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A Study on a Simplified Thermo-Mechanical Coupling Model Based on the Improved Local Linearization Method

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Abstract

The Absolute Nodal Coordinate Formulation (ANCF) is extensively utilized in the field of flexible multibody dynamics because it offers a constant mass matrix and inherently eliminates Coriolis forces. However, ANCF requires the computation of complex nonlinear elastic internal forces and thermal deformation forces at each time step, which imposes a significant computational burden. To alleviate this burden, researchers have developed local linearization (LL) methods. The local linearization method constructs constant elastic and thermal stiffness matrices within a small range by means of Taylor expansion, effectively reducing the number of stiffness matrix updates. But the method suffers from error accumulation and relies on displacement-based update criteria that are inefficient for systems with large rigid-body motion. This paper proposes an improved local linearization (I-LL) method to address these issues. Two key enhancements are introduced: (1) the update criterion for the elastic and thermal stiffness matrices is modified from displacement-based to total strain-based, enabling more accurate and size-independent updates; (2) accurate elastic or thermal deformation force calculations are inserted within the local linearization iteration cycle to mitigate error accumulation. These two improvements reduce the number of calculations of the nonlinear internal forces and, at the same time, lessen the error accumulation in the simplified model. The accuracy and effectiveness of the I-LL algorithm are demonstrated through three numerical examples.

Keywords: ANCF; local linearization; thermo-mechanical couple; model simplification; error accumulation

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1. Introduction

In the past few years, the ANCF has become a widely adopted approach in flexible multibody dynamics simulations. By incorporating both global displacement vectors and slope vectors as generalized coordinates, ANCF naturally yields a constant mass matrix and eliminates Coriolis and centrifugal force terms from the equations of motion [1]. To capture deformations, the formulation relies on the nonlinear Green–Lagrange strain measure, which leads to strongly nonlinear elastic force expressions [1,2]. Unlike conventional finite element analysis (FEA) that relies on nodal displacements and rotational degrees of freedom (DOFs), ANCF avoids the need for coordinate transformations and is particularly well-suited for large-displacement and large-rotation problems [3–6]. These distinctive features have motivated extensive research, with ANCF being successfully applied to the

modeling of flexible robotic manipulators, deployable space structures, and other complex systems [7–9].

Compared with traditional FEA, ANCF incorporates more generalized coordinates. This results in ANCF equations having much higher dimensionality than traditional finite element equations. In addition, as ANCF elements have progressed from rudimentary two-node beams to advanced elements including triangular, rectangular, tetrahedral, and hexahedral elements, the dimension of the dynamic equations and the required computational effort have grown substantially [10–16]. Thus, improving the computational efficiency of ANCF dynamic equations remains a central challenge in the field [17]. The local linearization method has emerged as a prominent focus in recent research on ANCF model order reduction.

A local linearization method was first introduced by Tang et al. in 2019 [18]. In their approach, linearization is performed at a series of quasi-static equilibrium configurations, with the intervals determined by the required accuracy of the dynamic computation. Through the application of the Craig-Bampton method, a reduced-order model was constructed. This was achieved by projecting the incremental displacements of the locally linearized system onto the local modal basis associated with the quasi-static equilibrium configurations. The efficiency and accuracy of the proposed method were demonstrated through three numerical examples. Tian, Lan, and colleagues performed a local linearization of the dynamic equations to derive a constant tangent stiffness matrix [19]. Subsequently, they applied a free-interface component mode synthesis technique to construct a reduced-order model. In contrast to the Craig-Bampton method, this approach eliminates the need to retain interface coordinates, enabling a more significant reduction in the system's DOFs. The validity and effectiveness of the proposed method were demonstrated through numerical examples, including deployable solar array systems. In 2022, Tian et al. linearized the elastic force and thermal deformation force and reduced system order via the free-interface component mode synthesis method. This method improved the computational efficiency of the thermo-mechanical coupling flexible multibody system [20]. In 2025, Tian et al. employed the Absolute Nodal Coordinate Formulation (ANCF) to model rigid–flexible–thermal coupled multibody systems [21]. Through the application of Taylor expansion, they locally linearized both the dynamic equations and the heat transfer equations, resulting in constant tangent stiffness and damping matrices. This approach was combined with dimensionality reduction to obtain simplified governing equations for numerical solution. Meanwhile, Yu et al. investigated nonproportionally damped viscoelastic flexible multibody systems using a complex modal synthesis method [22]. By performing local linearization around quasi-static equilibrium configurations, they also derived constant tangent stiffness and damping matrices. To maintain energy conservation and guarantee solution accuracy while utilizing a minimal set of retained modes, an energy-based criterion was developed for mode selection. The validity of this approach was confirmed through three numerical case studies. In 2026, Zhang et al. proposed an improved local linearization method at the 2026 6th International Conference on Mechanical Design and Simulation (MDS 2026). This method can effectively reduce error accumulation. However, this method was only adopted in purely dynamic problems and cannot be adapted to thermo-mechanical coupling models [23].

