

Article

# Fifth-Order Max-Type Fuzzy Difference Equations: Existence, Periodicity, Boundedness, and Persistence

Tao Yang <sup>1</sup>, Lirong Ma <sup>1</sup>, Run Yang <sup>1</sup> and Changyou Wang <sup>1,2,\*</sup> 

<sup>1</sup> Faculty of Mathematics and Statistics, Sichuan Minzu College, Kangding 626000, China; 240041@scun.edu.cn (T.Y.)

<sup>2</sup> College of Applied Mathematics, Chengdu University of Information Technology, Chengdu 610225, China

\* Correspondence: wangcy@cuit.edu.cn

## Abstract

Max-type fuzzy difference equations constitute an emerging research area arising from the integration of fuzzy mathematics and discrete dynamical systems. By characterizing uncertainty through fuzzy numbers, these equations provide rigorous mathematical modeling tools for practical problems involving both discreteness and uncertainty. This paper systematically investigates the dynamical properties of a class of max-type fuzzy difference equations. First, fuzzy set theory is used to transform the fuzzy difference equation into a corresponding system of parametric ordinary difference equations. Then, using the iterative method, inequality techniques, and mathematical induction, an expression for the periodic solutions of the max-type ordinary difference equation is derived, from which an expression for the periodic solutions of the max-type fuzzy difference equation is further obtained. In addition, the boundedness and persistence of the solutions to the fuzzy difference equation are proved. Finally, numerical simulations are conducted in MATLAB 2024, and the results illustrate the theoretical findings. This study not only enriches the theoretical framework of fuzzy difference equations but also provides new insights and methodological support for the modeling and analysis of uncertain discrete systems.

**Keywords:** fuzzy difference equation; max type; periodicity; boundedness; persistence

**MSC:** 39A23; 39A26

## 1. Introduction

Difference equations constitute an important class of classical dynamical system models in applied mathematics and have been widely applied in computer science, biology, economics, epidemiology, and other fields [1–4]. As an important subclass of difference equations, max-type difference equations have attracted considerable attention from researchers because of their significant applications in fields such as automatic control theory [5–10]. However, relatively few studies have been reported on fuzzy difference equations. In [11], H. D. Voulov investigated the periodicity of positive solutions of max-type difference Equation (1).

$$x_n = \max\left\{\frac{A}{x_{n-k}}, \frac{B}{x_{n-m}}\right\}, n = 0, 1, 2, \dots, \quad (1)$$

where  $k$  and  $m$  are positive integers,  $d = \max\{k, m\}$ , and the parameters  $A, B$  and initial conditions  $x_{-d}, x_{-d+1}, \dots, x_{-1}$  are positive real numbers. By means of a case-by-case analysis, the author proved the periodicity of the positive solutions of this equation.



Academic Editor: Patricia J. Y. Wong

Received: 24 May 2026

Revised: 20 June 2026

Accepted: 22 June 2026

Published: 26 June 2026

**Copyright:** © 2026 by the authors.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\) license](https://creativecommons.org/licenses/by/4.0/).

In [12], E. M. Elsayed studied the following class of third-order max-type and min-type difference equations:

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, x_{n-2}\right\}, x_{n+1} = \min\left\{\frac{A_n}{x_n}, x_{n-2}\right\}, n = 0, 1, 2, \dots, \tag{2}$$

where the parameter  $A_n$  is a positive periodic sequence with period 3. The author proved that every solution of these equations is periodic with period 3.

In [13], N. Fotiades investigated the periodicity of the following system of difference equations:

$$x_{n+1} = \max\left\{A, \frac{y_n}{x_{n-1}}\right\}, y_{n+1} = \max\left\{B, \frac{x_n}{y_{n-1}}\right\}, n = 0, 1, 2, \dots, \tag{3}$$

where the parameters  $A$  and  $B$ , together with the initial conditions  $x_{-1}, x_0, y_{-1}, y_0$  are positive constants.

As a further development, T. X. Sun studied the dynamical properties of Equation (4) in [14]:

$$x_n = \max\left\{\frac{A_n}{x_{n-r}}, x_{n-k}\right\}, n = 0, 1, 2, \dots, \tag{4}$$

where the parameter sequence  $\{A_n\}$  is periodic with period  $p$ ,  $k, r \in \{1, 2, 3, \dots\}$ ,  $\gcd\{k, r\} = 1$ , and  $k \neq r$ . Let  $d = \max\{k, r\}$  and let the initial values  $x_{-d}, x_{-d+1}, \dots, x_0$  be real numbers. The author proved that if  $p = 1$  (or if  $p \geq 2$  and  $k$  is odd), then the solutions of Equation (4) are periodic with period  $k$ . More recently, several studies have further enriched the qualitative theory of nonlinear and rational systems of difference equations. Ghezal and Zemouri investigated closed-form solutions and global stability for a balancing–bilinear system of difference equations [15], and further studied the global stability of a multi-dimensional higher-order rational difference system with Pell coefficients [16]. In addition, Ghezal, Salman, and Ghafli considered three-dimensional second-order rational difference equations and provided explicit formulas together with numerical simulations [17]. These works show that explicit solution formulas, periodic behavior, boundedness and stability remain central topics in the qualitative analysis of nonlinear difference equations.

With the development of the theory of difference equations, it has been found that the information required by difference-equation models to describe many practical problems is often incomplete. One effective approach to accounting for uncertainty and imprecision in real-world phenomena is to fuzzify the corresponding systems of difference equations. To date, a large body of research has been conducted on fuzzy difference equations (see, for example, [18–28] and the references cited therein). In recent decades, max-type fuzzy difference equations have attracted increasing attention. In [29], G. Stefanidou investigated the dynamical properties of fuzzy difference Equation (5) corresponding to Equation (1) with  $k = 0$  and  $m = 1$ :

$$x_{n+1} = \max\left\{\frac{A_0}{x_n}, \frac{A_1}{x_{n-1}}\right\}, n = 0, 1, 2, \dots, \tag{5}$$

where the parameters  $A_0$  and  $A_1$ , together with the initial values  $x_{-1}, x_0$  are positive fuzzy numbers. The author established the periodicity, unboundedness, and non-persistence of the solutions of Equation (5). In [30], G. Stefanidou further extended the form of Equation (5) and investigated the periodicity of the solutions of the following fuzzy difference equation:

$$x_{n+1} = \max\left\{\frac{A_0}{x_{n-k}}, \frac{A_1}{x_{n-m}}\right\}, n = 0, 1, 2, \dots, \tag{6}$$

where the parameters  $A_0$  and  $A_1$  are positive fuzzy numbers. Let  $d = \max\{k, m\}$ , and let the initial conditions  $x_{-d}, x_{-d+1}, \dots, x_0$  also be positive fuzzy numbers. In [28], Wang investigated the periodicity of the following class of max-type fuzzy difference equations, where the initial values  $z_i (0 \leq i \leq 3)$  and the parameter  $H$  are positive fuzzy numbers

$$z_{n+1} = \max\left\{\frac{H}{z_{n-1}}, \frac{H}{z_{n-2}}, z_{n-3}\right\} \tag{7}$$

Inspired by the above studies on max-type difference equations and max-type fuzzy difference equations, this paper investigates the dynamical properties of the following fifth-order max-type fuzzy difference equation:

$$x_{n+1} = \max\left\{\frac{A}{x_{n-1}}, x_{n-4}\right\}, n = 0, 1, 2, \dots, \tag{8}$$

where  $\{x_n\}$  is a sequence of positive fuzzy numbers, and the parameter  $A$  and the initial conditions  $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$  are positive fuzzy numbers.

**Remark 1.** *The practical significance of this fifth-order max-type fuzzy difference Equation (8) lies first in its capacity to serve as a general discrete-time modelling framework for systems where decisions or updates depend on the maximum value of previous states, a common mechanism in control, signal processing, and algorithmic analysis. Second, this class of equations has already found concrete application in the mathematical modelling of epileptic seizures, where parameter variations mimic abnormal neuronal discharges and reveal complex dynamical patterns such as rippled almost periodic behaviour. Third, the fuzzy setting explicitly accommodates imprecise or linguistically described parameters and initial conditions, making the model directly applicable to real-world problems where exact numerical values are unavailable. From a theoretical perspective, this work extends classical max-type difference equations with real coefficients to the fuzzy domain for high-order systems, and it also generalises lower-order fuzzy max-type equations to a fifth-order counterpart, thereby enriching the qualitative theory of such nonlinear recurrences.*

## 2. Definitions and Preliminaries

In order to establish the principal results of this paper, we begin by introducing some definitions and lemmas that will be employed throughout the entire study.

**Definition 1.** Suppose that  $A$  is a function from the real number set  $R$  into  $[0, 1]$ , namely  $A : R \rightarrow [0, 1], x \mapsto A(x), x \in R$ , then  $A$  is said to be a fuzzy set on  $R$ .

**Definition 2.** Fuzzy set  $A$  is said to be a fuzzy convex if, for every  $\alpha \in [0, 1]$  and  $\theta_1, \theta_2 \in R$  the inequality  $A(\alpha\theta_1 + (1 - \alpha)\theta_2) \geq \min\{A(\theta_1), A(\theta_2)\}$  is satisfied.

**Definition 3.** Fuzzy set  $A$  is said to be normal (regular) if  $\theta \in R$  exists, for which  $A(\theta) = 1$ .

**Definition 4.** The support of  $A$  is defined by  $\text{supp}A = \overline{U_{\alpha \in [0,1]} [A]_{\alpha}} = \overline{\{x : A(x) > 0\}}$ .

**Definition 5 ([31]).** Fuzzy set  $A$  is said to be a fuzzy number provided that it satisfies the following conditions:

- (1)  $A$  is normal (regular);
- (2)  $A$  is fuzzy convex;
- (3)  $A$  is upper semicontinuous;
- (4) The support of  $A$  is compact.

If  $\min(\text{supp}A) > 0$ , then  $A$  is referred to as a positive fuzzy number.

**Definition 6** ([31]). Let  $\{x_n\}_{n \in \mathbb{Z}}$  denote a sequence of fuzzy numbers defined on the positive integer set  $\mathbb{Z}$ . If there exists a positive integer  $T$  such that  $x_{n+T} = x_n$  for all  $n \in \mathbb{Z}$ , then this sequence is said to be a periodic fuzzy number sequence with period  $T$ , abbreviated as a periodic fuzzy sequence.

### 3. Main Results

In this section, we study the existence and uniqueness, periodicity, and boundedness of positive solutions for system (8).

**Theorem 1.** For fuzzy difference Equation (8), if  $A$  is a positive fuzzy number, then for any prescribed positive fuzzy initial values  $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ , Equation (8) admits a unique fuzzy solution sequence  $\{x_n\}$ .

**Proof.** The proof of this theorem is analogous to that of Proposition 3.1 in [29]; hence, it is omitted here.  $\square$

We now analyze the periodicity of positive solutions to Equation (8) given that  $A$  is a fuzzy number. For max-type fuzzy difference Equation (8), by applying  $\alpha$ -cut theory, it can be transformed into the following system of two ordinary difference equations

$$y_{n+1} = \max\left\{\frac{B}{z_{n-1}}, y_{n-4}\right\}, z_{n+1} = \max\left\{\frac{C}{y_{n-1}}, z_{n-4}\right\}, n = 0, 1, 2, \dots \tag{9}$$

In system (9),  $B$  and  $C$  denote the left and right endpoints, respectively, of the  $\alpha$ -cut interval of the fuzzy parameter  $A$  in Equation (8). Both  $B$  and  $C$  are positive real numbers, and they satisfy  $B \leq C$ . The sequence  $\{y_n\}$  consists of the left endpoints of the  $\alpha$ -cut intervals of the fuzzy sequence  $\{x_n\}$  from Equation (8), and  $\{z_n\}$  consists of the corresponding right endpoints. Both  $\{y_n\}$  and  $\{z_n\}$  are positive real sequences, and the inequality  $y_n \leq z_n$  holds for every  $n$ .

**Theorem 2.** For the ordinary difference system (9), given any positive real initial conditions,  $y_{-4}, y_{-3}, y_{-2}, y_{-1}, y_0$  and  $z_{-4}, z_{-3}, z_{-2}, z_{-1}, z_0$ , each solution of system (9) is periodic with period 5. That is, for any positive real initial values and all integers  $k \geq 0$ , it follows that

$$y_{5k+1} = y_{5k+6}, y_{5k+2} = y_{5k+7}, y_{5k+3} = y_{5k+8}, y_{5k+4} = y_{5k+9}, y_{5k+5} = y_{5k+10},$$

$$z_{5k+1} = z_{5k+6}, z_{5k+2} = z_{5k+7}, z_{5k+3} = z_{5k+8}, z_{5k+4} = z_{5k+9}, z_{5k+5} = z_{5k+10}.$$

**Proof.** Based on difference system (9), the following equations are derived

$$y_1 = \max\left\{\frac{B}{z_{-1}}, y_{-4}\right\}, z_1 = \max\left\{\frac{C}{y_{-1}}, z_{-4}\right\}. \tag{10}$$

(a<sub>1</sub>) From Equation (10), if  $\frac{B}{z_{-1}} > y_{-4}, \frac{C}{y_{-1}} > z_{-4}$ , then one obtains  $y_1 = \frac{B}{z_{-1}}, z_1 = \frac{C}{y_{-1}}$ . According to system (9), we have

$$y_2 = \max\left\{\frac{B}{z_0}, y_{-3}\right\}, z_2 = \max\left\{\frac{C}{y_0}, z_{-3}\right\}, \tag{11}$$

$$y_3 = \max\left\{\frac{B}{z_1}, y_{-2}\right\}, z_3 = \max\left\{\frac{C}{y_1}, z_{-2}\right\}, \tag{12}$$

$$y_4 = \max\left\{\frac{B}{z_2}, y_{-1}\right\}, z_4 = \max\left\{\frac{C}{y_2}, z_{-1}\right\}, \tag{13}$$

and

$$y_5 = \max\left\{\frac{B}{z_3}, y_0\right\}, z_5 = \max\left\{\frac{C}{y_3}, z_0\right\}. \tag{14}$$

( $a_{1-1}$ ) From Equation (11), if  $\frac{B}{z_0} > y_{-3}, \frac{C}{y_0} > z_{-3}$ , then one obtains  $y_2 = \frac{B}{z_0}, z_2 = \frac{C}{y_0}$ .

( $a_{1-1-1}$ ) From Equation (12), if  $\frac{B}{z_1} > y_{-2}, \frac{C}{y_1} > z_{-2}$ , then one obtains

$$y_3 = \frac{B}{z_1} = \frac{B}{C}y_{-1}, z_3 = \frac{C}{y_1} = \frac{C}{B}z_{-1}.$$

( $a_{1-1-1-1}$ ) From Equation (13), if  $\frac{B}{z_2} > y_{-1}, \frac{C}{y_2} > z_{-1}$ , then one obtains

$$y_4 = \frac{B}{z_2} = \frac{B}{C}y_0, z_4 = \frac{C}{y_2} = \frac{C}{B}z_0.$$

( $a_{1-1-1-1-1}$ ) From Equation (14), if  $\frac{B}{z_3} > y_0, \frac{C}{y_3} > z_0$ , then one obtains

$$y_5 = \frac{B}{z_3} = \frac{B^2}{Cz_{-1}}, z_5 = \frac{C}{y_3} = \frac{C^2}{By_{-1}}.$$

By ( $a_{1-1-1-1}$ ), it follows that  $\frac{B}{z_2} = \frac{B}{C}y_0 > y_{-1}, \frac{C}{y_2} = \frac{C}{B}z_0 > z_{-1}$ , which implies

$$By_0 > Cy_{-1}, Cz_0 > Bz_{-1},$$

therefore,

$$y_6 = \max\left\{\frac{B}{z_4}, y_1\right\} = \max\left\{\frac{B^2}{Cz_0}, \frac{B}{z_{-1}}\right\} = \frac{B}{z_{-1}}, z_6 = \max\left\{\frac{C}{y_4}, z_1\right\} = \max\left\{\frac{C^2}{By_0}, \frac{C}{y_{-1}}\right\} = \frac{C}{y_{-1}},$$

by ( $a_{1-1-1-1-1}$ ), it follows that  $\frac{B}{z_3} = \frac{B^2}{Cz_{-1}} > y_0, \frac{C}{y_3} = \frac{C^2}{By_{-1}} > z_0$ , which implies

$$\frac{B^2y_{-1}}{C^2} < \frac{B}{z_0}, \frac{C^2z_{-1}}{B^2} < \frac{C}{y_0},$$

hence,

$$\begin{aligned} y_7 &= \max\left\{\frac{B}{z_5}, y_2\right\} = \max\left\{\frac{B^2y_{-1}}{C^2}, \frac{B}{z_0}\right\} = \frac{B}{z_0}, z_7 = \max\left\{\frac{C}{y_5}, z_2\right\} = \max\left\{\frac{C^2z_{-1}}{B^2}, \frac{C}{y_0}\right\} = \frac{C}{y_0}, \\ y_8 &= \max\left\{\frac{B}{z_6}, y_3\right\} = \max\left\{\frac{B}{C}y_{-1}, \frac{B}{C}y_{-1}\right\} = \frac{B}{C}y_{-1}, z_8 = \max\left\{\frac{C}{y_6}, z_3\right\} = \max\left\{\frac{C}{B}z_{-1}, \frac{C}{B}z_{-1}\right\} = \frac{C}{B}z_{-1}, \\ y_9 &= \max\left\{\frac{B}{z_7}, y_4\right\} = \max\left\{\frac{B}{C}y_0, \frac{B}{C}y_0\right\} = \frac{B}{C}y_0, z_9 = \max\left\{\frac{C}{y_7}, z_4\right\} = \max\left\{\frac{C}{B}z_0, \frac{C}{B}z_0\right\} = \frac{C}{B}z_0, \end{aligned}$$

and

$$y_{10} = \max\left\{\frac{B}{z_8}, y_5\right\} = \max\left\{\frac{B^2}{Cz_{-1}}, \frac{B^2}{Cz_{-1}}\right\} = \frac{B^2}{Cz_{-1}}, z_{10} = \max\left\{\frac{C}{y_8}, z_5\right\} = \max\left\{\frac{C^2}{By_{-1}}, \frac{C^2}{By_{-1}}\right\} = \frac{C^2}{By_{-1}}.$$

If the relations

$$y_{5k+1} = \frac{B}{z_{-1}}, y_{5k+2} = \frac{B}{z_0}, y_{5k+3} = \frac{B}{C}y_{-1}, y_{5k+4} = \frac{B}{C}y_0, y_{5k+5} = \frac{B^2}{Cz_{-1}}$$

and

$$z_{5k+1} = \frac{C}{y_{-1}}, z_{5k+2} = \frac{C}{y_0}, z_{5k+3} = \frac{C}{B}z_{-1}, z_{5k+4} = \frac{C}{B}z_0, z_{5k+5} = \frac{C^2}{By_{-1}}$$

are satisfied, then the following expressions are derived

$$y_{5k+6} = \max\left\{\frac{B}{z_{5k+4}}, y_{5k+1}\right\} = \max\left\{\frac{B^2}{Cz_0}, \frac{B}{z_{-1}}\right\} = \frac{B}{z_{-1}}, z_{5k+6} = \max\left\{\frac{C}{y_{5k+4}}, z_{5k+1}\right\} = \max\left\{\frac{C^2}{By_0}, \frac{C}{y_{-1}}\right\} = \frac{C}{y_{-1}},$$

$$\begin{aligned}
 y_{5k+7} &= \max\left\{\frac{B}{z_{5k+5}}, y_{5k+2}\right\} = \max\left\{\frac{B^2}{C^2}y_{-1}, \frac{B}{z_0}\right\} = \frac{B}{z_0}, \quad z_{5k+7} = \max\left\{\frac{C}{y_{5k+5}}, z_{5k+2}\right\} = \max\left\{\frac{C^2}{B^2}z_{-1}, \frac{C}{y_0}\right\} = \frac{C}{y_0}, \\
 y_{5k+8} &= \max\left\{\frac{B}{z_{5k+6}}, y_{5k+3}\right\} = \max\left\{\frac{B}{C}y_{-1}, \frac{B}{C}y_{-1}\right\} = \frac{B}{C}y_{-1}, \quad z_{5k+8} = \max\left\{\frac{C}{y_{5k+6}}, z_{5k+3}\right\} = \max\left\{\frac{C}{B}z_{-1}, \frac{C}{B}z_{-1}\right\} = \frac{C}{B}z_{-1}, \\
 y_{5k+9} &= \max\left\{\frac{B}{z_{5k+7}}, y_{5k+4}\right\} = \max\left\{\frac{B}{C}y_0, \frac{B}{C}y_0\right\} = \frac{B}{C}y_0, \quad z_{5k+9} = \max\left\{\frac{C}{y_{5k+7}}, z_{5k+4}\right\} = \max\left\{\frac{C}{B}z_0, \frac{C}{B}z_0\right\} = \frac{C}{B}z_0,
 \end{aligned}$$

and

$$y_{5k+10} = \max\left\{\frac{B}{z_{5k+8}}, y_{5k+5}\right\} = \max\left\{\frac{B^2}{Cz_{-1}}, \frac{B^2}{Cz_{-1}}\right\} = \frac{B^2}{Cz_{-1}}, \quad z_{5k+10} = \max\left\{\frac{C}{y_{5k+8}}, z_{5k+5}\right\} = \max\left\{\frac{C^2}{By_{-1}}, \frac{C^2}{By_{-1}}\right\} = \frac{C^2}{By_{-1}}.$$

For case  $(a_{1-1-1-1-1})$ , assuming

$$\frac{B}{z_0} > y_{-3}, \frac{C}{y_0} > z_{-3}, \frac{B}{y_{-2}} > \frac{C}{y_{-1}}, \frac{C}{z_{-2}} > \frac{B}{z_{-1}}, \frac{B}{y_{-1}} > \frac{C}{y_0}, \frac{C}{z_{-1}} > \frac{B}{z_0}, \frac{B^2}{Cz_{-1}} > y_0, \frac{C^2}{By_{-1}} > z_0$$

and  $k \geq 0$ , mathematical induction yields the following equalities:

$$\begin{aligned}
 y_{5k+1} &= \frac{B}{z_{-1}}, y_{5k+2} = \frac{B}{z_0}, y_{5k+3} = \frac{B}{C}y_{-1}, y_{5k+4} = \frac{B}{C}y_0, y_{5k+5} = \frac{B^2}{Cz_{-1}}, \\
 z_{5k+1} &= \frac{C}{y_{-1}}, z_{5k+2} = \frac{C}{y_0}, z_{5k+3} = \frac{C}{B}z_{-1}, z_{5k+4} = \frac{C}{B}z_0, z_{5k+5} = \frac{C^2}{By_{-1}}.
 \end{aligned}$$

**Remark 2.** For the remaining 63 cases in  $(a_{1-1})$ , methods similar to those described above can be employed to prove the existence of periodic solutions of period 5 and to derive the corresponding explicit forms of the solutions.

$(a_{1-2})$  From Equation (11), if  $\frac{B}{z_0} > y_{-3}, \frac{C}{y_0} < z_{-3}$ , then one obtains  $y_2 = \frac{B}{z_0}, z_2 = z_{-3}$ .

