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# Pullback in the Category of Intuitionistic Fuzzy Modules

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**Abstract**

Pullbacks are fundamental limit constructions in category theory. Their study within intuitionistic fuzzy module categories, which provide a framework for handling mathematical uncertainty, remains underdeveloped. This work establishes their theoretical foundation in such categories. We first investigate the explicit construction of pullbacks and then study the properties of double pullbacks, monomorphisms, and epimorphisms. A classification of different pullback categories is provided. Key results include characterising pullback construction conditions, describing features of double pullbacks and related morphisms, and rigorously establishing the relationship between pullbacks and exactness. This work deepens the understanding of limit theory in intuitionistic fuzzy settings and provides a foundation for future research.

**Keywords:** intuitionistic fuzzy modules; category; pullback

**MSC:** 18A30, 03E72, 16D80, 18A25, 18A10, 18G10

## 1. Introduction

As a foundational framework within the field of mathematics, the study of category theory transcends the boundaries of a single discipline, exerting a profound influence on modern mathematics, computer science, theoretical physics, and numerous other fields. Pullback is a component of category theory, and scholars have investigated it within various categories such as set categories, group categories, and module categories. Within this framework, scholars have further investigated the pullback structures within these categories. For example, in [1], Klein studied pullbacks in a certain category of sets. In [2], Khalis and Abidine explored pullbacks in a certain category of groups. In [3], Land and Tamme studied pullbacks in a certain category of modules. For details on the specific nature of pullback, see References [4–6].

In [7], Zadeh was instrumental in introducing the theoretical framework of fuzzy set theory, laying the underlying basis for the mathematics of uncertainty. This theoretical framework was rapidly integrated into the scholarly examination of algebraic constructs, resulting in the development of areas such as fuzzy groups, rings, ideals, and modules. Subsequently. In [8–11], scholars conducted in-depth explorations of fuzzy submodules, fuzzy homomorphisms, fuzzy quotient modules, and exact sequences of fuzzy modules. These developments were almost always accompanied by a process of categorisation: fuzzy sets, fuzzy groups and fuzzy modules each formed their own categories (the domains encompassing fuzzy sets, fuzzy groups, and fuzzy modules can be respectively formalised as distinct categorical frameworks within the scope of category theory). Within this framework, scholars further studied pullback structures within these categories. For example,



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in [12], Kim et al. study pullbacks in a certain category of fuzzy sets. Resende et al. and Zaidi et al. studied pullbacks in a certain category of fuzzy groups in [13,14]. In [15], Popescu pullback structures in a certain category of fuzzy modules were explored. Beyond algebraic structures, fuzzy modeling has also been successfully applied to nonlinear dynamic systems. For instance. In [16], Q. T. Tran et al. developed a dynamic Takagi–Sugeno fuzzy model for antagonistic pneumatic artificial muscles, achieving low prediction errors without relying on predefined mathematical formulations.

In the work presented by K.T. Atanassov [17,18], the concept of Intuitionistic Fuzzy Sets (IFSs) was initially proposed as a significant extension of fuzzy set theory. As the theory matured and research into fuzzy algebra deepened, researchers naturally introduced IFS into group structures. R. Biswas [19] proposed the concept of Intuitionistic Fuzzy Subgroups (IFGs). Subsequently, scholars further applied this framework to module structures. B. Davvaz et al. [20] introduced the notion of intuitionistic fuzzy submodules (IFMs). Along this direction, Paul Isaac and Pearly P. John [21] carried out a detailed algebraic study, in which many properties of an intuitionistic fuzzy submodule of an $R$-module are discussed and investigated. Since then, the categorical framework for intuitionistic fuzzy modules has gradually developed. Gunduz and Davvaz [22] studied inverse and direct systems and the exactness of limits of exact sequences of IF modules, which naturally leads to the consideration of pullbacks as a fundamental type of limit. Sharma et al. [23] introduced kernels and cokernels in the category of IF modules and showed that the category is not Abelian, providing the necessary categorical background for pullback constructions. Talaee and Oskooie [24] investigated IF exact sequences, Hom functors, products and coproducts. This work provides important categorical background and exactness results that directly support our pullback-induced exact sequences. In a related direction, Sharma [25] developed the notion of exact sequences for intuitionistic fuzzy $G$-modules and studied their properties, further demonstrating the importance of exactness in this categorical setting. Although Kim et al. [26] studied pullback structures in the categories of intuitionistic fuzzy sets, respectively, research on the pullback structure of the category of intuitionistic fuzzy modules remains relatively scarce. The present paper, therefore, systematically studies pullback constructions, double pullback properties, the preservation of monomorphisms and epimorphisms, and exact sequences derived from pullbacks in this category.

This study establishes both the existence and the uniqueness of pullback constructions within the framework of the category constituted by intuitionistic fuzzy modules, and it provides an in-depth investigation of their double pullback and monomorphism properties. Furthermore, explicit examples of pullbacks in this category are constructed. Finally, the relationship between pullbacks and exact sequences in the intuitionistic fuzzy module category is presented.

### *1.1. Comparison with Classical Module Categories*

In the classical category $\mathbf{C_{R\text{-}M}}$, the pullback of two homomorphisms, $f : X \to Z$ and $g : Y \to Z$, is the submodule $\{(x, y) \mid f(x) = g(y)\}$ of $X \oplus Y$. In the intuitionistic fuzzy setting, the underlying set is identical, but the membership and non-membership functions introduce a graded structure. Consequently, an IF $R$-homomorphism must satisfy inequality conditions $\mu_B(f(x)) \geq \mu_A(x)$ and $\nu_B(f(x)) \leq \nu_A(x)$, which have no analogue in the crisp case. Moreover, while $\mathbf{C_{R\text{-}M}}$ is an Abelian category, the category $\mathbf{C_{R\text{-}IFM}}$ is not Abelian. This makes pullbacks particularly valuable as a tool to recover exactness properties in a non-Abelian environment.

*1.2. Main Contributions of This Paper*

To help readers identify the core novelties of this work, we explicitly list the main results and their contributions. The main results of this paper are as follows: Theorem 4 establishes the existence and uniqueness of pullbacks in the fuzzy module category, using the key property that the forgetful functor preserves pullbacks. Theorem 5 shows that, in a pullback diagram in the intuitionistic fuzzy category, if the fuzzy morphism $\bar{f}$ is an IF $R$-monomorphism, then the underlying $R$-homomorphism $f$ and the projection $p_2$ are both $R$-monomorphisms. Theorem 6 shows that, in a pullback diagram in the intuitionistic fuzzy category, if $\bar{f}$ is an IF $R$-epimorphism, then $f$ and the projection $p_2$ are both $R$-epimorphisms. Theorem 7 gives a condition for the existence of double pullbacks in the intuitionistic fuzzy category: with $G$ on $Y \times_M Z$ and $H$ on $X \times_Y (Y \times_M Z)$ as IFSMs, and the natural projections as IF $R$-homomorphisms, $H$ is a pullback of $\bar{g}$ and $\bar{p}_4$ if and only if the underlying module pullback exists and the membership function $\mu_H$ is defined as the minimum ($\wedge$) of $\mu_A, \mu_B, \mu_C$, while the non-membership function $\nu_H$ is defined as the maximum ($\vee$) of $\nu_A, \nu_B, \nu_C$. Theorem 8 characterises the connection between double pullbacks and fuzzy exactness: under suitable IF $R$-homomorphisms and the constructed IFSMs $G, H$, $G$ is a pullback of $\bar{h}$ and $\bar{f}$, and $H$ is a pullback of $\bar{g}$ and $\bar{p}_4$, if and only if the underlying module pullbacks exist and the membership and non-membership functions are given by the minimum and maximum, respectively, of the component functions. Theorem 9 shows that, for the diagram formed by a single IF $R$-homomorphism $\bar{f} : A \to B$ and itself, $A$ is a pullback of that diagram if and only if the underlying $R$-homomorphism $f$ is an $R$-monomorphism. Property 6 provides, under the pullback construction, an intuitionistic fuzzy exact sequence, $0 \to D \to A \oplus B \to C$, where $D$ is the pullback object and the morphisms are induced by the projections and by subtraction ($\psi(x, y) = f(x) - g(y)$). Property 7 provides, under the additional conditions that $p_2$ is surjective and membership function values compare appropriately, another intuitionistic fuzzy exact sequence involving kernels and cokernels. Together, the above results provide a rigorous categorical foundation for handling pullbacks, double pullbacks, monomorphisms, epimorphisms, and exactness in intuitionistic fuzzy module theory.

## 2. Materials and Methods

*2.1. Pullback Construction Procedure*

For IF $R$-homomorphisms $\bar{f} : A \to C, \bar{g} : B \to C$ (corresponding to $R$-homomorphisms $f : X \to Z, g : Y \to Z$):

(1) Underlying set $X \times_Z Y = \{(x, y) \mid f(x) = g(y)\}$ is the pullback in $\mathbf{C_{R\text{-}M}}$ with canonical projections $p_1, p_2$. Let $D$ be the intuitionistic fuzzy module on $X \times_Z Y$.
(2) Membership and non-membership: $\mu_D(x, y) = \mu_A(x) \wedge \mu_B(y), \nu_D(x, y) = \nu_A(x) \vee \nu_B(y)$.
(3) Projections $\bar{p}_1 : D \to A$, $\bar{p}_2 : D \to B$ induced by $p_1, p_2$ are IF $R$-homomorphisms, and $(D, \bar{p}_1, \bar{p}_2)$ is the pullback of $\bar{f}$ and $\bar{g}$ (Theorem 4).

*2.2. Double Pullback Construction Procedure*

2.2.1. Intermediate Pullback $G$

For IF $R$-homomorphisms $\bar{h} : B \to F$ and $\bar{f} : C \to F$ (underlying $h : Y \to M$, $f : Z \to M$), the pullback in $\mathbf{C_{R\text{-}IFM}}$ is given by $G = (\mu_G, \nu_G)$ on the module pullback $Y \times_M Z = \{(y, z) \mid h(y) = f(z)\}$ with

$$\mu_G(y, z) = \mu_B(y) \wedge \mu_C(z), \qquad \nu_G(y, z) = \nu_B(y) \vee \nu_C(z).$$

The projections $\bar{p}_4 : G \to B$ and $\bar{p}_2 : G \to C$ are the IF $R$-homomorphisms induced by the natural projections (Theorem 4).

2.2.2. Final Pullback $H$

Given an additional IF $R$-homomorphism, $\bar{g} : A \to B$ (underlying $g : X \to Y$), the pullback of $\bar{g}$ and the projection $\bar{p}_4 : G \to B$ are $H = (\mu_H, \nu_H)$ on the iterated pullback

$$X \times_Y (Y \times_M Z) = \{(x, y, z) \mid g(x) = y,\ h(y) = f(z)\}$$

with

$$\mu_H(x, y, z) = \mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z), \qquad \nu_H(x, y, z) = \nu_A(x) \vee \nu_B(y) \vee \nu_C(z).$$

The projections $\bar{p}_3 : H \to A$ and $\bar{p}_1 : H \to G$ are IF $R$-homomorphisms. By Theorem 4 (and the transitivity of pullbacks, Theorem 8), $H$ is a pullback of the given diagram.

### *2.3. Preservation of Monomorphisms and Epimorphisms Under Pullback Construction*

In the classical module category, if $f : X \to Z$ is a monomorphism (resp. epimorphism), then in a pullback diagram of $f$ and $g : Y \to Z$, the projection $p_2 : P \to Y$ is also a monomorphism (resp. epimorphism). In the intuitionistic fuzzy setting, if $D$ is a pullback and $\bar{f} : A \to C$ is an IF monomorphism (resp. epimorphism), then the underlying $R$-homomorphism $f$ and the projection $p_2$ are $R$-module monomorphisms (resp. epimorphisms).

### *2.4. Construction and Classification of Pullbacks*

2.4.1. Equaliser Type

In the classical module category, for a monomorphism, $f : X \to Y$, the pullback of two identical morphisms, $f, f$, is isomorphic to $X$ via the diagonal embedding $x \mapsto (x, x)$. In the intuitionistic fuzzy setting, let $\bar{f} : A \to B$ correspond to $f : X \to Y$. Then, $A$ itself (with identity projections) is a pullback of the diagram

$$A \xrightarrow{\bar{f}} B \xleftarrow{\bar{f}} A$$

in $\mathbf{C}_{\mathbf{R\text{-}IFM}}$ if and only if $f$ is an $R$-module monomorphism. In this case, the pullback object is exactly $X$ (the underlying module of $A$), and the intuitionistic fuzzy structure remains unchanged.

2.4.2. Direct Sum Type

In the classical module category, for submodules $X, Y$ of the direct sum $X + Y$ with inclusions $i_X : X \to X + Y$ and $i_Y : Y \to X + Y$, their pullback is the intersection $X \cap Y$. In the fuzzy setting, define $C$ on $X + Y$ by

$$\mu_C(t) = \sup_{\substack{t = x + y \\ x \in X,\, y \in Y}} (\mu_A(x) \wedge \mu_B(y)), \quad \nu_C(t) = \inf_{\substack{t = x + y \\ x \in X,\, y \in Y}} (\nu_A(x) \vee \nu_B(y)).$$

Then, the pullback of the induced IF $R$-homomorphisms $\bar{i}_X : A \to C$ and $\bar{i}_Y : B \to C$ is the IFSM $D$ on $X \cap Y$ with

$$\mu_D(t) = \mu_A(t) \wedge \mu_B(t), \quad \nu_D(t) = \nu_A(t) \vee \nu_B(t).$$

2.4.3. Product Type

In the classical module category, if $T$ is a terminal object, the pullback of $f : X \to T$ and $g : Y \to T$ is the direct product $X \times Y$. In the fuzzy setting, $D = X \times Y$, $\mu_D = \mu_A \wedge \mu_B$, $\nu_D = \nu_A \vee \nu_B$.

2.4.4. Kernel-Terminal Pullback

In the classical module category, for a terminal object, $T$, and a morphism, $g: T \to Z$, the pullback of $f: X \to Z$ and $g$ is $\ker f \times T$. Now, in the fuzzy setting, let $B$ be an IF module on $T$, $\bar{f}: A \to C$ corresponding to $f: X \to Z$, and $\bar{g}: B \to C$ corresponding to $g: T \to Z$. Then, $D$ defined on $\operatorname{Ker} f \times T$ by

$$\mu_D(x,t) = \mu_A(x) \wedge \mu_B(t), \quad \nu_D(x,t) = \nu_A(x) \vee \nu_B(t)$$

is a pullback of $\bar{f}$ and $\bar{g}$ in $\mathbf{C_{R\text{-}IFM}}$.

*2.5. Exact Sequences from a Pullback*

2.5.1. Short Exact Sequence

For $\bar{f}: A \to C$, $\bar{g}: B \to C$ and pullback $D$, define $A \oplus B$ by $\mu_{A \oplus B}(x,y) = \mu_A(x) \wedge \mu_B(y)$, $\nu_{A\oplus B}(x,y) = \nu_A(x) \vee \nu_B(y)$. Let $\bar{\varphi}: D \to A \oplus B$, $(x,y) \mapsto (x,y)$, and $\bar{\psi}: A \oplus B \to C$, $(x,y) \mapsto \bar{f}(x) - \bar{g}(y)$. Then,

$$0 \longrightarrow D \xrightarrow{\bar{\varphi}} A \oplus B \xrightarrow{\bar{\psi}} C$$

is exact ($\bar{\varphi}$ injective, $\operatorname{Im}(\bar{\varphi}) = \operatorname{Ker}(\bar{\psi})$ because $f(x) = g(y)$ holds exactly on $D$).

2.5.2. Kernel–Cokernel Long Exact Sequence

In the classical module category $\mathbf{C_{R\text{-}M}}$, for a pullback, $P$, of $f: X \to Z$ and $g: Y \to Z$, if $p_2: P \to Y$ is surjective, then the sequence

$$0 \longrightarrow \operatorname{Ker} p_2 \xrightarrow{i} P \xrightarrow{p_2} Y \xrightarrow{j} \operatorname{Coker} \bar{g}$$

is exact, where $i$ is the inclusion and $j$ the natural projection.

Now consider the intuitionistic fuzzy setting. Let $D$ be the pullback constructed as before. Assume that $p_2: P \to Y$ is surjective and, for every $b \in B$, there exists $a \in A$, such that $\bar{f}(a) = \bar{g}(b)$ with $\mu_A(a) \geq \mu_B(b)$ and $\nu_A(a) \leq \nu_B(b)$. Then, the following sequence is exact in $\mathbf{C_{R\text{-}IFM}}$:

$$0 \longrightarrow \widetilde{\operatorname{Ker}} \bar{p}_2 \xrightarrow{\bar{i}} D \xrightarrow{\bar{p}_2} B \xrightarrow{\bar{j}} \widetilde{\operatorname{Coker}} \bar{g}$$

where $\bar{i}$ is the inclusion, and $\bar{j}$ the projection to the cokernel. Key facts: $\widetilde{\operatorname{Ker}}\, \bar{p}_2 \cong \widetilde{\operatorname{Ker}}\, \bar{f}$ (via $(x,0) \mapsto x$) and $\bar{i}$ is an IF monomorphism. By the pullback definition, $\operatorname{Im}(\bar{i}) = \operatorname{Ker}(\bar{p}_2)$. Surjectivity and the lifting condition guarantee $\operatorname{Im}(\bar{p}_2) = \operatorname{Ker}(\bar{j})$ with equality of membership/non-membership functions under sup/inf.

