
Failure of element 1 (bottom chord)	2.18×10^{-5}	1.09×10^{-3}	3.07	−1.13	medium
Failure of element 2 (tension diagonal)	1.99×10^{-7}	9.95×10^{-6}	4.27	+0.91	ideal
Failure of element 3 (compression diagonal)	7.38×10^{-7}	3.69×10^{-5}	3.96	+0.34	ideal
Failure of element 4 (top chord)	4.79×10^{-3}	2.13×10^{-1}	0.80	−3.43	none

It is instructive that the published model retains only four failure modes although the truss comprises many more members; this is itself an instance of the dominant-set reduction of Propositions 6–8 applied to a real structure. A series system's failure probability is monotone in its component set (Proposition 7), so the omitted members can only increase p_{Sf}, and the consistency of the reported intact index $\beta_S = 5.33$ ($p_{Sf} = 5.0 \times 10^{-8}$) with the four-mode model implies that their aggregate contribution to the intact failure probability lies below this level—otherwise the reported index would be markedly lower. The robustness assessment is, moreover, governed by the worst-case scenario, top-chord failure, whose system failure probability (4.79×10^{-3}) exceeds any such omitted contribution by more than four orders of magnitude; the governing $F_{rob} = -3.43$ and its no-robustness classification are therefore insensitive to the inclusion of the remaining members. The published four-mode analysis thus already embodies the dominant-set reduction and F_{rob} inherits it directly—which is precisely the property that makes the reduction reliable for the larger systems targeted by the framework.

8.7. Limitations and Scope

Four modelling assumptions delimit the scope of the framework and deserve to be stated together. (i) Full component removal versus partial strength reduction. The propositions of Section 3 and the worked example treat damage as the full removal of a component (a failed or notionally removed member, as in the EN 1991-1-7 notional-removal procedure). Many real damage processes are better represented as a partial loss of strength, in which a member survives with a reduced reliability index $\beta_i - \delta\beta_i$ rather than disappearing from the system. As noted in Section 2.2, the framework extends to this case by redefining the damage scenario as a vector of strength reductions rather than a subset of removed indices; the propositions then hold with this redefinition. Partial reduction is in general the less severe and the more realistic idealisation, so the full-removal results of this paper are conservative with respect to it, but the magnitude of $\delta\beta_i$ must be supplied by a damage model external to the framework. (ii) The damage-consistency assumption (A0). Proposition 2 (monotonicity in damage severity) and Proposition 5 (boundedness) rest on assumption (A0): that adding components to a damage scenario cannot reduce the system failure probability. As the proof of Proposition 2 makes explicit, (A0) is not automatic in a surviving-series representation, because removing a lightly loaded member also removes its failure mode and can lower p_{Sf}; (A0) excludes such configurations. In practice (A0) holds whenever load redistribution does not relieve any surviving component, which is the normal situation for the governing members of a statically efficient structure but should be checked for members carrying little force. Where (A0) fails, Propositions 2 and 5 are to be read as applying to the monotone envelope obtained by retaining the worst sub-scenario.

(iii) Independence in Proposition 3. The combination bounds of Proposition 3 assume that the failure events of disjoint damage scenarios are independent (assumption A3) and that the combined failure event is exactly their union (A2). As Remark 1 emphasises, both are idealisations: in real structures the scenarios share load and material variables and are positively dependent, and combined damage aggravates redistribution beyond the union, so that $F(D_1 \cup D_2) \supseteq F(D_1) \cup F(D_2)$ and the resulting F_{rob} is an optimistic estimate for coupled scenarios. For this reason the safety-critical assessment in this work relies on the worst single-element-removal scenario ($|D| = 1$) rather than on scenario combination, and Proposition 3 is offered as a set of reference bounds for the idealised model, not as a guaranteed result for physically coupled multi-element damage. (iv) Scalability. The per-configuration cost is O(j) arithmetic operations for the mean-bound estimate (Section 7.2) and O(m) for each one-factor envelope integral (Proposition 11(iii)), with j and m the dominant-set and full-system sizes; the enumeration of single-element damage scenarios adds a factor $|D| \leq m$, giving an overall $O(m^2)$ component-level analyses for $k = 1$ damage. This polynomial growth, in place of the substantially higher cost of repeating direct m-dimensional integration—which moreover requires the full correlation matrix—across all scenarios and configurations, is what makes the framework applicable to the $m = 20$–50 member systems that motivated it, and, with the warm-starting and parallelisation of Section 7.2, to networks with m of order 100. The limiting cost for very large systems is the FORM analyses themselves rather than the system aggregation, a regime in which surrogate-model acceleration (Section 7.2) becomes the appropriate tool; a formal complexity analysis for $m \gg 100$ is identified as future work (Section 9.2, item 9).

Three short examples make these limitations concrete. (i) Partial strength reduction: a glulam chord that loses one of two bolt rows retains roughly half its connection capacity rather than failing outright; this is modelled not as removal but as a reduction in its reliability index, for example from $\beta_i = 4.0$ to $\beta_i - \delta\beta_i = 3.0$ ($\delta\beta_i = 1.0$), and the framework is applied to the reduced-index configuration exactly as for removal, giving a less severe—hence non-conservative relative to full removal—F_{rob}. (ii) Failure of (A0): removing a lightly loaded secondary tie that carries almost no force deletes its (negligible) failure mode without increasing the demand on any surviving member, so p_{Sf} decreases and $F_{rob}(D) > F_{rob}(\emptyset)$, violating monotonicity; by contrast, removing a governing bottom chord forces redistribution onto the remaining members, raises p_{Sf}, and satisfies (A0)—the monotone behaviour that the propositions assume. The remedy in practice is to report F_{rob} over the actual scenario set and to read Propositions 2 and 5 as applying to its worst-case sub-scenario. (iii) Correlated damage scenarios: if two damage scenarios are triggered by the same event (e.g., a single impact damaging two adjacent diagonals), assumption (A3) of Proposition 3 fails; a defensible treatment is to bound the combined robustness below by min over the constituent scenarios (an upper bound on robustness that ignores aggravating coupling), or, when a positive-dependence model is available, to condition on the common triggering variable and combine the conditional probabilities, which is exact and never optimistic. The single-element-removal assessment used in this paper sidesteps the issue by construction.