However, in these studies employing the local linearization method, the simulation duration for flexible body motion is typically short. For simpler models, the simulation duration generally does not exceed 20 s, while for more complex models, it lasts only a few seconds [18–21]. When the simulation time is extended, error accumulation from the local linearization method can severely affect the computational accuracy of the simplified mod-

els. Therefore, controlling error accumulation over longer simulation times and ensuring the computational accuracy of the simplified models hold significant research value.

2. Implementation of ANCF Planar Triangular Element

For meshing and modeling purposes, this study utilizes the Bézier-triangle-based ANCF planar element originally developed by Wang [24,25] and the method of thermo-mechanical coupled analysis developed by Qi [26]. The present section provides a concise description of the implementation procedure for this thermo-mechanical coupled planar triangular element.

2.1. ANCF Triangular Element Coordinate Transformation

The location of any point P within a planar triangular element is characterized by the area coordinates $\xi = (\xi_1, \xi_2, \xi_3)$. Defined as $\xi_i = A_i/A$ ($i = 1, 2, 3$), these coordinates rely on the total triangle area A and the areas A_i of the three sub-triangles created when point P is joined to each vertex.

From the above definition, it follows that the area coordinates are inherently subject to the following constraint:

$$\xi_1 + \xi_2 + \xi_3 = 1 \quad (1)$$

For a planar triangular element, the position vector $\mathbf{r}$ and temperature T of any point P can be written in the form:

$$\left. \begin{aligned} \mathbf{r} &= \mathbf{S}_2 \mathbf{e} \\ T &= \mathbf{S}_T \mathbf{e}_T \end{aligned} \right\} \quad (2)$$

Here, $\mathbf{S}_2 = [S_1 \mathbf{I}_{2 \times 2}, S_2 \mathbf{I}_{2 \times 2}, S_3 \mathbf{I}_{2 \times 2} \cdots S_9 \mathbf{I}_{2 \times 2}]$ and $\mathbf{S} = [S_1, S_2, S_3 \cdots S_9]$ represent the shape function matrix. For a unified representation, the same set of shape functions is adopted for both position coordinates and temperature. $\mathbf{e}$ is the element's nodal coordinate vector and $\mathbf{e}_T$ is the temperature and temperature gradient at the element nodes, expressed as:

$$\begin{aligned} \mathbf{e} &= [\mathbf{r}_1^1, \mathbf{r}_{\xi_2}^1, \mathbf{r}_{\xi_3}^1, \mathbf{r}_2^2, \mathbf{r}_{\xi_3}^2, \mathbf{r}_{\xi_1}^2, \mathbf{r}_3^3, \mathbf{r}_{\xi_1}^3, \mathbf{r}_{\xi_2}^3]^T \\ \mathbf{e}_T &= [T^1, T_{\xi_2}^1, T_{\xi_3}^1, T^2, T_{\xi_3}^2, T_{\xi_1}^2, T^3, T_{\xi_1}^3, T_{\xi_2}^3]^T \end{aligned} \quad (3)$$

where $\mathbf{r}^i, i = 1, 2, 3$ are the nodal position vectors, $\mathbf{r}_{\xi_j}^i, i, j, k = 1, 2, 3$ represent the partial derivatives of the position vector with respect to the area coordinates, $T^i, i = 1, 2, 3$ are the nodal temperatures, and $T_{\xi_j}^i, i, j, k = 1, 2, 3$ represent the partial derivatives of the temperatures with respect to the area coordinates. The vector $\mathbf{e}$ contains only two gradient entries per node because the three area coordinates are not independent; since they are subject to a constraint (see Equation (1)), the third gradient coordinate can be derived from the other two.

Within the Bézier triangle, the coordinates of point P are determined through interpolation of its control points:

$$\mathbf{r} = \sum_{k=1}^{10} g_k \mathbf{q}_k \quad (4)$$

In this context, $\mathbf{q}_k, k = 1, 2, \cdots 10$ represent the control points of the Bézier triangle, as illustrated in Figure 1b, while $g_k, k = 1, 2, \cdots 10$ correspond to the Bernstein basis functions. The transformation linking the coordinates of these control points to the element's nodal coordinates is established as follows:

$$\left. \begin{aligned} \mathbf{r}^1 &= \mathbf{q}_1, \mathbf{r}_{\xi_2}^1 = 3(\mathbf{q}_4 - \mathbf{q}_1), \mathbf{r}_{\xi_3}^1 = 3(\mathbf{q}_9 - \mathbf{q}_1) \\ \mathbf{r}^2 &= \mathbf{q}_2, \mathbf{r}_{\xi_3}^2 = 3(\mathbf{q}_6 - \mathbf{q}_2), \mathbf{r}_{\xi_1}^2 = 3(\mathbf{q}_5 - \mathbf{q}_2) \\ \mathbf{r}^3 &= \mathbf{q}_3, \mathbf{r}_{\xi_1}^3 = 3(\mathbf{q}_8 - \mathbf{q}_3), \mathbf{r}_{\xi_2}^3 = 3(\mathbf{q}_7 - \mathbf{q}_3) \end{aligned} \right\} \quad (5)$$