## 3. Preliminaries

**Definition 1** ([23]). *A category, $(Ob, Hom, id, \circ)$, where:*

(1) *Ob denotes a class of objects;*

(2) *For every ordered pair $(X, Y)$ of objects, there is an associated set, $Hom(X, Y)$, comprised of morphisms from $X$ to $Y$;*

(3) *For each object $X \in Ob$, there exists an identity morphism, $id_X \in Hom(X, X)$;*

(4) *A composition operation $\circ$ is defined, such that, for any morphisms $f \in Hom(X, Y)$ and $g \in Hom(Y, Z)$, the composition $g \circ f$ belongs to $Hom(X, Z)$.*

*These components must satisfy the following axioms:*

(1) *For all morphisms, $f \in Hom(X, Y)$, $g \in Hom(Y, Z)$, and $h \in Hom(Z, W)$, it holds that $h \circ (g \circ f) = (h \circ g) \circ f$.*

(2) *The identity morphisms act as neutral elements with respect to composition, meaning, for every $f \in Hom(X, Y)$, the equalities $id_Y \circ f = f$ and $f \circ id_X = f$ are satisfied.*

**Example 1** ([22]). *The set of all R-modules, together with all R-homomorphisms, constitutes a category, which we denote by $\boldsymbol{C_{R\text{-}M}}$.*

**Definition 2** ([6]). *In a category, $\boldsymbol{C}$, an object, $T \in Ob(\boldsymbol{C})$, is called terminal if, for each object $X \in Ob(\boldsymbol{C})$, there exists exactly one morphism from X to T. Equivalently, for every $X \in Ob(\boldsymbol{C})$, there is a unique morphism, $f : X \to T$.*

**Example 2** ([5]). *In $\boldsymbol{C_{R\text{-}M}}$, its terminal object is $\{0\}$ (the zero module).*

**Definition 3** ([6]). *In a category, $\boldsymbol{C}$, a morphism, $f : X \to Y$, is called a monomorphism if, for any object, Z, and any pair of parallel morphisms, $g, h : Z \to X$, the equality $f \circ g = f \circ h$ implies $g = h$.*

**Definition 4** ([6]). *In a category $\boldsymbol{C}$, a morphism $f : X \to Y$ is called an epimorphism if, for any object, Z, and any pair of parallel morphisms, $g, h : Y \to Z$, the equality $g \circ f = h \circ f$ implies $g = h$.*

**Definition 5** ([6]). *In a category, $\boldsymbol{C}$, given morphisms $f : X \to Z$ and $g : Y \to Z$, a pullback of the pair $\{f, g\}$ consists of:*

(1) *An object, $X \times_Z Y$;*
(2) *Morphisms $p_1 : X \times_Z Y \to X$ and $p_2 : X \times_Z Y \to Y$ satisfying $f \circ p_1 = g \circ p_2$,*

*and possessing the following universal property: For any object, W, and any morphisms, $m_1 : W \to X$, $m_2 : W \to Y$, with $f \circ m_1 = g \circ m_2$, there exists a unique morphism, $u : W \to X \times_Z Y$, such that $p_1 \circ u = m_1$ and $p_2 \circ u = m_2$.*

**Example 3** ([5]). *In $\boldsymbol{C_{R\text{-}M}}$, the pullback of the morphism pair $f : X \to Z$ and $g : 0 \to Z$ is the module: $P = \{(x, y) \in X \oplus Y \mid f(x) = 0\}$ along with the projection morphisms $\pi_A(x, y) = x$ and $\pi_B(x, y) = y$. When f and g are inclusion morphisms, then* Ker $f(a)$ *is a pullback in $\boldsymbol{C_{R\text{-}M}}$.*

**Example 4** ([5]). *In $\boldsymbol{C_{R\text{-}M}}$, the pullback of the morphism pair $f : X \to 0$ and $g : Y \to 0$ is the module: $P = \{(a, b) \in X \oplus Y \mid f(x) = g(y) = 0\}$ along with the projection morphisms $\pi_A(x, y) = x$ and $\pi_B(x, y) = y$. When f and g are zero morphisms, then $X \times Y$ is a pullback in $\boldsymbol{C_{R\text{-}M}}$.*

**Example 5** ([5]). *In $\boldsymbol{C_{R\text{-}M}}$, let X and Y be submodules of an R-module M. The sum $X + Y$ is the smallest submodule containing both X and Y. Let $i_X : X \to X + Y$ and $i_Y : Y \to X + Y$, where $i_X$ and $i_Y$ are inclusion morphisms. Then, $X \cap Y$ is a pullback in $\boldsymbol{C_{R\text{-}M}}$.*

**Definition 6** ([5]). *In $\boldsymbol{C_{R\text{-}M}}$, where: objects are R-modules, morphisms are R-module homomorphisms, and a terminal object is an R-module, T, such that: for every R-module, M, there exists a unique R-module homomorphism, $f : M \to T$.*

**Definition 7** ([5]). *In $\boldsymbol{C_{R\text{-}M}}$, let X and Y be submodules of an R-module M. Let $f : X \to Y$ be an R-module homomorphism. Its cokernel is defined as* Coker $f = Y/ \operatorname{Im} f$, *and let $\pi : Y \to$* Coker $f$ *be the canonical projection. Then, the composite map $j = \pi \circ f : X \to$* Coker $f$ *is the zero homomorphism.*

**Theorem 1** ([6]). *In $\mathbf{C_{R\text{-}M}}$, consider morphisms $f : X \to Z$ and $g : Y \to Z$. Let $\{X \times_Z Y, p_1, p_2\}$ be a pullback of the pair $\{f, g\}$. The following property holds: If $f$ is a monomorphism(epimorphism), then the projection $p_2 : X \times_Z Y \to Y$ is also a monomorphism(epimorphism).*

**Theorem 2** ([6]). *Let $X$, $Y$, $Z$ and $M$ be objects in a category $\mathbf{C}$, and consider the following morphisms: $f : Z \to M$, $g : X \to Y$ and $h : Y \to M$. Suppose that:*

*(1) Let $\{Y \times_M Z, p_2, p_4\}$ be a pullback of the pair $\{f, h\}$;*
*(2) Let $\{X \times_Y (Y \times_M Z), p_1, p_3\}$ be a pullback of the pair $\{g, p_4\}$.*

*Then, $\{X \times_Y (Y \times_M Z), p_2 \circ p_1, p_3\}$ is a pullback of the pair $\{h \circ g, f\}$.*

**Theorem 3** ([6]). *Let $X$, $Y$, $Z$ and $M$ be objects in a category $\mathbf{C}$, and consider morphisms: $f : Z \to M$, $g : X \to Y$, $h : Y \to M$. Suppose $\{X \times_Y (Y \times_M Z), p_2 \circ p_1, p_3\}$ is a pullback of the pair $\{h \circ g, f\}$, with projection morphisms:*

*(1) $p_1 : X \times_Y (Y \times_M Z) \to X$;*
*(2) $p_2 : Y \times_M Z \to Z$ (where $p_2$ factors through $Y \times_M Z$).*

*If $p_2$ is a monomorphism, then $\{X \times_Y (Y \times_M Z), g, p_4\}$ is a pullback of the pair $\{p_1, p_3\}$.*

**Definition 8** ([5]). *Let $M_1$, $M_2$, $M_3$ be R-modules, with R-homomorphisms $f : M_1 \longrightarrow M_2$ and $g : M_2 \longrightarrow M_3$. If the sequence of R-modules and R-homomorphisms*

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$$

$\mathrm{Im} f = \mathrm{Ker}\, g$*, then the sequence is said to be exact at $M_2$. A longer sequence*

$$\cdots \longrightarrow M_{n-1} \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \longrightarrow \cdots$$

*is called an exact sequence if it is exact at every module $M_n$ in the sequence.*

**Property 1** ([5]). *In $\mathbf{C_{R\text{-}M}}$, let $X$ and $Y$ be submodules of an R-module $M$. Let $f : X \to Y$ be an R-module homomorphism. Then,*

*(1) $0 \longrightarrow X \xrightarrow{f} Y$ is exact if and only if $f$ is a monomorphism;*
*(2) $X \xrightarrow{f} Y \longrightarrow 0$ is exact if and only if $f$ is an epimorphism.*

**Definition 9** ([17,18]). *An intuitionistic fuzzy set (IFS) $A$ in $X$ can be expressed as $A = \{\langle x, \mu_A, \nu_A \rangle : x \in X\}$, where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ are functions representing, respectively, the membership degree, $\mu_A(x)$, and the non-membership degree, $\nu_A(x)$, of each $x \in X$ to $A$, satisfying $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.*

**Definition 10** ([20,22]). *An IFS, $A = (\mu_A, \nu_A)$, of R-module $M$ is called an intuitionistic fuzzy submodule (IFSM) if*

*(1) $\mu_A(\theta) = 1, \nu_A(\theta) = 0$;*
*(2) $\mu_A(a+b) \geq \mu_A(a) \wedge \mu_A(b)$ and $\nu_A(a+b) \leq \nu_A(a) \vee \nu_A(b)$, for all $a, b \in M$;*
*(3) $\mu_A(ra) \geq \mu_A(a)$ and $\nu_A(ra) \leq \nu_A(a)$, for all $a \in M, r \in R$.*

**Definition 11** ([25]). *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSMs of the R-modules $M$ and $N$, respectively. A mapping, $f : A \to B$, is called an intuitionistic fuzzy R-homomorphism from $A$ to $B$ if the following conditions hold:*

*(1) $f : M \to N$ is an R-homomorphism;*
*(2) For every $a \in M$, $\mu_B(f(a)) \geq \mu_A(a)$ and $\nu_B(f(a)) \leq \nu_A(a)$.*

*To distinguish an R-homomorphism $f : M \to N$ from an intuitionistic fuzzy R-homomorphism $f : A \to B$, we denote the latter by $\bar{f} : A \to B$. Thus, for an IF R-homomorphism $\bar{f} : A \to B$, the map $f : M \to N$ is its underlying R-homomorphism. The collection of all IF R-homomorphisms from A to B is denoted by* $\mathrm{Hom}(A, B)$.

## 4. Results

### 4.1. Pullback Construction

**Definition 12.** *In the category $\mathbf{C_{R\text{-}IFM}}$, given intuitionistic fuzzy R-homomorphisms $\bar{f} : A \to C$ and $\bar{g} : B \to C$, a pullback of the pair $\{\bar{f}, \bar{g}\}$ consists of:*

*(1) An intuitionistic fuzzy R-module D;*

*(2) Intuitionistic fuzzy R-homomorphisms $\bar{p}_1 : D \to A$ and $\bar{p}_2 : D \to B$ satisfying $\bar{f} \circ \bar{p}_1 = \bar{g} \circ \bar{p}_2$,*

*and possessing the following universal property: For any intuitionistic fuzzy R-module E and any intuitionistic fuzzy R-homomorphisms $\bar{m}_1 : E \to A$, $\bar{m}_2 : E \to B$ with $\bar{f} \circ \bar{m}_1 = \bar{g} \circ \bar{m}_2$, there exists a unique intuitionistic fuzzy R-homomorphism $\bar{u} : E \to D$ such that $\bar{p}_1 \circ \bar{u} = \bar{m}_1$ and $\bar{p}_2 \circ \bar{u} = \bar{m}_2$.*

**Theorem 4.** *Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$ and $C = (\mu_C, \nu_C)$ be IFSM of R-modules X,Y and Z, respectively, and let $\bar{f} : A \to C$ and $\bar{g} : B \to C$ be IF $R-$homomorphism corresponding to the R-homomorphism $f : X \to Z$ and $g : Y \to Z$, with $p_1$ and $p_2$ being the projections from $X \times_Z Y$ to X and Y. Then, the object D is called a pullback in $\mathbf{C_{R\text{-}IFM}}$ if the following conditions hold:*

*(1) $X \times_Z Y$ is a pullback in $\mathbf{C_{R\text{-}M}}$ ;*

*(2) $\mu_D(x, y) = \mu_A(x) \wedge \mu_B(y)$ and $\nu_D(x, y) = \nu_A(x) \vee \nu_B(y)$.*

**Proof of Theorem 4.** Sufficiency: Since $X \times_Z Y$ is a pullback in $\mathbf{C_{R\text{-}M}}$, let $m_1 : W \to X$ and $m_2 : W \to Y$ be $R$-homomorphisms with $f \circ m_1 = g \circ m_2$. Then, there exists a unique $R$-homomorphism, $u : W \to X \times_Z Y$, such that $m_1 = p_1 \circ u$ and $m_2 = p_2 \circ u$; i.e., Figure 1a commutes. From (2), we have for any $(x, y) \in X \times_Z Y$:

$$\mu_D(x, y) = \mu_A(x) \wedge \mu_B(y) \leq \mu_A(x) = \mu_A \circ p_1(x, y), \quad \mu_D(x, y) \leq \mu_B(y) = \mu_B \circ p_2(x, y),$$

$$\nu_D(x, y) = \nu_A(x) \vee \nu_B(y) \geq \nu_A(x) = \nu_A \circ p_1(x, y), \quad \nu_D(x, y) \geq \nu_B(y) = \nu_B \circ p_2(x, y).$$

Hence, $\bar{p}_1 : D \to A$ and $\bar{p}_2 : D \to B$ are IF $R$-homomorphisms corresponding to $p_1, p_2$.

Let $E = (\mu_E, \nu_E)$ be an IFSM of $W$ and let $\bar{m}_1 : E \to A$, $\bar{m}_2 : E \to B$ be IF $R$-homomorphisms satisfying $\bar{f} \circ \bar{m}_1 = \bar{g} \circ \bar{m}_2$. Define $u : W \to X \times_Z Y$ by $u(w) = (m_1(w), m_2(w))$. For any $w \in W$, we have

$$\mu_E(w) \leq \mu_A(m_1(w)),\ \mu_E(w) \leq \mu_B(m_2(w)),\ \nu_E(w) \geq \nu_A(m_1(w)),\ \nu_E(w) \geq \nu_B(m_2(w)),$$

so $\mu_E(w) \leq \mu_D(u(w))$ and $\nu_E(w) \geq \nu_D(u(w))$; therefore, $\bar{u} : E \to D$ is an IF $R$-homomorphism. Since $u$ in Figure 1a is unique, the morphism $\bar{u}$ in Figure 1b is also unique: if another IF $R$-homomorphism $\bar{u'} : E \to D$ satisfies the same commutativity, its underlying map, $u'$, must equal $u$ by the uniqueness in $\mathbf{C_{R\text{-}M}}$, and hence, $\bar{u'} = \bar{u}$. Thus, $D$ is a pullback in $\mathbf{C_{R\text{-}IFM}}$.

Necessity: Let $D$ be a pullback in $\mathbf{C_{R\text{-}IFM}}$. Write $M = U(D)$, and denote the projections by $p_1^0 : M \to X$, $p_2^0 : M \to Y$. The forgetful functor $U : \mathbf{C_{R\text{-}IFM}} \to \mathbf{C_{R\text{-}M}}$ maps the pullback square in the IF category to a commutative square in the module category, as shown in Figure 2.

We first show that $M$ is a pullback in $\mathbf{C_{R\text{-}M}}$.

$$\begin{tikzcd}
W \arrow[drr, bend left, "m_2"] \arrow[dr, dashed, "u"] \arrow[ddr, bend right, "m_1"'] & & \\
& X \times_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[drr, bend left, "\bar{m}_2"] \arrow[dr, dashed, "\bar{u}"] \arrow[ddr, bend right, "\bar{m}_1"'] & & \\
& D \arrow[r, "\bar{p}_2"] \arrow[d, "\bar{p}_1"'] & B \arrow[d, "\bar{g}"] \\
& A \arrow[r, "\bar{f}"'] & C
\end{tikzcd}$$

(**a**) Pullback in $\mathbf{C_{R\text{-}M}}$ (**b**) Pullback in $\mathbf{C_{R\text{-}IFM}}$

**Figure 1.** Pullback diagrams in the categories of $R$-modules and intuitionistic fuzzy $R$-modules.

Existence: For any $R$-module $W$ and $R$-homomorphisms $m_1 : W \to X$, $m_2 : W \to Y$ with $f \circ m_1 = g \circ m_2$, define an IFSM $E = (\mu_E, \nu_E)$ on $W$ by

$$\mu_E(w) = \mu_A(m_1(w)) \wedge \mu_B(m_2(w)), \qquad \nu_E(w) = \nu_A(m_1(w)) \vee \nu_B(m_2(w)).$$

Then, $\bar{m}_1 : E \to A$ and $\bar{m}_2 : E \to B$ are IF $R$-homomorphisms and $\bar{f} \circ \bar{m}_1 = \bar{g} \circ \bar{m}_2$. Because $D$ is a pullback in $\mathbf{C_{R\text{-}IFM}}$, there exists a unique IF $R$-homomorphism $\bar{u} : E \to D$ with $\bar{m}_1 = \bar{p}_1 \circ \bar{u}$, $\bar{m}_2 = \bar{p}_2 \circ \bar{u}$. Taking underlying maps gives $u = U(\bar{u}) : W \to M$, such that $m_1 = p_1^0 \circ u$ and $m_2 = p_2^0 \circ u$. Hence, the required $u$ exists.

$$\begin{tikzcd}
E \arrow[drr, bend left, "\bar{m}_2"] \arrow[dr, dashed, "\bar{u}"] \arrow[ddr, bend right, "\bar{m}_1"'] & & \\
& D \arrow[r, "\bar{p}_2"] \arrow[d, "\bar{p}_1"'] & B \arrow[d, "\bar{g}"] \\
& A \arrow[r, "\bar{f}"'] & C
\end{tikzcd}
\quad \overset{U}{\rightsquigarrow} \quad
\begin{tikzcd}
W \arrow[drr, bend left, "m_2"] \arrow[dr, dashed, "u"] \arrow[ddr, bend right, "m_1"'] & & \\
& M \arrow[r, "p_2^0"] \arrow[d, "p_1^0"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}$$

**Figure 2.** Action of the forgetful functor on the pullback. The forgetful functor $U$ sends the IF pullback to its underlying square in $\mathbf{C_{R\text{-}M}}$.

Uniqueness and isomorphism: Since $f \circ p_1^0 = g \circ p_2^0$ and $X \times_Z Y$ is a pullback in $\mathbf{C_{R\text{-}M}}$ (it always exists), there is a unique $R$-homomorphism $\varphi : M \to X \times_Z Y$ with $p_1^0 = p_1 \circ \varphi$, $p_2^0 = p_2 \circ \varphi$, as shown in Figure 3a.