$(a_{1-2-1})$  From Equation (12), if  $\frac{B}{z_1} > y_{-2}, \frac{C}{y_1} > z_{-2}$ , then one obtains

$$y_3 = \frac{B}{z_1} = \frac{B}{C}y_{-1}, z_3 = \frac{C}{y_1} = \frac{C}{B}z_{-1}.$$

$(a_{1-2-1-1})$  From Equation (13), if  $\frac{B}{z_2} > y_{-1}, \frac{C}{y_2} > z_{-1}$ , then one obtains

$$y_4 = \frac{B}{z_2} = \frac{B}{z_{-3}}, z_4 = \frac{C}{y_2} = \frac{C}{B}z_0.$$

$(a_{1-2-1-1-1})$  From Equation (14), if  $\frac{B}{z_3} > y_0, \frac{C}{y_3} > z_0$ , then one obtains

$$y_5 = \frac{B}{z_3} = \frac{B^2}{Cz_{-1}}, z_5 = \frac{C}{y_3} = \frac{C^2}{By_{-1}}.$$

By  $(a_{1-2-1-1-1})$ , it follows that  $\frac{B}{z_2} = \frac{B}{z_{-3}} > y_{-1}, \frac{C}{y_2} = \frac{C}{B}z_0 > z_{-1}$ , which implies

$$\frac{B}{Cz_{-3}} > \frac{y_{-1}}{C}, Cz_0 > Bz_{-1},$$

therefore,

$$y_6 = \max\left\{\frac{B}{z_4}, y_1\right\} = \max\left\{\frac{B^2}{Cz_0}, \frac{B}{z_{-1}}\right\} = \frac{B}{z_{-1}}, \quad z_6 = \max\left\{\frac{C}{y_4}, z_1\right\} = \max\left\{\frac{C}{B}z_{-3}, \frac{C}{y_{-1}}\right\} = \frac{C}{y_{-1}},$$

by  $(a_{1-2-1-1-1})$ , it follows that  $\frac{B}{z_3} = \frac{B^2}{Cz_{-1}} > y_0, \frac{C}{y_3} = \frac{C^2}{By_{-1}} > z_0$ , which implies

$$\frac{B^2 y_{-1}}{C^2} < \frac{B}{z_0}, \frac{C^2 z_{-1}}{B^2} < \frac{C}{y_0},$$

hence,

$$y_7 = \max\left\{\frac{B}{z_5}, y_2\right\} = \max\left\{\frac{B^2 y_{-1}}{C^2}, \frac{B}{z_0}\right\} = \frac{B}{z_0}, z_7 = \max\left\{\frac{C}{y_5}, z_2\right\} = \max\left\{\frac{C^2 z_{-1}}{B^2}, \frac{C}{y_0}\right\} = \frac{C}{y_0},$$

$$y_8 = \max\left\{\frac{B}{z_6}, y_3\right\} = \max\left\{\frac{B}{C} y_{-1}, \frac{B}{C} y_{-1}\right\} = \frac{B}{C} y_{-1}, z_8 = \max\left\{\frac{C}{y_6}, z_3\right\} = \max\left\{\frac{C}{B} z_{-1}, \frac{C}{B} z_{-1}\right\} = \frac{C}{B} z_{-1},$$

$$y_9 = \max\left\{\frac{B}{z_7}, y_4\right\} = \max\left\{\frac{B}{z_{-3}}, \frac{B}{z_{-3}}\right\} = \frac{B}{z_{-3}}, z_9 = \max\left\{\frac{C}{y_7}, z_4\right\} = \max\left\{\frac{C}{B} z_0, \frac{C}{B} z_0\right\} = \frac{C}{B} z_0,$$

and

$$y_{10} = \max\left\{\frac{B}{z_8}, y_5\right\} = \max\left\{\frac{B^2}{Cz_{-1}}, \frac{B^2}{Cz_{-1}}\right\} = \frac{B^2}{Cz_{-1}}, z_{10} = \max\left\{\frac{C}{y_8}, z_5\right\} = \max\left\{\frac{C^2}{By_{-1}}, \frac{C^2}{By_{-1}}\right\} = \frac{C^2}{By_{-1}}.$$

If the relations

$$y_{5k+1} = \frac{B}{z_{-1}}, y_{5k+2} = \frac{B}{z_0}, y_{5k+3} = \frac{B}{C} y_{-1}, y_{5k+4} = \frac{B}{z_{-3}}, y_{5k+5} = \frac{B^2}{Cz_{-1}}$$

and

$$z_{5k+1} = \frac{C}{y_{-1}}, z_{5k+2} = z_{-3}, z_{5k+3} = \frac{C}{B} z_{-1}, z_{5k+4} = \frac{C}{B} z_0, z_{5k+5} = \frac{C^2}{By_{-1}}$$

are satisfied, then the following expressions are derived

$$y_{5k+6} = \max\left\{\frac{B}{z_{5k+4}}, y_{5k+1}\right\} = \max\left\{\frac{B^2}{Cz_0}, \frac{B}{z_{-1}}\right\} = \frac{B}{z_{-1}}, z_{5k+6} = \max\left\{\frac{C}{y_{5k+4}}, z_{5k+1}\right\} = \max\left\{\frac{Cz_{-3}}{B}, \frac{C}{y_{-1}}\right\} = \frac{C}{y_{-1}},$$

$$y_{5k+7} = \max\left\{\frac{B}{z_{5k+5}}, y_{5k+2}\right\} = \max\left\{\frac{B^2}{C^2} y_{-1}, \frac{B}{z_0}\right\} = \frac{B}{z_0}, z_{5k+7} = \max\left\{\frac{C}{y_{5k+5}}, z_{5k+2}\right\} = \max\left\{\frac{C^2}{B^2} z_{-1}, z_{-3}\right\} = z_{-3},$$

$$y_{5k+8} = \max\left\{\frac{B}{z_{5k+6}}, y_{5k+3}\right\} = \max\left\{\frac{B}{C} y_{-1}, \frac{B}{C} y_{-1}\right\} = \frac{B}{C} y_{-1}, z_{5k+8} = \max\left\{\frac{C}{y_{5k+6}}, z_{5k+3}\right\} = \max\left\{\frac{C}{B} z_{-1}, \frac{C}{B} z_{-1}\right\} = \frac{C}{B} z_{-1},$$

$$y_{5k+9} = \max\left\{\frac{B}{z_{5k+7}}, y_{5k+4}\right\} = \max\left\{\frac{B}{z_{-3}}, \frac{B}{z_{-3}}\right\} = \frac{B}{z_{-3}}, z_{5k+9} = \max\left\{\frac{C}{y_{5k+7}}, z_{5k+4}\right\} = \max\left\{\frac{C}{B} z_0, \frac{C}{B} z_0\right\} = \frac{C}{B} z_0,$$

and

$$y_{5k+10} = \max\left\{\frac{B}{z_{5k+8}}, y_{5k+5}\right\} = \max\left\{\frac{B^2}{Cz_{-1}}, \frac{B^2}{Cz_{-1}}\right\} = \frac{B^2}{Cz_{-1}}, z_{5k+10} = \max\left\{\frac{C}{y_{5k+8}}, z_{5k+5}\right\} = \max\left\{\frac{C^2}{By_{-1}}, \frac{C^2}{By_{-1}}\right\} = \frac{C^2}{By_{-1}}.$$

For case  $(a_{1-2-1-1-1})$ , assuming

$$\frac{B}{z_0} > y_{-3}, \frac{C}{y_0} < z_{-3}, \frac{B}{y_{-2}} > \frac{C}{y_{-1}}, \frac{C}{z_{-2}} > \frac{B}{z_{-1}}, \frac{B}{z_{-3}} > y_{-1}, \frac{C}{z_{-1}} > \frac{B}{z_0}, \frac{B^2}{Cz_{-1}} > y_0, \frac{C^2}{By_{-1}} > z_0$$

and  $k \geq 0$ , mathematical induction yields the following equalities:

$$y_{5k+1} = \frac{B}{z_{-1}}, y_{5k+2} = \frac{B}{z_0}, y_{5k+3} = \frac{B}{C} y_{-1}, y_{5k+4} = \frac{B}{z_{-3}}, y_{5k+5} = \frac{B^2}{Cz_{-1}},$$

$$z_{5k+1} = \frac{C}{y_{-1}}, z_{5k+2} = z_{-3}, z_{5k+3} = \frac{C}{B} z_{-1}, z_{5k+4} = \frac{C}{B} z_0, z_{5k+5} = \frac{C^2}{By_{-1}}.$$

**Remark 3.** For the remaining 63 cases in  $(a_{1-2})$ , methods similar to those described above can be employed to prove the existence of periodic solutions of period 5 and to derive the corresponding explicit forms of the solutions.

$(a_{1-3})$  From Equation (11), if  $\frac{B}{z_0} < y_{-3}, \frac{C}{y_0} > z_{-3}$ , then one obtains  $y_2 = y_{-3}, z_2 = \frac{C}{y_0}$ .  
 $(a_{1-3-1})$  From Equation (12), if  $\frac{B}{z_1} > y_{-2}, \frac{C}{y_1} > z_{-2}$ , then one obtains

$$y_3 = \frac{B}{z_1} = \frac{B}{C}y_{-1}, z_3 = \frac{C}{y_1} = \frac{C}{B}z_{-1}.$$

$(a_{1-3-1-1})$  From Equation (13), if  $\frac{B}{z_2} > y_{-1}, \frac{C}{y_2} > z_{-1}$ , then one obtains

$$y_4 = \frac{B}{z_2} = \frac{B}{C}y_0, z_4 = \frac{C}{y_2} = \frac{C}{y_{-3}}.$$

$(a_{1-3-1-1-1})$  From Equation (14), if  $\frac{B}{z_3} > y_0, \frac{C}{y_3} > z_0$ , then one obtains

$$y_5 = \frac{B}{z_3} = \frac{B^2}{Cz_{-1}}, z_5 = \frac{C}{y_3} = \frac{C^2}{By_{-1}}.$$

By  $(a_{1-3-1-1})$ , it follows that  $\frac{C}{y_2} = \frac{C}{y_{-3}} > z_{-1}, \frac{B}{C}y_0 > y_{-1}$ , which implies

$$\frac{B}{z_{-1}} > \frac{B}{C}y_{-3}, By_0 > Cy_{-1},$$

therefore,

$$y_6 = \max\left\{\frac{B}{z_4}, y_1\right\} = \max\left\{\frac{B}{C}y_{-3}, \frac{B}{z_{-1}}\right\} = \frac{B}{z_{-1}}, z_6 = \max\left\{\frac{C}{y_4}, z_1\right\} = \max\left\{\frac{C^2}{By_0}, \frac{C}{y_{-1}}\right\} = \frac{C}{y_{-1}},$$

by  $(a_{1-3-1-1-1})$ , it follows that  $\frac{B}{z_3} = \frac{B^2}{Cz_{-1}} > y_0, \frac{C}{y_3} = \frac{C^2}{By_{-1}} > z_0$ , which implies

$$\frac{B^2}{C^2}y_{-1} < \frac{B}{z_0} < y_{-3}, \frac{C^2}{B^2}z_{-1} < \frac{C}{y_0},$$

hence,

$$y_7 = \max\left\{\frac{B}{z_5}, y_2\right\} = \max\left\{\frac{B^2}{C^2}y_{-1}, y_{-3}\right\} = y_{-3}, z_7 = \max\left\{\frac{C}{y_5}, z_2\right\} = \max\left\{\frac{C^2}{B^2}z_{-1}, \frac{C}{y_0}\right\} = \frac{C}{y_0},$$

$$y_8 = \max\left\{\frac{B}{z_6}, y_3\right\} = \max\left\{\frac{B}{C}y_{-1}, \frac{B}{C}y_{-1}\right\} = \frac{B}{C}y_{-1}, z_8 = \max\left\{\frac{C}{y_6}, z_3\right\} = \max\left\{\frac{C}{B}z_{-1}, \frac{C}{B}z_{-1}\right\} = \frac{C}{B}z_{-1},$$

$$y_9 = \max\left\{\frac{B}{z_7}, y_4\right\} = \max\left\{\frac{B}{C}y_0, \frac{B}{C}y_0\right\} = \frac{B}{C}y_0, z_9 = \max\left\{\frac{C}{y_7}, z_4\right\} = \max\left\{\frac{C}{y_{-3}}, \frac{C}{y_{-3}}\right\} = \frac{C}{y_{-3}},$$

and

$$y_{10} = \max\left\{\frac{B}{z_8}, y_5\right\} = \max\left\{\frac{B^2}{Cz_{-1}}, \frac{B^2}{Cz_{-1}}\right\} = \frac{B^2}{Cz_{-1}}, z_{10} = \max\left\{\frac{C}{y_8}, z_5\right\} = \max\left\{\frac{C^2}{By_{-1}}, \frac{C^2}{By_{-1}}\right\} = \frac{C^2}{By_{-1}}.$$

If the relations

$$y_{5k+1} = \frac{B}{z_{-1}}, y_{5k+2} = y_{-3}, y_{5k+3} = \frac{B}{C}y_{-1}, y_{5k+4} = \frac{B}{C}y_0, y_{5k+5} = \frac{B^2}{Cz_{-1}}$$

and

$$z_{5k+1} = \frac{C}{y_{-1}}, z_{5k+2} = \frac{C}{y_0}, z_{5k+3} = \frac{C}{B}z_{-1}, z_{5k+4} = \frac{C}{B}z_0, z_{5k+5} = \frac{C^2}{By_{-1}}$$

are satisfied, then the following expressions are derived

$$\begin{aligned}
 y_{5k+6} &= \max\left\{\frac{B}{z_{5k+4}}, y_{5k+1}\right\} = \max\left\{\frac{B}{C}y_{-3}, \frac{B}{z_{-1}}\right\} = \frac{B}{z_{-1}}, \quad z_{5k+6} = \max\left\{\frac{C}{y_{5k+4}}, z_{5k+1}\right\} = \max\left\{\frac{C^2}{By_0}, \frac{C}{y_{-1}}\right\} = \frac{C}{y_{-1}}, \\
 y_{5k+7} &= \max\left\{\frac{B}{z_{5k+5}}, y_{5k+2}\right\} = \max\left\{\frac{B^2}{C^2}y_{-1}, y_{-3}\right\} = y_{-3}, \quad z_{5k+7} = \max\left\{\frac{C}{y_{5k+5}}, z_{5k+2}\right\} = \max\left\{\frac{C^2}{B^2}z_{-1}, \frac{C}{y_0}\right\} = \frac{C}{y_0}, \\
 y_{5k+8} &= \max\left\{\frac{B}{z_{5k+6}}, y_{5k+3}\right\} = \max\left\{\frac{B}{C}y_{-1}, \frac{B}{C}y_{-1}\right\} = \frac{B}{C}y_{-1}, \quad z_{5k+8} = \max\left\{\frac{C}{y_{5k+6}}, z_{5k+3}\right\} = \max\left\{\frac{C}{B}z_{-1}, \frac{C}{B}z_{-1}\right\} = \frac{C}{B}z_{-1}, \\
 y_{5k+9} &= \max\left\{\frac{B}{z_{5k+7}}, y_{5k+4}\right\} = \max\left\{\frac{B}{C}y_0, \frac{B}{C}y_0\right\} = \frac{B}{C}y_0, \quad z_{5k+9} = \max\left\{\frac{C}{y_{5k+7}}, z_{5k+4}\right\} = \max\left\{\frac{C}{y_{-3}}, \frac{C}{y_{-3}}\right\} = \frac{C}{y_{-3}},
 \end{aligned}$$

and

$$y_{5k+10} = \max\left\{\frac{B}{z_{5k+8}}, y_{5k+5}\right\} = \max\left\{\frac{B^2}{Cz_{-1}}, \frac{B^2}{Cz_{-1}}\right\} = \frac{B^2}{Cz_{-1}}, \quad z_{5k+10} = \max\left\{\frac{C}{y_{5k+8}}, z_{5k+5}\right\} = \max\left\{\frac{C^2}{By_{-1}}, \frac{C^2}{By_{-1}}\right\} = \frac{C^2}{By_{-1}}.$$

For case  $(a_{1-3-1-1-1})$ , assuming

$$\frac{B}{z_0} < y_{-3}, \frac{C}{y_0} > z_{-3}, \frac{B}{y_{-2}} > \frac{C}{y_{-1}}, \frac{C}{z_{-2}} > \frac{B}{z_{-1}}, \frac{B}{y_{-1}} > \frac{C}{y_0}, \frac{C}{y_{-3}} > z_{-1}, \frac{B^2}{Cz_{-1}} > y_0, \frac{C^2}{By_{-1}} > z_0$$

and  $k \geq 0$ , mathematical induction yields the following equalities:

$$\begin{aligned}
 y_{5k+1} &= \frac{B}{z_{-1}}, y_{5k+2} = y_{-3}, y_{5k+3} = \frac{B}{C}y_{-1}, y_{5k+4} = \frac{B}{C}y_0, y_{5k+5} = \frac{B^2}{Cz_{-1}}, \\
 z_{5k+1} &= \frac{C}{y_{-1}}, z_{5k+2} = \frac{C}{y_0}, z_{5k+3} = \frac{C}{B}z_{-1}, z_{5k+4} = \frac{C}{y_{-3}}, z_{5k+5} = \frac{C^2}{By_{-1}}.
 \end{aligned}$$

**Remark 4.** For the remaining 63 cases in  $(a_{1-3})$ , methods similar to those described above can be employed to prove the existence of periodic solutions of period 5 and to derive the corresponding explicit forms of the solutions.

$(a_{1-4})$  From Equation (11), if  $\frac{B}{z_0} < y_{-3}, \frac{C}{y_0} < z_{-3}$ , then one obtains  $y_2 = y_{-3}, z_2 = z_{-3}$ .

$(a_{1-4-1})$  From Equation (12), if  $\frac{B}{z_1} > y_{-2}, \frac{C}{y_1} > z_{-2}$ , then one obtains

$$y_3 = \frac{B}{z_1} = \frac{B}{C}y_{-1}, z_3 = \frac{C}{y_1} = \frac{C}{B}z_{-1}.$$

$(a_{1-4-1-1})$  From Equation (13), if  $\frac{B}{z_2} > y_{-1}, \frac{C}{y_2} > z_{-1}$ , then one obtains

$$y_4 = \frac{B}{z_2} = \frac{B}{z_{-3}}, z_4 = \frac{C}{y_2} = \frac{C}{y_{-3}}.$$

$(a_{1-4-1-1-1})$  From Equation (14), if  $\frac{B}{z_3} > y_0, \frac{C}{y_3} > z_0$ , then one obtains

$$y_5 = \frac{B}{z_3} = \frac{B^2}{Cz_{-1}}, z_5 = \frac{C}{y_3} = \frac{C^2}{By_{-1}}.$$

By  $(a_{1-4-1-1-1})$ , it follows that  $\frac{C}{y_2} = \frac{C}{y_{-3}} > z_{-1}, \frac{B}{z_2} = \frac{B}{z_{-3}} > y_{-1}$ , which implies

$$\frac{B}{z_{-1}} > \frac{B}{C}y_{-3}, \frac{C}{y_{-1}} > \frac{C}{B}z_{-3},$$

therefore,

$$y_6 = \max\left\{\frac{B}{z_4}, y_1\right\} = \max\left\{\frac{B}{C}y_{-3}, \frac{B}{z_{-1}}\right\} = \frac{B}{z_{-1}}, z_6 = \max\left\{\frac{C}{y_4}, z_1\right\} = \max\left\{\frac{C}{B}z_{-3}, \frac{C}{y_{-1}}\right\} = \frac{C}{y_{-1}},$$

by  $(a_{1-4-1-1-1})$ , it follows that  $\frac{B}{z_3} = \frac{B^2}{Cz_{-1}} > y_0, \frac{C}{y_3} = \frac{C^2}{By_{-1}} > z_0$ , which implies

$$\frac{B^2}{C^2}y_{-1} < \frac{B}{z_0} < y_{-3}, \frac{C^2}{B^2}z_{-1} < \frac{C}{y_0} < z_{-3},$$

hence,

$$y_7 = \max\left\{\frac{B}{z_5}, y_2\right\} = \max\left\{\frac{B^2}{C^2}y_{-1}, y_{-3}\right\} = y_{-3}, z_7 = \max\left\{\frac{C}{y_5}, z_2\right\} = \max\left\{\frac{C^2}{B^2}z_{-1}, z_{-3}\right\} = z_{-3},$$

$$y_8 = \max\left\{\frac{B}{z_6}, y_3\right\} = \max\left\{\frac{B}{C}y_{-1}, \frac{B}{C}y_{-1}\right\} = \frac{B}{C}y_{-1}, z_8 = \max\left\{\frac{C}{y_6}, z_3\right\} = \max\left\{\frac{C}{B}z_{-1}, \frac{C}{B}z_{-1}\right\} = \frac{C}{B}z_{-1},$$

$$y_9 = \max\left\{\frac{B}{z_7}, y_4\right\} = \max\left\{\frac{B}{z_{-3}}, \frac{B}{z_{-3}}\right\} = \frac{B}{z_{-3}}, z_9 = \max\left\{\frac{C}{y_7}, z_4\right\} = \max\left\{\frac{C}{y_{-3}}, \frac{C}{y_{-3}}\right\} = \frac{C}{y_{-3}},$$

and

$$y_{10} = \max\left\{\frac{B}{z_8}, y_5\right\} = \max\left\{\frac{B^2}{Cz_{-1}}, \frac{B^2}{Cz_{-1}}\right\} = \frac{B^2}{Cz_{-1}}, z_{10} = \max\left\{\frac{C}{y_8}, z_5\right\} = \max\left\{\frac{C^2}{By_{-1}}, \frac{C^2}{By_{-1}}\right\} = \frac{C^2}{By_{-1}}.$$

If the relations

$$y_{5k+1} = \frac{B}{z_{-1}}, y_{5k+2} = y_{-3}, y_{5k+3} = \frac{B}{C}y_{-1}, y_{5k+4} = \frac{B}{z_{-3}}, y_{5k+5} = \frac{B^2}{Cz_{-1}}$$

and

$$z_{5k+1} = \frac{C}{y_{-1}}, z_{5k+2} = z_{-3}, z_{5k+3} = \frac{C}{B}z_{-1}, z_{5k+4} = \frac{C}{y_{-3}}, z_{5k+5} = \frac{C^2}{By_{-1}}$$

are satisfied, then the following expressions are derived

$$y_{5k+6} = \max\left\{\frac{B}{z_{5k+4}}, y_{5k+1}\right\} = \max\left\{\frac{B}{C}y_{-3}, \frac{B}{z_{-1}}\right\} = \frac{B}{z_{-1}}, z_{5k+6} = \max\left\{\frac{C}{y_{5k+4}}, z_{5k+1}\right\} = \max\left\{\frac{C}{B}z_{-3}, \frac{C}{y_{-1}}\right\} = \frac{C}{y_{-1}},$$

$$y_{5k+7} = \max\left\{\frac{B}{z_{5k+5}}, y_{5k+2}\right\} = \max\left\{\frac{B^2}{C^2}y_{-1}, y_{-3}\right\} = y_{-3}, z_{5k+7} = \max\left\{\frac{C}{y_{5k+5}}, z_{5k+2}\right\} = \max\left\{\frac{C^2}{B^2}z_{-1}, z_{-3}\right\} = z_{-3},$$

$$y_{5k+8} = \max\left\{\frac{B}{z_{5k+6}}, y_{5k+3}\right\} = \max\left\{\frac{B}{C}y_{-1}, \frac{B}{C}y_{-1}\right\} = \frac{B}{C}y_{-1}, z_{5k+8} = \max\left\{\frac{C}{y_{5k+6}}, z_{5k+3}\right\} = \max\left\{\frac{C}{B}z_{-1}, \frac{C}{B}z_{-1}\right\} = \frac{C}{B}z_{-1},$$

$$y_{5k+9} = \max\left\{\frac{B}{z_{5k+7}}, y_{5k+4}\right\} = \max\left\{\frac{B}{z_{-3}}, \frac{B}{z_{-3}}\right\} = \frac{B}{z_{-3}}, z_{5k+9} = \max\left\{\frac{C}{y_{5k+7}}, z_{5k+4}\right\} = \max\left\{\frac{C}{y_{-3}}, \frac{C}{y_{-3}}\right\} = \frac{C}{y_{-3}},$$

and

$$y_{5k+10} = \max\left\{\frac{B}{z_{5k+8}}, y_{5k+5}\right\} = \max\left\{\frac{B^2}{Cz_{-1}}, \frac{B^2}{Cz_{-1}}\right\} = \frac{B^2}{Cz_{-1}}, z_{5k+10} = \max\left\{\frac{C}{y_{5k+8}}, z_{5k+5}\right\} = \max\left\{\frac{C^2}{By_{-1}}, \frac{C^2}{By_{-1}}\right\} = \frac{C^2}{By_{-1}}.$$

For case  $(a_{1-4-1-1-1})$ , assuming

$$\frac{B}{z_0} < y_{-3}, \frac{C}{y_0} < z_{-3}, \frac{B}{y_{-2}} > \frac{C}{y_{-1}}, \frac{C}{z_{-2}} > \frac{B}{z_{-1}}, \frac{B}{z_{-3}} > y_{-1}, \frac{C}{y_{-3}} > z_{-1}, \frac{B^2}{Cz_{-1}} > y_0, \frac{C^2}{By_{-1}} > z_0$$

and  $k \geq 0$ , mathematical induction yields the following equalities:

$$y_{5k+1} = \frac{B}{z_{-1}}, y_{5k+2} = y_{-3}, y_{5k+3} = \frac{B}{C}y_{-1}, y_{5k+4} = \frac{B}{z_{-3}}, y_{5k+5} = \frac{B^2}{Cz_{-1}},$$

$$z_{5k+1} = \frac{C}{y_{-1}}, z_{5k+2} = z_{-3}, z_{5k+3} = \frac{C}{B}z_{-1}, z_{5k+4} = \frac{C}{y_{-3}}, z_{5k+5} = \frac{C^2}{By_{-1}}.$$

**Remark 5.** For the remaining 63 cases in  $(a_{1-4})$ , methods similar to those described above can be employed to prove the existence of periodic solutions of period 5 and to derive the corresponding explicit forms of the solutions.