9. Conclusions and Future Research

9.1. Conclusions

This paper has developed a unified probabilistic framework for the simultaneous assessment of the reliability and robustness of series structural systems. The principal conclusions are:

(1) The robustness factor F_{rob}, defined as the negative base-10 logarithm of the ratio between damaged-state failure probability and a normalised target failure probability,

satisfies five axiomatic properties: normalisation (Proposition 1), monotonicity in damage severity (Proposition 2), exact combination bounds for independent damage scenarios (Proposition 3), invariance under reliability-preserving transformations (Proposition 4), and asymptotic boundedness (Proposition 5).

(2) The geometric mean-bound approximation for series system reliability satisfies four formal properties: consistency with bounds (Proposition 6), monotonicity in component failure probabilities (Proposition 7), convergence as $j \to m$ (Proposition 8), and a complete characterisation of the error direction under positive equicorrelation, with a provably conservative (safe-side) regime above a threshold correlation ρ^* and with the upper simple bound available as a universally conservative estimate (Proposition 9).

(3) The unified computational framework combines these results into a single algorithm requiring only FORM-level computational cost. The error in F_{rob} admits an explicit, computable bound (Proposition 10), with a corollary guaranteeing that the robustness classification is unaffected whenever the bound is smaller than the margin to the nearest classification threshold. A further proposition (Proposition 11) bounds the robustness factor two-sidedly for any valid correlation matrix with non-negative off-diagonal entries using only its smallest and largest pairwise correlations, so that the classification can be certified without the exact correlation structure.

(4) The framework is applicable to any structural system with series logic, including timber trusses [7,8,24], steel lattice girders [25], and reinforced concrete frames [19]. Its application to a published timber-truss reliability analysis [24] is demonstrated in Section 8.6.

9.2. Future Research Directions

Several extensions of the present work merit investigation.

The future research directions listed below are organised in increasing order of complexity and ambition. The first four extensions (parallel systems, time-variant reliability, software implementation, non-Gaussian variables) are straightforward developments of the existing framework. The remaining extensions (Bayesian updating, multi-hazard robustness, discrete optimisation, complexity bounds) represent more ambitious research directions that would substantially expand the scope of probabilistic structural assessment.

(1) Parallel and series–parallel systems. The framework is restricted to pure series systems. Extension to general topologies via the sequential compounding method [13] would broaden applicability.

(2) Time-variant reliability. The treatment here is time-invariant. Extending the axioms to time-dependent system reliability, accounting for degradation processes, would enable life-cycle robustness assessment [20].

(3) Computational software. Implementation of the framework as a Python or R package, distributed as open-source software, would facilitate adoption.

(4) Non-Gaussian basic variables. Generalisation to non-Gaussian probability distributions and non-linear limit-state functions through second-order reliability methods (SORMs) would extend the framework's accuracy.

(5) Application to other materials and topologies. Validation of the framework through case studies on steel, concrete, and masonry structures would test the universality of the propositions established here.

(6) Bayesian updating of reliability indices. The framework assumes that component reliability indices β are fixed inputs from FORM analysis. In practice, inspection and monitoring data update these indices through Bayesian inference. Integrating Bayesian updating into the F_{rob} framework would enable real-time robustness assessment of monitored structures, including bridges with structural health monitoring systems and high-rise buildings with embedded sensors.

(7) Multi-hazard robustness. The framework treats damage as a generic event affecting a subset of components. Real damage events have associated probabilities and may be caused by distinct hazards (earthquake, fire, blast, impact, deterioration). A natural extension would weight the robustness factors of different damage scenarios by their respective probabilities of occurrence, yielding a hazard-weighted robustness measure. This would provide a tractable bridge from purely structural robustness assessment to multi-hazard risk analysis.

(8) Discrete optimisation. The framework can serve as the constraint function in discrete optimisation problems where design variables are choices among catalogued elements (member sizes, connection types). The closed-form expression of Equation (9) is differentiable, enabling gradient-based optimisation with continuous relaxations of discrete variables. Integration with mixed-integer programming solvers would extend the framework to optimal design of robust structures.

(9) Computational complexity bounds. The mean-bound approximation has O(j) arithmetic complexity, with j typically in the range 4 to 10 for engineering systems. A formal complexity analysis would establish the limits of computational tractability and identify the system sizes for which the framework remains practical. This is particularly relevant for very large infrastructure networks where m may exceed 100.

(10) Coupling with structural health monitoring. As sensor data from monitored structures becomes increasingly available, the framework should be extended to incorporate evidence-based updating of damage scenarios. A monitored structure may exhibit indicators of partial damage to specific elements, raising the prior probability of those damage scenarios in the robustness assessment. Formal Bayesian updating of the damage-scenario distribution, combined with sensor reliability models, would close the loop from monitoring data to robustness reassessment.

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