Take $W = X \times_Z Y$, $m_1 = p_1$, $m_2 = p_2$. By the existence part, we obtain $v : X \times_Z Y \to M$, such that $p_1 = p_1^0 \circ v$, $p_2 = p_2^0 \circ v$ , as shown in Figure 3b.

$$\begin{tikzcd}
M \arrow[drr, bend left, "p_2^0"] \arrow[dr, dashed, "\varphi"] \arrow[ddr, bend right, "p_1^0"'] & & \\
& X \times_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times_Z Y \arrow[drr, bend left, "p_2"] \arrow[dr, dashed, "v"] \arrow[ddr, bend right, "p_1"'] & & \\
& M \arrow[r, "p_2^0"] \arrow[d, "p_1^0"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

(**a**) The unique map $\varphi : M \to X \times_Z Y$ (**b**) Map $v$ from existence.

**Figure 3.** Maps $\varphi$ and $v$ establishing the isomorphism.

Then, $p_1 = p_1 \circ (\varphi \circ v)$, $p_2 = p_2 \circ (\varphi \circ v)$, and the uniqueness of the identity in the pullback $X \times_Z Y$ yields $\varphi \circ v = \mathrm{id}_{X \times_Z Y}$.

To prove $v \circ \varphi = \mathrm{id}_M$, we work in the category $\mathbf{C_{R\text{-}IFM}}$. Equip $X \times_Z Y$ with the intuitionistic fuzzy submodule $D_0 = (\mu_{D_0}, \nu_{D_0})$, defined by

$$\mu_{D_0}(x, y) = \mu_A(x) \wedge \mu_B(y), \qquad \nu_{D_0}(x, y) = \nu_A(x) \vee \nu_B(y).$$

Then, $\bar{p}_1 : D_0 \to A$ and $\bar{p}_2 : D_0 \to B$ are IF $R$-homomorphisms. From $\bar{p}_i : D \to A$, we have $\mu_D(m) \leq \mu_A(p_i^0(m))$ and $\nu_D(m) \geq \nu_A(p_i^0(m))$, and similarly for $B$. Since $p_i^0 = p_i \circ \varphi$, it follows that

$$\mu_D(m) \leq \mu_A(p_1(\varphi(m))) \wedge \mu_B(p_2(\varphi(m))) = \mu_{D_0}(\varphi(m)),$$

$$\nu_D(m) \geq \nu_A(p_1(\varphi(m))) \vee \nu_B(p_2(\varphi(m))) = \nu_{D_0}(\varphi(m)).$$

Hence, the underlying map $\varphi$ lifts to an IF $R$-homomorphism $\bar{\varphi} : D \to D_0$, satisfying $\bar{p}_i \circ \bar{\varphi} = \bar{p}_i$, as shown in Figure 4a.

On the other hand, the map $v$ obtained in the existence part was induced by the pullback property of $D$ applied to the IFSM $E$ on $W = X \times_Z Y$ with $\mu_E(x,y) = \mu_A(p_1(x,y)) \wedge \mu_B(p_2(x,y)) = \mu_{D_0}(x,y)$ and $\nu_E(x,y) = \mu_A(p_1(x,y)) \vee \mu_B(p_2(x,y)) = \nu_{D_0}(x,y)$. Thus, $E = D_0$ and the morphism $\bar{v} : D_0 \to D$ are an IF $R$-homomorphism with $\bar{p}_i \circ \bar{v} = \bar{p}_i$, as shown in Figure 4b.

Now consider the two IF $R$-homomorphisms $\bar{v} \circ \bar{\varphi} : D \to D$ and $\bar{\mathrm{id}}_D : D \to D$. Both satisfy

$$\bar{p}_i \circ (\bar{v} \circ \bar{\varphi}) = \bar{p}_i, \qquad \bar{p}_i \circ \bar{\mathrm{id}}_D = \bar{p}_i,$$

and clearly, $\bar{f} \circ \bar{p}_1 = \bar{g} \circ \bar{p}_2$. Because $D$ is a pullback in $\mathbf{C_{R\text{-}IFM}}$, the mediating morphism from $D$ to $D$ is unique; therefore, $\bar{v} \circ \bar{\varphi} = \bar{\mathrm{id}}_D$ , as shown in Figure 5.

$$\begin{tikzcd}
D \arrow[dr, dashed, "\bar{\varphi}"] \arrow[drr, bend left, "\bar{p}_2"] \arrow[ddr, bend right, "\bar{p}_1"'] & & \\
& D_0 \arrow[r, "\bar{p}_2"] \arrow[d, "\bar{p}_1"'] & B \arrow[d, "\bar{g}"] \\
& A \arrow[r, "\bar{f}"] & C
\end{tikzcd}$$

(**a**) The IF $R$-homomorphism $\bar{\varphi} : D \to D_0$

$$\begin{tikzcd}
D_0 \arrow[dr, dashed, "\bar{v}"] \arrow[drr, bend left, "\bar{p}_2"] \arrow[ddr, bend right, "\bar{p}_1"'] & & \\
& D \arrow[r, "\bar{p}_2"] \arrow[d, "\bar{p}_1"'] & B \arrow[d, "\bar{g}"] \\
& A \arrow[r, "\bar{f}"] & C
\end{tikzcd}$$

(**b**) The IF $R$-homomorphism $\bar{v} : D_0 \to D$

**Figure 4.** Lifts of $\varphi$ and $v$ to the IF category.

$$\begin{tikzcd}
D \arrow[dr, bend left, "\bar{v}\circ\bar{\varphi}"] \arrow[dr, bend right, "\bar{\mathrm{id}}_D"'] \arrow[drr, bend left, "\bar{p}_2"] \arrow[ddr, bend right=60, "\bar{p}_1"'] & & \\
& D \arrow[r, "\bar{p}_2"] \arrow[d, "\bar{p}_1"'] & B \arrow[d, "\bar{g}"] \\
& A \arrow[r, "\bar{f}"'] & C
\end{tikzcd}$$

**Figure 5.** Uniqueness argument in the pullback $D$. Both $\bar{v} \circ \bar{\varphi}$ and $\bar{\mathrm{id}}_D$ make the diagram commute; uniqueness forces equality.

Applying the forgetful functor $U$ gives $v \circ \varphi = \mathrm{id}_M$.

Thus, $\varphi$ and $v$ are inverse isomorphisms, so $M \cong X \times_Z Y$. We identify $M$ with $X \times_Z Y$ and $p_1^0, p_2^0$ with $p_1, p_2$. Condition (1) holds.

Now, we verify condition (2). Define $D^* = (\mu_{D^*}, \nu_{D^*})$ on $X \times_Z Y$ by

$$\mu_{D^*}(x,y) = \mu_A(x) \wedge \mu_B(y), \qquad \nu_{D^*}(x,y) = \nu_A(x) \vee \nu_B(y).$$

Clearly, $\bar{p}_1 : D^* \to A$ and $\bar{p}_2 : D^* \to B$ are IF $R$-homomorphisms. Since $\bar{p}_1 : D \to A$ is an IF $R$-homomorphism, we have $\mu_D(x,y) \leq \mu_A(p_1(x,y)) = \mu_A(x)$; similarly, $\mu_D(x,y) \leq \mu_B(y)$. Hence, $\mu_D(x,y) \leq \mu_A(x) \wedge \mu_B(y) = \mu_{D^*}(x,y)$. Likewise, $\nu_D(x,y) \geq \nu_A(x) \vee \nu_B(y) = \nu_{D^*}(x,y)$.

For the reverse inequalities, consider $E = D^*$ and the IF $R$-homomorphisms $\bar{m}_1 = \bar{p}_1 : D^* \to A$, $\bar{m}_2 = \bar{p}_2 : D^* \to B$. They satisfy $\bar{f} \circ \bar{m}_1 = \bar{g} \circ \bar{m}_2$ because $f \circ p_1 = g \circ p_2$ and the fuzzy conditions are inherited. Since $D$ is a pullback, there exists a unique IF $R$-homomorphism $\bar{u} : D^* \to D$, such that $\bar{p}_1 = \bar{p}_1 \circ \bar{u}$ and $\bar{p}_2 = \bar{p}_2 \circ \bar{u}$. Let $u = U(\bar{u}) : X \times_Z Y \to X \times_Z Y$ be its underlying map. Then, $p_1 = p_1 \circ u$, $p_2 = p_2 \circ u$, so by the uniqueness property of the pullback $X \times_Z Y$, we obtain $u = \mathrm{id}_{X \times_Z Y}$ (Figure 6a). Because $\bar{u}$ is an IF $R$-homomorphism,

$$\mu_{D^*}(x,y) \leq \mu_D(u(x,y)) = \mu_D(x,y), \qquad \nu_{D^*}(x,y) \geq \nu_D(u(x,y)) = \nu_D(x,y).$$

Combining both directions gives $\mu_D = \mu_{D^*}$ and $\nu_D = \nu_{D^*}$. Thus, condition (2) holds (Figure 6b). $\square$

$$\begin{tikzcd}
X \times_Z Y \arrow[dr, dashed, "\mathrm{id}_{X \times_Z Y}"] \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\
& X \times_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}$$

(**a**) The identity map on $X \times_Z Y$

$$\begin{tikzcd}
D^* \arrow[dr, dashed, "\bar{u}"] \arrow[drr, bend left, "\bar{p}_2"] \arrow[ddr, bend right, "\bar{p}_1"'] & & \\
& D \arrow[r, "\bar{p}_2"] \arrow[d, "\bar{p}_1"'] & B \arrow[d, "\bar{g}"] \\
& A \arrow[r, "\bar{f}"'] & C
\end{tikzcd}$$

(**b**) The unique morphism $\bar{u} : D^* \to D$

**Figure 6.** Final step in verifying condition (2).

**Corollary 1.** *The forgetful functor $U : \mathbf{C_{R\text{-}IFM}} \to \mathbf{C_{R\text{-}M}}$ preserves pullbacks. That is, if $(D, \bar{p}_1, \bar{p}_2)$ is a pullback of $\bar{f}$ and $\bar{g}$ in $\mathbf{C_{R\text{-}IFM}}$, then $(U(D), U(\bar{p}_1), U(\bar{p}_2))$ is a pullback of $U(\bar{f})$ and $U(\bar{g})$ in $\mathbf{C_{R\text{-}M}}$.*

**Proof of Corollary 1.** By the necessity part of Theorem 4, if $D$ is a pullback, then conditions (1) and (2) hold. Condition (1) implies that $U(D) \cong X \times_Z Y$ and, under this isomorphism, $U(\bar{p}_1), U(\bar{p}_2)$ correspond to the canonical projections. Thus, $U(D)$ is indeed a pullback of $U(\bar{f})$ and $U(\bar{g})$. $\square$

**Example 6.** *Let $R = \mathbb{Z}$, $X = \mathbb{Z}_2$, $Y = \mathbb{Z}_2$, $Z = \{0\}$, and let $f : X \to Z$, $g : Y \to Z$ be the zero homomorphisms. Define IFSMs $A = (\mu_A, \nu_A)$ on $X$ and $B = (\mu_B, \nu_B)$ on $Y$ by*

$$\mu_A(0) = 1,\ \mu_A(1) = 0.5, \quad \nu_A(0) = 0,\ \nu_A(1) = 0.3;$$

$$\mu_B(0) = 1,\ \mu_B(1) = 0.4, \quad \nu_B(0) = 0,\ \nu_B(1) = 0.5.$$

*$C$ is the unique IFSM on $Z$ with $\mu_C(0) = 1$, $\nu_C(0) = 0$. Then, $\bar{f} : A \to C$, $\bar{g} : B \to C$ are trivial IF $R$-homomorphisms. The pullback in $\mathbf{C_{R\text{-}M}}$ is $X \times_Z Y = X \times Y \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. According to Theorem 4, the pullback $D$ in $\mathbf{C_{R\text{-}IFM}}$ is given by*

$$\mu_D(x,y) = \mu_A(x) \wedge \mu_B(y), \quad \nu_D(x,y) = \nu_A(x) \vee \nu_B(y).$$

*For instance, $\mu_D(1,1) = 0.5 \wedge 0.4 = 0.4$, $\nu_D(1,1) = 0.3 \vee 0.5 = 0.5$. It is straightforward to check the universal property: any IF cone over $A$ and $B$ factors uniquely through $D$.*

### *4.2. Preservation of Monomorphisms and Epimorphisms*

**Theorem 5.** *Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$, and $C = (\mu_C, \nu_C)$ be IFSMs of $R$-modules $X$, $Y$, and $Z$, respectively. Let $\bar{f} : A \to C$ and $\bar{g} : B \to C$ be IFR-homomorphisms corresponding* *to $R$-homomorphisms $f : X \to Z$ and $g : Y \to Z$, with $p_1$ and $p_2$ being the projections from $X \times_Z Y$ to $X$ and $Y$. If $D$ is a pullback in $\mathbf{C_{R\text{-}IFM}}$ and $\bar{f}$ is an IFR-monomorphism, then $f$ is an $R$-monomorphism, and $p_2$ is an $R$-monomorphism.*

**Proof of Theorem 5.** Since $D$ is a pullback in $\mathbf{C_{R\text{-}IFM}}$, we first show that $\bar{p}_2$ is an IF$R$-monomorphism. Suppose that $\bar{u}_1, \bar{u}_2 : E \to D$ are IF$R$-homomorphisms, such that $\bar{p}_2 \circ \bar{u}_1 = \bar{p}_2 \circ \bar{u}_2$. In Figure 7, the pullback diagram commutes, so we have $\bar{f} \circ \bar{p}_1 \circ \bar{u}_1 = \bar{g} \circ \bar{p}_2 \circ \bar{u}_1 = \bar{g} \circ \bar{p}_2 \circ \bar{u}_2 = \bar{f} \circ \bar{p}_1 \circ \bar{u}_2$. Since $\bar{f}$ is an IF$R$-monomorphism, it follows that $\bar{p}_1 \circ \bar{u}_1 = \bar{p}_1 \circ \bar{u}_2$.Let $\bar{m}_1 = \bar{p}_1 \circ \bar{u}_1$ and $\bar{m}_2 = \bar{p}_2 \circ \bar{u}_1$. Then, $\bar{f} \circ \bar{m}_1 = \bar{g} \circ \bar{m}_2$. By the pullback property of $D$, there exists a unique IF$R$-homomorphism, $\bar{u} : E \to D$, such that $\bar{p}_1 \circ \bar{u} = \bar{m}_1 \quad \textit{and} \quad \bar{p}_2 \circ \bar{u} = \bar{m}_2$. But both $\bar{u}_1$ and $\bar{u}_2$ satisfy these conditions, so $\bar{u}_1 = \bar{u}_2$. Hence, $\bar{p}_2$ is an IF$R$-monomorphism.

$$\begin{tikzcd}
W \arrow[dr, bend left, "u_2"] \arrow[dr, bend right, "u_1"'] \arrow[dr, dashed, "u"] \arrow[drr, bend left, "m_2"] \arrow[ddr, bend right, "m_1"'] & & \\
& X \times_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[dr, bend left, "\bar{u}_2"] \arrow[dr, bend right, "\bar{u}_1"'] \arrow[dr, dashed, "\bar{u}"] \arrow[drr, bend left, "\bar{m}_2"] \arrow[ddr, bend right, "\bar{m}_1"'] & & \\
& D \arrow[r, "\bar{p}_2"] \arrow[d, "\bar{p}_1"'] & B \arrow[d, "\bar{g}"] \\
& A \arrow[r, "\bar{f}"'] & C
\end{tikzcd}$$

(**a**) Pullback for monomorphism in $\mathbf{C_{R\text{-}M}}$ (**b**) Pullback for monomorphism in $\mathbf{C_{R\text{-}IFM}}$

**Figure 7.** Pullback diagrams for monomorphism preservation in the categories of $R$-modules and intuitionistic fuzzy submodules.

Now, since $\bar{p}_2$ is an IF$R$-homomorphism corresponding to $p_2$, and we will show in Part 2 that an IF$R$-monomorphism implies that the underlying $R$-homomorphism is a monomorphism, it follows that $p_2$ is an $R$-monomorphism. Suppose, as illustrated in Figure 8a. for contradiction, that $f$ is not an $R$-monomorphism. Then, there exist $x, y \in X$ with $x \neq y$, such that $f(x) = f(y)$. Let $N$ be an $R$-module, and define $m, h : N \to X$ by

$$m(n) = x, \quad h(n) = y \quad \text{for all } n \in N.$$

Let $M$ be the IFSM of $N$ defined by

$$\mu_M(n) = 0, \quad \nu_M(n) = 1 \quad \text{for all } n \in N.$$

Then, $\bar{m}, \bar{h} : M \to A$ are IF$R$-homomorphisms because, for all $n \in N$:

$$\mu_A(m(n)) = \mu_A(x) \geq 0 = \mu_M(n), \quad \nu_A(m(n)) = \nu_A(x) \leq 1 = \nu_M(n),$$

and similarly for $\bar{h}$.

$$\begin{tikzcd}
N \arrow[r, shift left, "m"] \arrow[r, shift right, "h"'] & X \arrow[r, "f"] & Z
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
M \arrow[r, shift left, "\bar{m}"] \arrow[r, shift right, "\bar{h}"'] & A \arrow[r, "\bar{f}"] & C
\end{tikzcd}$$

(**a**) Monomorphism condition in $\mathbf{C_{R\text{-}M}}$ (**b**) Monomorphism condition in $\mathbf{C_{R\text{-}IFM}}$

**Figure 8.** Monomorphism condition in the categories of$R$-modules and intuitionistic fuzzy submodules.

Now, as shown in Figure 8b, $\bar{f} \circ \bar{m} = \bar{f} \circ \bar{h}$ because $f(x) = f(y)$, but $\bar{m} \neq \bar{h}$ since $x \neq y$. This contradicts the assumption that $\bar{f}$ is an IF$R$-monomorphism. Hence, $f$ must be an $R$-monomorphism. $\square$

**Example 7.** *Let $R = \mathbb{Z}$, $X = \mathbb{Z}_2$, $Y = \mathbb{Z}_2$, $Z = \mathbb{Z}_2 \times \mathbb{Z}_2$, and let $f : X \to Z$, $g : Y \to Z$ be the diagonal embeddings $f(x) = (x, x)$, $g(y) = (y, y)$. Define IFSMs:*

$$\mu_A(0) = 1,\ \mu_A(1) = 0.6, \quad \nu_A(0) = 0,\ \nu_A(1) = 0.3;$$

$$\mu_B(0) = 1,\ \mu_B(1) = 0.6, \quad \nu_B(0) = 0,\ \nu_B(1) = 0.3.$$

*Thus $A = B$. On $Z$, define*

$$\mu_C(z_1, z_2) = \mu_A(z_1) \wedge \mu_A(z_2), \qquad \nu_C(z_1, z_2) = \nu_A(z_1) \vee \nu_A(z_2).$$

*Then the induced $\bar{f} : A \to C$ and $\bar{g} : B \to C$ are IF $R$-homomorphisms. Indeed, for $x \in X$,*

$$\mu_C(f(x)) = \mu_C(x, x) = \mu_A(x) \wedge \mu_A(x) = \mu_A(x),$$

*and similarly $\nu_C(f(x)) = \nu_A(x)$, so the inequalities hold with equality; the same holds for $g$. Moreover, $\bar{f}$ is an IF monomorphism because its underlying $f$ is injective. The pullback $D$ exists by Theorem 4. By the theorem, $p_2 : D \to B$ (the projection onto $B$) must be an $R$-monomorphism. Indeed, the underlying pullback is*

$$X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\} = \{(0, 0), (1, 1)\},$$

*and $p_2$ restricted to this set, given by $(x, x) \mapsto x$, is injective. Hence the conclusion of Theorem 5 is verified.*

**Theorem 6.** *Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$, and $C = (\mu_C, \nu_C)$ be IFSMs of $R$-modules $X$, $Y$, and $Z$, respectively. Let $\bar{f} : A \to C$ and $\bar{g} : B \to C$ be IF$R$-homomorphisms corresponding to $R$-homomorphisms $f : X \to Z$ and $g : Y \to Z$, with $p_1$ and $p_2$ being the projections from $X \times_Z Y$ to $X$ and $Y$. If $D$ is a pullback in $\mathbf{C_{R\text{-}IFM}}$ and $\bar{f}$ is an IF$R$-epimorphism, then $f$ is an $R$-epimorphism and $p_2$ is an $R$-epimorphism.*

**Proof of Theorem 6.** Assume that $\bar{f}$ is an IF$R$-epimorphism but $f$ is not an $R$-epimorphism. Then, there exists some $z_0 \in Z$, such that $z_0 \notin \mathrm{Im}(f)$.