- $(a_2)$  From Equation (10), if  $\frac{B}{z_{-1}} < y_{-4}, \frac{C}{y_{-1}} > z_{-4}$ , then one obtains  $y_1 = y_{-4}, z_1 = \frac{C}{y_{-1}}$ .
- $(a_{2-1})$  From Equation (11), if  $\frac{B}{z_0} > y_{-3}, \frac{C}{y_0} > z_{-3}$ , then one obtains  $y_2 = \frac{B}{z_0}, z_2 = \frac{C}{y_0}$ .
- $(a_{2-1-1})$  From Equation (12), if  $\frac{B}{z_1} > y_{-2}, \frac{C}{y_1} > z_{-2}$ , then one obtains

$$y_3 = \frac{B}{z_1} = \frac{B}{C}y_{-1}, z_3 = \frac{C}{y_1} = \frac{C}{y_{-4}}.$$

- $(a_{2-1-1-1})$  From Equation (13), if  $\frac{B}{z_2} > y_{-1}, \frac{C}{y_2} > z_{-1}$ , then one obtains

$$y_4 = \frac{B}{z_2} = \frac{B}{C}y_0, z_4 = \frac{C}{y_2} = \frac{C}{B}z_0.$$

- $(a_{2-1-1-1-1})$  From Equation (14), if  $\frac{B}{z_3} > y_0, \frac{C}{y_3} > z_0$ , then one obtains

$$y_5 = \frac{B}{z_3} = \frac{B}{C}y_{-4}, z_5 = \frac{C}{y_3} = \frac{C^2}{By_{-1}}.$$

By  $(a_{2-1-1-1})$ , it follows that  $By_0 > Cy_{-1}, Cz_0 > Bz_{-1}$ , which implies

$$\frac{B}{z_4} = \frac{B^2}{Cz_0} < \frac{B}{z_{-1}} < y_{-4}, \frac{C}{y_{-1}} > \frac{C^2}{By_0},$$

therefore,

$$y_6 = \max\left\{\frac{B}{z_4}, y_1\right\} = \max\left\{\frac{B^2}{Cz_0}, y_{-4}\right\} = y_{-4}, z_6 = \max\left\{\frac{C}{y_4}, z_1\right\} = \max\left\{\frac{C^2}{By_0}, \frac{C}{y_{-1}}\right\} = \frac{C}{y_{-1}}.$$

By  $(a_{2-1-1-1-1})$ , it follows that  $\frac{C^2}{By_{-1}} > z_0, \frac{B}{C}y_{-4} > y_0$ , which implies

$$\frac{B^2y_{-1}}{C^2} < \frac{B}{z_0}, \frac{C^2}{By_{-4}} < \frac{C}{y_0},$$

hence,

$$y_7 = \max\left\{\frac{B}{z_5}, y_2\right\} = \max\left\{\frac{B^2y_{-1}}{C^2}, \frac{B}{z_0}\right\} = y_2, z_7 = \max\left\{\frac{C}{y_5}, z_2\right\} = \max\left\{\frac{C^2}{By_{-4}}, \frac{C}{y_0}\right\} = \frac{C}{y_0},$$

$$y_8 = \max\left\{\frac{B}{z_6}, y_3\right\} = \max\left\{\frac{B}{C}y_{-1}, \frac{B}{C}y_{-1}\right\} = \frac{B}{C}y_{-1}, z_8 = \max\left\{\frac{C}{y_6}, z_3\right\} = \max\left\{\frac{C}{y_{-4}}, \frac{C}{y_{-4}}\right\} = \frac{C}{y_{-4}},$$

$$y_9 = \max\left\{\frac{B}{z_7}, y_4\right\} = \max\left\{\frac{B}{C}y_0, \frac{B}{C}y_0\right\} = \frac{B}{C}y_0, z_9 = \max\left\{\frac{C}{y_7}, z_4\right\} = \max\left\{\frac{C}{B}z_0, \frac{C}{B}z_0\right\} = \frac{C}{B}z_0,$$

and

$$y_{10} = \max\left\{\frac{B}{z_8}, y_5\right\} = \max\left\{\frac{B}{C}y_{-4}, \frac{B}{C}y_{-4}\right\} = \frac{B}{C}y_{-4}, z_{10} = \max\left\{\frac{C}{y_8}, z_5\right\} = \max\left\{\frac{C^2}{By_{-1}}, \frac{C^2}{By_{-1}}\right\} = \frac{C^2}{By_{-1}}.$$

If the relations

$$y_{5k+1} = y_{-4}, y_{5k+2} = \frac{B}{z_0}, y_{5k+3} = \frac{B}{C}y_{-1}, y_{5k+4} = \frac{B}{C}y_0, y_{5k+5} = \frac{B}{C}y_{-4}$$

and

$$z_{5k+1} = \frac{C}{y_{-1}}, z_{5k+2} = \frac{C}{y_0}, z_{5k+3} = \frac{C}{y_{-4}}, z_{5k+4} = \frac{C}{B}z_0, z_{5k+5} = \frac{C^2}{By_{-1}}$$

are satisfied, then the following expressions are derived

$$y_{5k+6} = \max\left\{\frac{B}{z_{5k+4}}, y_{5k+1}\right\} = \max\left\{\frac{B^2}{Cz_0}, y_{-4}\right\} = y_{-4}, z_{5k+6} = \max\left\{\frac{C}{y_{5k+4}}, z_{5k+1}\right\} = \max\left\{\frac{C^2}{By_0}, \frac{C}{y_{-1}}\right\} = \frac{C}{y_{-1}},$$

$$y_{5k+7} = \max\left\{\frac{B}{z_{5k+5}}, y_{5k+2}\right\} = \max\left\{\frac{B^2y_{-1}}{C^2}, \frac{B}{z_0}\right\} = \frac{B}{z_0}, z_{5k+7} = \max\left\{\frac{C}{y_{5k+5}}, z_{5k+2}\right\} = \max\left\{\frac{C^2}{By_{-4}}, \frac{C}{y_0}\right\} = \frac{C}{y_0},$$

$$y_{5k+8} = \max\left\{\frac{B}{z_{5k+6}}, y_{5k+3}\right\} = \max\left\{\frac{B}{C}y_{-1}, \frac{B}{C}y_{-1}\right\} = \frac{B}{C}y_{-1}, z_{5k+8} = \max\left\{\frac{C}{y_{5k+6}}, z_{5k+3}\right\} = \max\left\{\frac{C}{y_{-4}}, \frac{C}{y_{-4}}\right\} = \frac{C}{y_{-4}},$$

$$y_{5k+9} = \max\left\{\frac{B}{z_{5k+7}}, y_{5k+4}\right\} = \max\left\{\frac{B}{C}y_0, \frac{B}{C}y_0\right\} = \frac{B}{C}y_0, z_{5k+9} = \max\left\{\frac{C}{y_{5k+7}}, z_{5k+4}\right\} = \max\left\{\frac{C}{B}z_0, \frac{C}{B}z_0\right\} = \frac{C}{B}z_0,$$

and

$$y_{5k+10} = \max\left\{\frac{B}{z_{5k+8}}, y_{5k+5}\right\} = \max\left\{\frac{B}{C}y_{-4}, \frac{B}{C}y_{-4}\right\} = \frac{B}{C}y_{-4}, z_{5k+10} = \max\left\{\frac{C}{y_{5k+8}}, z_{5k+5}\right\} = \max\left\{\frac{C^2}{By_{-1}}, \frac{C^2}{By_{-1}}\right\} = \frac{C^2}{By_{-1}}.$$

For case  $(a_{2-1-1-1-1})$ , assuming

$$\frac{B}{z_0} > y_{-3}, \frac{C}{y_0} > z_{-3}, \frac{B}{z_1} > y_{-2}, \frac{C}{y_1} > z_{-2}, \frac{B}{z_2} > y_{-1}, \frac{C}{y_2} > z_{-1}, \frac{B}{z_3} > y_0, \frac{C}{y_3} > z_0$$

and  $k \geq 0$ , mathematical induction yields the following equalities:

$$y_{5k+1} = y_{-4}, y_{5k+2} = \frac{B}{z_0}, y_{5k+3} = \frac{B}{C}y_{-1}, y_{5k+4} = \frac{B}{C}y_0, y_{5k+5} = \frac{B}{C}y_{-4},$$

$$z_{5k+1} = \frac{C}{y_{-1}}, z_{5k+2} = \frac{C}{y_0}, z_{5k+3} = \frac{C}{y_{-4}}, z_{5k+4} = \frac{C}{B}z_0, z_{5k+5} = \frac{C^2}{By_{-1}}.$$

**Remark 6.** For the remaining 63 cases in  $(a_{2-1})$ , methods similar to those described above can be employed to prove the existence of periodic solutions of period 5 and to derive the corresponding explicit forms of the solutions.

$(a_{2-2})$  From Equation (11), if  $\frac{B}{z_0} > y_{-3}, \frac{C}{y_0} < z_{-3}$ , then one obtains  $y_2 = \frac{B}{z_0}, z_2 = z_{-3}$ .

$(a_{2-2-1})$  From Equation (12), if  $\frac{B}{z_1} > y_{-2}, \frac{C}{y_1} > z_{-2}$ , then one obtains

$$y_3 = \frac{B}{z_1} = \frac{B}{C}y_{-1}, z_3 = \frac{C}{y_1} = \frac{C}{y_{-4}}.$$

$(a_{2-2-1-1})$  From Equation (13), if  $\frac{B}{z_2} > y_{-1}, \frac{C}{y_2} > z_{-1}$ , then one obtains

$$y_4 = \frac{B}{z_2} = \frac{B}{z_{-3}}, z_4 = \frac{C}{y_2} = \frac{C}{B}z_0.$$

( $a_{2-2-1-1-1}$ ) From Equation (14), if  $\frac{B}{z_3} > y_0, \frac{C}{y_3} > z_0$ , then one obtains

$$y_5 = \frac{B}{z_3} = \frac{B}{C}y_{-4}, z_5 = \frac{C}{y_3} = \frac{C^2}{By_{-1}}.$$

By ( $a_{2-2-1-1}$ ), it follows that  $\frac{B}{z_{-3}} > y_{-1}, Cz_0 > Bz_{-1}$ , which implies

$$\frac{B}{z_4} = \frac{B^2}{Cz_0} < \frac{B}{z_{-1}} < y_{-4}, \frac{C}{y_{-1}} > \frac{C}{B}z_{-3},$$

therefore,

$$y_6 = \max\left\{\frac{B}{z_4}, y_1\right\} = \max\left\{\frac{B^2}{Cz_0}, y_{-4}\right\} = y_{-4}, z_6 = \max\left\{\frac{C}{y_4}, z_1\right\} = \max\left\{\frac{C}{B}z_{-3}, \frac{C}{y_{-1}}\right\} = \frac{C}{y_{-1}}.$$

By ( $a_{2-2-1-1-1}$ ), it follows that  $\frac{C^2}{By_{-1}} > z_0, \frac{B}{C}y_{-4} > y_0$ , which implies

$$\frac{B^2}{C^2}y_{-1} < \frac{B}{z_0}, \frac{C^2}{By_{-4}} < \frac{C}{y_0} < z_{-3},$$

hence,

$$\begin{aligned} y_7 &= \max\left\{\frac{B}{z_5}, y_2\right\} = \max\left\{\frac{B^2y_{-1}}{C^2}, \frac{B}{z_0}\right\} = \frac{B}{z_0}, z_7 = \max\left\{\frac{C}{y_5}, z_2\right\} = \max\left\{\frac{C^2}{By_{-4}}, z_{-3}\right\} = z_{-3}, \\ y_8 &= \max\left\{\frac{B}{z_6}, y_3\right\} = \max\left\{\frac{B}{C}y_{-1}, \frac{B}{C}y_{-1}\right\} = \frac{B}{C}y_{-1}, z_8 = \max\left\{\frac{C}{y_6}, z_3\right\} = \max\left\{\frac{C}{y_{-4}}, \frac{C}{y_{-4}}\right\} = \frac{C}{y_{-4}}, \\ y_9 &= \max\left\{\frac{B}{z_7}, y_4\right\} = \max\left\{\frac{B}{z_{-3}}, \frac{B}{z_{-3}}\right\} = \frac{B}{z_{-3}}, z_9 = \max\left\{\frac{C}{y_7}, z_4\right\} = \max\left\{\frac{C}{B}z_0, \frac{C}{B}z_0\right\} = \frac{C}{B}z_0, \end{aligned}$$

and

$$y_{10} = \max\left\{\frac{B}{z_8}, y_5\right\} = \max\left\{\frac{B}{C}y_{-4}, \frac{B}{C}y_{-4}\right\} = \frac{B}{C}y_{-4}, z_{10} = \max\left\{\frac{C}{y_8}, z_5\right\} = \max\left\{\frac{C^2}{By_{-1}}, \frac{C^2}{By_{-1}}\right\} = \frac{C^2}{By_{-1}}.$$

If the relations

$$y_{5k+1} = y_{-4}, y_{5k+2} = \frac{B}{z_0}, y_{5k+3} = \frac{B}{C}y_{-1}, y_{5k+4} = \frac{B}{z_{-3}}, y_{5k+5} = \frac{B}{C}y_{-4}$$

and

$$z_{5k+1} = \frac{C}{y_{-1}}, z_{5k+2} = z_{-3}, z_{5k+3} = \frac{C}{y_{-4}}, z_{5k+4} = \frac{C}{B}z_0, z_{5k+5} = \frac{C^2}{By_{-1}}$$

are satisfied, then the following expressions are derived

$$\begin{aligned} y_{5k+6} &= \max\left\{\frac{B}{z_{5k+4}}, y_{5k+1}\right\} = \max\left\{\frac{B^2}{Cz_0}, y_{-4}\right\} = y_{-4}, z_{5k+6} = \max\left\{\frac{C}{y_{5k+4}}, z_{5k+1}\right\} = \max\left\{\frac{C}{B}z_{-3}, \frac{C}{y_{-1}}\right\} = \frac{C}{y_{-1}}, \\ y_{5k+7} &= \max\left\{\frac{B}{z_{5k+5}}, y_{5k+2}\right\} = \max\left\{\frac{B^2}{C^2}y_{-1}, \frac{B}{z_0}\right\} = \frac{B}{z_0}, z_{5k+7} = \max\left\{\frac{C}{y_{5k+5}}, z_{5k+2}\right\} = \max\left\{\frac{C^2}{By_{-4}}, z_{-3}\right\} = z_{-3}, \\ y_{5k+8} &= \max\left\{\frac{B}{z_{5k+6}}, y_{5k+3}\right\} = \max\left\{\frac{B}{C}y_{-1}, \frac{B}{C}y_{-1}\right\} = \frac{B}{C}y_{-1}, z_{5k+8} = \max\left\{\frac{C}{y_{5k+6}}, z_{5k+3}\right\} = \max\left\{\frac{C}{y_{-4}}, \frac{C}{y_{-4}}\right\} = \frac{C}{y_{-4}}, \\ y_{5k+9} &= \max\left\{\frac{B}{z_{5k+7}}, y_{5k+4}\right\} = \max\left\{\frac{B}{z_{-3}}, \frac{B}{z_{-3}}\right\} = \frac{B}{z_{-3}}, z_{5k+9} = \max\left\{\frac{C}{y_{5k+7}}, z_{5k+4}\right\} = \max\left\{\frac{C}{B}z_0, \frac{C}{B}z_0\right\} = \frac{C}{B}z_0, \\ y_{5k+10} &= \max\left\{\frac{B}{z_{5k+8}}, y_{5k+5}\right\} = \max\left\{\frac{B}{C}y_{-4}, \frac{B}{C}y_{-4}\right\} = \frac{B}{C}y_{-4}, z_{5k+10} = \max\left\{\frac{C}{y_{5k+8}}, z_{5k+5}\right\} = \max\left\{\frac{C^2}{By_{-1}}, \frac{C^2}{By_{-1}}\right\} = \frac{C^2}{By_{-1}}. \end{aligned}$$

For case  $(a_{2-2-1-1-1})$ , assuming

$$\frac{B}{z_0} > y_{-3}, \frac{C}{y_0} < z_{-3}, \frac{B}{y_{-2}} > \frac{C}{y_{-1}}, \frac{C}{y_{-4}} > z_{-2}, \frac{B}{z_{-3}} > y_{-1}, \frac{C}{z_{-1}} > \frac{B}{z_0}, \frac{B}{y_0} > \frac{C}{y_{-4}}, \frac{C^2}{By_{-1}} > z_0$$

and  $k \geq 0$ , mathematical induction yields the following equalities:

$$y_{5k+1} = y_{-4}, y_{5k+2} = \frac{B}{z_0}, y_{5k+3} = \frac{B}{C}y_{-1}, y_{5k+4} = \frac{B}{z_{-3}}, y_{5k+5} = \frac{B}{C}y_{-4},$$

$$z_{5k+1} = \frac{C}{y_{-1}}, z_{5k+2} = z_{-3}, z_{5k+3} = \frac{C}{y_{-4}}, z_{5k+4} = \frac{C}{B}z_0, z_{5k+5} = \frac{C^2}{By_{-1}}.$$

**Remark 7.** For the remaining 63 cases in  $(a_{2-2})$ , methods similar to those described above can be employed to prove the existence of periodic solutions of period 5 and to derive the corresponding explicit forms of the solutions.

$(a_{2-3})$  From Equation (11), if  $\frac{B}{z_0} < y_{-3}, \frac{C}{y_0} > z_{-3}$ , then one obtains  $y_2 = y_{-3}, z_2 = \frac{C}{y_0}$ .

$(a_{2-3-1})$  From Equation (12), if  $\frac{B}{z_1} > y_{-2}, \frac{C}{y_1} > z_{-2}$ , then one obtains

$$y_3 = \frac{B}{z_1} = \frac{B}{C}y_{-1}, z_3 = \frac{C}{y_1} = \frac{C}{y_{-4}}.$$

$(a_{2-3-1-1})$  From Equation (13), if  $\frac{B}{z_2} > y_{-1}, \frac{C}{y_2} > z_{-1}$ , then one obtains

$$y_4 = \frac{B}{z_2} = \frac{B}{C}y_0, z_4 = \frac{C}{y_2} = \frac{C}{y_{-3}}.$$

$(a_{2-3-1-1-1})$  From Equation (14), if  $\frac{B}{z_3} > y_0, \frac{C}{y_3} > z_0$ , then one obtains

$$y_5 = \frac{B}{z_3} = \frac{B}{C}y_{-4}, z_5 = \frac{C}{y_3} = \frac{C^2}{By_{-1}}.$$

By  $(a_{2-3-1-1-1})$ , it follows that  $\frac{B}{C}y_0 > y_{-1}, \frac{C}{y_{-3}} > z_{-1}$ , which implies

$$\frac{B}{z_4} = \frac{B}{C}y_{-3} < \frac{B}{z_{-1}} < y_{-4}, \frac{C}{y_{-1}} > \frac{C^2}{By_0},$$

therefore,

$$y_6 = \max\left\{\frac{B}{z_4}, y_1\right\} = \max\left\{\frac{B}{C}y_{-3}, y_{-4}\right\} = y_{-4}, z_6 = \max\left\{\frac{C}{y_4}, z_1\right\} = \max\left\{\frac{C^2}{By_0}, \frac{C}{y_{-1}}\right\} = \frac{C}{y_{-1}}.$$

By  $(a_{2-3-1-1-1-1})$ , it follows that  $\frac{C^2}{By_{-1}} > z_0, \frac{B}{C}y_{-4} > y_0$ , which implies

$$\frac{B^2}{C^2}y_{-1} < \frac{B}{z_0} < y_{-3}, \frac{C^2}{By_{-4}} < \frac{C}{y_0},$$

hence,

$$y_7 = \max\left\{\frac{B}{z_5}, y_2\right\} = \max\left\{\frac{B^2}{C^2}y_{-1}, y_{-3}\right\} = y_{-3}, z_7 = \max\left\{\frac{C}{y_5}, z_2\right\} = \max\left\{\frac{C^2}{By_{-4}}, \frac{C}{y_0}\right\} = \frac{C}{y_0},$$

$$y_8 = \max\left\{\frac{B}{z_6}, y_3\right\} = \max\left\{\frac{B}{C}y_{-1}, \frac{B}{C}y_{-1}\right\} = \frac{B}{C}y_{-1}, z_8 = \max\left\{\frac{C}{y_6}, z_3\right\} = \max\left\{\frac{C}{y_{-4}}, \frac{C}{y_{-4}}\right\} = \frac{C}{y_{-4}},$$

$$y_9 = \max\left\{\frac{B}{z_7}, y_4\right\} = \max\left\{\frac{B}{C}y_0, \frac{B}{C}y_0\right\} = \frac{B}{C}y_0, z_9 = \max\left\{\frac{C}{y_7}, z_4\right\} = \max\left\{\frac{C}{y_{-3}}, \frac{C}{y_{-3}}\right\} = \frac{C}{y_{-3}},$$

$$y_{10} = \max\left\{\frac{B}{z_8}, y_5\right\} = \max\left\{\frac{B}{C}y_{-4}, \frac{B}{C}y_{-4}\right\} = \frac{B}{C}y_{-4}, z_{10} = \max\left\{\frac{C}{y_8}, z_5\right\} = \max\left\{\frac{C^2}{By_{-1}}, \frac{C^2}{By_{-1}}\right\} = \frac{C^2}{By_{-1}}.$$