Let $N = Z/\mathrm{Im}(f)$ be the quotient module, and let $\pi : Z \to N$ be the natural projection map. Define the IFSM $M$ of $N$ as follows:

$$\mu_M(n) = 1, \quad \nu_M(n) = 0 \quad \text{for all } n \in N.$$

Now define two IF$R$-homomorphisms, $\bar{\pi}, \bar{0} : C \to M$, corresponding to the $R$-homomorphisms $\pi : Z \to N$ and $0 : Z \to N$ (the zero map), respectively, as shown in Figure 9. For any $x \in X$, we have:

$$\pi(f(x)) = 0 \text{ since } f(x) \in \mathrm{Im}(f), \quad 0(f(x)) = 0$$

$$X \xrightarrow{f} Z \underset{0}{\overset{\pi}{\rightrightarrows}} N \qquad\qquad A \xrightarrow{\bar{f}} C \underset{\bar{0}}{\overset{\bar{\pi}}{\rightrightarrows}} M$$

(**a**) Epimorphism condition in $\mathbf{C_{R\text{-}M}}$ (**b**) Epimorphism condition in $\mathbf{C_{R\text{-}IFM}}$

**Figure 9.** Monomorphism condition in the categories of $R$-modules and intuitionistic fuzzy submodules.

Therefore, $\pi \circ f = 0 \circ f$ as $R$-homomorphisms, which implies $\bar{\pi} \circ \bar{f} = \bar{0} \circ \bar{f}$ as IF$R$-homomorphisms.

However, $\pi \neq 0$ as $R$-homomorphisms (since $\pi(z_0) \neq 0$), so $\bar{\pi} \neq \bar{0}$ as IF$R$-homomorphisms.

This contradicts the assumption that $\bar{f}$ is an IF$R$-epimorphism. Hence, $f$ must be an $R$-epimorphism.

Since $f$ is an $R$-epimorphism, for any $y \in Y$, there exists some $x \in X$, such that $f(x) = g(y)$. Therefore, $(x, y) \in X \times_Z Y$ and $p_2(x, y) = y$, as shown in Figure 10.

This shows that $p_2$ is surjective and, hence, an $R$-epimorphism. This completes the proof of Theorem 6. □

$$\begin{tikzcd}
W \arrow[drr, bend left, "m_2"] \arrow[dr, dashed, "u"] \arrow[ddr, bend right, "m_1"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[drr, bend left, "\bar{m}_2"] \arrow[dr, dashed, "\bar{u}"] \arrow[ddr, bend right, "\bar{m}_1"'] & & \\
& D \arrow[r, "\bar{p}_2"] \arrow[d, "\bar{p}_1"'] & B \arrow[d, "\bar{g}"] \\
& A \arrow[r, "\bar{f}"'] & C
\end{tikzcd}$$

(**a**) Pullback for epimorphism in $\mathbf{C_{R\text{-}M}}$ (**b**) Pullback for epimorphism in $\mathbf{C_{R\text{-}IFM}}$

**Figure 10.** Pullback diagrams for epimorphism preservation in the categories of $R$-modules and intuitionistic fuzzy submodules.

**Example 8.** *Let $R = \mathbb{Z}$, $X = \mathbb{Z}_2$, $Y = \mathbb{Z}$, $Z = \mathbb{Z}_2$, with $f : X \to Z$ being the identity and $g : Y \to Z$ being the natural surjection $y \mapsto y \bmod 2$. Define IFSMs:*

$$\mu_A(0) = 1,\ \mu_A(1) = 0.7, \quad \nu_A(0) = 0,\ \nu_A(1) = 0.1;$$

$$\mu_B(y) = \begin{cases} 1, & y = 0, \\ 0.4, & y \neq 0, \end{cases} \quad \nu_B(y) = \begin{cases} 0, & y = 0, \\ 0.5, & y \neq 0. \end{cases}$$

*Let $\mu_C(z) = \max_{x \in f^{-1}(z)} \mu_A(x)$, $\nu_C(z) = \min_{x \in f^{-1}(z)} \nu_A(x)$, making $\bar{f}$ an IF epimorphism (since $f$ is surjective and the membership of $C$ is the image fuzzy set). The pullback $D$ in $\mathbf{C_{R\text{-}M}}$ is $X \times_Z Y = \{(0, even), (1, odd)\}$. By the theorem, $p_2$ is an $R$-epimorphism, which is true because every integer $y$ has a parity matching some $x$, so $p_2$ is surjective onto $Y$.*

### *4.3. Double Pullback Construction*

**Theorem 7.** *Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$, $C = (\mu_C, \nu_C)$ be IFSMs of $R$-modules $X$, $Y$, $Z$, respectively. Define $G = (\mu_G, \nu_G)$ as the IFSM of the $R$-module $Y \times_M Z$ by $\mu_G(y, z) = \mu_B(y) \wedge \mu_C(z), \nu_G(y, z) = \nu_B(y) \vee \nu_C(z)$, and let $\bar{p}_4 : G \to B$, $\bar{p}_2 : G \to C$ be the IF $R$-homomorphisms induced by the natural projections. Let $\bar{g} : A \to B$ be the IF $R$-homomorphism corresponding to the $R$-homomorphism $g : X \to Y$, and let $p_1$, $p_3$ be the projections from the pullback $X \times_Y (Y \times_M Z)$ to $Y \times_M Z$ and $X$ in the category of modules. Then, an IFSM $H$ on $X \times_Y (Y \times_M Z)$ is a pullback of $\bar{g}$ and $\bar{p}_4$ in $\mathbf{C_{R\text{-}IFM}}$ if and only if*

*(1) $X \times_Y (Y \times_M Z)$ is a pullback of $g$ and $p_4$ in $\mathbf{C_{R\text{-}M}}$;*

*(2) $H = (\mu_H, \nu_H)$ is an IFSM of $X \times_Y (Y \times_M Z)$ defined by $\mu_H(x, y, z) = \mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z), \nu_H(x, y, z) = \nu_A(x) \vee \nu_B(y) \vee \nu_C(z)$.*

**Proof of Theorem 7.** *Sufficiency:* By condition (1) , as shown in Figure 11, the underlying module $X \times_Y (Y \times_M Z)$ is a pullback of $g$ and $p_4$, and condition (2) provides the same form of fuzzy structure as in Theorem 4. For any $R$-module $W$ and homomorphisms $m_3 : W \to X$, $m_2 : W \to Y \times_M Z$ satisfying $g \circ m_3 = p_4 \circ m_2$, the universal property of the pullback yields a unique $u : W \to X \times_Y (Y \times_M Z)$ such that $p_3 \circ u = m_3$ and $p_1 \circ u = m_2$. Let $E$ be an IFSM on $W$, and let $\bar{m}_3 : E \to A$, $\bar{m}_2 : E \to G$ be IF $R$-homomorphisms satisfying $\bar{g} \circ \bar{m}_3 = \bar{p}_4 \circ \bar{m}_2$. Define $\bar{u} : E \to H$ with underlying map $u$, as shown in Figure 12. For any $w \in W$, write $u(w) = (x, y, z)$; then $x = m_3(w)$ and $(y, z) = m_2(w)$.

Since $\bar{m}_3$ is an IF $R$-homomorphism,

$$\mu_E(w) \leq \mu_A(m_3(w)) = \mu_A(x), \qquad \nu_E(w) \geq \nu_A(m_3(w)) = \nu_A(x).$$

Since $\bar{m}_2$ is an IF $R$-homomorphism,

$$\mu_E(w) \leq \mu_G(m_2(w)) = \mu_B(y) \wedge \mu_C(z), \qquad \nu_E(w) \geq \nu_G(m_2(w)) = \nu_B(y) \vee \nu_C(z).$$

From $\mu_E(w) \leq \mu_B(y) \wedge \mu_C(z)$ we obtain $\mu_E(w) \leq \mu_B(y)$ and $\mu_E(w) \leq \mu_C(z)$. Combining with $\mu_E(w) \leq \mu_A(x)$ gives

$$\mu_E(w) \leq \mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z).$$

By condition (2), the right-hand side is exactly $\mu_H(u(w))$; hence, $\mu_E(w) \leq \mu_H(u(w))$. For the non-membership part, $\nu_E(w) \geq \nu_A(x)$ and $\nu_E(w) \geq \nu_B(y) \vee \nu_C(z)$; therefore,

$$\nu_E(w) \geq \nu_A(x) \vee (\nu_B(y) \vee \nu_C(z)) = \nu_A(x) \vee \nu_B(y) \vee \nu_C(z) = \nu_H(u(w)).$$

Thus, $\bar{u}$ is an IF $R$-homomorphism; its uniqueness follows from the uniqueness of $u$ at the module level. Consequently, $H$ is a pullback of $\bar{g}$ and $\bar{p}_4$ in $\mathbf{C_{R\text{-}IFM}}$.

Necessity: Suppose that $H$ is a pullback of $\bar{g}$ and $\bar{p}_4$ in $\mathbf{C_{R\text{-}IFM}}$. By Corollary 1, the forgetful functor $U$ sends this pullback to a pullback in $\mathbf{C_{R\text{-}M}}$; hence, $U(H)$ is a pullback of $g$ and $p_4$. Pullbacks are unique up to isomorphism, so there exists an isomorphism, $\varphi : U(H) \to X \times_Y (Y \times_M Z)$, such that $p_1 \circ \varphi = U(\bar{p}_1)$ and $p_3 \circ \varphi = U(\bar{p}_3)$. Identifying $U(H)$ with $X \times_Y (Y \times_M Z)$ via $\varphi$ reveals that the projections match; this gives condition (1).

$$\begin{tikzcd}
W \arrow[dr, dashed, "u"] \arrow[drr, bend left, "\varepsilon"] \arrow[drrr, bend left, "m_1"] \arrow[ddrr, "m_2"] \arrow[ddr, bend right, "m_3"'] & & & \\
& X \times_Y (Y \times_M Z) \arrow[r, "p_1"] \arrow[d, "p_3"'] & Y \times_M Z \arrow[r, "p_2"] \arrow[d, "p_4"'] & Z \arrow[d, "f"] \\
& X \arrow[r, "g"'] & Y \arrow[r, "h"'] & M
\end{tikzcd}$$

**Figure 11.** Double Pullback diagrams in the categories of $R$-modules.

$$\begin{tikzcd}
E \arrow[dr, dashed, "u"] \arrow[drr, bend left, "\varepsilon"] \arrow[drrr, bend left, "m_1"] \arrow[ddrr, "m_2"] \arrow[ddr, bend right, "m_3"'] & & & \\
& H \arrow[r, "p_1"] \arrow[d, "p_3"'] & G \arrow[r, "p_2"] \arrow[d, "p_4"'] & C \arrow[d, "f"] \\
& A \arrow[r, "g"'] & B \arrow[r, "h"'] & F
\end{tikzcd}$$

**Figure 12.** Double Pullback diagrams in the categories of intuitionistic fuzzy submodules.

Now define $H^* = (\mu_{H^*}, \nu_{H^*})$ on $X \times_Y (Y \times_M Z)$ exactly as in condition (2):

$$\mu_{H^*}(x,y,z) = \mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z), \qquad \nu_{H^*}(x,y,z) = \nu_A(x) \vee \nu_B(y) \vee \nu_C(z).$$

A direct verification shows that $\bar{p}_1 : H^* \to G$ and $\bar{p}_3 : H^* \to A$ are IF $R$-homomorphisms: for any $(x,y,z)$,

$$\mu_{H^*}(x,y,z) = \mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z) \leq \mu_B(y) \wedge \mu_C(z) = \mu_G(p_1(x,y,z)),$$

$$\nu_{H^*}(x,y,z) = \nu_A(x) \vee \nu_B(y) \vee \nu_C(z) \geq \nu_B(y) \vee \nu_C(z) = \nu_G(p_1(x,y,z)),$$

and similarly $\mu_{H^*}(x,y,z) \le \mu_A(p_3(x,y,z))$, $\nu_{H^*}(x,y,z) \ge \nu_A(p_3(x,y,z))$. Moreover, $\bar{g} \circ \bar{p}_3 = \bar{p}_4 \circ \bar{p}_1$ holds by the underlying commutativity and the fuzzy structure.

Since $H$ is a pullback of $\bar{g}$ and $\bar{p}_4$, for the cone $(H^*, \bar{p}_3, \bar{p}_1)$ there exists a unique IF $R$-homomorphism, $\bar{u} : H^* \to H$, such that $\bar{p}_3 \circ \bar{u} = \bar{p}_3$ and $\bar{p}_1 \circ \bar{u} = \bar{p}_1$. Let $u$ be the underlying map of $\bar{u}$; then, $p_3 \circ u = p_3$ and $p_1 \circ u = p_1$. By the universal property of the pullback $X \times_Y (Y \times_M Z)$, the only map satisfying these two equations is the identity, so $u = \mathrm{id}_{X \times_Y (Y \times_M Z)}$.

Because $\bar{u}$ is an IF $R$-homomorphism, for any $(x,y,z) \in X \times_Y (Y \times_M Z)$, we have

$$\mu_{H^*}(x,y,z) \le \mu_H(u(x,y,z)) = \mu_H(x,y,z), \qquad \nu_{H^*}(x,y,z) \ge \nu_H(u(x,y,z)) = \nu_H(x,y,z).$$

On the other hand, $\bar{p}_1 : H \to G$ is an IF $R$-homomorphism, so

$$\mu_H(x,y,z) \le \mu_G(p_1(x,y,z)) = \mu_B(y) \wedge \mu_C(z),$$

and $\bar{p}_3 : H \to A$ is an IF $R$-homomorphism, yielding

$$\mu_H(x,y,z) \le \mu_A(p_3(x,y,z)) = \mu_A(x).$$

Combining these, $\mu_H(x,y,z) \le \mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z) = \mu_{H^*}(x,y,z)$. For the non-membership direction, from $\bar{p}_1$, we get $\nu_H(x,y,z) \ge \nu_G(p_1(x,y,z)) = \nu_B(y) \vee \nu_C(z)$; from $\bar{p}_3$, we get $\nu_H(x,y,z) \ge \nu_A(x)$. Hence,

$$\nu_H(x,y,z) \ge \nu_A(x) \vee \nu_B(y) \vee \nu_C(z) = \nu_{H^*}(x,y,z).$$

Thus, $\mu_H(x,y,z) = \mu_{H^*}(x,y,z)$ and $\nu_H(x,y,z) = \nu_{H^*}(x,y,z)$, which is exactly condition (2). $\square$

**Example 9.** *Let $X = \mathbb{Z}$, $Y = \mathbb{Z}_4$, $Z = \mathbb{Z}_2$, $M = \{0\}$. Define $g : X \to Y$ by $g(x) = x \bmod 4$, with $h : Y \to M$ and $f : Z \to M$ being the zero maps. Set IFSMs:*

$$\mu_A(x) = \begin{cases} 1, & x = 0, \\ 0.6, & x \neq 0, \end{cases} \quad \nu_A(x) = \begin{cases} 0, & x = 0, \\ 0.3, & x \neq 0; \end{cases}$$

$$\mu_B(y) = \begin{cases} 1, & y = 0, \\ 0.8, & y \neq 0, \end{cases} \quad \nu_B(y) = \begin{cases} 0, & y = 0, \\ 0.1, & y \neq 0; \end{cases}$$

$$\mu_C(z) = \begin{cases} 1, & z = 0, \\ 0.5, & z = 1, \end{cases} \quad \nu_C(z) = \begin{cases} 0, & z = 0, \\ 0.5, & z = 1. \end{cases}$$

*Then $G$ on $Y \times_M Z = Y \times Z$ has $\mu_G(y,z) = \mu_B(y) \wedge \mu_C(z)$, $\nu_G(y,z) = \nu_B(y) \vee \nu_C(z)$. The pullback $X \times_Y (Y \times_M Z)$ consists of triples $(x,y,z)$ with $g(x) = y$ and $f(z) = h(y)$; here $y = x \bmod 4$, $z$ arbitrary. Condition (1) of Theorem 7 holds. By Theorem 7, the IFSM $H$ on this set defined by*

$$\mu_H(x,y,z) = \mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z), \quad \nu_H(x,y,z) = \nu_A(x) \vee \nu_B(y) \vee \nu_C(z)$$

*is the pullback of $\bar{g}$ and $\bar{p}_4$. For a concrete element, take $x = 1, y = 1, z = 0$; then*

$$\mu_H(1,1,0) = 0.6 \wedge 0.8 \wedge 1 = 0.6, \qquad \nu_H(1,1,0) = 0.3 \vee 0.1 \vee 0 = 0.4.$$

**Theorem 8.** *Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$, $C = (\mu_C, \nu_C)$ and $F = (\mu_F, \nu_F)$ be IFSMs of $R$-modules $X$, $Y$, $Z$ and $M$, respectively. Let $\bar{f} : C \to F$, $\bar{g} : A \to B$, and $\bar{h} : B \to F$ be IF $R$-*

*homomorphisms corresponding to the R-homomorphisms* $f : Z \to M$, $g : X \to Y$, *and* $h : Y \to M$. *Define* $G = (\mu_G, \nu_G)$ *on* $Y \times_M Z$ *by* $\mu_G(y,z) = \mu_B(y) \wedge \mu_C(z), \nu_G(y,z) = \nu_B(y) \vee \nu_C(z)$, *and let* $\bar{p}_4 : G \to B$, $\bar{p}_2 : G \to C$ *be the IF R-homomorphisms induced by the natural projections. Define* $H = (\mu_H, \nu_H)$ *on* $X \times_Y (Y \times_M Z)$. *Then, G is a pullback of* $\bar{h}$, *and* $\bar{f}$, *and H is a pullback of* $\bar{g}$ *and* $\bar{p}_4$ *in* $\mathbf{C_{R\text{-}IFM}}$ *if and only if*

(1) $Y \times_M Z$ *is a pullback of h and f in* $\mathbf{C_{R\text{-}M}}$, *and* $X \times_Y (Y \times_M Z)$ *is a pullback of g and* $p_4$ *in* $\mathbf{C_{R\text{-}M}}$;

(2) $\mu_H(x,y,z) = \mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z)$ *and* $\nu_H(x,y,z) = \nu_A(x) \vee \nu_B(y) \vee \nu_C(z)$.

**Proof of Theorem8.** Sufficiency: First, by condition (1), $Y \times_M Z$ is a pullback of $h$ and $f$ in $\mathbf{C_{R\text{-}M}}$, and the definition of $G$ satisfies

$$\mu_G(y,z) = \mu_B(y) \wedge \mu_C(z), \quad \nu_G(y,z) = \nu_B(y) \vee \nu_C(z).$$

Thus, by Theorem 4, $G$ is a pullback of $\bar{h}$ and $\bar{f}$ in $\mathbf{C_{R\text{-}IFM}}$.

Now, we prove that $H$ is a pullback of $\bar{g}$ and $\bar{p}_4$. Condition (1), as shown in Figure 13, also states that $X \times_Y (Y \times_M Z)$ is a pullback of $g$ and $p_4$ in $\mathbf{C_{R\text{-}M}}$. Let $W$ be any $R$-module and let $m_3 : W \to X$, $m_2 : W \to Y \times_M Z$ be $R$-homomorphisms satisfying $g \circ m_3 = p_4 \circ m_2$. By the pullback property, there exists a unique $R$-homomorphism, $u : W \to X \times_Y (Y \times_M Z)$, such that

$$p_3 \circ u = m_3, \qquad p_1 \circ u = m_2.$$

Now, let $E$ be an IFSM on $W$, and let $\bar{m}_3 : E \to A$, $\bar{m}_2 : E \to G$ be IF $R$-homomorphisms satisfying $\bar{g} \circ \bar{m}_3 = \bar{p}_4 \circ \bar{m}_2$. Define $\bar{u} : E \to H$, whose underlying map is $u$; we show $\bar{u}$ is an IF $R$-homomorphism, as shown in Figure 14.

For any $w \in W$, write $u(w) = (x,y,z)$. Then, $x = m_3(w)$ and $(y,z) = m_2(w)$. Since $\bar{m}_3$ is an IF $R$-homomorphism,

$$\mu_E(w) \le \mu_A(m_3(w)) = \mu_A(x), \qquad \nu_E(w) \ge \nu_A(m_3(w)) = \nu_A(x).$$

Since $\bar{m}_2$ is an IF $R$-homomorphism,

$$\mu_E(w) \le \mu_G(m_2(w)) = \mu_B(y) \wedge \mu_C(z), \qquad \nu_E(w) \ge \nu_G(m_2(w)) = \nu_B(y) \vee \nu_C(z).$$

From $\mu_E(w) \le \mu_B(y) \wedge \mu_C(z)$, we obtain $\mu_E(w) \le \mu_B(y)$ and $\mu_E(w) \le \mu_C(z)$. Combining with $\mu_E(w) \le \mu_A(x)$ gives

$$\mu_E(w) \le \mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z).$$

The right-hand side is exactly $\mu_H(u(w))$ by condition (2), so $\mu_E(w) \le \mu_H(u(w))$.

For the non-membership part, $\nu_E(w) \ge \nu_A(x)$ and $\nu_E(w) \ge \nu_B(y) \vee \nu_C(z)$; therefore,

$$\nu_E(w) \ge \nu_A(x) \vee (\nu_B(y) \vee \nu_C(z)) = \nu_A(x) \vee \nu_B(y) \vee \nu_C(z) = \nu_H(u(w)).$$

Hence, $\bar{u}$ is an IF $R$-homomorphism. Its uniqueness follows from the uniqueness of $u$ at the module level. Consequently, $H$ is a pullback of $\bar{g}$ and $\bar{p}_4$ in $\mathbf{C_{R\text{-}IFM}}$.

$$\begin{tikzcd}
W \arrow[dr, dashed, "u"] \arrow[drr, bend left, "\varepsilon"] \arrow[drrr, bend left, "m_1"] \arrow[ddrr, "m_2"] \arrow[ddr, bend right, "m_3"'] & & & \\
& X \times_Y (Y \times_M Z) \arrow[r, "p_1"] \arrow[d, "p_3"'] & Y \times_M Z \arrow[r, "p_2"] \arrow[d, "p_4"'] & Z \arrow[d, "f"] \\
& X \arrow[r, "g"'] & Y \arrow[r, "h"'] & M
\end{tikzcd}$$

**Figure 13.** Double pullback construction in the categories of $R$-modules.

$$
\begin{tikzcd}
E \arrow[drrr, bend left, "m_1"] \arrow[drr, bend left, "\varepsilon"'] \arrow[dr, dashed, "u"] \arrow[ddrr, "m_2"] \arrow[ddr, bend right, "m_3"'] & & & \\
& H \arrow[r, "p_1"] \arrow[d, "p_3"'] & G \arrow[r, "p_2"] \arrow[d, "p_4"'] & C \arrow[d, "f"] \\
& A \arrow[r, "g"'] & B \arrow[r, "h"'] & F
\end{tikzcd}
$$

**Figure 14.** Double pullback construction in the categories of intuitionistic fuzzy submodules.

Necessity: Assume that $G$ is a pullback of $\bar{h}$ and $\bar{f}$, and $H$ is a pullback of $\bar{g}$ and $\bar{p}_4$ in $\mathbf{C_{R\text{-}IFM}}$. By Corollary 1, the forgetful functor $U$ sends these pullbacks to pullbacks in $\mathbf{C_{R\text{-}M}}$; therefore, $U(G) \cong Y \times_M Z$ is a pullback of $h$ and $f$, while $U(H) \cong X \times_Y (Y \times_M Z)$ is a pullback of $g$ and $p_4$. This establishes condition (1).

Now we verify the membership condition (2). Since $G$ is a pullback of $\bar{h}$ and $\bar{f}$, applying the necessity part of Theorem 4 to $G$ immediately yields

$$\mu_G(y,z) = \mu_B(y) \wedge \mu_C(z), \quad \nu_G(y,z) = \nu_B(y) \vee \nu_C(z),$$

which is already the definition.

It remains to prove that $H$ must have the membership functions as in condition (2). Because $U(H)$ is a pullback of $g$ and $p_4$, and the underlying module of $G$ is $Y \times_M Z$, the morphisms $\bar{p}_4 : G \to B$ and $\bar{g} : A \to B$ are in place. Define $H^*$ on $X \times_Y (Y \times_M Z)$ exactly as in condition (2):

$$\mu_{H^*}(x,y,z) = \mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z), \quad \nu_{H^*}(x,y,z) = \nu_A(x) \vee \nu_B(y) \vee \nu_C(z).$$

We verify that $\bar{p}_1 : H^* \to G$ is an IF $R$-homomorphism: for any $(x,y,z)$,

$$\mu_{H^*}(x,y,z) \leq \mu_B(y) \wedge \mu_C(z) = \mu_G(p_1(x,y,z)),$$

$$\nu_{H^*}(x,y,z) \geq \nu_B(y) \vee \nu_C(z) = \nu_G(p_1(x,y,z)).$$

Similarly, $\bar{p}_3 : H^* \to A$ is an IF $R$-homomorphism. One easily checks that $\bar{g} \circ \bar{p}_3 = \bar{p}_4 \circ \bar{p}_1$.

Since $H$ is a pullback of $\bar{g}$ and $\bar{p}_4$, for the cone $(H^*, \bar{p}_3, \bar{p}_1)$, there exists a unique IF $R$-homomorphism, $\bar{u} : H^* \to H$, such that $\bar{p}_3 \circ \bar{u} = \bar{p}_3$ and $\bar{p}_1 \circ \bar{u} = \bar{p}_1$. Let $u$ be its underlying map; then, $p_3 \circ u = p_3$, and $p_1 \circ u = p_1$. By the universal property of the pullback $X \times_Y (Y \times_M Z)$, the only such map is the identity, so $u = \mathrm{id}_{X \times_Y (Y \times_M Z)}$.

Because $\bar{u}$ is an IF $R$-homomorphism, for any $(x,y,z)$ we have

$$\mu_{H^*}(x,y,z) \leq \mu_H(u(x,y,z)) = \mu_H(x,y,z), \qquad \nu_{H^*}(x,y,z) \geq \nu_H(u(x,y,z)) = \nu_H(x,y,z).$$

On the other hand, $\bar{p}_1 : H \to G$ is an IF $R$-homomorphism, so

$$\mu_H(x,y,z) \leq \mu_G(p_1(x,y,z)) = \mu_B(y) \wedge \mu_C(z),$$

$$\nu_H(x,y,z) \geq \nu_G(p_1(x,y,z)) = \nu_B(y) \vee \nu_C(z).$$

Also, $\bar{p}_3 : H \to A$ is an IF $R$-homomorphism, giving

$$\mu_H(x,y,z) \leq \mu_A(p_3(x,y,z)) = \mu_A(x), \qquad \nu_H(x,y,z) \geq \nu_A(p_3(x,y,z)) = \nu_A(x).$$

Putting these together reveals,

$$\mu_H(x,y,z) \leq \mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z) = \mu_{H^*}(x,y,z),$$

$$\nu_H(x,y,z) \geq \nu_A(x) \vee \nu_B(y) \vee \nu_C(z) = \nu_{H^*}(x,y,z).$$

Therefore, $\mu_H(x,y,z) = \mu_{H^*}(x,y,z)$ and $\nu_H(x,y,z) = \nu_{H^*}(x,y,z)$, i.e., condition (2) holds. □

**Example 10.** *In the setting of Example 9, define $F$ as the IFSM on $M = \{0\}$ with $\mu_F(0) = 1$, $\nu_F(0) = 0$. The maps $\bar{h} : B \to F$, $\bar{f} : C \to F$ are trivial. Condition (1) of Theorem 8 is satisfied, and the intermediate pullback $G$ is a pullback of $\bar{h}$ and $\bar{f}$ by Theorem 4. Consequently, Theorem 8 asserts that $H$ is the pullback of the entire diagram, i.e., the pullback of $\bar{g}$ and $\bar{p}_4$, and simultaneously the pullback of the composed morphisms. This illustrates the transitivity of pullbacks in the intuitionistic fuzzy setting.*

### *4.4. Pullback Constructions: Theorem and Classification*

**Theorem 9.** *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSMs of $R$-modules $X$ and $Y$, respectively. Let $\bar{f} : A \to B$ be an intuitionistic fuzzy $R$-homomorphism corresponding to the $R$-homomorphism $f : X \to Y$. Then, $A$ is a pullback of the diagram in the category $\mathbf{C_{R\text{-}IFM}}$ formed by the two intuitionistic fuzzy morphisms $\bar{f} : A \to B$ and $\bar{f} : A \to B$, if and only if the underlying homomorphism $f$ is an $R$-monomorphism.*

**Proof of Theorem 9.** Sufficiency: Assume $f$ is a monomorphism. We show that $A$ satisfies the universal property of the pullback. Consider any object $E$ in $\mathbf{C_{R\text{-}IFM}}$ and morphisms $\bar{m}_1, \bar{m}_2 : E \to A$, such that $\bar{f} \circ \bar{m}_1 = \bar{f} \circ \bar{m}_2$. Since $\bar{f}$ corresponds to $f$ and $f$ is a monomorphism, as shown in Figure 15a, the underlying $R$-homomorphisms satisfy $m_1 = m_2$.

Now, define $\bar{u} = \bar{m}_1 = \bar{m}_2$. Then, clearly:

$$\bar{1}_X \circ \bar{u} = \bar{m}_1, \qquad \bar{1}_X \circ \bar{u} = \bar{m}_2.$$

Moreover, $\bar{u}$ is unique with this property. Therefore, $A$ is indeed the pullback in $\mathbf{C_{R\text{-}IFM}}$.

Necessity: Assume that $A$ is a pullback in $\mathbf{C_{R\text{-}IFM}}$, as shown in Figure 15b. For any object $E$ and morphisms $\bar{m}_1, \bar{m}_2 : E \to A$ with $\bar{f} \circ \bar{m}_1 = \bar{f} \circ \bar{m}_2$, the pullback property guarantees a unique morphism, $\bar{u} : E \to A$, such that $\bar{m}_1 = \bar{1}_X \circ \bar{u}$, and $\bar{m}_2 = \bar{1}_X \circ \bar{u}$. Since $\bar{1}_X$ is the identity morphism, this implies $\bar{m}_1 = \bar{u} = \bar{m}_2$. Hence, $\bar{f}$ is a monomorphism in $\mathbf{C_{R\text{-}IFM}}$.

To see that $f$ is an $R$-monomorphism, let $h_1, h_2 : W \to X$ be $R$-homomorphisms with $f \circ h_1 = f \circ h_2$. Equip $W$ with the IFSM $E = (\mu_E, \nu_E)$, defined by

$$\mu_E(w) = \mu_A(h_1(w)) \wedge \mu_A(h_2(w)), \qquad \nu_E(w) = \nu_A(h_1(w)) \vee \nu_A(h_2(w)).$$

For any $w \in W$,

$$\mu_E(w) \leq \mu_A(h_i(w)), \quad \nu_E(w) \geq \nu_A(h_i(w)) \qquad (i = 1, 2),$$

hence the maps $\bar{h}_1, \bar{h}_2 : E \to A$ with underlying $R$-homomorphisms $h_1, h_2$ are IF $R$-homomorphisms. Moreover,

$$\bar{f} \circ \bar{h}_1 = \bar{f} \circ \bar{h}_2$$

because their underlying maps coincide and the fuzzy conditions are automatically satisfied. Since $\bar{f}$ is a monomorphism, we obtain $\bar{h}_1 = \bar{h}_2$, and in particular $h_1 = h_2$ as $R$-homomorphisms. Thus, $f$ is an $R$-monomorphism. □

$$\begin{tikzcd}
W \arrow[dr, dashed, "u"] \arrow[drr, bend left, "m_2"] \arrow[ddr, bend right, "m_1"'] & & \\
& X \arrow[r, "1_X"] \arrow[d, "1_X"'] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

(**a**) Pullback for equaliser in $\mathbf{C_{R\text{-}M}}$

$$\begin{tikzcd}
E \arrow[dr, dashed, "\bar{u}"] \arrow[drr, bend left, "\bar{m}_2"] \arrow[ddr, bend right, "\bar{m}_1"'] & & \\
& A \arrow[r, "\bar{1}_X"] \arrow[d, "\bar{1}_X"'] & A \arrow[d, "\bar{f}"] \\
& A \arrow[r, "\bar{f}"'] & B
\end{tikzcd}$$

(**b**) Pullback for equaliser in $\mathbf{C_{R\text{-}IFM}}$

**Figure 15.** Pullback diagrams for equaliser-type pullback in the categories of $R$-modules and intuitionistic fuzzy submodules.

**Example 11.** *Let $R = \mathbb{Z}$, $X = \mathbb{Z}$, $Y = \mathbb{Z}$. Define the $R$-monomorphism $f : X \to Y$ by $f(n) = 2n$. Define intuitionistic fuzzy submodules $A$ on $X$ and $B$ on $Y$ as follows:*

$$\mu_A(n) = \begin{cases} 1, & n = 0, \\ 0.6, & n \neq 0, \end{cases} \qquad \nu_A(n) = \begin{cases} 0, & n = 0, \\ 0.3, & n \neq 0, \end{cases}$$

$$\mu_B(m) = \begin{cases} 1, & m = 0, \\ 0.6, & m \neq 0,\ m \text{ even}, \\ 0, & m \text{ odd}, \end{cases} \qquad \nu_B(m) = \begin{cases} 0, & m = 0, \\ 0.3, & m \neq 0,\ m \text{ even}, \\ 1, & m \text{ odd}. \end{cases}$$

*It is straightforward to verify that $A$ and $B$ satisfy all conditions of an intuitionistic fuzzy submodule. For the map $f$ we have:*

(1) $n = 0$: $f(0) = 0$, $\mu_B(0) = 1 \geq \mu_A(0) = 1$, $\nu_B(0) = 0 \leq \nu_A(0) = 0$.
(2) $n \neq 0$: $f(n) = 2n$ *is a non-zero even integer, so* $\mu_B(2n) = 0.6 \geq \mu_A(n) = 0.6$ *and* $\nu_B(2n) = 0.3 \leq \nu_A(n) = 0.3$.