If the relations

$$y_{5k+1} = y_{-4}, y_{5k+2} = y_{-3}, y_{5k+3} = \frac{B}{C}y_{-1}, y_{5k+4} = \frac{B}{C}y_0, y_{5k+5} = \frac{B}{C}y_{-4}$$

and

$$z_{5k+1} = \frac{C}{y_{-1}}, z_{5k+2} = \frac{C}{y_0}, z_{5k+3} = \frac{C}{y_{-4}}, z_{5k+4} = \frac{C}{y_{-3}}, z_{5k+5} = \frac{C^2}{By_{-1}}$$

are satisfied, then the following expressions are derived

$$y_{5k+6} = \max\left\{\frac{B}{z_{5k+4}}, y_{5k+1}\right\} = \max\left\{\frac{B}{C}y_{-3}, y_{-4}\right\} = y_{-4}, z_{5k+6} = \max\left\{\frac{C}{y_{5k+4}}, z_{5k+1}\right\} = \max\left\{\frac{C^2}{By_0}, \frac{C}{y_{-1}}\right\} = \frac{C}{y_{-1}},$$

$$y_{5k+7} = \max\left\{\frac{B}{z_{5k+5}}, y_{5k+2}\right\} = \max\left\{\frac{B^2}{C^2}y_{-1}, y_{-3}\right\} = y_{-3}, z_{5k+7} = \max\left\{\frac{C}{y_{5k+5}}, z_{5k+2}\right\} = \max\left\{\frac{C^2}{By_{-4}}, \frac{C}{y_0}\right\} = \frac{C}{y_0},$$

$$y_{5k+8} = \max\left\{\frac{B}{z_{5k+6}}, y_{5k+3}\right\} = \max\left\{\frac{B}{C}y_{-1}, \frac{B}{C}y_{-1}\right\} = \frac{B}{C}y_{-1}, z_{5k+8} = \max\left\{\frac{C}{y_{5k+6}}, z_{5k+3}\right\} = \max\left\{\frac{C}{y_{-4}}, \frac{C}{y_{-4}}\right\} = \frac{C}{y_{-4}},$$

$$y_{5k+9} = \max\left\{\frac{B}{z_{5k+7}}, y_{5k+4}\right\} = \max\left\{\frac{B}{C}y_0, \frac{B}{C}y_0\right\} = \frac{B}{C}y_0, z_{5k+9} = \max\left\{\frac{C}{y_{5k+7}}, z_{5k+4}\right\} = \max\left\{\frac{C}{y_{-3}}, \frac{C}{y_{-3}}\right\} = \frac{C}{y_{-3}},$$

$$y_{5k+10} = \max\left\{\frac{B}{z_{5k+8}}, y_{5k+5}\right\} = \max\left\{\frac{B}{C}y_{-4}, \frac{B}{C}y_{-4}\right\} = \frac{B}{C}y_{-4}, z_{5k+10} = \max\left\{\frac{C}{y_{5k+8}}, z_{5k+5}\right\} = \max\left\{\frac{C^2}{By_{-1}}, \frac{C^2}{By_{-1}}\right\} = \frac{C^2}{By_{-1}}.$$

For case  $(a_{2-3-1-1-1})$ , assuming

$$\frac{B}{z_0} < y_{-3}, \frac{C}{y_0} > z_{-3}, \frac{B}{y_{-2}} > \frac{C}{y_{-1}}, \frac{C}{y_{-4}} > z_{-2}, \frac{B}{y_{-1}} > \frac{C}{y_0}, \frac{C}{y_{-3}} > z_{-1}, \frac{B}{y_0} > \frac{C}{y_{-4}}, \frac{C^2}{By_{-1}} > z_0$$

and  $k \geq 0$ , mathematical induction yields the following equalities:

$$y_{5k+1} = y_{-4}, y_{5k+2} = y_{-3}, y_{5k+3} = \frac{B}{C}y_{-1}, y_{5k+4} = \frac{B}{C}y_0, y_{5k+5} = \frac{B}{C}y_{-4},$$

$$z_{5k+1} = \frac{C}{y_{-1}}, z_{5k+2} = \frac{C}{y_0}, z_{5k+3} = \frac{C}{y_{-4}}, z_{5k+4} = \frac{C}{y_{-3}}, z_{5k+5} = \frac{C^2}{By_{-1}}.$$

**Remark 8.** For the remaining 63 cases in  $(a_{2-3})$ , methods similar to those described above can be employed to prove the existence of periodic solutions of period 5 and to derive the corresponding explicit forms of the solutions.

$(a_{2-4})$  From Equation (11), if  $\frac{B}{z_0} < y_{-3}, \frac{C}{y_0} < z_{-3}$ , then one obtains  $y_2 = y_{-3}, z_2 = z_{-3}$ .

$(a_{2-4-1})$  From Equation (12), if  $\frac{B}{z_1} > y_{-2}, \frac{C}{y_1} > z_{-2}$ , then one obtains

$$y_3 = \frac{B}{z_1} = \frac{B}{C}y_{-1}, z_3 = \frac{C}{y_1} = \frac{C}{y_{-4}}.$$

$(a_{2-4-1-1})$  From Equation (13), if  $\frac{B}{z_2} > y_{-1}, \frac{C}{y_2} > z_{-1}$ , then one obtains

$$y_4 = \frac{B}{z_2} = \frac{B}{z_{-3}}, z_4 = \frac{C}{y_2} = \frac{C}{y_{-3}}.$$

( $a_{2-4-1-1-1}$ ) From Equation (14), if  $\frac{B}{z_3} > y_0, \frac{C}{y_3} > z_0$ , then one obtains

$$y_5 = \frac{B}{z_3} = \frac{B}{C}y_{-4}, z_5 = \frac{C}{y_3} = \frac{C^2}{By_{-1}}.$$

By ( $a_{2-4-1-1}$ ), it follows that  $\frac{B}{z_{-3}} > y_{-1}, \frac{C}{y_{-3}} > z_{-1}$ , which implies

$$\frac{B}{z_4} = \frac{B}{C}y_{-3} < \frac{B}{z_{-1}} < y_{-4}, \frac{C}{y_{-1}} > \frac{C}{B}z_{-3},$$

therefore,

$$y_6 = \max\left\{\frac{B}{z_4}, y_1\right\} = \max\left\{\frac{B}{C}y_{-3}, y_{-4}\right\} = y_{-4}, z_6 = \max\left\{\frac{C}{y_4}, z_1\right\} = \max\left\{\frac{C}{B}z_{-3}, \frac{C}{y_{-1}}\right\} = \frac{C}{y_{-1}}.$$

By ( $a_{2-4-1-1-1}$ ), it follows that  $\frac{C^2}{By_{-1}} > z_0, \frac{B}{C}y_{-4} > y_0$ , which implies

$$\frac{B^2}{C^2}y_{-1} < \frac{B}{z_0} < y_{-3}, \frac{C^2}{By_{-4}} < \frac{C}{y_0},$$

hence,

$$\begin{aligned} y_7 &= \max\left\{\frac{B}{z_5}, y_2\right\} = \max\left\{\frac{B^2}{C^2}y_{-1}, y_{-3}\right\} = y_{-3}, z_7 = \max\left\{\frac{C}{y_5}, z_2\right\} = \max\left\{\frac{C^2}{By_{-4}}, z_{-3}\right\} = z_{-3}, \\ y_8 &= \max\left\{\frac{B}{z_6}, y_3\right\} = \max\left\{\frac{B}{C}y_{-1}, \frac{B}{C}y_{-1}\right\} = \frac{B}{C}y_{-1}, z_8 = \max\left\{\frac{C}{y_6}, z_3\right\} = \max\left\{\frac{C}{y_{-4}}, \frac{C}{y_{-4}}\right\} = \frac{C}{y_{-4}}, \\ y_9 &= \max\left\{\frac{B}{z_7}, y_4\right\} = \max\left\{\frac{B}{z_{-3}}, \frac{B}{z_{-3}}\right\} = \frac{B}{z_{-3}}, z_9 = \max\left\{\frac{C}{y_7}, z_4\right\} = \max\left\{\frac{C}{y_{-3}}, \frac{C}{y_{-3}}\right\} = \frac{C}{y_{-3}}, \end{aligned}$$

and

$$y_{10} = \max\left\{\frac{B}{z_8}, y_5\right\} = \max\left\{\frac{B}{C}y_{-4}, \frac{B}{C}y_{-4}\right\} = \frac{B}{C}y_{-4}, z_{10} = \max\left\{\frac{C}{y_8}, z_5\right\} = \max\left\{\frac{C^2}{By_{-1}}, \frac{C^2}{By_{-1}}\right\} = \frac{C^2}{By_{-1}}.$$

If the relations

$$y_{5k+1} = y_{-4}, y_{5k+2} = y_{-3}, y_{5k+3} = \frac{B}{C}y_{-1}, y_{5k+4} = \frac{B}{z_{-3}}, y_{5k+5} = \frac{B}{C}y_{-4}$$

and

$$z_{5k+1} = \frac{C}{y_{-1}}, z_{5k+2} = z_{-3}, z_{5k+3} = \frac{C}{y_{-4}}, z_{5k+4} = \frac{C}{y_{-3}}, z_{5k+5} = \frac{C^2}{By_{-1}}$$

are satisfied, then the following expressions are derived

$$\begin{aligned} y_{5k+6} &= \max\left\{\frac{B}{z_{5k+4}}, y_{5k+1}\right\} = \max\left\{\frac{B}{C}y_{-3}, y_{-4}\right\} = y_{-4}, z_{5k+6} = \max\left\{\frac{C}{y_{5k+4}}, z_{5k+1}\right\} = \max\left\{\frac{C}{B}z_{-3}, \frac{C}{y_{-1}}\right\} = \frac{C}{y_{-1}}, \\ y_{5k+7} &= \max\left\{\frac{B}{z_{5k+5}}, y_{5k+2}\right\} = \max\left\{\frac{B^2}{C^2}y_{-1}, y_{-3}\right\} = y_{-3}, z_{5k+7} = \max\left\{\frac{C}{y_{5k+5}}, z_{5k+2}\right\} = \max\left\{\frac{C^2}{By_{-4}}, z_{-3}\right\} = z_{-3}, \\ y_{5k+8} &= \max\left\{\frac{B}{z_{5k+6}}, y_{5k+3}\right\} = \max\left\{\frac{B}{C}y_{-1}, \frac{B}{C}y_{-1}\right\} = \frac{B}{C}y_{-1}, z_{5k+8} = \max\left\{\frac{C}{y_{5k+6}}, z_{5k+3}\right\} = \max\left\{\frac{C}{y_{-4}}, \frac{C}{y_{-4}}\right\} = \frac{C}{y_{-4}}, \\ y_{5k+9} &= \max\left\{\frac{B}{z_{5k+7}}, y_{5k+4}\right\} = \max\left\{\frac{B}{z_{-3}}, \frac{B}{z_{-3}}\right\} = \frac{B}{z_{-3}}, z_{5k+9} = \max\left\{\frac{C}{y_{5k+7}}, z_{5k+4}\right\} = \max\left\{\frac{C}{y_{-3}}, \frac{C}{y_{-3}}\right\} = \frac{C}{y_{-3}}, \end{aligned}$$

and

$$y_{5k+10} = \max\left\{\frac{B}{z_{5k+8}}, y_{5k+5}\right\} = \max\left\{\frac{B}{C}y_{-4}, \frac{B}{C}y_{-4}\right\} = \frac{B}{C}y_{-4}, z_{5k+10} = \max\left\{\frac{C}{y_{5k+8}}, z_{5k+5}\right\} = \max\left\{\frac{C^2}{By_{-1}}, \frac{C^2}{By_{-1}}\right\} = \frac{C^2}{By_{-1}}.$$

For case  $(a_{2-4-1-1-1})$ , assuming

$$\frac{B}{z_0} < y_{-3}, \frac{C}{y_0} < z_{-3}, \frac{B}{y_{-2}} > \frac{C}{y_{-1}}, \frac{C}{y_{-4}} > z_{-2}, \frac{B}{z_{-3}} > y_{-1}, \frac{C}{y_{-3}} > z_{-1}, \frac{B}{y_0} > \frac{C}{y_{-4}}, \frac{C^2}{By_{-1}} > z_0$$

and  $k \geq 0$ , mathematical induction yields the following equalities:

$$y_{5k+1} = y_{-4}, y_{5k+2} = y_{-3}, y_{5k+3} = \frac{B}{C}y_{-1}, y_{5k+4} = \frac{B}{z_{-3}}, y_{5k+5} = \frac{B}{C}y_{-4},$$

$$z_{5k+1} = \frac{C}{y_{-1}}, z_{5k+2} = z_{-3}, z_{5k+3} = \frac{C}{y_{-4}}, z_{5k+4} = \frac{C}{y_{-3}}, z_{5k+5} = \frac{C^2}{By_{-1}}.$$

**Remark 9.** For the remaining 63 cases in  $(a_{2-4})$ , methods similar to those described above can be employed to prove the existence of periodic solutions of period 5 and to derive the corresponding explicit forms of the solutions.

$(a_3)$  From Equation (10), if  $\frac{B}{z_{-1}} < y_{-4}, \frac{C}{y_{-1}} < z_{-4}$ , then one obtains  $y_1 = y_{-4}, z_1 = z_{-4}$ .

$(a_{3-1})$  From Equation (11), if  $\frac{B}{z_0} > y_{-3}, \frac{C}{y_0} > z_{-3}$ , then one obtains  $y_2 = \frac{B}{z_0}, z_2 = \frac{C}{y_0}$ .

$(a_{3-1-1})$  From Equation (12), if  $\frac{B}{z_1} > y_{-2}, \frac{C}{y_1} > z_{-2}$ , then one obtains

$$y_3 = \frac{B}{z_1} = \frac{B}{z_{-4}}, z_3 = \frac{C}{y_1} = \frac{C}{y_{-4}}.$$

$(a_{3-1-1-1})$  From Equation (13), if  $\frac{B}{z_2} > y_{-1}, \frac{C}{y_2} > z_{-1}$ , then one obtains

$$y_4 = \frac{B}{z_2} = \frac{B}{C}y_0, z_4 = \frac{C}{y_2} = \frac{C}{B}z_0.$$

$(a_{3-1-1-1-1})$  From Equation (14), if  $\frac{B}{z_3} > y_0, \frac{C}{y_3} > z_0$ , then one obtains

$$y_5 = \frac{B}{z_3} = \frac{B}{C}y_{-4}, z_5 = \frac{C}{y_3} = \frac{C}{B}z_{-4}.$$

By  $(a_{3-1-1-1-1})$ , it follows that  $\frac{B}{C}y_0 > y_{-1}, \frac{C}{B}z_0 > z_{-1}$ , which implies

$$\frac{B^2}{Cz_0} < \frac{B}{z_{-1}} < y_{-4}, \frac{C^2}{By_0} < \frac{C}{y_{-1}} < z_{-4},$$

therefore,

$$y_6 = \max\left\{\frac{B}{z_4}, y_1\right\} = \max\left\{\frac{B^2}{Cz_0}, y_{-4}\right\} = y_{-4}, z_6 = \max\left\{\frac{C}{y_4}, z_1\right\} = \max\left\{\frac{C^2}{By_0}, z_{-4}\right\} = z_{-4}$$

By  $(a_{3-1-1-1-1-1})$ , it follows that  $\frac{C}{B}z_{-4} > z_0, \frac{B}{C}y_{-4} > y_0$ , which implies

$$\frac{B^2}{Cz_{-4}} < \frac{B}{z_0}, \frac{C^2}{By_{-4}} < \frac{C}{y_0},$$

hence,

$$y_7 = \max\left\{\frac{B}{z_5}, y_2\right\} = \max\left\{\frac{B^2}{Cz_{-4}}, \frac{B}{z_0}\right\} = \frac{B}{z_0}, z_7 = \max\left\{\frac{C}{y_5}, z_2\right\} = \max\left\{\frac{C^2}{By_{-4}}, \frac{C}{y_0}\right\} = \frac{C}{y_0},$$

$$y_8 = \max\left\{\frac{B}{z_6}, y_3\right\} = \max\left\{\frac{B}{z_{-4}}, \frac{B}{z_{-4}}\right\} = \frac{B}{z_{-4}}, z_8 = \max\left\{\frac{C}{y_6}, z_3\right\} = \max\left\{\frac{C}{y_{-4}}, \frac{C}{y_{-4}}\right\} = \frac{C}{y_{-4}},$$

$$y_9 = \max\left\{\frac{B}{z_7}, y_4\right\} = \max\left\{\frac{B}{C}y_0, \frac{B}{C}y_0\right\} = \frac{B}{C}y_0, z_9 = \max\left\{\frac{C}{y_7}, z_4\right\} = \max\left\{\frac{C}{B}z_0, \frac{C}{B}z_0\right\} = \frac{C}{B}z_0,$$

and

$$y_{10} = \max\left\{\frac{B}{z_8}, y_5\right\} = \max\left\{\frac{B}{C}y_{-4}, \frac{B}{C}y_{-4}\right\} = \frac{B}{C}y_{-4}, z_{10} = \max\left\{\frac{C}{y_8}, z_5\right\} = \max\left\{\frac{C}{B}z_{-4}, \frac{C}{B}z_{-4}\right\} = \frac{C}{B}z_{-4}.$$

If the relations

$$y_{5k+1} = y_{-4}, y_{5k+2} = \frac{B}{z_0}, y_{5k+3} = \frac{B}{z_{-4}}, y_{5k+4} = \frac{B}{C}y_0, y_{5k+5} = \frac{B}{C}y_{-4}$$

and

$$z_{5k+1} = z_{-4}, z_{5k+2} = \frac{C}{y_0}, z_{5k+3} = \frac{C}{y_{-4}}, z_{5k+4} = \frac{C}{B}z_0, z_{5k+5} = \frac{C}{B}z_{-4}$$

are satisfied, then the following expressions are derived

$$y_{5k+6} = \max\left\{\frac{B}{z_{5k+4}}, y_{5k+1}\right\} = \max\left\{\frac{B^2}{Cz_0}, y_{-4}\right\} = y_{-4}, z_{5k+6} = \max\left\{\frac{C}{y_{5k+4}}, z_{5k+1}\right\} = \max\left\{\frac{C^2}{By_0}, z_{-4}\right\} = z_{-4},$$

$$y_{5k+7} = \max\left\{\frac{B}{z_{5k+5}}, y_{5k+2}\right\} = \max\left\{\frac{B^2}{Cz_{-4}}, \frac{B}{z_0}\right\} = \frac{B}{z_0}, z_{5k+7} = \max\left\{\frac{C}{y_{5k+5}}, z_{5k+2}\right\} = \max\left\{\frac{C^2}{By_{-4}}, \frac{C}{y_0}\right\} = \frac{C}{y_0},$$

$$y_{5k+8} = \max\left\{\frac{B}{z_{5k+6}}, y_{5k+3}\right\} = \max\left\{\frac{B}{z_{-4}}, \frac{B}{z_{-4}}\right\} = \frac{B}{z_{-4}}, z_{5k+8} = \max\left\{\frac{C}{y_{5k+6}}, z_{5k+3}\right\} = \max\left\{\frac{C}{y_{-4}}, \frac{C}{y_{-4}}\right\} = \frac{C}{y_{-4}},$$

$$y_{5k+9} = \max\left\{\frac{B}{z_{5k+7}}, y_{5k+4}\right\} = \max\left\{\frac{B}{C}y_0, \frac{B}{C}y_0\right\} = \frac{B}{C}y_0, z_{5k+9} = \max\left\{\frac{C}{y_{5k+7}}, z_{5k+4}\right\} = \max\left\{\frac{C}{B}z_0, \frac{C}{B}z_0\right\} = \frac{C}{B}z_0,$$

and

$$y_{5k+10} = \max\left\{\frac{B}{z_{5k+8}}, y_{5k+5}\right\} = \max\left\{\frac{B}{C}y_{-4}, \frac{B}{C}y_{-4}\right\} = \frac{B}{C}y_{-4}, z_{5k+10} = \max\left\{\frac{C}{y_{5k+8}}, z_{5k+5}\right\} = \max\left\{\frac{C}{B}z_{-4}, \frac{C}{B}z_{-4}\right\} = \frac{C}{B}z_{-4}.$$

For case  $(a_{3-1-1-1-1})$ , assuming

$$\frac{B}{z_0} > y_{-3}, \frac{C}{y_0} > z_{-3}, \frac{B}{z_{-4}} > y_{-2}, \frac{C}{y_{-4}} > z_{-2}, \frac{B}{y_{-1}} > \frac{C}{y_0}, \frac{C}{z_{-1}} > \frac{B}{z_0}, \frac{B}{y_0} > \frac{C}{y_{-4}}, \frac{C}{z_0} > \frac{B}{z_{-4}}$$

and  $k \geq 0$ , mathematical induction yields the following equalities:

$$y_{5k+1} = y_{-4}, y_{5k+2} = \frac{B}{z_0}, y_{5k+3} = \frac{B}{z_{-4}}, y_{5k+4} = \frac{B}{C}y_0, y_{5k+5} = \frac{B}{C}y_{-4},$$

$$z_{5k+1} = z_{-4}, z_{5k+2} = \frac{C}{y_0}, z_{5k+3} = \frac{C}{y_{-4}}, z_{5k+4} = \frac{C}{B}z_0, z_{5k+5} = \frac{C}{B}z_{-4}.$$

**Remark 10.** For the remaining 63 cases in  $(a_{3-1})$ , methods similar to those described above can be employed to prove the existence of periodic solutions of period 5 and to derive the corresponding explicit forms of the solutions.

$(a_{3-2})$  From Equation (11), if  $\frac{B}{z_0} > y_{-3}, \frac{C}{y_0} < z_{-3}$ , then one obtains  $y_2 = \frac{B}{z_0}, z_2 = z_{-3}$ .