*Thus $\bar{f} : A \to B$ is an intuitionistic fuzzy $R$-homomorphism.*

*Now consider the pullback of $\bar{f}$ with itself. The underlying set of the pullback object $P$ is*

$$\{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}.$$

*Because $f$ is injective, $f(x_1) = f(x_2)$ implies $x_1 = x_2$, so $P \cong X$ and the two projections act as the diagonal map. The fuzzy structure on $P$ is given by the pullback of the product:*

$$\mu_P(x, x) = \mu_A(x) \wedge \mu_A(x) = \mu_A(x), \qquad \nu_P(x, x) = \nu_A(x) \vee \nu_A(x) = \nu_A(x).$$

*Therefore, $P$ is isomorphic to $A$ as an intuitionistic fuzzy $R$-module; that is, the pullback of $\bar{f}$ with itself is exactly $A$. This example confirms that when $f$ is a monomorphism, the pullback coincides with the domain, as stated in the corresponding theorem.*

**Property 2.** *Let $X$ and $Y$ be $R$-modules, and consider the module $X + Y$ with the canonical inclusion maps $i_X : X \to X + Y$ and $i_Y : Y \to X + Y$. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSMs of $X$ and $Y$, respectively. Define an IFSM $C = (\mu_C, \nu_C)$ on $X + Y$ by*

$$\mu_C(t) = \sup_{\substack{t = x + y \\ x \in X, y \in Y}} (\mu_A(x) \wedge \mu_B(y)), \quad \nu_C(t) = \inf_{\substack{t = x + y \\ x \in X, y \in Y}} (\nu_A(x) \vee \nu_B(y)).$$

*Let $\bar{i}_X : A \to C$ and $\bar{i}_Y : B \to C$ be the IF R-homomorphisms induced by the inclusions $i_X$ and $i_Y$, respectively. Then, the object D, defined as the IFSM of the pullback module $X \cap Y$ by*

$$\mu_D(t) = \mu_A(t) \wedge \mu_B(t), \quad \nu_D(t) = \nu_A(t) \vee \nu_B(t) \quad \textit{for } t \in X \cap Y,$$

*is a pullback of $\bar{i}_X$ and $\bar{i}_Y$ in $\mathbf{C_{R\text{-}IFM}}$.*

**Proof of Property 2.** We verify that the given construction satisfies the two conditions of Theorem 4.

As established in Example 5, in $\mathbf{C_{R\text{-}M}}$, the module $X \cap Y$ together with the inclusion maps $p_1 : X \cap Y \hookrightarrow X$, and $p_2 : X \cap Y \hookrightarrow Y$ forms a pullback of the inclusions $i_X : X \hookrightarrow X + Y$ and $i_Y : Y \hookrightarrow X + Y$. As shown in Figure 16a, this satisfies condition (1) of Theorem 4.

Moreover, the membership functions of $D$ on $X \cap Y$ are explicitly given by

$$\mu_D(t) = \mu_A(t) \wedge \mu_B(t), \qquad \nu_D(t) = \nu_A(t) \vee \nu_B(t).$$

This matches the form required for the pullback object in condition (2) of Theorem 4.

Since both conditions of Theorem 4 are met, we conclude that $D$ is a pullback in $\mathbf{C_{R\text{-}IFM}}$, as shown in Figure 16b. □

$$\begin{tikzcd}
W \arrow[drr, bend left, "m_2"] \arrow[dr, dashed, "u"] \arrow[ddr, bend right, "m_1"'] & & \\
& X \cap Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "i_Y"] \\
& X \arrow[r, "i_X"'] & X + Y
\end{tikzcd}$$

(**a**) Pullback for direct sum in $\mathbf{C_{R\text{-}M}}$

$$\begin{tikzcd}
E \arrow[drr, bend left, "\bar{m}_2"] \arrow[dr, dashed, "\bar{u}"] \arrow[ddr, bend right, "\bar{m}_1"'] & & \\
& D \arrow[r, "\bar{p}_2"] \arrow[d, "\bar{p}_1"'] & B \arrow[d, "\bar{i}_Y"] \\
& A \arrow[r, "\bar{i}_X"'] & C
\end{tikzcd}$$

(**b**) Pullback for direct sum in $\mathbf{C_{R\text{-}IFM}}$

**Figure 16.** Pullback diagrams for direct sum type pullback in the categories of $R$-modules and intuitionistic fuzzy submodules.

**Example 12.** *Let $R = \mathbb{Z}$, $X = 2\mathbb{Z}$, $Y = 3\mathbb{Z}$ . Then, $X + Y = \mathbb{Z}$ and $X \cap Y = 6\mathbb{Z}$. Define intuitionistic fuzzy submodules A on X and B on Y by*

$$\mu_A(2k) = \frac{1}{2+|k|}, \quad \nu_A(2k) = 1 - \mu_A(2k); \qquad \mu_B(3k) = \frac{1}{2+3|k|}, \quad \nu_B(3k) = 1 - \mu_B(3k).$$

*According to the property, the IFSM C on $X + Y = \mathbb{Z}$ is given by*

$$\mu_C(t) = \sup_{\substack{t=x+y \\ x\in 2\mathbb{Z},\, y\in 3\mathbb{Z}}} (\mu_A(x) \wedge \mu_B(y)), \quad \nu_C(t) = \inf_{\substack{t=x+y \\ x\in 2\mathbb{Z},\, y\in 3\mathbb{Z}}} (\nu_A(x) \vee \nu_B(y)).$$

*For instance, $t = 1$ can be expressed as $1 = (-2) + 3$ (with $x = -2, y = 3$)etc. The supremum in $\mu_C(1)$ is taken over all such decompositions. The IFSM D on $X \cap Y = 6\mathbb{Z}$ is defined by*

$$\mu_D(t) = \mu_A(t) \wedge \mu_B(t), \quad \nu_D(t) = \nu_A(t) \vee \nu_B(t), \qquad t \in 6\mathbb{Z}.$$

*For $t = 6k$,*

$$\mu_D(6k) = \frac{1}{2+|k|} \wedge \frac{1}{2+3|k|}, \qquad \nu_D(6k) = \left(\frac{1}{2+|k|}\right) \vee \left(1 - \frac{1}{2+|3k|}\right).$$

*The inclusion maps $i_X : X \to X + Y$, $i_Y : Y \to X + Y$ induce IF R-homomorphisms $\bar{i}_X : A \to C$, $\bar{i}_Y : B \to C$ because, e.g., for $x \in 2\mathbb{Z}$,*

$$\mu_A(x) = \mu_A(x) \wedge \mu_B(0) \leq \sup_{x=x+0} (\mu_A(x) \wedge \mu_B(0)) = \mu_C(x),$$

*and similarly for $\nu$. The underlying pullback of $i_X$ and $i_Y$ in $\mathbf{C_{R\text{-}M}}$ is exactly $X \cap Y = 6\mathbb{Z}$, and the fuzzy structure on D matches condition (ii) of Theorem 4. Hence, D is the pullback of $\bar{i}_X$ and $\bar{i}_Y$ in $\mathbf{C_{R\text{-}IFM}}$, as asserted.*

**Property 3.** *Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$, $C = (1, 0)$ and $D = (\mu_D, \nu_D)$ be IFSMs of R-modules X, Y, T and $X \times Y$, respectively. If T is a terminal object in $\mathbf{C_{R\text{-}M}}$, let $f : X \to T$ and $g : Y \to T$ be the unique R-homomorphisms, and let $\bar{f} : A \to C$, $\bar{g} : B \to C$ be the corresponding IF R-homomorphisms. Then, D is a pullback of $\bar{f}$ and $\bar{g}$ in $\mathbf{C_{R\text{-}IFM}}$, where $\mu_D(x, y) = \mu_A(x) \wedge \mu_B(y)$ and $\nu_D(x, y) = \nu_A(x) \vee \nu_B(y)$.*

**Proof of Property 3.** In $\mathbf{C_{R\text{-}M}}$, since $T$ is terminal, there exist unique homomorphisms, $f : X \to T$ and $g : Y \to T$. The pullback of $f$ and $g$, as shown in Figure 17a, is precisely the product $X \times Y$ with the canonical projections $p_1 : X \times Y \to X$, $p_2 : X \times Y \to Y$ (the commutativity $f \circ p_1 = g \circ p_2$ holds automatically because any two maps into a terminal object coincide). This satisfies condition (1) of Theorem 4.

Moreover, $D$ is defined on $X \times Y$ by

$$\mu_D(x, y) = \mu_A(x) \wedge \mu_B(y), \qquad \nu_D(x, y) = \nu_A(x) \vee \nu_B(y),$$

which is exactly the form required in condition (2) of Theorem 4. Therefore, by Theorem 4, $D$ is a pullback of $\bar{f}$ and $\bar{g}$ in $\mathbf{C_{R\text{-}IFM}}$, as shown in Figure 17b. □

$$\begin{tikzcd}
W \arrow[dr, dashed, "u"] \arrow[drr, bend left, "m_2"] \arrow[ddr, bend right, "m_1"'] & & \\
& X \times Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & T
\end{tikzcd}$$

(**a**) Pullback for product in $\mathbf{C_{R\text{-}M}}$

$$\begin{tikzcd}
E \arrow[dr, dashed, "\bar{u}"] \arrow[drr, bend left, "\bar{m}_2"] \arrow[ddr, bend right, "\bar{m}_1"'] & & \\
& D \arrow[r, "\bar{p}_2"] \arrow[d, "\bar{p}_1"'] & B \arrow[d, "\bar{g}"] \\
& A \arrow[r, "\bar{f}"'] & C
\end{tikzcd}$$

(**b**) Pullback for product in $\mathbf{C_{R\text{-}IFM}}$

**Figure 17.** Pullback diagrams for product type pullback in the categories of $R$-modules and intuitionistic fuzzy submodules.

**Example 13.** *Let $R = \mathbb{Z}$ and $X = Y = \mathbb{Z}$. Define intuitionistic fuzzy submodules A and B on $\mathbb{Z}$ by*

$$\mu_A(n) = \begin{cases} 1, & n = 0, \\ 0.6, & n \neq 0 \text{ and } n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \qquad \nu_A(n) = \begin{cases} 0, & n = 0, \\ 0, & n \neq 0 \text{ and } n \text{ even}, \\ 0.6, & n \text{ odd}, \end{cases}$$

$$\mu_B(n) = \begin{cases} 1, & n = 0, \\ 0.6, & n \neq 0 \text{ and } n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \qquad \nu_B(n) = \begin{cases} 0, & n = 0, \\ 0, & n \neq 0 \text{ and } n \text{ even}, \\ 0.6, & n \text{ odd}. \end{cases}$$

*Let $C = (\mu_C, \nu_C)$ be the intuitionistic fuzzy submodule on the zero module $\{0\}$ given by $\mu_C(0) = 1$, $\nu_C(0) = 0$; this is the terminal object in the category $\mathbf{C_{R\text{-}IFM}}$. Define $D$ on $X \times Y$ by the fibre product over $C$:*

$$\mu_D(x,y) = \mu_A(x) \wedge \mu_B(y), \qquad \nu_D(x,y) = \nu_A(x) \vee \nu_B(y).$$

*The explicit form of $D$ is*

$$D(x,y) = \begin{cases} (1,0), & x = 0 \text{ and } y = 0, \\ (0.6,0), & x \text{ and } y \text{ are both even, but not both zero,} \\ (0,0.6), & \text{otherwise (i.e. at least one of } x, y \text{ is odd).} \end{cases}$$

*Then $D$ is an intuitionistic fuzzy submodule of $X \times Y$, and it is the pullback of the unique morphisms $A \to C$ and $B \to C$ in $\mathbf{C_{R\text{-}IFM}}$. This example confirms that the pullback over the terminal object coincides with the direct product equipped with the meet–join fuzzy structure.*

**Property 4.** *Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$, $C = (\mu_C, \nu_C)$, and $D = (\mu_D, \nu_D)$ be IFSMs of the $R$-modules $X$, $T$, $Z$, and $P = \operatorname{Ker} f \times T$, respectively. Let $\bar{f} : A \to C$ and $\bar{g} : B \to C$ be intuitionistic fuzzy $R$-homomorphisms corresponding to the $R$-homomorphisms $f : X \to Z$ and $g : T \to Z$. If $T$ is a terminal object in $\mathbf{C_{R\text{-}M}}$, then $D$ is a pullback in $\mathbf{C_{R\text{-}IFM}}$, where, for $(x,t) \in P$,*

$$\mu_D(x,t) = \mu_A(x) \wedge \mu_B(t), \quad \nu_D(x,t) = \nu_A(x) \vee \nu_B(t).$$

**Proof of Property 4.** In $\mathbf{C_{R\text{-}M}}$, since $T$ is terminal, the unique homomorphism $g : T \to Z$ is the zero map. The pullback of $f$ and $g$ is, therefore,

$$P = \{(x,t) \in X \times T \mid f(x) = g(t) = 0\} = \operatorname{Ker} f \times T,$$

with canonical projections $p_1 : P \to X$, $p_2 : P \to T$. This establishes condition (1) of Theorem 4, as shown in Figure 18a.

The IFSM $D$ is defined on $P$ exactly by

$$\mu_D(x,t) = \mu_A(x) \wedge \mu_B(t), \qquad \nu_D(x,t) = \nu_A(x) \vee \nu_B(t),$$

which coincides with condition (2) of Theorem 4. Hence, by Theorem 4, $D$ together with the induced projection IF $R$-homomorphisms is a pullback of $\bar{f}$ and $\bar{g}$ in $\mathbf{C_{R\text{-}IFM}}$, as shown in Figure 18b. $\square$

$$\begin{tikzcd} W \arrow[dr, dashed, "u"] \arrow[drr, bend left, "m_2"] \arrow[ddr, bend right, "m_1"'] & & \\ & \operatorname{Ker} f \times T \arrow[r, "p_2"] \arrow[d, "p_1"'] & T \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd}$$

(**a**) Pullback for kernel-terminal in $\mathbf{C_{R\text{-}M}}$

$$\begin{tikzcd} E \arrow[dr, dashed, "\bar{u}"] \arrow[drr, bend left, "\bar{m}_2"] \arrow[ddr, bend right, "\bar{m}_1"'] & & \\ & D \arrow[r, "\bar{p}_2"] \arrow[d, "\bar{p}_1"'] & B \arrow[d, "\bar{g}"] \\ & A \arrow[r, "\bar{f}"'] & C \end{tikzcd}$$

(**b**) Pullback for kernel-terminal in $\mathbf{C_{R\text{-}IFM}}$

**Figure 18.** Pullback diagrams for kernel-terminal type pullback in the categories of $R$-modules and intuitionistic fuzzy submodules.

**Example 14.** *Let $R = \mathbb{Z}$, $X = \mathbb{Z}$, $Z = \mathbb{Z}/2\mathbb{Z}$, and let $f : X \to Z$ be the natural projection $f(n) = n \bmod 2$. Then, $\operatorname{Ker} f = 2\mathbb{Z}$. Let $T = \{0\}$ be a terminal object in $\mathbf{C_{R\text{-}M}}$.*

*Define the following intuitionistic fuzzy submodules:*

$$\mu_A(n) = \begin{cases} 1, & n \text{ is even}, \\ 0.4, & n \text{ is odd}, \end{cases} \qquad \nu_A(n) = \begin{cases} 0, & n \text{ is even}, \\ 0.5, & n \text{ is odd}, \end{cases}$$

$$\mu_B(0) = 1, \qquad \nu_B(0) = 0,$$

$$\mu_C(0) = 1, \qquad \nu_C(0) = 0, \qquad \mu_C(1) = 0.4, \qquad \nu_C(1) = 0.5.$$

*One checks that $\bar{f} : A \to C$ (induced by $f$) and $\bar{g} : B \to C$ (induced by the zero homomorphism $g : T \to Z$, $g(0) = 0$) are well-defined intuitionistic fuzzy R-homomorphisms. Indeed:*

(1) *For an even $n$, $f(n) = 0$, $\mu_C(0) = 1 \geq \mu_A(n) = 1$, $\nu_C(0) = 0 \leq \nu_A(n) = 0$;*

(2) *For an odd $n$, $f(n) = 1$, $\mu_C(1) = 0.4 \geq \mu_A(n) = 0.4$, $\nu_C(1) = 0.5 \leq \nu_A(n) = 0.5$;*

(3) *For $g$, $g(0) = 0$, $\mu_C(0) = 1 \geq \mu_B(0) = 1$, $\nu_C(0) = 0 \leq \nu_B(0) = 0$.*

*Now $\operatorname{Ker} f \times T \cong 2\mathbb{Z}$. For any even $x = 2k$, the pullback $D$ is given by*

$$\mu_D(2k, 0) = \min\big(\mu_A(2k), \mu_B(0)\big) = \min(1, 1) = 1,$$

$$\nu_D(2k, 0) = \max\big(\nu_A(2k), \nu_B(0)\big) = \max(0, 0) = 0.$$

*Hence, $D(2k, 0) = (1, 0)$ for all $k \in \mathbb{Z}$. This coincides with the restriction of $A$ to $\operatorname{Ker} f$, and the diagram is a pullback in $\boldsymbol{C_{R\text{-}IFM}}$.*

### *4.5. Pullbacks and Exact Sequences*

**Lemma 1** ([22]). *Let $M$ and $N$ be R-modules, and let $f : M \to N$ be an R-module homomorphism. If $A = (\mu_A, \nu_A)$ is an IFSM on $M$, then its image under $f$, denoted by $f(A) = (\mu_{f(A)}, \nu_{f(A)})$, is an IFSM on $N$, where the membership and non-membership functions are defined as follows: for any $n \in N$,*

$$\mu_{f(A)}(n) = \begin{cases} \sup\{\mu_A(m) \mid m \in f^{-1}(n)\}, & f^{-1}(n) \neq \varnothing, \\ 0, & f^{-1}(n) = \varnothing, \end{cases}$$

$$\nu_{f(A)}(n) = \begin{cases} \inf\{\nu_A(m) \mid m \in f^{-1}(n)\}, & f^{-1}(n) \neq \varnothing, \\ 1, & f^{-1}(n) = \varnothing. \end{cases}$$

**Property 5.** *Let $f : M \to N$ be an injective function and $A$ an intuitionistic fuzzy set on $M$. For any $n \in \operatorname{Im}(f)$, there exists a unique $m \in M$, such that $f(m) = n$, and*

$$\mu_{f(A)}(n) = \mu_A(m), \quad \nu_{f(A)}(n) = \nu_A(m).$$

**Proof of Property 5.** Since $f$ is injective, if $f(m_1) = f(m_2)$ then $m_1 = m_2$.

Take $n \in \operatorname{Im}(f)$. By definition, there exists at least one $m \in M$, such that $f(m) = n$. Suppose that there is another $m' \in M$ with $f(m') = n$. Then, $f(m) = f(m')$, and by injectivity, $m = m'$. Hence, the preimage $f^{-1}(n) = m_n$ is a singleton.