$(a_{3-2-1})$  From Equation (12), if  $\frac{B}{z_1} > y_{-2}, \frac{C}{y_1} > z_{-2}$ , then one obtains

$$y_3 = \frac{B}{z_1} = \frac{B}{z_{-4}}, z_3 = \frac{C}{y_1} = \frac{C}{y_{-4}}.$$

( $a_{3-2-1-1}$ ) From Equation (13), if  $\frac{B}{z_2} > y_{-1}, \frac{C}{y_2} > z_{-1}$ , then one obtains

$$y_4 = \frac{B}{z_2} = \frac{B}{z_{-3}}, z_4 = \frac{C}{y_2} = \frac{C}{B}z_0.$$

( $a_{3-2-1-1-1}$ ) From Equation (14), if  $\frac{B}{z_3} > y_0, \frac{C}{y_3} > z_0$ , then one obtains

$$y_5 = \frac{B}{z_3} = \frac{B}{C}y_{-4}, z_5 = \frac{C}{y_3} = \frac{C}{B}z_{-4}.$$

By ( $a_{3-2-1-1}$ ), it follows that  $\frac{B}{z_{-3}} > y_{-1}, \frac{C}{B}z_0 > z_{-1}$ , which implies

$$\frac{B^2}{Cz_0} < \frac{B}{z_{-1}} < y_{-4}, \frac{C}{B}z_{-3} < \frac{C}{y_{-1}} < z_{-4},$$

therefore,

$$y_6 = \max\left\{\frac{B}{z_4}, y_1\right\} = \max\left\{\frac{B^2}{Cz_0}, y_{-4}\right\} = y_{-4}, z_6 = \max\left\{\frac{C}{y_4}, z_1\right\} = \max\left\{\frac{C}{B}z_{-3}, z_{-4}\right\} = z_{-4}.$$

By ( $a_{3-2-1-1-1}$ ), it follows that  $\frac{C}{B}z_{-4} > z_0, \frac{B}{C}y_{-4} > y_0$ , which implies

$$\frac{B^2}{Cz_{-4}} < \frac{B}{z_0}, \frac{C^2}{By_{-4}} < \frac{C}{y_0} < z_{-3},$$

hence,

$$y_7 = \max\left\{\frac{B}{z_5}, y_2\right\} = \max\left\{\frac{B^2}{Cz_{-4}}, \frac{B}{z_0}\right\} = \frac{B}{z_0}, z_7 = \max\left\{\frac{C}{y_5}, z_2\right\} = \max\left\{\frac{C^2}{By_{-4}}, z_{-3}\right\} = z_{-3},$$

$$y_8 = \max\left\{\frac{B}{z_6}, y_3\right\} = \max\left\{\frac{B}{z_{-4}}, \frac{B}{z_{-4}}\right\} = \frac{B}{z_{-4}}, z_8 = \max\left\{\frac{C}{y_6}, z_3\right\} = \max\left\{\frac{C}{y_{-4}}, \frac{C}{y_{-4}}\right\} = \frac{C}{y_{-4}},$$

$$y_9 = \max\left\{\frac{B}{z_7}, y_4\right\} = \max\left\{\frac{B}{z_{-3}}, \frac{B}{z_{-3}}\right\} = \frac{B}{z_{-3}}, z_9 = \max\left\{\frac{C}{y_7}, z_4\right\} = \max\left\{\frac{C}{B}z_0, \frac{C}{B}z_0\right\} = \frac{C}{B}z_0,$$

and

$$y_{10} = \max\left\{\frac{B}{z_8}, y_5\right\} = \max\left\{\frac{B}{C}y_{-4}, \frac{B}{C}y_{-4}\right\} = \frac{B}{C}y_{-4}, z_{10} = \max\left\{\frac{C}{y_8}, z_5\right\} = \max\left\{\frac{C}{B}z_{-4}, \frac{C}{B}z_{-4}\right\} = \frac{C}{B}z_{-4}.$$

If the relations

$$y_{5k+1} = y_{-4}, y_{5k+2} = \frac{B}{z_0}, y_{5k+3} = \frac{B}{z_{-4}}, y_{5k+4} = \frac{B}{z_{-3}}, y_{5k+5} = \frac{B}{C}y_{-4}$$

and

$$z_{5k+1} = z_{-4}, z_{5k+2} = z_{-3}, z_{5k+3} = \frac{C}{y_{-4}}, z_{5k+4} = \frac{C}{B}z_0, z_{5k+5} = \frac{C}{B}z_{-4}$$

are satisfied, then the following expressions are derived

$$y_{5k+6} = \max\left\{\frac{B}{z_{5k+4}}, y_{5k+1}\right\} = \max\left\{\frac{B^2}{Cz_0}, y_{-4}\right\} = y_{-4}, z_{5k+6} = \max\left\{\frac{C}{y_{5k+4}}, z_{5k+1}\right\} = \max\left\{\frac{C}{B}z_{-3}, z_{-4}\right\} = z_{-4},$$

$$y_{5k+7} = \max\left\{\frac{B}{z_{5k+5}}, y_{5k+2}\right\} = \max\left\{\frac{B^2}{Cz_{-4}}, \frac{B}{z_0}\right\} = \frac{B}{z_0}, z_{5k+7} = \max\left\{\frac{C}{y_{5k+5}}, z_{5k+2}\right\} = \max\left\{\frac{C^2}{By_{-4}}, z_{-3}\right\} = z_{-3},$$

$$y_{5k+8} = \max\left\{\frac{B}{z_{5k+6}}, y_{5k+3}\right\} = \max\left\{\frac{B}{z_{-4}}, \frac{B}{z_{-4}}\right\} = \frac{B}{z_{-4}}, z_{5k+8} = \max\left\{\frac{C}{y_{5k+6}}, z_{5k+3}\right\} = \max\left\{\frac{C}{y_{-4}}, \frac{C}{y_{-4}}\right\} = \frac{C}{y_{-4}},$$

$$y_{5k+9} = \max\left\{\frac{B}{z_{5k+7}}, y_{5k+4}\right\} = \max\left\{\frac{B}{z_{-3}}, \frac{B}{z_{-3}}\right\} = \frac{B}{z_{-3}}, z_{5k+9} = \max\left\{\frac{C}{y_{5k+7}}, z_{5k+4}\right\} = \max\left\{\frac{C}{B}z_0, \frac{C}{B}z_0\right\} = \frac{C}{B}z_0,$$

and

$$y_{5k+10} = \max\left\{\frac{B}{z_{5k+8}}, y_{5k+5}\right\} = \max\left\{\frac{B}{C}y_{-4}, \frac{B}{C}y_{-4}\right\} = \frac{B}{C}y_{-4}, z_{5k+10} = \max\left\{\frac{C}{y_{5k+8}}, z_{5k+5}\right\} = \max\left\{\frac{C}{B}z_{-4}, \frac{C}{B}z_{-4}\right\} = \frac{C}{B}z_{-4}.$$

For case  $(a_{3-2-1-1-1})$ , assuming

$$\frac{B}{z_0} > y_{-3}, \frac{C}{y_0} < z_{-3}, \frac{B}{z_{-4}} > y_{-2}, \frac{C}{y_{-4}} > z_{-2}, \frac{B}{z_{-3}} > y_{-1}, \frac{C}{z_{-1}} > \frac{B}{z_0}, \frac{B}{y_0} > \frac{C}{y_{-4}}, \frac{C}{z_0} > \frac{B}{z_{-4}}$$

and  $k \geq 0$ , mathematical induction yields the following equalities:

$$y_{5k+1} = y_{-4}, y_{5k+2} = \frac{B}{z_0}, y_{5k+3} = \frac{B}{z_{-4}}, y_{5k+4} = \frac{B}{z_{-3}}, y_{5k+5} = \frac{B}{C}y_{-4},$$

$$z_{5k+1} = z_{-4}, z_{5k+2} = z_{-3}, z_{5k+3} = \frac{C}{y_{-4}}, z_{5k+4} = \frac{C}{B}z_0, z_{5k+5} = \frac{C}{B}z_{-4}.$$

**Remark 11.** For the remaining 63 cases in  $(a_{3-2})$ , methods similar to those described above can be employed to prove the existence of periodic solutions of period 5 and to derive the corresponding explicit forms of the solutions.

$(a_{3-3})$  From Equation (11), if  $\frac{B}{z_0} < y_{-3}, \frac{C}{y_0} > z_{-3}$ , then one obtains  $y_2 = y_{-3}, z_2 = \frac{C}{y_0}$ .

$(a_{3-3-1})$  From Equation (12), if  $\frac{B}{z_1} > y_{-2}, \frac{C}{y_1} > z_{-2}$ , then one obtains

$$y_3 = \frac{B}{z_1} = \frac{B}{z_{-4}}, z_3 = \frac{C}{y_1} = \frac{C}{y_{-4}}.$$

$(a_{3-3-1-1})$  From Equation (13), if  $\frac{B}{z_2} > y_{-1}, \frac{C}{y_2} > z_{-1}$ , then one obtains

$$y_4 = \frac{B}{z_2} = \frac{B}{C}y_0, z_4 = \frac{C}{y_2} = \frac{C}{y_{-3}}.$$

$(a_{3-3-1-1-1})$  From Equation (14), if  $\frac{B}{z_3} > y_0, \frac{C}{y_3} > z_0$ , then one obtains

$$y_5 = \frac{B}{z_3} = \frac{B}{C}y_{-4}, z_5 = \frac{C}{y_3} = \frac{C}{B}z_{-4}.$$

By  $(a_{3-3-1-1-1})$ , it follows that  $\frac{B}{C}y_0 > y_{-1}, \frac{C}{y_{-3}} > z_{-1}$ , which implies

$$\frac{B}{C}y_{-3} < \frac{B}{z_{-1}} < y_{-4}, \frac{C^2}{By_0} < \frac{C}{y_{-1}} < z_{-4},$$

therefore,

$$y_6 = \max\left\{\frac{B}{z_4}, y_1\right\} = \max\left\{\frac{B}{C}y_{-3}, y_{-4}\right\} = y_{-4}, z_6 = \max\left\{\frac{C}{y_4}, z_1\right\} = \max\left\{\frac{C^2}{By_0}, z_{-4}\right\} = z_{-4}.$$

By  $(a_{3-3-1-1-1-1})$ , it follows that  $\frac{C}{B}z_{-4} > z_0, \frac{B}{C}y_{-4} > y_0$ , which implies

$$\frac{B^2}{Cz_{-4}} < \frac{B}{z_0} < y_{-3}, \frac{C^2}{By_{-4}} < \frac{C}{y_0},$$

hence,

$$y_7 = \max\left\{\frac{B}{z_5}, y_2\right\} = \max\left\{\frac{B^2}{Cz_{-4}}, y_{-3}\right\} = y_{-3}, z_7 = \max\left\{\frac{C}{y_5}, z_2\right\} = \max\left\{\frac{C^2}{By_{-4}}, \frac{C}{y_0}\right\} = \frac{C}{y_0},$$

$$y_8 = \max\left\{\frac{B}{z_6}, y_3\right\} = \max\left\{\frac{B}{z_{-4}}, \frac{B}{z_{-4}}\right\} = \frac{B}{z_{-4}}, z_8 = \max\left\{\frac{C}{y_6}, z_3\right\} = \max\left\{\frac{C}{y_{-4}}, \frac{C}{y_{-4}}\right\} = \frac{C}{y_{-4}},$$

$$y_9 = \max\left\{\frac{B}{z_7}, y_4\right\} = \max\left\{\frac{B}{C}y_0, \frac{B}{C}y_0\right\} = \frac{B}{C}y_0, z_9 = \max\left\{\frac{C}{y_7}, z_4\right\} = \max\left\{\frac{C}{y_{-3}}, \frac{C}{y_{-3}}\right\} = \frac{C}{y_{-3}},$$

and

$$y_{10} = \max\left\{\frac{B}{z_8}, y_5\right\} = \max\left\{\frac{B}{C}y_{-4}, \frac{B}{C}y_{-4}\right\} = \frac{B}{C}y_{-4}, z_{10} = \max\left\{\frac{C}{y_8}, z_5\right\} = \max\left\{\frac{C}{B}z_{-4}, \frac{C}{B}z_{-4}\right\} = \frac{C}{B}z_{-4}.$$

If the relations

$$y_{5k+1} = y_{-4}, y_{5k+2} = y_{-3}, y_{5k+3} = \frac{B}{z_{-4}}, y_{5k+4} = \frac{B}{C}y_0, y_{5k+5} = \frac{B}{C}y_{-4}$$

and

$$z_{5k+1} = z_{-4}, z_{5k+2} = \frac{C}{y_0}, z_{5k+3} = \frac{C}{y_{-4}}, z_{5k+4} = \frac{C}{y_{-3}}, z_{5k+5} = \frac{C}{B}z_{-4}$$

are satisfied, then the following expressions are derived

$$y_{5k+6} = \max\left\{\frac{B}{z_{5k+4}}, y_{5k+1}\right\} = \max\left\{\frac{B}{C}y_{-3}, y_{-4}\right\} = y_{-4}, z_{5k+6} = \max\left\{\frac{C}{y_{5k+4}}, z_{5k+1}\right\} = \max\left\{\frac{C^2}{By_0}, z_{-4}\right\} = z_{-4},$$

$$y_{5k+7} = \max\left\{\frac{B}{z_{5k+5}}, y_{5k+2}\right\} = \max\left\{\frac{B^2}{Cz_{-4}}, y_{-3}\right\} = y_{-3}, z_{5k+7} = \max\left\{\frac{C}{y_{5k+5}}, z_{5k+2}\right\} = \max\left\{\frac{C^2}{By_{-4}}, \frac{C}{y_0}\right\} = \frac{C}{y_0},$$

$$y_{5k+8} = \max\left\{\frac{B}{z_{5k+6}}, y_{5k+3}\right\} = \max\left\{\frac{B}{z_{-4}}, \frac{B}{z_{-4}}\right\} = \frac{B}{z_{-4}}, z_{5k+8} = \max\left\{\frac{C}{y_{5k+6}}, z_{5k+3}\right\} = \max\left\{\frac{C}{y_{-4}}, \frac{C}{y_{-4}}\right\} = \frac{C}{y_{-4}},$$

$$y_{5k+9} = \max\left\{\frac{B}{z_{5k+7}}, y_{5k+4}\right\} = \max\left\{\frac{B}{C}y_0, \frac{B}{C}y_0\right\} = \frac{B}{C}y_0, z_{5k+9} = \max\left\{\frac{C}{y_{5k+7}}, z_{5k+4}\right\} = \max\left\{\frac{C}{y_{-3}}, \frac{C}{y_{-3}}\right\} = \frac{C}{y_{-3}},$$

and

$$y_{5k+10} = \max\left\{\frac{B}{z_{5k+8}}, y_{5k+5}\right\} = \max\left\{\frac{B}{C}y_{-4}, \frac{B}{C}y_{-4}\right\} = \frac{B}{C}y_{-4}, z_{5k+10} = \max\left\{\frac{C}{y_{5k+8}}, z_{5k+5}\right\} = \max\left\{\frac{C}{B}z_{-4}, \frac{C}{B}z_{-4}\right\} = \frac{C}{B}z_{-4}.$$

For case  $(a_{3-3-1-1-1})$ , assuming

$$\frac{B}{z_0} < y_{-3}, \frac{C}{y_0} > z_{-3}, \frac{B}{z_{-4}} > y_{-2}, \frac{C}{y_{-4}} > z_{-2}, \frac{B}{y_{-1}} > \frac{C}{y_0}, \frac{C}{y_{-3}} > z_{-1}, \frac{B}{y_0} > \frac{C}{y_{-4}}, \frac{C}{z_0} > \frac{B}{z_{-4}}$$

and  $k \geq 0$ , mathematical induction yields the following equalities:

$$y_{5k+1} = y_{-4}, y_{5k+2} = y_{-3}, y_{5k+3} = \frac{B}{z_{-4}}, y_{5k+4} = \frac{B}{C}y_0, y_{5k+5} = \frac{B}{C}y_{-4},$$

$$z_{5k+1} = z_{-4}, z_{5k+2} = \frac{C}{y_0}, z_{5k+3} = \frac{C}{y_{-4}}, z_{5k+4} = \frac{C}{y_{-3}}, z_{5k+5} = \frac{C}{B}z_{-4}.$$

**Remark 12.** For the remaining 63 cases in  $(a_{3-3})$ , methods similar to those described above can be employed to prove the existence of periodic solutions of period 5 and to derive the corresponding explicit forms of the solutions.

$(a_{3-4})$  From Equation (11), if  $\frac{B}{z_0} < y_{-3}, \frac{C}{y_0} < z_{-3}$ , then one obtains  $y_2 = y_{-3}, z_2 = z_{-3}$ .

( $a_{3-4-1}$ ) From Equation (12), if  $\frac{B}{z_1} > y_{-2}, \frac{C}{y_1} > z_{-2}$ , then one obtains

$$y_3 = \frac{B}{z_1} = \frac{B}{z_{-4}}, z_3 = \frac{C}{y_1} = \frac{C}{y_{-4}}.$$

( $a_{3-4-1-1}$ ) From Equation (13), if  $\frac{B}{z_2} > y_{-1}, \frac{C}{y_2} > z_{-1}$ , then one obtains

$$y_4 = \frac{B}{z_2} = \frac{B}{z_{-3}}, z_4 = \frac{C}{y_2} = \frac{C}{y_{-3}}.$$

( $a_{3-4-1-1-1}$ ) From Equation (14), if  $\frac{B}{z_3} > y_0, \frac{C}{y_3} > z_0$ , then one obtains

$$y_5 = \frac{B}{z_3} = \frac{B}{C}y_{-4}, z_5 = \frac{C}{y_3} = \frac{C}{B}z_{-4}.$$

By ( $a_{3-4-1-1-1}$ ), it follows that  $\frac{B}{z_{-3}} > y_{-1}, \frac{C}{y_{-3}} > z_{-1}$ , which implies

$$\frac{B}{C}y_{-3} < \frac{B}{z_{-1}} < y_{-4}, \frac{C}{B}z_{-3} < \frac{C}{y_{-1}} < z_{-4},$$

therefore,

$$y_6 = \max\left\{\frac{B}{z_4}, y_1\right\} = \max\left\{\frac{B}{C}y_{-3}, y_{-4}\right\} = y_{-4}, z_6 = \max\left\{\frac{C}{y_4}, z_1\right\} = \max\left\{\frac{C}{B}z_{-3}, z_{-4}\right\} = z_{-4}.$$

By ( $a_{3-4-1-1-1}$ ), it follows that  $\frac{C}{B}z_{-4} > z_0, \frac{B}{C}y_{-4} > y_0$ , which implies

$$\frac{B^2}{Cz_{-4}} < \frac{B}{z_0} < y_{-3}, \frac{C^2}{By_{-4}} < \frac{C}{y_0} < z_{-3},$$

hence,

$$y_7 = \max\left\{\frac{B}{z_5}, y_2\right\} = \max\left\{\frac{B^2}{Cz_{-4}}, y_{-3}\right\} = y_{-3}, z_7 = \max\left\{\frac{C}{y_5}, z_2\right\} = \max\left\{\frac{C^2}{By_{-4}}, z_{-3}\right\} = z_{-3},$$

$$y_8 = \max\left\{\frac{B}{z_6}, y_3\right\} = \max\left\{\frac{B}{z_{-4}}, \frac{B}{z_{-4}}\right\} = \frac{B}{z_{-4}}, z_8 = \max\left\{\frac{C}{y_6}, z_3\right\} = \max\left\{\frac{C}{y_{-4}}, \frac{C}{y_{-4}}\right\} = \frac{C}{y_{-4}},$$

$$y_9 = \max\left\{\frac{B}{z_7}, y_4\right\} = \max\left\{\frac{B}{z_{-3}}, \frac{B}{z_{-3}}\right\} = \frac{B}{z_{-3}}, z_9 = \max\left\{\frac{C}{y_7}, z_4\right\} = \max\left\{\frac{C}{y_{-3}}, \frac{C}{y_{-3}}\right\} = \frac{C}{y_{-3}},$$

and

$$y_{10} = \max\left\{\frac{B}{z_8}, y_5\right\} = \max\left\{\frac{B}{C}y_{-4}, \frac{B}{C}y_{-4}\right\} = \frac{B}{C}y_{-4}, z_{10} = \max\left\{\frac{C}{y_8}, z_5\right\} = \max\left\{\frac{C}{B}z_{-4}, \frac{C}{B}z_{-4}\right\} = \frac{C}{B}z_{-4}.$$

If the relations

$$y_{5k+1} = y_{-4}, y_{5k+2} = y_{-3}, y_{5k+3} = \frac{B}{z_{-4}}, y_{5k+4} = \frac{B}{z_{-3}}, y_{5k+5} = \frac{B}{C}y_{-4}$$

and

$$z_{5k+1} = z_{-4}, z_{5k+2} = z_{-3}, z_{5k+3} = \frac{C}{y_{-4}}, z_{5k+4} = \frac{C}{y_{-3}}, z_{5k+5} = \frac{C}{B}z_{-4}$$

are satisfied, then the following expressions are derived

$$y_{5k+6} = \max\left\{\frac{B}{z_{5k+4}}, y_{5k+1}\right\} = \max\left\{\frac{B}{C}y_{-3}, y_{-4}\right\} = y_{-4}, z_{5k+6} = \max\left\{\frac{C}{y_{5k+4}}, z_{5k+1}\right\} = \max\left\{\frac{C}{B}z_{-3}, z_{-4}\right\} = z_{-4},$$

$$y_{5k+7} = \max\left\{\frac{B}{z_{5k+5}}, y_{5k+2}\right\} = \max\left\{\frac{B^2}{Cz_{-4}}, y_{-3}\right\} = y_{-3}, z_{5k+7} = \max\left\{\frac{C}{y_{5k+5}}, z_{5k+2}\right\} = \max\left\{\frac{C^2}{By_{-4}}, z_{-3}\right\} = z_{-3},$$

$$y_{5k+8} = \max\left\{\frac{B}{z_{5k+6}}, y_{5k+3}\right\} = \max\left\{\frac{B}{z_{-4}}, \frac{B}{z_{-4}}\right\} = \frac{B}{z_{-4}}, z_{5k+8} = \max\left\{\frac{C}{y_{5k+6}}, z_{5k+3}\right\} = \max\left\{\frac{C}{y_{-4}}, \frac{C}{y_{-4}}\right\} = \frac{C}{y_{-4}},$$

$$y_{5k+9} = \max\left\{\frac{B}{z_{5k+7}}, y_{5k+4}\right\} = \max\left\{\frac{B}{z_{-3}}, \frac{B}{z_{-3}}\right\} = \frac{B}{z_{-3}}, z_{5k+9} = \max\left\{\frac{C}{y_{5k+7}}, z_{5k+4}\right\} = \max\left\{\frac{C}{y_{-3}}, \frac{C}{y_{-3}}\right\} = \frac{C}{y_{-3}},$$

and

$$y_{5k+10} = \max\left\{\frac{B}{z_{5k+8}}, y_{5k+5}\right\} = \max\left\{\frac{B}{C}y_{-4}, \frac{B}{C}y_{-4}\right\} = \frac{B}{C}y_{-4}, z_{5k+10} = \max\left\{\frac{C}{y_{5k+8}}, z_{5k+5}\right\} = \max\left\{\frac{C}{B}z_{-4}, \frac{C}{B}z_{-4}\right\} = \frac{C}{B}z_{-4}.$$

For case  $(a_{3-4-1-1-1})$ , assuming

$$\frac{B}{z_0} < y_{-3}, \frac{C}{y_0} < z_{-3}, \frac{B}{z_{-4}} > y_{-2}, \frac{C}{y_{-4}} > z_{-2}, \frac{B}{z_{-3}} > y_{-1}, \frac{C}{y_{-3}} > z_{-1}, \frac{B}{y_0} > \frac{C}{y_{-4}}, \frac{C}{z_0} > \frac{B}{z_{-4}}$$

and  $k \geq 0$ , mathematical induction yields the following equalities:

$$y_{5k+1} = y_{-4}, y_{5k+2} = y_{-3}, y_{5k+3} = \frac{B}{z_{-4}}, y_{5k+4} = \frac{B}{z_{-3}}, y_{5k+5} = \frac{B}{C}y_{-4},$$

$$z_{5k+1} = z_{-4}, z_{5k+2} = z_{-3}, z_{5k+3} = \frac{C}{y_{-4}}, z_{5k+4} = \frac{C}{y_{-3}}, z_{5k+5} = \frac{C}{B}z_{-4}.$$

**Remark 13.** For the remaining 63 cases in  $(a_{3-4})$ , methods similar to those described above can be employed to prove the existence of periodic solutions of period 5 and to derive the corresponding explicit forms of the solutions.

- $(a_4)$  From Equation (10), if  $\frac{B}{z_{-1}} > y_{-4}, \frac{C}{y_{-1}} < z_{-4}$ , then one obtains  $y_1 = \frac{B}{z_{-1}}, z_1 = z_{-4}$ .
- $(a_{4-1})$  From Equation (11), if  $\frac{B}{z_0} > y_{-3}, \frac{C}{y_0} > z_{-3}$ , then one obtains  $y_2 = \frac{B}{z_0}, z_2 = \frac{C}{y_0}$ .
- $(a_{4-1-1})$  From Equation (12), if  $\frac{B}{z_1} > y_{-2}, \frac{C}{y_1} > z_{-2}$ , then one obtains

$$y_3 = \frac{B}{z_1} = \frac{B}{z_{-4}}, z_3 = \frac{C}{y_1} = \frac{C}{B}z_{-1}.$$

- $(a_{4-1-1-1})$  From Equation (13), if  $\frac{B}{z_2} > y_{-1}, \frac{C}{y_2} > z_{-1}$ , then one obtains

$$y_4 = \frac{B}{z_2} = \frac{B}{C}y_0, z_4 = \frac{C}{y_2} = \frac{C}{B}z_0.$$

- $(a_{4-1-1-1-1})$  From Equation (14), if  $\frac{B}{z_3} > y_0, \frac{C}{y_3} > z_0$ , then one obtains

$$y_5 = \frac{B}{z_3} = \frac{B^2}{Cz_{-1}}, z_5 = \frac{C}{y_3} = \frac{C}{B}z_{-4}.$$

By  $(a_{4-1-1-1-1})$ , it follows that  $\frac{Cz_0}{B} > z_{-1}, \frac{By_0}{C} > y_{-1}$ , which implies

$$y_6 = \max\left\{\frac{B}{z_4}, y_1\right\} = \max\left\{\frac{B^2}{Cz_0}, \frac{B}{z_{-1}}\right\} = \frac{B}{z_{-1}}, z_6 = \max\left\{\frac{C}{y_4}, z_1\right\} = \max\left\{\frac{C^2}{By_0}, z_{-4}\right\} = z_{-4}$$

By  $(a_{4-1-1-1-1-1})$ , it follows that  $\frac{C}{B}z_{-4} > z_0, \frac{B^2}{Cz_{-1}} > y_0$ , which implies

$$\frac{B^2}{Cz_{-4}} < \frac{B}{z_0}, \frac{C^2}{B^2}z_{-1} < \frac{C}{y_0},$$

hence,

$$\begin{aligned}
 y_7 &= \max\left\{\frac{B}{z_5}, y_2\right\} = \max\left\{\frac{B^2}{Cz_{-4}}, \frac{B}{z_0}\right\} = y_{-3}, \quad z_7 = \max\left\{\frac{C}{y_5}, z_2\right\} = \max\left\{\frac{C^2}{B^2z_{-1}}, \frac{C}{y_0}\right\} = \frac{C}{y_0}, \\
 y_8 &= \max\left\{\frac{B}{z_6}, y_3\right\} = \max\left\{\frac{B}{z_{-4}}, \frac{B}{z_{-4}}\right\} = \frac{B}{z_{-4}}, \quad z_8 = \max\left\{\frac{C}{y_6}, z_3\right\} = \max\left\{\frac{C}{B}z_{-1}, \frac{C}{B}z_{-1}\right\} = \frac{C}{B}z_{-1}, \\
 y_9 &= \max\left\{\frac{B}{z_7}, y_4\right\} = \max\left\{\frac{B}{C}y_0, \frac{B}{C}y_0\right\} = \frac{B}{C}y_0, \quad z_9 = \max\left\{\frac{C}{y_7}, z_4\right\} = \max\left\{\frac{C}{B}z_0, \frac{C}{B}z_0\right\} = \frac{C}{B}z_0,
 \end{aligned}$$

and

$$y_{10} = \max\left\{\frac{B}{z_8}, y_5\right\} = \max\left\{\frac{B^2}{Cz_{-1}}, \frac{B^2}{Cz_{-1}}\right\} = \frac{B^2}{Cz_{-1}}, \quad z_{10} = \max\left\{\frac{C}{y_8}, z_5\right\} = \max\left\{\frac{C}{B}z_{-4}, \frac{C}{B}z_{-4}\right\} = \frac{C}{B}z_{-4}.$$

If the relations

$$y_{5k+1} = \frac{B}{z_{-1}}, y_{5k+2} = \frac{B}{z_0}, y_{5k+3} = \frac{B}{z_{-4}}, y_{5k+4} = \frac{B}{C}y_0, y_{5k+5} = \frac{B^2}{Cz_{-1}}$$

and

$$z_{5k+1} = z_{-4}, z_{5k+2} = \frac{C}{y_0}, z_{5k+3} = \frac{C}{B}z_{-1}, z_{5k+4} = \frac{C}{B}z_0, z_{5k+5} = \frac{C}{B}z_{-4}$$

are satisfied, then the following expressions are derived

$$\begin{aligned}
 y_{5k+6} &= \max\left\{\frac{B}{z_{5k+4}}, y_{5k+1}\right\} = \max\left\{\frac{B^2}{Cz_0}, \frac{B}{z_{-1}}\right\} = \frac{B}{z_{-1}}, \quad z_{5k+6} = \max\left\{\frac{C}{y_{5k+4}}, z_{5k+1}\right\} = \max\left\{\frac{C^2}{By_0}, z_{-4}\right\} = z_{-4}, \\
 y_{5k+7} &= \max\left\{\frac{B}{z_{5k+5}}, y_{5k+2}\right\} = \max\left\{\frac{B^2}{Cz_{-4}}, \frac{B}{z_0}\right\} = \frac{B}{z_0}, \quad z_{5k+7} = \max\left\{\frac{C}{y_{5k+5}}, z_{5k+2}\right\} = \max\left\{\frac{C^2}{B^2z_{-1}}, \frac{C}{y_0}\right\} = \frac{C}{y_0}, \\
 y_{5k+8} &= \max\left\{\frac{B}{z_{5k+6}}, y_{5k+3}\right\} = \max\left\{\frac{B}{z_{-4}}, \frac{B}{z_{-4}}\right\} = \frac{B}{z_{-4}}, \quad z_{5k+8} = \max\left\{\frac{C}{y_{5k+6}}, z_{5k+3}\right\} = \max\left\{\frac{C}{B}z_{-1}, \frac{C}{B}z_{-1}\right\} = \frac{C}{B}z_{-1}, \\
 y_{5k+9} &= \max\left\{\frac{B}{z_{5k+7}}, y_{5k+4}\right\} = \max\left\{\frac{B}{C}y_0, \frac{B}{C}y_0\right\} = \frac{B}{C}y_0, \quad z_{5k+9} = \max\left\{\frac{C}{y_{5k+7}}, z_{5k+4}\right\} = \max\left\{\frac{C}{B}z_0, \frac{C}{B}z_0\right\} = \frac{C}{B}z_0,
 \end{aligned}$$

and

$$y_{5k+10} = \max\left\{\frac{B}{z_{5k+8}}, y_{5k+5}\right\} = \max\left\{\frac{B^2}{Cz_{-1}}, \frac{B^2}{Cz_{-1}}\right\} = \frac{B^2}{Cz_{-1}}, \quad z_{5k+10} = \max\left\{\frac{C}{y_{5k+8}}, z_{5k+5}\right\} = \max\left\{\frac{C}{B}z_{-4}, \frac{C}{B}z_{-4}\right\} = \frac{C}{B}z_{-4}.$$

For case  $(a_{4-1-1-1-1})$ , assuming

$$\frac{B}{z_0} > y_{-3}, \frac{C}{y_0} > z_{-3}, \frac{B}{z_{-4}} > y_{-2}, \frac{C}{z_{-2}} > \frac{B}{z_{-1}}, \frac{B}{y_{-1}} > \frac{C}{y_0}, \frac{C}{z_{-1}} > \frac{B}{z_0}, \frac{B^2}{Cz_{-1}} > y_0, \frac{C}{z_0} > \frac{B}{z_{-4}}$$

and  $k \geq 0$ , mathematical induction yields the following equalities:

$$\begin{aligned}
 y_{5k+1} &= \frac{B}{z_{-1}}, y_{5k+2} = \frac{B}{z_0}, y_{5k+3} = \frac{B}{z_{-4}}, y_{5k+4} = \frac{B}{C}y_0, y_{5k+5} = \frac{B^2}{Cz_{-1}}, \\
 z_{5k+1} &= z_{-4}, z_{5k+2} = \frac{C}{y_0}, z_{5k+3} = \frac{C}{B}z_{-1}, z_{5k+4} = \frac{C}{B}z_0, z_{5k+5} = \frac{C}{B}z_{-4}.
 \end{aligned}$$

**Remark 14.** For the remaining 63 cases in  $(a_{4-1})$ , methods similar to those described above can be employed to prove the existence of periodic solutions of period 5 and to derive the corresponding explicit forms of the solutions.

$(a_{4-2})$  From Equation (11), if  $\frac{B}{z_0} > y_{-3}, \frac{C}{y_0} < z_{-3}$ , then one obtains  $y_2 = \frac{B}{z_0}, z_2 = z_{-3}$ .

( $a_{4-2-1}$ ) From Equation (12), if  $\frac{B}{z_1} > y_{-2}, \frac{C}{y_1} > z_{-2}$ , then one obtains

$$y_3 = \frac{B}{z_1} = \frac{B}{z_{-4}}, z_3 = \frac{C}{y_1} = \frac{C}{B}z_{-1}.$$

( $a_{4-2-1-1}$ ) From Equation (13), if  $\frac{B}{z_2} > y_{-1}, \frac{C}{y_2} > z_{-1}$ , then one obtains

$$y_4 = \frac{B}{z_2} = \frac{B}{z_{-3}}, z_4 = \frac{C}{y_2} = \frac{C}{B}z_0.$$

( $a_{4-2-1-1-1}$ ) From Equation (14), if  $\frac{B}{z_3} > y_0, \frac{C}{y_3} > z_0$ , then one obtains

$$y_5 = \frac{B}{z_3} = \frac{B^2}{Cz_{-1}}, z_5 = \frac{C}{y_3} = \frac{C}{B}z_{-4}.$$

By ( $a_{4-2-1-1}$ ), it follows that  $\frac{Cz_0}{B} > z_{-1}, \frac{B}{z_{-3}} > y_{-1}$ , which implies

$$y_6 = \max\left\{\frac{B}{z_4}, y_1\right\} = \max\left\{\frac{B^2}{Cz_0}, \frac{B}{z_{-1}}\right\} = \frac{B}{z_{-1}}, z_6 = \max\left\{\frac{C}{y_4}, z_1\right\} = \max\left\{\frac{Cz_{-3}}{B}, z_{-4}\right\} = z_{-4}$$

By ( $a_{4-2-1-1-1}$ ), it follows that  $\frac{C}{B}z_{-4} > z_0, \frac{B^2}{Cz_{-1}} > y_0$ , which implies

$$\frac{B^2}{Cz_{-4}} < \frac{B}{z_0}, \frac{C^2}{B^2}z_{-1} < \frac{C}{y_0},$$

hence,

$$\begin{aligned} y_7 &= \max\left\{\frac{B}{z_5}, y_2\right\} = \max\left\{\frac{B^2}{Cz_{-4}}, \frac{B}{z_0}\right\} = \frac{B}{z_0}, z_7 = \max\left\{\frac{C}{y_5}, z_2\right\} = \max\left\{\frac{C^2}{B^2}z_{-1}, z_{-3}\right\} = z_{-3}, \\ y_8 &= \max\left\{\frac{B}{z_6}, y_3\right\} = \max\left\{\frac{B}{z_{-4}}, \frac{B}{z_{-4}}\right\} = \frac{B}{z_{-4}}, z_8 = \max\left\{\frac{C}{y_6}, z_3\right\} = \max\left\{\frac{C}{B}z_{-1}, \frac{C}{B}z_{-1}\right\} = \frac{C}{B}z_{-1}, \\ y_9 &= \max\left\{\frac{B}{z_7}, y_4\right\} = \max\left\{\frac{B}{z_{-3}}, \frac{B}{z_{-3}}\right\} = \frac{B}{z_{-3}}, z_9 = \max\left\{\frac{C}{y_7}, z_4\right\} = \max\left\{\frac{C}{B}z_0, \frac{C}{B}z_0\right\} = \frac{C}{B}z_0, \end{aligned}$$

and

$$y_{10} = \max\left\{\frac{B}{z_8}, y_5\right\} = \max\left\{\frac{B^2}{Cz_{-1}}, \frac{B^2}{Cz_{-1}}\right\} = \frac{B^2}{Cz_{-1}}, z_{10} = \max\left\{\frac{C}{y_8}, z_5\right\} = \max\left\{\frac{C}{B}z_{-4}, \frac{C}{B}z_{-4}\right\} = \frac{C}{B}z_{-4}.$$

If the relations

$$y_{5k+1} = \frac{B}{z_{-1}}, y_{5k+2} = \frac{B}{z_0}, y_{5k+3} = \frac{B}{z_{-4}}, y_{5k+4} = \frac{B}{z_{-3}}, y_{5k+5} = \frac{B^2}{Cz_{-1}}$$

and

$$z_{5k+1} = z_{-4}, z_{5k+2} = z_{-3}, z_{5k+3} = \frac{C}{B}z_{-1}, z_{5k+4} = \frac{C}{B}z_0, z_{5k+5} = \frac{C}{B}z_{-4}$$

are satisfied, then the following expressions are derived

$$\begin{aligned} y_{5k+6} &= \max\left\{\frac{B}{z_{5k+4}}, y_{5k+1}\right\} = \max\left\{\frac{B^2}{Cz_0}, \frac{B}{z_{-1}}\right\} = \frac{B}{z_{-1}}, z_{5k+6} = \max\left\{\frac{C}{y_{5k+4}}, z_{5k+1}\right\} = \max\left\{\frac{Cz_{-3}}{B}, z_{-4}\right\} = z_{-4}, \\ y_{5k+7} &= \max\left\{\frac{B}{z_{5k+5}}, y_{5k+2}\right\} = \max\left\{\frac{B^2}{Cz_{-4}}, \frac{B}{z_0}\right\} = \frac{B}{z_0}, z_{5k+7} = \max\left\{\frac{C}{y_{5k+5}}, z_{5k+2}\right\} = \max\left\{\frac{C^2}{B^2}z_{-1}, z_{-3}\right\} = z_{-3}, \\ y_{5k+8} &= \max\left\{\frac{B}{z_{5k+6}}, y_{5k+3}\right\} = \max\left\{\frac{B}{z_{-4}}, \frac{B}{z_{-4}}\right\} = \frac{B}{z_{-4}}, z_{5k+8} = \max\left\{\frac{C}{y_{5k+6}}, z_{5k+3}\right\} = \max\left\{\frac{C}{B}z_{-1}, \frac{C}{B}z_{-1}\right\} = \frac{C}{B}z_{-1}, \end{aligned}$$

$$y_{5k+9} = \max\left\{\frac{B}{z_{5k+7}}, y_{5k+4}\right\} = \max\left\{\frac{B}{z_{-3}}, \frac{B}{z_{-3}}\right\} = \frac{B}{z_{-3}}, z_{5k+9} = \max\left\{\frac{C}{y_{5k+7}}, z_{5k+4}\right\} = \max\left\{\frac{C}{B}z_0, \frac{C}{B}z_0\right\} = \frac{C}{B}z_0,$$

and

$$y_{5k+10} = \max\left\{\frac{B}{z_{5k+8}}, y_{5k+5}\right\} = \max\left\{\frac{B^2}{Cz_{-1}}, \frac{B^2}{Cz_{-1}}\right\} = \frac{B^2}{Cz_{-1}}, z_{5k+10} = \max\left\{\frac{C}{y_{5k+8}}, z_{5k+5}\right\} = \max\left\{\frac{C}{B}z_{-4}, \frac{C}{B}z_{-4}\right\} = \frac{C}{B}z_{-4}.$$

For case  $(a_{4-2-1-1-1})$ , assuming

$$\frac{B}{z_0} > y_{-3}, \frac{C}{y_0} < z_{-3}, \frac{B}{z_{-4}} > y_{-2}, \frac{C}{z_{-2}} > \frac{B}{z_{-1}}, \frac{B}{z_{-3}} > y_{-1}, \frac{C}{z_{-1}} > \frac{B}{z_0}, \frac{B^2}{Cz_{-1}} > y_0, \frac{C}{z_0} > \frac{B}{z_{-4}}$$

and  $k \geq 0$ , mathematical induction yields the following equalities:

$$y_{5k+1} = \frac{B}{z_{-1}}, y_{5k+2} = \frac{B}{z_0}, y_{5k+3} = \frac{B}{z_{-4}}, y_{5k+4} = \frac{B}{z_{-3}}, y_{5k+5} = \frac{B^2}{Cz_{-1}},$$

$$z_{5k+1} = z_{-4}, z_{5k+2} = z_{-3}, z_{5k+3} = \frac{C}{B}z_{-1}, z_{5k+4} = \frac{C}{B}z_0, z_{5k+5} = \frac{C}{B}z_{-4}.$$

**Remark 15.** For the remaining 63 cases in  $(a_{4-2})$ , methods similar to those described above can be employed to prove the existence of periodic solutions of period 5 and to derive the corresponding explicit forms of the solutions.

$(a_{4-3})$  From Equation (11), if  $\frac{B}{z_0} < y_{-3}, \frac{C}{y_0} < z_{-3}$ , then one obtains  $y_2 = y_{-3}, z_2 = z_{-3}$ .

$(a_{4-3-1})$  From Equation (12), if  $\frac{B}{z_1} > y_{-2}, \frac{C}{y_1} > z_{-2}$ , then one obtains

$$y_3 = \frac{B}{z_1} = \frac{B}{z_{-4}}, z_3 = \frac{C}{y_1} = \frac{C}{B}z_{-1}.$$

$(a_{4-3-1-1})$  From Equation (13), if  $\frac{B}{z_2} > y_{-1}, \frac{C}{y_2} > z_{-1}$ , then one obtains

$$y_4 = \frac{B}{z_2} = \frac{B}{z_{-3}}, z_4 = \frac{C}{y_2} = \frac{C}{y_{-3}}.$$

$(a_{4-3-1-1-1})$  From Equation (14), if  $\frac{B}{z_3} > y_0, \frac{C}{y_3} > z_0$ , then one obtains

$$y_5 = \frac{B}{z_3} = \frac{B^2}{Cz_{-1}}, z_4 = \frac{C}{y_3} = \frac{C}{B}z_{-4}.$$

By  $(a_{4-3-1-1})$ , it follows that  $\frac{C}{y_{-3}} > z_{-1}, \frac{B}{z_{-3}} > y_{-1}$ , which implies

$$y_6 = \max\left\{\frac{B}{z_4}, y_1\right\} = \max\left\{\frac{By_{-3}}{C}, \frac{B}{z_{-1}}\right\} = \frac{B}{z_{-1}}, z_6 = \max\left\{\frac{C}{y_4}, z_1\right\} = \max\left\{\frac{Cz_{-3}}{B}, z_{-4}\right\} = z_{-4}.$$

By  $(a_{4-3-1-1-1})$ , it follows that  $\frac{C}{B}z_{-4} > z_0, \frac{B^2}{Cz_{-1}} > y_0$ , which implies

$$\frac{B^2}{Cz_{-4}} < \frac{B}{z_0}; \frac{C^2}{B^2}z_{-1} < \frac{C}{y_0},$$

hence,

$$y_7 = \max\left\{\frac{B}{z_5}, y_2\right\} = \max\left\{\frac{B^2}{Cz_{-4}}, y_{-3}\right\} = y_{-3}, z_7 = \max\left\{\frac{C}{y_5}, z_2\right\} = \max\left\{\frac{C^2}{B^2}z_{-1}, z_{-3}\right\} = z_{-3},$$

$$y_8 = \max\left\{\frac{B}{z_6}, y_3\right\} = \max\left\{\frac{B}{z_{-4}}, \frac{B}{z_{-4}}\right\} = \frac{B}{z_{-4}}, z_8 = \max\left\{\frac{C}{y_6}, z_3\right\} = \max\left\{\frac{C}{B}z_{-1}, \frac{C}{B}z_{-1}\right\} = \frac{C}{B}z_{-1},$$

$$y_9 = \max\left\{\frac{B}{z_7}, y_4\right\} = \max\left\{\frac{B}{z_{-3}}, \frac{B}{z_{-3}}\right\} = \frac{B}{z_{-3}}, z_9 = \max\left\{\frac{C}{y_7}, z_4\right\} = \max\left\{\frac{C}{y_{-3}}, \frac{C}{y_{-3}}\right\} = \frac{C}{y_{-3}},$$

and

$$y_{10} = \max\left\{\frac{B}{z_8}, y_5\right\} = \max\left\{\frac{B^2}{Cz_{-1}}, \frac{B^2}{Cz_{-1}}\right\} = \frac{B^2}{Cz_{-1}}, z_{10} = \max\left\{\frac{C}{y_8}, z_5\right\} = \max\left\{\frac{C}{B^{z_{-4}}}, \frac{C}{B^{z_{-4}}}\right\} = \frac{C}{B^{z_{-4}}}.$$

If the relations

$$y_{5k+1} = \frac{B}{z_{-1}}, y_{5k+2} = y_{-3}, y_{5k+3} = \frac{B}{z_{-4}}, y_{5k+4} = \frac{B}{z_{-3}}, y_{5k+5} = \frac{B^2}{Cz_{-1}}$$

and

$$z_{5k+1} = z_{-4}, z_{5k+2} = z_{-3}, z_{5k+3} = \frac{C}{B^{z_{-1}}}, z_{5k+4} = \frac{C}{y_{-3}}, z_{5k+5} = \frac{C}{B^{z_{-4}}}$$

are satisfied, then the following expressions are derived

$$y_{5k+6} = \max\left\{\frac{B}{z_{5k+4}}, y_{5k+1}\right\} = \max\left\{\frac{By_{-3}}{C}, \frac{B}{z_{-1}}\right\} = \frac{B}{z_{-1}}, z_{5k+6} = \max\left\{\frac{C}{y_{5k+4}}, z_{5k+1}\right\} = \max\left\{\frac{Cz_{-3}}{B}, z_{-4}\right\} = z_{-4},$$

$$y_{5k+7} = \max\left\{\frac{B}{z_{5k+5}}, y_{5k+2}\right\} = \max\left\{\frac{B^2}{Cz_{-4}}, y_{-3}\right\} = y_{-3}, z_{5k+7} = \max\left\{\frac{C}{y_{5k+5}}, z_{5k+2}\right\} = \max\left\{\frac{C^2}{B^2 z_{-1}}, z_{-3}\right\} = z_{-3},$$

$$y_{5k+8} = \max\left\{\frac{B}{z_{5k+6}}, y_{5k+3}\right\} = \max\left\{\frac{B}{z_{-4}}, \frac{B}{z_{-4}}\right\} = \frac{B}{z_{-4}}, z_{5k+8} = \max\left\{\frac{C}{y_{5k+6}}, z_{5k+3}\right\} = \max\left\{\frac{C}{B^{z_{-1}}}, \frac{C}{B^{z_{-1}}}\right\} = \frac{C}{B^{z_{-1}}},$$

$$y_{5k+9} = \max\left\{\frac{B}{z_{5k+7}}, y_{5k+4}\right\} = \max\left\{\frac{B}{z_{-3}}, \frac{B}{z_{-3}}\right\} = \frac{B}{z_{-3}}, z_{5k+9} = \max\left\{\frac{C}{y_{5k+7}}, z_{5k+4}\right\} = \max\left\{\frac{C}{y_{-3}}, \frac{C}{y_{-3}}\right\} = \frac{C}{y_{-3}},$$

and

$$y_{5k+10} = \max\left\{\frac{B}{z_{5k+8}}, y_{5k+5}\right\} = \max\left\{\frac{B^2}{Cz_{-1}}, \frac{B^2}{Cz_{-1}}\right\} = \frac{B^2}{Cz_{-1}}, z_{5k+10} = \max\left\{\frac{C}{y_{5k+8}}, z_{5k+5}\right\} = \max\left\{\frac{C}{B^{z_{-4}}}, \frac{C}{B^{z_{-4}}}\right\} = \frac{C}{B^{z_{-4}}}.$$

For case  $(a_{4-3-1-1-1})$ , assuming

$$\frac{B}{z_0} < y_{-3}, \frac{C}{y_0} < z_{-3}, \frac{B}{z_{-4}} > y_{-2}, \frac{C}{z_{-2}} > \frac{B}{z_{-1}}, \frac{B}{z_{-3}} > y_{-1}, \frac{C}{y_{-3}} > z_{-1}, \frac{B^2}{Cz_{-1}} > y_0, \frac{C}{z_0} > \frac{B}{z_{-4}}$$

and  $k \geq 0$ , mathematical induction yields the following equalities:

$$y_{5k+1} = \frac{B}{z_{-1}}, y_{5k+2} = y_{-3}, y_{5k+3} = \frac{B}{z_{-4}}, y_{5k+4} = \frac{B}{z_{-3}}, y_{5k+5} = \frac{B^2}{Cz_{-1}},$$

$$z_{5k+1} = z_{-4}, z_{5k+2} = z_{-3}, z_{5k+3} = \frac{C}{B^{z_{-1}}}, z_{5k+4} = \frac{C}{y_{-3}}, z_{5k+5} = \frac{C}{B^{z_{-4}}}.$$

**Remark 16.** For the remaining 63 cases in  $(a_{4-3})$ , methods similar to those described above can be employed to prove the existence of periodic solutions of period 5 and to derive the corresponding explicit forms of the solutions.