The intuitionistic fuzzy image set $f(A)$ is defined by:

$$\mu_{f(A)}(n) = \begin{cases} \sup\{\mu_A(m) \mid m \in f^{-1}(n)\}, & f^{-1}(n) \neq \varnothing, \\ 0, & f^{-1}(n) = \varnothing, \end{cases}$$

$$\nu_{f(A)}(n) = \begin{cases} \inf\{\nu_A(m) \mid m \in f^{-1}(n)\}, & f^{-1}(n) \neq \varnothing, \\ 1, & f^{-1}(n) = \varnothing. \end{cases}$$

Since $n \in \mathrm{Im}(f)$, we have$f^{-1}(n) = m_n \neq \varnothing$. Then:

$$\mu_{f(A)}(n) = \sup\{\mu_A(m_n)\} = \mu_A(m_n), \quad \nu_{f(A)}(n) = \inf\{\nu_A(m_n)\} = \nu_A(m_n).$$

Thus, for $n \in \mathrm{Im}(f)$, the unique $m_n \in M$ with $f(m_n) = n$ satisfies

$$\mu_{f(A)}(n) = \mu_A(m_n), \quad \nu_{f(A)}(n) = \nu_A(m_n).$$

□

**Lemma 2.** *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSM of the $R$-modules $X$ and $Y$, respectively. Let $\bar{f} : A \to B$ be an intuitionistic fuzzy $R$-homomorphism induced by the ordinary $R$-homomorphism $f : X \to Y$. Then, the intuitionistic fuzzy kernel of $\bar{f}$, denoted by $\widetilde{\mathrm{Ker}}\bar{f} = (\mu_{\mathrm{Ker}\, f}, \nu_{\mathrm{Ker}\, f})$, is an IFSM of* $\mathrm{Ker}\, f$, *and its membership functions are given by:*

$$\mu_{\mathrm{Ker}\, f}(x) = \mu_A(x), \qquad \nu_{\mathrm{Ker}\, f}(x) = \nu_A(x) \qquad \text{for all } x \in \mathrm{Ker}\, f.$$

**Proof of Lemma 2.** Let $\mathrm{Ker}\, f = \{x \in X \mid f(x) = 0\}$, as shown in Figure 19a. Define an intuitionistic fuzzy set, $\widetilde{\mathrm{Ker}}\bar{f} = (\mu_{\mathrm{Ker}\, f}, \nu_{\mathrm{Ker}\, f})$ on $\mathrm{Ker}\, f$ by restricting $\mu_A$ and $\nu_A$ to $\mathrm{Ker}\, f$, i.e., $\mu_{\mathrm{Ker}\, f}(x) = \mu_A(x), \nu_{\mathrm{Ker}\, f}(x) = \nu_A(x)$ for all $x \in \mathrm{Ker}\, f$. Since $A$ is an IFSM of $X$ and $\mathrm{Ker}\, f$ is a submodule of $X$, $\widetilde{\mathrm{Ker}}\bar{f}$ is an IFSM of $\mathrm{Ker}\, f$.

Let $i : \mathrm{Ker}\, f \hookrightarrow X$ be the inclusion map. Because $\mu_{\mathrm{Ker}\, f}(x) = \mu_A(i(x))$ and $\nu_{\mathrm{Ker}\, f}(x) = \nu_A(i(x))$ for all $x \in \mathrm{Ker}\, f$, the map $i$ induces an intuitionistic fuzzy $R$-homomorphism $\bar{i} : \widetilde{\mathrm{Ker}}\bar{f} \to A$ in the category $\mathbf{C_{R\text{-}IFM}}$, as shown in Figure 19b. Clearly, $\bar{i}$ is a monomorphism.

For any $x \in \mathrm{Ker}\, f$, we have $f(i(x)) = f(x) = 0$, whence $\bar{f} \circ \bar{i} = \bar{0}$ (the zero intuitionistic fuzzy homomorphism).

We now verify the universal property of the kernel. Suppose that $C = (\mu_C, \nu_C)$ is another intuitionistic fuzzy module with underlying set $P$, and $\bar{h} : C \to A$ is an intuitionistic fuzzy homomorphism such that $\bar{f} \circ \bar{h} = \bar{0}$. Let $h : P \to X$ be the ordinary $R$-homomorphism inducing $\bar{h}$. The condition $\bar{f} \circ \bar{h} = \bar{0}$ implies $f(h(p)) = 0$ for every $p \in P$; hence, $h(p) \in \mathrm{Ker}\, f$. By the universal property of the ordinary kernel $\mathrm{Ker}\, f$, there exists a unique $R$-homomorphism, $u : P \to \mathrm{Ker}\, f$, satisfying $i \circ u = h$.

It remains to show that $u$ induces an intuitionistic fuzzy homomorphism, $\bar{u} : C \to \widetilde{\mathrm{Ker}}\bar{f}$. For any $p \in P$, since $\bar{h}$ is an intuitionistic fuzzy homomorphism, we have

$$\mu_C(p) \leq \mu_A(h(p)), \qquad \nu_C(p) \geq \nu_A(h(p)).$$

Using $h = i \circ u$ and the definition of $\mu_{\mathrm{Ker}\, f}, \nu_{\mathrm{Ker}\, f}$, we obtain

$$\mu_A(h(p)) = \mu_A(i(u(p))) = \mu_{\mathrm{Ker}\, f}(u(p)), \qquad \nu_A(h(p)) = \nu_A(i(u(p))) = \nu_{\mathrm{Ker}\, f}(u(p)).$$

Therefore,

$$\mu_C(p) \leq \mu_{\mathrm{Ker}\, f}(u(p)), \qquad \nu_C(p) \geq \nu_{\mathrm{Ker}\, f}(u(p)),$$

which means that $\bar{u}$ is an intuitionistic fuzzy homomorphism. The uniqueness of $\bar{u}$ follows from the uniqueness of the ordinary map $u$.

Thus, $\left(\widetilde{\mathrm{Ker}}\bar{f}, \bar{i}\right)$ is the kernel of $\bar{f}$ in the category $\mathbf{C_{R\text{-}IFM}}$. In particular, the intuitionistic fuzzy kernel is exactly the restriction of $(\mu_A, \nu_A)$ to $\mathrm{Ker}\, f$. □

$$\begin{tikzcd}
P \arrow[d, "u"'] \arrow[dr, "h"] & & \\
\mathrm{Ker}\, f \arrow[r, "i"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

(**a**) $\mathrm{Ker}\, f$ in $\mathbf{C_{R\text{-}M}}$

$$\begin{tikzcd}
C \arrow[d, "\bar{u}"'] \arrow[dr, "\bar{h}"] & & \\
\widetilde{\mathrm{Ker}}\bar{f} \arrow[r, "\bar{i}"'] & B \arrow[r, "\bar{f}"'] & C
\end{tikzcd}$$

(**b**) $\widetilde{\mathrm{Ker}}\bar{f}$ in $\mathbf{C_{R\text{-}IFM}}$

**Figure 19.** Kernel constructions in the categories of $R$-modules and IFSMs.

**Definition 13.** *Let $A, B, C$ be IF modules, with IF $R$-homomorphisms $\bar{f} : A \to B$ and $\bar{g} : B \to C$. If* $\operatorname{Im} \bar{f} = \operatorname{Ker} \bar{g}$, *then the sequence*

$$A \xrightarrow{\bar{f}} B \xrightarrow{\bar{g}} C$$

*is said to be exact at $B$. A longer sequence*

$$\cdots \longrightarrow A_{n-1} \xrightarrow{\bar{f}_{n-1}} A_n \xrightarrow{\bar{f}_n} A_{n+1} \longrightarrow \cdots$$

*is called an exact sequence if it is exact at every $A_n$ in the sequence.*

**Property 6.** *Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$, $C = (\mu_C, \nu_C)$, and $D = (\mu_D, \nu_D)$ be IFSMs of the $R$-modules $X$, $Y$, $Z$, and $P$, respectively. Let $\bar{f} : A \to C$ and $\bar{g} : B \to C$ be IF $R$-homomorphisms induced by the ordinary $R$-homomorphisms $f : X \to Z$ and $g : Y \to Z$, with $p_1$ and $p_2$ being the projections from $P$ to $X$ and $Y$. Suppose that $D$ is the pullback in $\boldsymbol{C_{R\text{-}IFM}}$, where*

$$P = \{(x, y) \in X \times Y \mid f(x) = g(y)\},$$

*and the membership and non-membership functions are defined by*

$$\mu_D(x, y) = \mu_A(x) \wedge \mu_B(y), \qquad \nu_D(x, y) = \nu_A(x) \vee \nu_B(y).$$

*Then, there exists an intuitionistic fuzzy exact sequence*

$$0 \longrightarrow D \xrightarrow{\bar{\varphi}} A \oplus B \xrightarrow{\bar{\psi}} C$$

*where $\bar{\varphi}$ and $\bar{\psi}$ are induced, respectively, by $\varphi(p) = (p_1(p), p_2(p))$ and $\psi(x, y) = f(x) - g(y)$, and $A \oplus B$ is the intuitionistic fuzzy direct sum with*

$$\mu_{A\oplus B}(x, y) = \mu_A(x) \wedge \mu_B(y), \qquad \nu_{A\oplus B}(x, y) = \nu_A(x) \vee \nu_B(y).$$

**Proof of Property 6.** We first establish exactness at the level of underlying modules and then show compatibility with the fuzzy structures.

Exactness of the underlying module sequence, as shown in Figure 20a: Define $\varphi : P \to X \oplus Y$ by $\varphi(x, y) = (x, y)$ and $\psi : X \oplus Y \to Z$ by $\psi(x, y) = f(x) - g(y)$. If $\varphi(p) = 0$, then $(x, y) = (0, 0)$, and hence, $p = 0$; thus, $\varphi$ is injective. If $(x, y) \in \operatorname{Ker} \psi$, then $f(x) = g(y)$, so $(x, y) \in P$ and $\varphi(x, y) = (x, y)$; therefore, $\operatorname{Ker} \psi \subseteq \operatorname{Im} \varphi$. The reverse inclusion, $\operatorname{Im} \varphi \subseteq \operatorname{Ker} \psi$, is obvious. Hence, $\operatorname{Im} \varphi = \operatorname{Ker} \psi$. Consequently, the sequence

$$0 \longrightarrow P \xrightarrow{\varphi} X \oplus Y \xrightarrow{\psi} Z$$

is exact in the category of $R$-modules.

Compatibility with the intuitionistic fuzzy structures, as shown in Figure 20b: For any $p = (x, y) \in P$, we directly have

$$\mu_D(p) = \mu_A(x) \wedge \mu_B(y) = \mu_{A\oplus B}(\varphi(p)), \qquad \nu_D(p) = \nu_A(x) \vee \nu_B(y) = \nu_{A\oplus B}(\varphi(p)).$$

Thus, $\mu_D = \mu_{A\oplus B} \circ \varphi$ and $\nu_D = \nu_{A\oplus B} \circ \varphi$. Together with the injectivity of $\varphi$, Lemma 1 and Lemma 2 imply that $\bar{\varphi}$ is an intuitionistic fuzzy monomorphism, and $\operatorname{Im} \bar{\varphi} = \widetilde{\operatorname{Ker}} \bar{\psi}$. Explicitly:

(1) For any $q = \varphi(p) \in \operatorname{Im} \varphi$, Lemma 1 gives $\mu_{\operatorname{Im} \bar{\varphi}}(q) = \mu_D(p) = \mu_{A\oplus B}(q)$ and $\nu_{\operatorname{Im} \bar{\varphi}}(q) = \nu_D(p) = \nu_{A\oplus B}(q)$.

(2) For any $q = (x, y) \in \operatorname{Ker}\psi$, Lemma 2 yields $\mu_{\widetilde{\operatorname{Ker}\bar{\psi}}}(q) = \mu_{A\oplus B}(q)$ and $\nu_{\widetilde{\operatorname{Ker}\bar{\psi}}}(q) = \nu_{A\oplus B}(q)$.

Since $\operatorname{Im}\varphi = \operatorname{Ker}\psi$ as sets and the fuzzy functions coincide on these elements, we obtain $\operatorname{Im}\bar{\varphi} = \widetilde{\operatorname{Ker}\bar{\psi}}$ as intuitionistic fuzzy submodules.

Therefore, the sequence

$$0 \longrightarrow D \xrightarrow{\bar{\varphi}} A \oplus B \xrightarrow{\bar{\psi}} C$$

is exact in $\mathbf{C_{R\text{-}IFM}}$, according to Definition 13. $\square$

$$\begin{tikzcd} & & X \arrow[dr, "f"] & \\ 0 \arrow[r] & P \arrow[ur, "p_1"] \arrow[r, "\varphi"] \arrow[dr, "p_2"'] & X \oplus Y \arrow[r, "\psi"] & Z \\ & & Y \arrow[ur, "g"'] & \end{tikzcd}$$

(**a**) Pullback exact sequence in $\mathbf{C_{R\text{-}M}}$

$$\begin{tikzcd} & & A \arrow[dr, "\bar{f}"] & \\ 0 \arrow[r] & D \arrow[ur, "\bar{p}_1"] \arrow[r, "\bar{\varphi}"] \arrow[dr, "\bar{p}_2"'] & A \oplus B \arrow[r, "\bar{\psi}"] & C \\ & & B \arrow[ur, "\bar{g}"'] & \end{tikzcd}$$

(**b**) Pullback exact sequence in $\mathbf{C_{R\text{-}IFM}}$

**Figure 20.** Pullback-induced exact sequences.

**Example 15.** *Take $R = \mathbb{Z}_2$. All R-modules are vector spaces over $\mathbb{Z}_2$. Let $X = Y = Z = \mathbb{Z}_2 = \{0, 1\}$, and define the ordinary R-homomorphisms $f : X \to Z$ and $g : Y \to Z$ as the identity maps, i.e., $f(0) = 0$, $f(1) = 1$ and $g(0) = 0$, $g(1) = 1$.*

*On each of $X, Y, Z$ we define an IFSM, $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$, $C = (\mu_C, \nu_C)$, by*

$$\mu(0) = 1, \quad \nu(0) = 0, \qquad \mu(1) = 0.8, \quad \nu(1) = 0.1,$$

*where $\mu + \nu \le 1$ is satisfied. These are clearly IFSMs of the corresponding $\mathbb{Z}_2$-modules.*

*The pullback D. Since $f = g = \mathrm{id}$, the fibre product is*

$$P = \{(x, y) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \mid f(x) = g(y)\} = \{(0, 0), (1, 1)\}.$$

*Following the construction in Property 6, we define $D = (\mu_D, \nu_D)$ on P by*

$$\mu_D(x, x) = \mu_A(x) \wedge \mu_B(x), \qquad \nu_D(x, x) = \nu_A(x) \vee \nu_B(x).$$

*Explicitly,*

$$\mu_D(0, 0) = 1, \;\; \mu_D(1, 1) = 0.8, \qquad \nu_D(0, 0) = 0, \;\; \nu_D(1, 1) = 0.1.$$

*By the pullback theorem, D is the pullback in $\mathbf{C_{R\text{-}IFM}}$.*

*On $X \times Y = \mathbb{Z}_2 \times \mathbb{Z}_2$ define*

$$\mu_{A\oplus B}(x, y) = \mu_A(x) \wedge \mu_B(y), \qquad \nu_{A\oplus B}(x, y) = \nu_A(x) \vee \nu_B(y).$$

*This gives*

$$\mu_{A\oplus B}(x, y) = \begin{cases} 1, & (x, y) = (0, 0), \\ 0.8, & (x, y) \in \{(0, 1), (1, 0), (1, 1)\}, \end{cases}$$

$$\nu_{A\oplus B}(x, y) = \begin{cases} 0, & (x, y) = (0, 0), \\ 0.1, & (x, y) \in \{(0, 1), (1, 0), (1, 1)\}. \end{cases}$$

*Consider the ordinary maps*

$$\varphi : P \to X \times Y, \quad \varphi(x, x) = (x, x),$$

$$\psi : X \times Y \to Z, \quad \psi(x, y) = f(x) - g(y) = x - y.$$

*One checks that $\bar{\varphi} : D \to A \oplus B$ and $\bar{\psi} : A \oplus B \to C$ are IF R-homomorphisms: For $\bar{\varphi}$, $\mu_D(x, x) \le \mu_{A\oplus B}(\varphi(x, x))$ and $\nu_D(x, x) \ge \nu_{A\oplus B}(\varphi(x, x))$ hold trivially with equality. For $\bar{\psi}$, if $x = y$, then $\psi(x, y) = 0$, and $\mu_C(0) = 1$, so $\mu_{A\oplus B}(x, y) \le 1 = \mu_C(\psi(x, y))$; if $x \ne y$, then $\psi(x, y) = 1$, and $\mu_C(1) = 0.8$, while $\mu_{A\oplus B}(x, y) = 0.8$, so the inequality is an equality. The non-membership part is analogous.*

*The underlying sequence of ordinary R-modules*

$$0 \longrightarrow P \xrightarrow{\varphi} \mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{\psi} \mathbb{Z}_2$$

*is exact because* $\operatorname{Im}\varphi = \{(0, 0), (1, 1)\} = \operatorname{Ker}\psi$.