$(a_{4-4})$  From Equation (11), if  $\frac{B}{z_0} < y_{-3}, \frac{C}{y_0} > z_{-3}$ , then one obtains  $y_2 = y_{-3}, z_2 = \frac{C}{y_0}$ .

$(a_{4-4-1})$  From Equation (12), if  $\frac{B}{z_1} > y_{-2}, \frac{C}{y_1} > z_{-2}$ , then one obtains

$$y_3 = \frac{B}{z_1} = \frac{B}{z_{-4}}, z_3 = \frac{C}{y_1} = \frac{C}{B^{z_{-1}}}.$$

( $a_{4-4-1-1}$ ) From Equation (13), if  $\frac{B}{z_2} > y_{-1}, \frac{C}{y_2} > z_{-1}$ , then one obtains

$$y_4 = \frac{B}{z_2} = \frac{B}{C}y_0, z_4 = \frac{C}{y_2} = \frac{C}{y_{-3}}.$$

( $a_{4-4-1-1-1}$ ) From Equation (14), if  $\frac{B}{z_3} > y_0, \frac{C}{y_3} > z_0$ , then one obtains

$$y_5 = \frac{B}{z_3} = \frac{B^2}{Cz_{-1}}, z_4 = \frac{C}{y_3} = \frac{C}{B}z_{-4}.$$

By ( $a_{4-4-1-1}$ ), it follows that  $\frac{C}{y_{-3}} > z_{-1}, \frac{By_0}{C} > y_{-1}$ , which implies

$$y_6 = \max\left\{\frac{B}{z_4}, y_1\right\} = \max\left\{\frac{B}{C}y_{-3}, \frac{B}{z_{-1}}\right\} = \frac{B}{z_{-1}}, z_6 = \max\left\{\frac{C}{y_4}, z_1\right\} = \max\left\{\frac{C^2}{By_0}, z_{-4}\right\} = z_{-4}.$$

By ( $a_{4-4-1-1-1}$ ), it follows that  $\frac{C}{B}z_{-4} > z_0, \frac{B^2}{Cz_{-1}} > y_0$ , which implies

$$\frac{B^2}{Cz_{-4}} < \frac{B}{z_0}, \frac{C^2}{B^2}z_{-1} < \frac{C}{y_0},$$

hence,

$$\begin{aligned} y_7 &= \max\left\{\frac{B}{z_5}, y_2\right\} = \max\left\{\frac{B^2}{Cz_{-4}}, y_{-3}\right\} = y_{-3}, z_7 = \max\left\{\frac{C}{y_5}, z_2\right\} = \max\left\{\frac{C^2}{B^2}z_{-1}, \frac{C}{y_0}\right\} = \frac{C}{y_0}, \\ y_8 &= \max\left\{\frac{B}{z_6}, y_3\right\} = \max\left\{\frac{B}{z_{-4}}, \frac{B}{z_{-4}}\right\} = \frac{B}{z_{-4}}, z_8 = \max\left\{\frac{C}{y_6}, z_3\right\} = \max\left\{\frac{C}{B}z_{-1}, \frac{C}{B}z_{-1}\right\} = \frac{C}{B}z_{-1}, \\ y_9 &= \max\left\{\frac{B}{z_7}, y_4\right\} = \max\left\{\frac{B}{C}y_0, \frac{B}{C}y_0\right\} = \frac{B}{C}y_0, z_9 = \max\left\{\frac{C}{y_7}, z_4\right\} = \max\left\{\frac{C}{y_{-3}}, \frac{C}{y_{-3}}\right\} = \frac{C}{y_{-3}}, \end{aligned}$$

and

$$y_{10} = \max\left\{\frac{B}{z_8}, y_5\right\} = \max\left\{\frac{B^2}{Cz_{-1}}, \frac{B^2}{Cz_{-1}}\right\} = \frac{B^2}{Cz_{-1}}, z_{10} = \max\left\{\frac{C}{y_8}, z_5\right\} = \max\left\{\frac{C}{B}z_{-4}, \frac{C}{B}z_{-4}\right\} = \frac{C}{B}z_{-4}.$$

If the relations

$$y_{5k+1} = \frac{B}{z_{-1}}, y_{5k+2} = y_{-3}, y_{5k+3} = \frac{B}{z_{-4}}, y_{5k+4} = \frac{B}{C}y_0, y_{5k+5} = \frac{B^2}{Cz_{-1}}$$

and

$$z_{5k+1} = z_{-4}, z_{5k+2} = \frac{C}{y_0}, z_{5k+3} = \frac{C}{B}z_{-1}, z_{5k+4} = \frac{C}{y_{-3}}, z_{5k+5} = \frac{C}{B}z_{-4}$$

are satisfied, then the following expressions are derived

$$\begin{aligned} y_{5k+6} &= \max\left\{\frac{B}{z_{5k+4}}, y_{5k+1}\right\} = \max\left\{\frac{B}{C}y_{-3}, \frac{B}{z_{-1}}\right\} = \frac{B}{z_{-1}}, z_{5k+6} = \max\left\{\frac{C}{y_{5k+4}}, z_{5k+1}\right\} = \max\left\{\frac{C^2}{By_0}, z_{-4}\right\} = z_{-4}, \\ y_{5k+7} &= \max\left\{\frac{B}{z_{5k+5}}, y_{5k+2}\right\} = \max\left\{\frac{B^2}{Cz_{-4}}, y_{-3}\right\} = y_{-3}, z_{5k+7} = \max\left\{\frac{C}{y_{5k+5}}, z_{5k+2}\right\} = \max\left\{\frac{C^2}{B^2}z_{-1}, \frac{C}{y_0}\right\} = \frac{C}{y_0}, \\ y_{5k+8} &= \max\left\{\frac{B}{z_{5k+6}}, y_{5k+3}\right\} = \max\left\{\frac{B}{z_{-4}}, \frac{B}{z_{-4}}\right\} = \frac{B}{z_{-4}}, z_{5k+8} = \max\left\{\frac{C}{y_{5k+6}}, z_{5k+3}\right\} = \max\left\{\frac{C}{B}z_{-1}, \frac{C}{B}z_{-1}\right\} = \frac{C}{B}z_{-1}, \\ y_{5k+9} &= \max\left\{\frac{B}{z_{5k+7}}, y_{5k+4}\right\} = \max\left\{\frac{B}{C}y_0, \frac{B}{C}y_0\right\} = \frac{B}{C}y_0, z_{5k+9} = \max\left\{\frac{C}{y_{5k+7}}, z_{5k+4}\right\} = \max\left\{\frac{C}{y_{-3}}, \frac{C}{y_{-3}}\right\} = \frac{C}{y_{-3}}, \end{aligned}$$

and

$$y_{5k+10} = \max\left\{\frac{B}{z_{5k+8}}, y_{5k+5}\right\} = \max\left\{\frac{B^2}{Cz_{-1}}, \frac{B^2}{Cz_{-1}}\right\} = \frac{B^2}{Cz_{-1}}, z_{5k+10} = \max\left\{\frac{C}{y_{5k+8}}, z_{5k+5}\right\} = \max\left\{\frac{C}{B}z_{-4}, \frac{C}{B}z_{-4}\right\} = \frac{C}{B}z_{-4}.$$

For case  $(a_{4-4-1-1-1})$ , assuming

$$\frac{B}{z_0} < y_{-3}, \frac{C}{y_0} > z_{-3}, \frac{B}{z_{-4}} > y_{-2}, \frac{C}{z_{-2}} > \frac{B}{z_{-1}}, \frac{B}{y_{-1}} > \frac{C}{y_0}, \frac{C}{y_{-3}} > z_{-1}, \frac{B^2}{Cz_{-1}} > y_0, \frac{C}{z_0} > \frac{B}{z_{-4}}$$

and  $k \geq 0$ , mathematical induction yields the following equalities:

$$y_{5k+1} = \frac{B}{z_{-1}}, y_{5k+2} = y_{-3}, y_{5k+3} = \frac{B}{z_{-4}}, y_{5k+4} = \frac{B}{C}y_0, y_{5k+5} = \frac{B^2}{Cz_{-1}},$$

$$z_{5k+1} = z_{-4}, z_{5k+2} = \frac{C}{y_0}, z_{5k+3} = \frac{C}{B}z_{-1}, z_{5k+4} = \frac{C}{y_{-3}}, z_{5k+5} = \frac{C}{B}z_{-4}.$$

**Remark 17.** For the remaining 63 cases in  $(a_{4-4})$ , methods similar to those described above can be employed to prove the existence of periodic solutions of period 5 and to derive the corresponding explicit forms of the solutions.

In conclusion, for any prescribed positive real initial values,  $y_{-4}, y_{-3}, y_{-2}, y_{-1}, y_0$  and  $z_{-4}, z_{-3}, z_{-2}, z_{-1}, z_0$ , every solution of system (9) is periodic with period 5; that is,

$$y_{5k+1} = y_{5k+6}, y_{5k+2} = y_{5k+7}, y_{5k+3} = y_{5k+8}, y_{5k+4} = y_{5k+9}, y_{5k+5} = y_{5k+10}$$

$$z_{5k+1} = z_{5k+6}, z_{5k+2} = z_{5k+7}, z_{5k+3} = z_{5k+8}, z_{5k+4} = z_{5k+9}, z_{5k+5} = z_{5k+10}$$

for  $k \geq 0$ . This completes the proof.  $\square$

**Theorem 3.** For fuzzy difference Equation (8), suppose that the parameter  $A$  and the initial conditions  $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$  are positive fuzzy numbers. Then each solution of Equation (8) is periodic with period 5.

**Proof.** Let  $\{x_n\}$  be a positive solution of Equation (8) with initial values  $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ . Based on  $\alpha$ -cut theory, we have

$$[x_i]_\alpha = [L_{i,\alpha}, R_{i,\alpha}], i = -4, -3, -2, -1, 0, \alpha \in (0, 1], [x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], n = 1, 2, 3, \dots, \alpha \in (0, 1].$$

Since  $A$  is a positive fuzzy number, we have

$$[A]_\alpha = [A_{l,\alpha}, A_{r,\alpha}] = [B, C], \alpha \in (0, 1].$$

According to Theorem 1,  $(L_{n,\alpha}, R_{n,\alpha}), n = 1, 2, 3, \dots, \alpha \in (0, 1]$  satisfy the system

$$L_{n+1,\alpha} = \max\left\{\frac{B}{R_{n-1,\alpha}}, L_{n-4,\alpha}\right\}, R_{n+1,\alpha} = \max\left\{\frac{C}{L_{n-1,\alpha}}, R_{n-4,\alpha}\right\} n \geq 0.$$

Applying Theorem 2, we get

$$L_{5n+1,\alpha} = L_{6,\alpha}, L_{5n+2,\alpha} = L_{7,\alpha}, L_{5n+3,\alpha} = L_{8,\alpha}, L_{5n+4,\alpha} = L_{9,\alpha}, L_{5n+5,\alpha} = L_{10,\alpha},$$

$$R_{5n+1,\alpha} = R_{6,\alpha}, R_{5n+2,\alpha} = R_{7,\alpha}, R_{5n+3,\alpha} = R_{8,\alpha}, R_{5n+4,\alpha} = R_{9,\alpha}, R_{5n+5,\alpha} = R_{10,\alpha}, n \geq 0, \alpha \in (0, 1].$$

Therefore,  $\{x_n\}$  is ultimately a periodic solution with period 5. This completes the proof.  $\square$

**Theorem 4.** For fuzzy difference Equation (8), suppose that the parameter  $A$  and the initial conditions  $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$  are positive fuzzy numbers. Then every positive solution of Equation (8) is bounded and persistent.

**Proof.** Based on  $\alpha$ -cut theory, fuzzy difference Equation (8) can be converted into the form of difference system (9). Then, applying Theorem 2, for  $k \geq 0$ , the following equations are valid

$$\begin{aligned}
 y_{5k+1} &= \frac{B}{z_{-1}} \text{ or } y_{-4}, y_{5k+2} = \frac{B}{z_0} \text{ or } y_{-3}, y_{5k+3} = \frac{B}{C} y_{-1} \text{ or } \frac{B}{z_{-4}}, \\
 y_{5k+4} &= \frac{B}{C} y_0 \text{ or } \frac{B}{z_{-3}}, y_{5k+5} = \frac{B^2}{Cz_{-1}} \text{ or } \frac{B}{C} y_{-4}, \\
 z_{5k+1} &= \frac{C}{y_{-1}} \text{ or } z_{-4}, z_{5k+2} = \frac{C}{y_0} \text{ or } z_{-3}, z_{5k+3} = \frac{C}{B} z_{-1} \text{ or } \frac{C}{y_{-4}}, \\
 z_{5k+4} &= \frac{C}{B} z_0 \text{ or } \frac{C}{y_{-3}}, y_{5k+5} = \frac{C^2}{By_{-1}} \text{ or } \frac{C}{B} z_{-4}.
 \end{aligned}$$

Hence,  $P, Q \in (0, +\infty)$  exists with  $P < Q$  such that

$$\text{supp } y_n \in [P, Q], \text{supp } z_n \in [P, Q], \text{for } n = 1, 2, 3 \dots \tag{15}$$

From Equation (15), for any  $x_n, S, T \in (0, +\infty)$  exists with  $S < T$ , satisfying  $\text{supp } x_n \in [S, T]$ , for  $n = 1, 2, 3 \dots$ . This completes the proof.  $\square$

**Remark 18.** The present paper focuses on the case where  $A$  is a constant positive fuzzy number. This assumption is essential in the proof of Theorem 2, because the  $\alpha$ -cut endpoints of  $A$  are fixed positive real numbers  $B$  and  $C$ . If  $A$  is replaced by a periodic fuzzy sequence  $A_n$ , then the corresponding  $\alpha$ -cut endpoints become periodic real sequences  $B_n$  and  $C_n$ . In this case, the associated parametric ordinary difference system contains periodic coefficients, and the periodicity of its solutions depends not only on the fifth-order structure of the equation but also on the period of  $A_n$ . When the period of  $A_n$  is compatible with 5, the period-five property may still be preserved under suitable assumptions. For a general period  $p$ , the resulting periodic behavior may be related to the least common multiple of 5 and  $p$ . A complete treatment of this periodic-coefficient fuzzy case requires a separate analysis and will be considered in future work.

**Remark 19.** The positivity assumption on the parameter and initial values is essential in this paper. All the theoretical results are established in the setting of positive fuzzy numbers. After applying the  $\alpha$ -cut representation, the fuzzy equation is transformed into a parametric ordinary difference system whose endpoint sequences are positive real sequences. The proofs of Theorem 2 and Theorem 4 rely on this positive setting. If some initial endpoints are zero, the recurrence may contain zero denominators. In this case, the next iterate may be undefined, and the existence and uniqueness of the solution sequence cannot be guaranteed by Theorem 1. Therefore, zero initial values are excluded from the present framework. If some initial endpoints are negative but nonzero, the corresponding ordinary recurrence may still be algebraically meaningful in some special cases. However, such solutions no longer belong to the positive fuzzy-number setting considered in this paper. In particular, the  $\alpha$ -cut endpoints may fail to preserve the positive order structure required in the transformed system, and the persistence result is no longer meaningful. Moreover, the case analysis used to prove the period-five property is based on positive endpoint sequences. Hence, the same periodicity conclusion cannot be guaranteed for negative initial values. Furthermore, the assumption regarding the positive fuzzy initial value aligns with the actual problem being described.

### 4. Numerical Simulations

**Example 1.** Consider the following max-type fuzzy difference equation:

$$x_{n+1} = \max\left\{\frac{A}{x_{n-1}}, x_{n-4}\right\}, n = 0, 1, 2, \dots, \tag{16}$$

here,  $A$  is a positive fuzzy number, and its membership function is defined by

$$A(x) = \begin{cases} \frac{1}{3}x - 1, & 3 \leq x \leq 6, \\ -\frac{1}{2}x + 4, & 6 \leq x \leq 8. \end{cases} \tag{17}$$

The membership function of the initial values  $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$  are defined as follows.

$$\begin{aligned} x_0(x) &= \begin{cases} x - 1, & 1 \leq x \leq 2, \\ -\frac{1}{2}x + 2, & 2 \leq x \leq 4, \end{cases} & x_{-1}(x) &= \begin{cases} \frac{1}{3}x - 2, & 6 \leq x \leq 9, \\ -x + 10, & 9 \leq x \leq 10, \end{cases} \\ x_{-2}(x) &= \begin{cases} 2x - 3, & 1.5 \leq x \leq 2, \\ -\frac{1}{4}x + \frac{3}{2}, & 2 \leq x \leq 6, \end{cases} & x_{-3}(x) &= \begin{cases} \frac{1}{5}x - \frac{2}{5}, & 2 \leq x \leq 7, \\ -2x + 15, & 7 \leq x \leq 7.5, \end{cases} \\ x_{-4}(x) &= \begin{cases} \frac{1}{7}x - \frac{3}{7}, & 3 \leq x \leq 10, \\ -\frac{1}{5}x + 3, & 10 \leq x \leq 15. \end{cases} \end{aligned} \tag{18}$$

The membership functions in (17) and (18) are chosen as triangular fuzzy numbers. This choice is made because triangular fuzzy numbers are simple, commonly used, and convenient for numerical computation through  $\alpha$ -cuts. The parameters of these fuzzy numbers are selected so that the supports of  $A$  and all initial values are strictly positive. Hence, the assumptions of Theorem 2 are satisfied. These membership functions are not obtained by fitting a specific empirical data set, but are used as representative examples to illustrate the theoretical results.

It should also be noted that the theoretical conclusions of this paper are not restricted to these particular triangular membership functions. The essential requirement is that the fuzzy parameter and the fuzzy initial values are positive fuzzy numbers satisfying the stated assumptions. Therefore, other admissible membership functions may also be used. If nonlinear membership functions are adopted, such as Gaussian-type membership functions, the specific  $\alpha$ -cut endpoints, numerical trajectories, and interval widths may change. However, the qualitative conclusions on periodicity, boundedness, and persistence remain valid as long as the corresponding  $\alpha$ -cuts define positive fuzzy numbers and satisfy the assumptions of the theorems. For a standard Gaussian membership function with unbounded support, additional restrictions or truncation may be needed to fit the compact-support requirement of fuzzy numbers used in this paper.

From (17) and (18), it follows that  $[A]_\alpha = [3 + 3\alpha, 8 - 2\alpha], \alpha \in (0, 1]$  and

$$\begin{aligned} [x_0]_\alpha &= [1 + \alpha, 4 - 2\alpha], [x_{-1}]_\alpha = [6 + 3\alpha, 10 - \alpha], [x_{-2}]_\alpha = [1.5 + 0.5\alpha, 6 - 4\alpha], \\ [x_{-3}]_\alpha &= [2 + 5\alpha, 7.5 - 0.5\alpha], [x_{-4}]_\alpha = [3 + 7\alpha, 15 - 5\alpha], \alpha \in (0, 1]. \end{aligned}$$

Using (16) and the specified parameters and initial values, the following difference system can be constructed:

$$\begin{aligned} L_{n+1,\alpha} &= \max\left\{\frac{3+3\alpha}{R_{n-1,\alpha}}, L_{n-4,\alpha}\right\} \\ R_{n+1,\alpha} &= \max\left\{\frac{8-2\alpha}{L_{n-1,\alpha}}, R_{n-4,\alpha}\right\}, \alpha \in (0, 1], n = 0, 1, 2, \dots, \end{aligned} \tag{19}$$

where we may set  $y_i = L_{i,\alpha}, z_i = R_{i,\alpha} (i = -4, -3, -2, -1, \dots)$ . Then  $y_i$  and  $z_i$  are numerical expressions as functions of  $\alpha$ . Numerical simulations using MATLAB 2024 produce Figures 1–6.

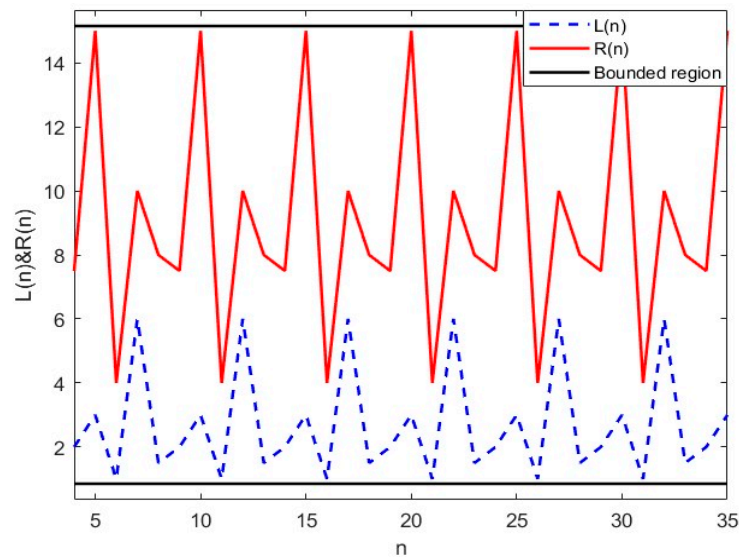


Figure 1. Solutions of system (19) when  $\alpha = 0$ .