*Now consider the image and kernel in the intuitionistic fuzzy sense.*

$\operatorname{Im}\bar{\varphi}$ *is the fuzzy submodule of $A \oplus B$ with*

$$\mu_{\operatorname{Im}\bar{\varphi}}(u, v) = \sup\{\mu_D(p) \mid \varphi(p) = (u, v)\} = \begin{cases} 1, & (u, v) = (0, 0), \\ 0.8, & (u, v) = (1, 1), \\ 0, & \textit{otherwise}. \end{cases}$$

$\operatorname{Ker}\bar{\psi}$ *is the fuzzy submodule of $A \oplus B$ whose support is $\{(u, v) \mid \psi(u, v) = 0\}$ and whose membership is the restriction of $\mu_{A\oplus B}$ to that support. Thus,*

$$\mu_{\operatorname{Ker}\bar{\psi}}(u, v) = \begin{cases} \mu_{A\oplus B}(u, v), & u = v, \\ 0, & u \ne v, \end{cases} = \begin{cases} 1, & (u, v) = (0, 0), \\ 0.8, & (u, v) = (1, 1), \\ 0, & \textit{otherwise}. \end{cases}$$

*Non-membership functions are computed analogously and coincide.*

*Hence,* $\operatorname{Im}\bar{\varphi} = \operatorname{Ker}\bar{\psi}$, *so the sequence*

$$0 \longrightarrow D \xrightarrow{\bar{\varphi}} A \oplus B \xrightarrow{\bar{\psi}} C$$

*is intuitionistic fuzzy exact. This verifies Property 6.*

**Property 7.** *Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$, $C = (\mu_C, \nu_C)$, and $D = (\mu_D, \nu_D)$ be IFSMs of the R-modules $X$, $Y$, $Z$, and $P$, respectively. Let $\bar{f} : A \to C$ and $\bar{g} : B \to C$ be IFs R-homomorphisms corresponding to the R-homomorphisms $f : X \to Z$ and $g : Y \to Z$, with $p_1$ and $p_2$ being the projections from $P$ to $X$ and $Y$. If $D$ is the pullback in $\boldsymbol{C_{R\text{-}IFM}}$, and if $p_2$ is surjective and satisfies for all $b \in B$, $a \in A$ exists, such that $\bar{f}(a) = \bar{g}(b)$ with $\mu_A(a) \ge \mu_B(b)$, $\nu_A(a) \le \nu_B(b)$, and then the following sequence is intuitionistic fuzzy exact:*

$$0 \longrightarrow \widetilde{\operatorname{Ker}}\bar{p}_2 \xrightarrow{\bar{i}} D \xrightarrow{\bar{p}_2} B \xrightarrow{\bar{j}} \widetilde{\operatorname{Coker}}\bar{g}$$

*Here,* $\widetilde{\operatorname{Ker}}$ *and* $\widetilde{\operatorname{Coker}}$ *denote the IFSM kernels and cokernels, respectively.*

**Proof of Property.** We first establish an isomorphism at the module level that will be lifted to the fuzzy setting.

Define the map $\varphi : \mathrm{Ker}(f) \to \mathrm{Ker}(p_2)$ as $\varphi(x) = (x, 0)$, as shown in Figure 21. If $\varphi(x) = (x, 0) = 0$, then $x = 0$, so $\varphi$ is injective. For any $(x, y) \in \mathrm{Ker}(p_2)$: $p_2(x, y) = y = 0$ and $(x, 0) \in P$, so $\bar{f}(x) = \bar{g}(0) = 0$. Thus, $x \in \mathrm{Ker}(f)$ and $\varphi(x) = (x, 0) = (x, y)$. Therefore, $\varphi$ is surjective. $\varphi$ is an isomorphism, i.e., $\mathrm{Ker}(p_2) \cong \mathrm{Ker}(f)$. If $x \in \mathrm{Ker}(f)$, then $\bar{f}(x) = 0 = \bar{g}(0)$, so $(x, 0) \in P$. Since $p_2(x, 0) = 0$, it follows that $(x, 0) \in \mathrm{Ker}(p_2)$. Because $\mu_{\mathrm{Ker}(p_2)}(\varphi(x)) = \mu_{\mathrm{Ker}(p_2)}(x, 0) = \mu_D(x, 0) = \mu_A(x) \wedge \mu_B(0) = \mu_A(x) = \mu_{\mathrm{Ker}(f)}(x)$, and $\nu_{\mathrm{Ker}(p_2)}(\varphi(x)) = \nu_{\mathrm{Ker}(p_2)}(x, 0) = \nu_D(x, 0) = \nu_A(x) \vee \nu_B(0) = \nu_A(x) = \nu_{\mathrm{Ker}(f)}(x)$, it follows that $\varphi$ is an intuitionistic fuzzy homomorphism.

Define $\psi : \mathrm{Ker}(p_2) \to \mathrm{Ker}(f)$ as $\psi(x, y) = x$. If $(x, y) \in \mathrm{Ker}(p_2)$, then $y = 0$, and $\bar{f}(x) = 0$, so $x \in \mathrm{Ker}(f)$. $\mu_{\mathrm{Ker}(f)}(\psi(x, y)) = \mu_A(x) = \mu_A(x) \wedge \mu_B(0) = \mu_D(x, 0) = \mu_{\mathrm{Ker}(p_2)}(x, y)$, $\nu_{\mathrm{Ker}(f)}(\psi(x, y)) = \nu_A(x) = \nu_A(x) \vee \nu_B(0) = \nu_D(x, 0) = \nu_{\mathrm{Ker}(p_2)}(x, y)$, $\psi \circ \varphi = \mathrm{id}_{\mathrm{Ker}(f)}$, $\varphi \circ \psi = \mathrm{id}_{\mathrm{Ker}(p_2)}$. Therefore, $\psi$ is an intuitionistic fuzzy homomorphism. Hence, $\mathrm{Ker}(p_2) \cong \mathrm{Ker}(f)$ as intuitionistic fuzzy modules.

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(p_2) \arrow[r, "i"] \arrow[d, shift left, "\psi"] & P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[r, "j"] \arrow[d, "g"] & \mathrm{Coker}(g) \\
& \mathrm{Ker}(f) \arrow[u, shift left, "\varphi"] \arrow[r] & X \arrow[r, "f"'] & Z \arrow[ur, "\pi"'] &
\end{tikzcd}
$$

**Figure 21.** Pullback construction and induced exact sequence at module level. Compatibility between the exact sequence and the pullback diagram at the module level.

According to Definition 13, we now prove the exactness of the sequence in Figure 22 in the category $\mathbf{C_{R\text{-}IFM}}$.

$$
\begin{tikzcd}
0 \arrow[r] & \widetilde{\mathrm{Ker}}(\bar{p}_2) \arrow[r, "\bar{i}"] \arrow[d, shift left, "\bar{\psi}"] & D \arrow[r, "\bar{p}_2"] \arrow[d, "\bar{p}_1"'] & B \arrow[r, "\bar{j}"] \arrow[d, "\bar{g}"] & \widetilde{\mathrm{Coker}}(\bar{g}) \\
& \widetilde{\mathrm{Ker}}(\bar{f}) \arrow[u, shift left, "\bar{\varphi}"] \arrow[r] & A \arrow[r, "\bar{f}"'] & C \arrow[ur, "\bar{\pi}"'] &
\end{tikzcd}
$$

**Figure 22.** Pullback construction and induced exact sequence in intuitionistic fuzzy modules. Compatibility between the intuitionistic fuzzy exact sequence and the pullback diagram.

Exactness at $\widetilde{\mathrm{Ker}}\bar{p}_2$: By definition, $\bar{i}$ is the inclusion map. For any $(x, 0) \in \mathrm{Ker}(p_2)$, using Lemma 2, we have $\mu_D(i(x, 0)) = \mu_D(x, 0) = \mu_A(x) \wedge \mu_B(0) = \mu_A(x) = \mu_{\mathrm{Ker}(p_2)}(x, 0)$ and $\nu_D(i(x, 0)) = \nu_D(x, 0) = \nu_A(x) \vee \nu_B(0) = \nu_A(x) = \nu_{\mathrm{Ker}(p_2)}(x, 0)$. Thus, $\bar{i}$ is an intuitionistic fuzzy monomorphism; hence, the sequence is exact at $\widetilde{\mathrm{Ker}}\bar{p}_2$.

Exactness at $D$: We need to show $\mathrm{Im}\,\bar{i} = \widetilde{\mathrm{Ker}}\bar{p}_2$. From the pullback property, $\bar{p}_2 \circ \bar{i} = 0$, so $\mathrm{Im}\,\bar{i} \subseteq \widetilde{\mathrm{Ker}}\bar{p}_2$. Conversely, for any $(x, 0) \in \mathrm{Ker}(p_2)$ (the underlying set of $\widetilde{\mathrm{Ker}}\bar{p}_2$), we have $(x, 0) \in \mathrm{Im}\,i$. Using Lemma 1, for such an element,

$$\mu_{\mathrm{Im}\,\bar{i}}(x, 0) = \sup\{\mu_{\widetilde{\mathrm{Ker}}\bar{p}_2}(x', 0) \mid i(x', 0) = (x, 0)\} = \mu_{\widetilde{\mathrm{Ker}}\bar{p}_2}(x, 0),$$

$$\nu_{\mathrm{Im}\,\bar{i}}(x, 0) = \inf\{\nu_{\widetilde{\mathrm{Ker}}\bar{p}_2}(x', 0) \mid i(x', 0) = (x, 0)\} = \nu_{\widetilde{\mathrm{Ker}}\bar{p}_2}(x, 0).$$

Thus, $\mathrm{Im}\,\bar{i} = \widetilde{\mathrm{Ker}}\bar{p}_2$ as IF submodules, so the sequence is exact at $D$.

Exactness at $B$: We prove $\mathrm{Im}\,\bar{p}_2 = \widetilde{\mathrm{Ker}}\bar{j}$. Let $\pi : Z \to \mathrm{Coker}\,g$ be the natural projection. By Definition 7, $j$ is a zero map, so $\widetilde{\mathrm{Ker}}\bar{j} = B$ (as an IF submodule). It remains to show that $\mathrm{Im}\,\bar{p}_2 = B$, i.e., $\bar{p}_2$ is an intuitionistic fuzzy epimorphism.

Since $p_2$ is surjective, for any $y \in Y$, there exists $(x, y) \in P$, such that $p_2(x, y) = y$ and $f(x) = g(y)$ (by the pullback definition). By Lemma 1,

$$\mu_{\mathrm{Im}\,\bar{p}_2}(y) = \sup\{\mu_D(x, y') \mid p_2(x, y') = y\} = \sup\{\mu_A(x) \wedge \mu_B(y) \mid f(x) = g(y)\},$$

$$\nu_{\mathrm{Im}\,\bar{p}_2}(y) = \inf\{\nu_D(x, y') \mid p_2(x, y') = y\} = \inf\{\nu_A(x) \vee \nu_B(y) \mid f(x) = g(y)\}.$$

The hypothesis states: for every $b \in B$ (i.e., every $y \in Y$ with membership $\mu_B(y), \nu_B(y)$), there exists $a \in A$, such that $\bar{f}(a) = \bar{g}(b)$ and $\mu_A(a) \geq \mu_B(y), \nu_A(a) \leq \nu_B(y)$. For such an $a$, we have

$$\mu_A(a) \wedge \mu_B(y) = \mu_B(y), \qquad \nu_A(a) \vee \nu_B(y) = \nu_B(y).$$

Moreover, for any $x$ with $f(x) = g(y)$, we always have $\mu_A(x) \wedge \mu_B(y) \leq \mu_B(y)$ and $\nu_A(x) \vee \nu_B(y) \geq \nu_B(y)$. Therefore,

$$\sup\{\mu_A(x) \wedge \mu_B(y) \mid f(x) = g(y)\} = \mu_B(y) = \mu_{\widetilde{\mathrm{Ker}\bar{j}}}(y),$$

$$\inf\{\nu_A(x) \vee \nu_B(y) \mid f(x) = g(y)\} = \nu_B(y) = \nu_{\widetilde{\mathrm{Ker}\bar{j}}}(y).$$

Thus, $\mathrm{Im}\, \bar{p}_2 = B = \widetilde{\mathrm{Ker}\bar{j}}$ as IF submodules, establishing exactness at $B$.

Since the sequence is exact at all three positions, it is an intuitionistic fuzzy exact sequence. □

**Example 16.** *Let $R = \mathbb{Z}_2$. Consider the $R$-modules $X = Y = Z = \mathbb{Z}_2 = \{0, 1\}$. Define IFSMs $A = (\mu_A, \nu_A)$ on $X$, $B = (\mu_B, \nu_B)$ on $Y$, and $C = (\mu_C, \nu_C)$ on $Z$ all equal, with*

$$\mu_A(0) = 1, \quad \mu_A(1) = 0.6, \qquad \nu_A(0) = 0, \quad \nu_A(1) = 0.3,$$

*and similarly $\mu_B = \mu_A$, $\nu_B = \nu_A$, $\mu_C = \mu_A$, $\nu_C = \nu_A$.*

*Let $f : X \to Z$ and $g : Y \to Z$ be the identity maps. Then $\bar{f} : A \to C$ and $\bar{g} : B \to C$ are IF $R$-homomorphisms.*

*The pullback $P$ is*

$$P = \{(x, y) \in X \times Y \mid f(x) = g(y)\} = \{(0, 0), (1, 1)\}.$$

*Define an IFSM $D = (\mu_D, \nu_D)$ on $P$ by*

$$\mu_D(x, x) = \mu_A(x) \wedge \mu_B(x), \qquad \nu_D(x, x) = \nu_A(x) \vee \nu_B(x).$$

*This gives $\mu_D(0, 0) = 1$, $\mu_D(1, 1) = 0.6$, $\nu_D(0, 0) = 0$, $\nu_D(1, 1) = 0.3$. The projection $p_2 : P \to Y$, $p_2(x, x) = x$, is surjective and clearly an IF $R$-homomorphism.*

*For every $b \in B$, choose $a = b$. Then $\bar{f}(a) = a = b = \bar{g}(b)$, and*

$$\mu_A(a) = \mu_B(b), \qquad \nu_A(a) = \nu_B(b),$$

*so the required inequalities hold.*

*Now consider the sequence*

$$0 \longrightarrow \mathrm{Ker}(p_2) \xrightarrow{\ i\ } P \xrightarrow{\ p_2\ } Y \xrightarrow{\ j\ } \mathrm{Coker}(g)$$

*where $i$ is the inclusion, and $j$ is the natural projection. Since $\ker(p_2) = \{(0, 0)\}$ and $\bar{g}$ is the identity, $\mathrm{coker}(g) = 0$ and $j$ is the zero map. The underlying module sequence reduces to*

$$0 \longrightarrow 0 \longrightarrow P \xrightarrow{\ \cong\ } Y \longrightarrow 0,$$

*and the induced fuzzy submodules on the kernel and image terms coincide with the corresponding restrictions, giving exactness in $\boldsymbol{C}_{\textbf{R-IFM}}$. Hence, all conditions of Property 7 are satisfied, and the sequence is exact in the category of intuitionistic fuzzy $R$-modules.*

## 5. Discussion

This study systematically constructs the theory of pullbacks within intuitionistic fuzzy module categories, addressing a research gap in this fundamental limit within uncertainty mathematics. Through explicit construction and classification, the research not only clarifies the specific conditions for the existence of pullbacks and reveals the unique properties of double pullbacks and related morphisms but also, more importantly, rigorously establishes the theoretical connection between pullbacks and exactness.

This work deepens the understanding of limit structures in fuzzy environments, with its core contribution lying in the successful adaptation of classical category theory tools to the intuitionistic fuzzy framework and the demonstration of their rigour. The foundation laid here provides a crucial prerequisite for the subsequent development of homological theory in intuitionistic fuzzy categories while also paving the way for investigating other limit constructions and exploring their applications in fuzzy logic and topology.

*Potential Applications*

The pullback constructions developed in this paper may find applications in several domains where intuitionistic fuzzy systems are employed:

**Database integration and information fusion:** Intuitionistic fuzzy databases store records with degrees of truth and falsity. Pullbacks can be used to merge two fuzzy relations along a common attribute, producing a consistent fuzzy relation that retains the original membership information. The double pullback provides a way to combine three sources with a chain of common attributes.

**Control systems and decision making:** In hierarchical fuzzy control, pullbacks allow the composition of IF rules from different levels while preserving consistency. The exactness properties guarantee that certain controllability and observability conditions are preserved under system interconnection.

**Image processing and pattern recognition:** Intuitionistic fuzzy representations of images often involve membership and non-membership functions. A pullback can serve as a feature fusion operator: given two feature spaces mapped to a common reference, the pullback yields a joint feature space with combined membership information.

**Categorical data analysis:** When dealing with uncertain or imprecise categories, the kernel-terminal pullback recovers the kernel of a fuzzy homomorphism, which can be interpreted as the collection of inputs that map to a “zero" output with high membership. This is useful for detecting redundant or invariant features.

These applications are beyond the scope of the present theoretical paper, but they indicate directions for future research.

## 6. Conclusions

In this work, we have addressed the underdeveloped study of pullbacks within intuitionistic fuzzy module categories, establishing their essential theoretical foundations. We systematically investigated the explicit construction of pullbacks and delved into the properties of related structures, including double pullbacks, monomorphisms, and epimorphisms. A classification scheme for different pullback categories was also provided.

The key theoretical contributions include a precise characterisation of the conditions for pullback construction, a detailed description of the features of double pullbacks and their associated morphisms, and a rigorous establishment of the fundamental relationship between pullbacks and exactness in this context.

Overall, this research significantly deepens the understanding of limit theory within intuitionistic fuzzy settings. The established framework and results provide a solid foundation for future explorations in categorical structures under mathematical uncertainty.

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