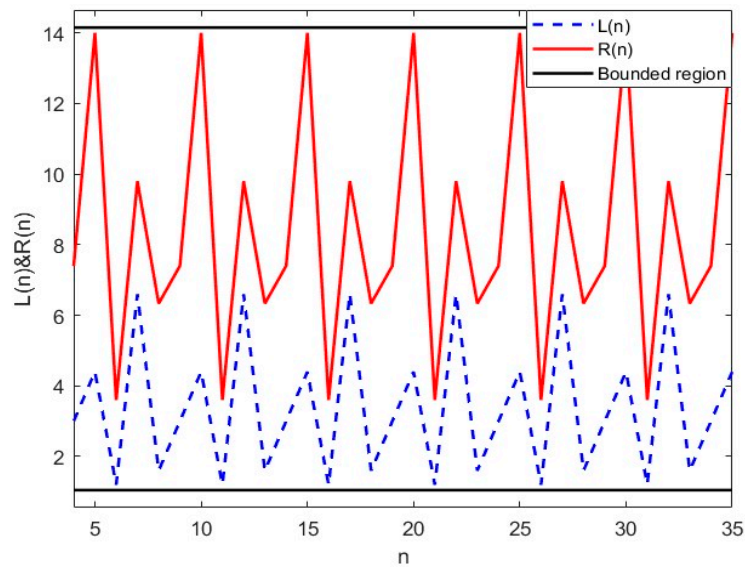


Figure 2. Solutions of system (19) when  $\alpha = 0.2$ .

By simple calculation, system (19), with the parameter and initial values specified by (17) and (18), satisfies the assumptions of Theorem 2. Hence, according to Theorem 2, every positive solution of system (19) is eventually periodic with period five and is both bounded and persistent (see Figures 1–6). The numerical results show that the periodic behavior is preserved for different  $\alpha$ -levels. For each fixed  $\alpha$ , the  $\alpha$ -cut gives an interval-valued trajectory. When  $\alpha$  increases, the interval width decreases and the lower and upper trajectories become closer. Nevertheless, the same periodic pattern can be observed for all selected  $\alpha$ -levels. This indicates that the value of  $\alpha$  changes the uncertainty range of the fuzzy solution, but it does not alter the qualitative periodicity established in the theoretical analysis.

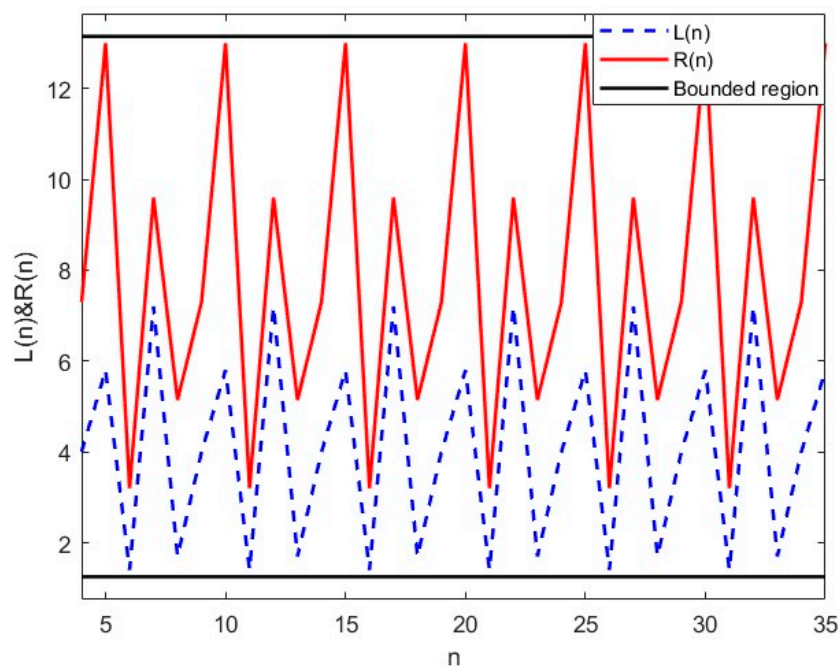


Figure 3. Solutions of system (19) when  $\alpha = 0.4$ .

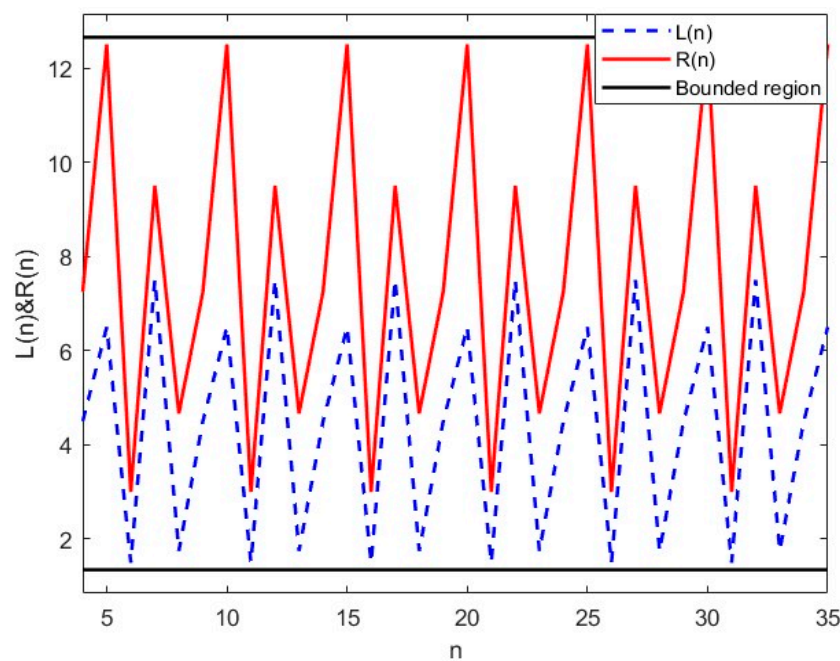


Figure 4. Solutions of system (19) when  $\alpha = 0.5$ .

To compare the theoretical and numerical behavior more quantitatively, we further compute several numerical indicators for Example 1. For each selected  $\alpha$ -level, the observed eventual period, the numerical lower bound, the numerical upper bound, and the maximum width of the  $\alpha$ -cut intervals are recorded over the plotted interval from  $n = 4$  to  $n = 35$ . The numerical results are summarized in Table 1.

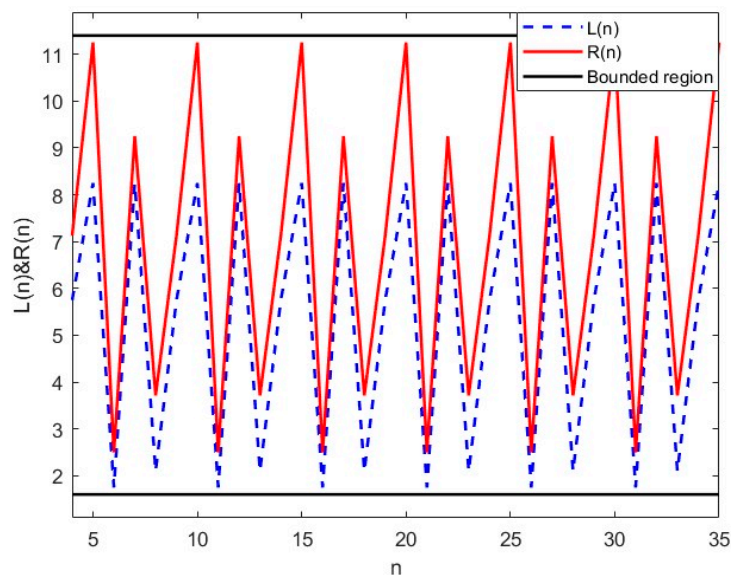


Figure 5. Solutions of system (19) when  $\alpha = 0.75$ .

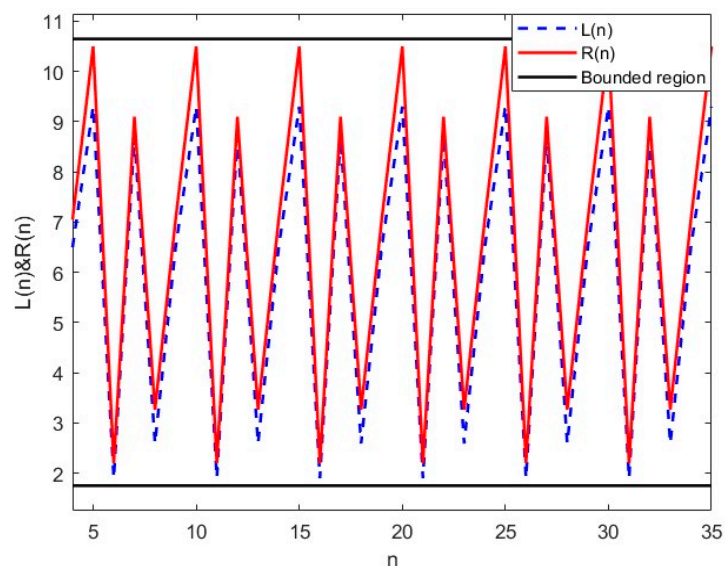


Figure 6. Solutions of system (19) when  $\alpha = 0.9$ .

Table 1. Quantitative comparison between theoretical and numerical results for Example 1.

$\alpha$	Theoretical Eventual Period	Observed Eventual Period	Numerical Lower Bound	Numerical Upper Bound	Maximum $\alpha$ -Cut Width
0	5	5	1.00	15.00	12.00
0.20	5	5	1.20	14.00	9.60
0.40	5	5	1.40	13.00	7.20
0.50	5	5	1.50	12.50	6.00
0.75	5	5	1.75	11.25	3.00
0.90	5	5	1.90	10.50	1.20

The numerical results in Table 1 are consistent with Theorem 2. For all selected  $\alpha$ -levels, the theoretical eventual period is five, and the observed eventual period is also five. The numerical lower bounds remain strictly positive, which illustrates persistence, while

the numerical upper bounds remain finite, which illustrates boundedness. Moreover, the maximum  $\alpha$ -cut width decreases as  $\alpha$  increases. This shows that higher  $\alpha$ -levels correspond to narrower uncertainty intervals. However, the change in the  $\alpha$ -cut width does not affect the eventual period-five structure of the solutions. Thus, the numerical results provide a quantitative confirmation of the theoretical conclusions, rather than merely a qualitative graphical illustration.

To further examine the influence of initial values on the eventual periodic behavior, we perform an initial-value sensitivity analysis based on Example 1. In this analysis, the fuzzy parameter A and the structure of system (19) are kept unchanged, while only the positive initial values are perturbed. Five groups of admissible positive initial values are considered. The first group, denoted by G0, is the original initial-value group. In G1 and G2, all initial values are multiplied by 0.8 and 1.2, respectively. In G3, a mixed positive multiplicative perturbation is imposed. More precisely, the first to fifth initial endpoint pairs are multiplied by 0.9, 1.1, 0.85, 1.15, and 1.05, respectively. That is, both the lower and upper endpoints of each initial  $\alpha$ -cut are multiplied by the same corresponding factor. In G4, a positive additive perturbation is imposed. Specifically, 0.5, 0.2, 0.3, 0.4, and 0.1 are added to the first to fifth initial endpoint pairs, respectively. In all cases, the positivity of the initial values and the endpoint ordering condition are preserved.

As a representative case, Figures 7 and 8 present the numerical results for  $\alpha = 0.75$ . Figure 7 shows the lower endpoint sequence under different initial-value groups, while Figure 8 shows the corresponding upper endpoint sequence. From Figures 7 and 8, it can be observed that changing the initial values leads to different numerical profiles of the lower and upper endpoint sequences. In particular, the amplitude and the specific values of the eventual periodic orbit vary with the initial-value group. This indicates that the initial values affect the transient behavior and the numerical form of the solution. However, the periodic structure is not destroyed. For all five initial-value groups, both endpoint sequences still exhibit a repeating pattern with period five after a finite number of iterations. Therefore, the numerical results show that admissible positive perturbations of the initial values do not change the eventual period-five behavior of the solution.

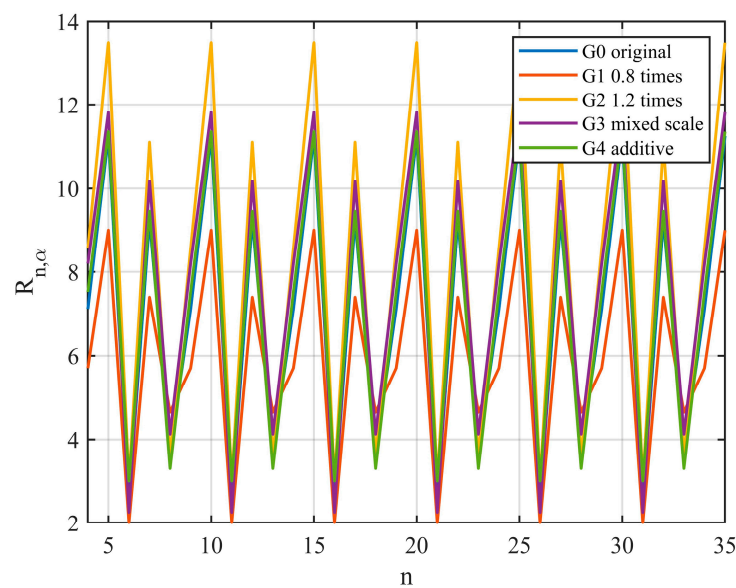
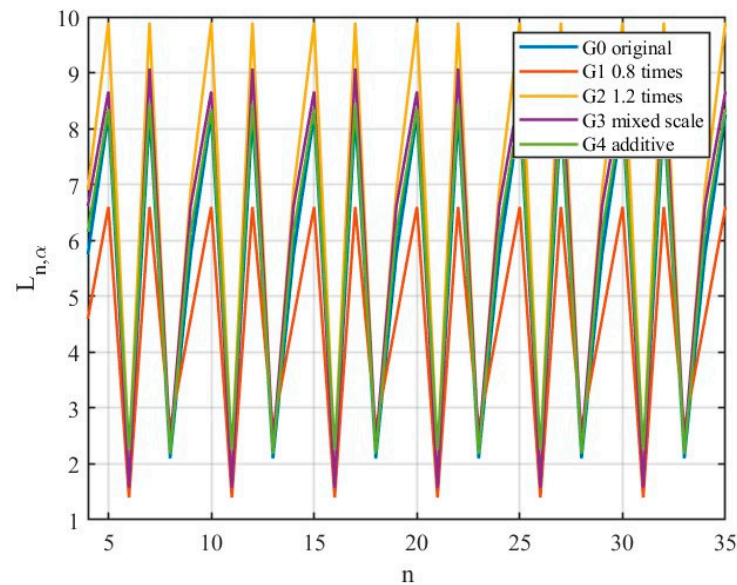


Figure 7. Initial-value sensitivity of the right endpoint sequence at  $\alpha = 0.75$ .

This observation is consistent with Theorem 2, which states that, for any prescribed positive real initial values, every solution of system (9) is eventually periodic with period

five. Hence, the initial-value sensitivity analysis further confirms the robustness of the eventual periodicity result.



**Figure 8.** Initial-value sensitivity of the left endpoint sequence at  $\alpha = 0.75$ .

## 5. Conclusions and Future Expectations

The present work investigates a class of max-type fuzzy difference equations and provides a systematic analysis of their dynamical properties. Initially, by employing fuzzy set theory, the fuzzy difference equation is reformulated into a corresponding system of parametric ordinary difference equations. Subsequently, using iterative techniques, inequality methods, and mathematical induction, the explicit forms of the periodic solutions of the max-type ordinary difference equations are established, from which the periodic solutions of the associated fuzzy difference equations are further obtained. Finally, based on the obtained periodic solutions, the boundedness and persistence of the solutions are established. In Section 4, numerical simulations performed using MATLAB 2024 confirm the validity of the theoretical findings.

Building on the contributions of this work, several promising directions for future research emerge. First, the parametric reduction framework developed here can be extended to more general classes of fuzzy difference equations, such as those involving min max or combination operators, the parameter  $A$  represents the case of periodic fuzzy numbers, as well as equations with time delays or variable coefficients. Second, the current analysis focuses on periodic behavior; however, the methodology could be adapted to investigate other dynamical properties including asymptotic stability, bifurcation phenomena, and chaotic dynamics in higher-order fuzzy systems. Third, incorporating uncertainty propagation techniques, such as interval analysis or fuzzy arithmetic with correlated parameters, may lead to more robust numerical simulations and tighter bounds on solution trajectories. Fourth, from an application perspective, the results can be utilized in fields like population dynamics, economic modeling, and control engineering where max-type nonlinearities naturally arise. Future work could also consider the development of adaptive numerical algorithms that automatically handle parameter-dependent discontinuities. Moreover, the interplay between fuzzy membership functions and the periodicity of solutions deserves further exploration, potentially leading to design criteria for fuzzy systems with prescribed oscillatory behavior. Lastly, combining the present approach with machine learning tools, such as neural network-based surrogate models, might enable efficient real-time prediction and control of fuzzy dynamical systems with complex nonlinear interactions.

**Author Contributions:** Conceptualization, T.Y. and C.W.; Methodology, T.Y., L.M. and C.W.; Software, T.Y.; Investigation, T.Y., L.M., R.Y. and C.W.; Writing—review & editing, L.M., R.Y. and C.W.; Funding acquisition, L.M., R.Y. and C.W. All authors have read and agreed to the published version of the manuscript.

**Funding:** This work is supported by the Key Laboratory of Social Computing and Intelligent Governance in Western Sichuan Plateau at Sichuan Minzu College, the Research Team on Differential Equations and Dynamical Systems of Sichuan Minzu College and the Sichuan Provincial Natural Science Foundation (Grant no. 2026NSFSC0138) of China.

**Data Availability Statement:** The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding authors.

**Conflicts of Interest:** The authors declare no conflicts of interest.

## References

- Kocic, V.L.; Ladas, G. *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1993.
- Migda, J. Approximative solutions to difference equations of neutral type. *Appl. Math. Comput.* **2015**, *268*, 763–774. [[CrossRef](#)]
- Jamieson, W.T.; Merino, O. Asymptotic behavior results for solutions to some nonlinear difference equations. *J. Math. Anal. Appl.* **2015**, *430*, 614–632. [[CrossRef](#)]
- Liu, X.; Zhang, Y.; Shi, H.; Deng, X. Existence of periodic solutions for a 2nth-order difference equation involving p-Laplacian. *Bull. Malays. Math. Sci. Soc.* **2015**, *38*, 1107–1125.
- Sun, T.; Zhou, Y.; Su, G.; Qin, B. Eventual periodicity of a max-type difference equation system. *J. Comput. Anal. Appl.* **2018**, *24*, 976–983.
- Sun, T.; Su, G. Dynamics of a difference equation with maximum. *J. Comput. Anal. Appl.* **2017**, *23*, 401–407.
- Su, G.; Sun, T.; Qin, B. On the solutions of a max-type system of difference equations with period-two parameters. *Adv. Differ. Equ.* **2018**, *2018*, 358.
- Dekkar, I.; Touafek, N. Existence and global attractivity of periodic solutions in a max type system of difference equations. *Turk. J. Math.* **2017**, *41*, 412–425. [[CrossRef](#)]
- Wang, C.Y.; Zhou, Y.Q.; Pan, S.; Li, R. On a system of three max-type nonlinear difference equations. *J. Comput. Anal. Appl.* **2018**, *25*, 1463–1479.
- Su, G.W.; Sun, T.X.; Qin, B. Eventually periodic solutions of a max-type system of difference equations of higher order. *Discret. Dyn. Nat. Soc.* **2018**, *2018*, 8467682. [[CrossRef](#)]
- Voulov, H.D. On the periodic character of some difference equation. *J. Differ. Equ. Appl.* **2002**, *8*, 799–810. [[CrossRef](#)]
- Elsayed, E.-M.; Iričanin, B.D. On a max-type and a min-type difference equation. *Appl. Math. Comput.* **2009**, *215*, 608–614.
- Fotiades, N.; Papaschinopoulos, G. On a system of difference equations with maximum. *Appl. Math. Comput.* **2013**, *221*, 684–690. [[CrossRef](#)]
- Sun, T.X.; Liu, J.; He, Q.L.; Liu, X.-H.; Tao, C. Eventually periodic solutions of a max-type difference equation. *Sci. World J.* **2014**, *2014*, 219437. [[CrossRef](#)]
- Ghezal, A.; Zemmouri, I. On the balancing-bilinear system of difference equations: Solutions via some number sequences and global stability. *Miskolc Math. Notes* **2025**, *26*, 785–795. [[CrossRef](#)]
- Ghezal, A.; Zemmouri, I. Global stability of multi-dimensional system of rational difference equations of higher-order with Pell-coefficients. *Bol. Soc. Parana. Matemática* **2025**, *43*, 1–9.
- Ghezal, A.; Al Salman, H.J.; Al Ghafli, A.A. Three-dimensional second-order rational difference equations: Explicit formulas and simulations. *Mathematics* **2026**, *14*, 876. [[CrossRef](#)]
- Papaschinopoulos, G.; Papadopoulos, B.K. On the fuzzy difference equation  $x_{n+1} = A + x_n/x_{n-m}$ . *Fuzzy Sets Syst.* **2002**, *129*, 73–81. [[CrossRef](#)]
- Zhang, Q.H.; Yang, L.H.; Liao, D.X. On first order fuzzy Riccati difference equation. *Inf. Sci.* **2014**, *270*, 226–236. [[CrossRef](#)]
- Zhang, Q.H.; Liu, J.Z.; Luo, Z.G. Dynamical behavior of a third-order rational fuzzy difference equation. *Adv. Differ. Equ.* **2015**, *2015*, 108. [[CrossRef](#)]
- Zhang, Q.H.; Yang, L.H.; Liao, D.X. Behavior of solutions to a fuzzy nonlinear difference equation. *Iran. J. Fuzzy Syst.* **2012**, *9*, 1–12. [[CrossRef](#)]
- Papaschinopoulos, G.; Stefanidou, G. Boundedness and asymptotic behavior of the solutions of a fuzzy difference equation. *Fuzzy Sets Syst.* **2003**, *140*, 523–539. [[CrossRef](#)]
- Deeba, E.Y.; De Korvin, A.; Koh, E.L. On a fuzzy logistic difference equation. *Differ. Equ. Dyn. Syst.* **1996**, *4*, 149–156.

24. Hatir, E.; Mansour, T.; Yalcinkaya, I. On a fuzzy difference equation. *Util. Math.* **2014**, *93*, 135–151.
25. Papaschinopoulos, G.; Papadopoulos, B.K. On the fuzzy difference equation  $x_{n+1} = A + B/x_n$ . *Soft Comput.* **2002**, *6*, 456–461. [[CrossRef](#)]
26. Khastan, A. New solutions for first order linear fuzzy difference equations. *J. Comput. Appl. Math.* **2017**, *312*, 156–166. [[CrossRef](#)]
27. Pakdaman, M.; Effati, S. On fuzzy linear projection equation and applications. *Fuzzy Optim. Decis. Mak.* **2016**, *15*, 219–236.
28. He, Q.L.; Tao, C.Y.; Sun, T.X.; Liu, X.; Su, D. Periodicity of the positive solutions of a fuzzy max-difference equation. *Abstr. Appl. Anal.* **2014**, *2014*, 760247. [[CrossRef](#)]
29. Stefanidou, G.; Papaschinopoulos, G. Behavior of the positive solutions of fuzzy max-difference equations. *Adv. Differ. Equ.* **2005**, *2*, 153–172.
30. Stefanidou, G.; Papaschinopoulos, G. The periodic nature of the positive solutions of a nonlinear fuzzy max-difference equation. *Inf. Sci.* **2006**, *176*, 3694–3710. [[CrossRef](#)]
31. Wang, C.; Wang, Q.; Zhang, Q.; Meng, J. Periodicity of a four-order maximum fuzzy difference equation. *IAENG Int. J. Appl. Math.* **2023**, *53*, 1617–1627.

**Disclaimer/Publisher’s Note:** The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.