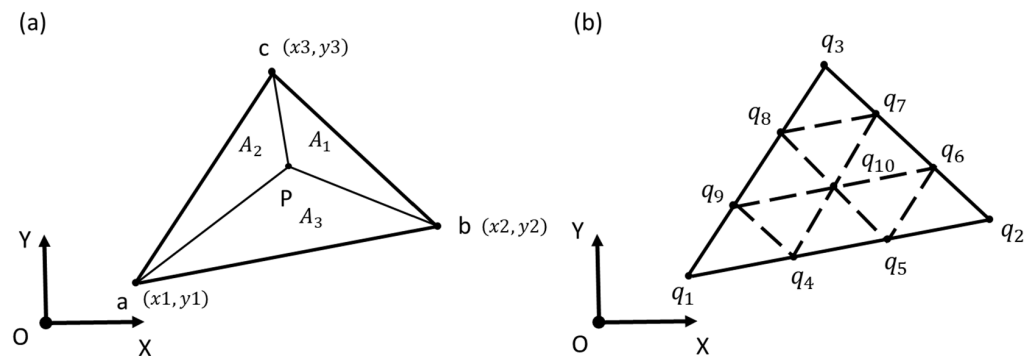


Figure 1. (a) Schematic of area coordinates for a triangular element. (b) Control point configuration for a Bézier triangle.

Combining Equation (2) with Equation (4), the shape function $\mathbf{S}$ expressed in terms of area coordinates can be obtained. Although the shape function $\mathbf{S}$ is formally concise, the discontinuity of its area gradient vectors at the nodes affects the assembly between elements. Wang [22–24] introduced a coordinate transformation technique to convert the element nodal coordinates from the form $\mathbf{e}$ (which incorporates area gradient coordinates) to the form $\mathbf{p}$ (which incorporates Cartesian gradient coordinates), thereby resolving the issue of discontinuity in the nodal gradient coordinates. The resulting nodal coordinates $\mathbf{p}$ are given by:

$$\left. \begin{aligned} \mathbf{e} &= \mathbf{T} \mathbf{p} \\ \mathbf{p} &= [\mathbf{r}^1 \ \mathbf{r}_x^1 \ \mathbf{r}_y^1 \ \mathbf{r}^2 \ \mathbf{r}_x^2 \ \mathbf{r}_y^2 \ \mathbf{r}^3 \ \mathbf{r}_x^3 \ \mathbf{r}_y^3]^T \end{aligned} \right\} \quad (6)$$

The following expression relates the nodal coordinates $\mathbf{e}$ to the nodal coordinates $\mathbf{p}$:

$$\left. \begin{aligned} \mathbf{T} &= \begin{bmatrix} \mathbf{T}^1 & 0 & 0 \\ 0 & \mathbf{T}^2 & 0 \\ 0 & 0 & \mathbf{T}^3 \end{bmatrix} \\ \mathbf{T}^1 &= \begin{bmatrix} \mathbf{I}_2 & 0 & 0 \\ 0 & 3\mathbf{r}_{0,x}^{1T}(\mathbf{q}_0^4 - \mathbf{q}_0^1)\mathbf{I}_2 & 3\mathbf{r}_{0,y}^{1T}(\mathbf{q}_0^4 - \mathbf{q}_0^1)\mathbf{I}_2 \\ 0 & 3\mathbf{r}_{0,x}^{1T}(\mathbf{q}_0^9 - \mathbf{q}_0^1)\mathbf{I}_2 & 3\mathbf{r}_{0,y}^{1T}(\mathbf{q}_0^9 - \mathbf{q}_0^1)\mathbf{I}_2 \end{bmatrix} \\ \mathbf{T}^2 &= \begin{bmatrix} \mathbf{I}_2 & 0 & 0 \\ 0 & 3\mathbf{r}_{0,x}^{2T}(\mathbf{q}_0^4 - \mathbf{q}_0^1)\mathbf{I}_2 & 3\mathbf{r}_{0,y}^{2T}(\mathbf{q}_0^4 - \mathbf{q}_0^1)\mathbf{I}_2 \\ 0 & 3\mathbf{r}_{0,x}^{2T}(\mathbf{q}_0^9 - \mathbf{q}_0^1)\mathbf{I}_2 & 3\mathbf{r}_{0,y}^{2T}(\mathbf{q}_0^9 - \mathbf{q}_0^1)\mathbf{I}_2 \end{bmatrix} \\ \mathbf{T}^3 &= \begin{bmatrix} \mathbf{I}_2 & 0 & 0 \\ 0 & 3\mathbf{r}_{0,x}^{3T}(\mathbf{q}_0^8 - \mathbf{q}_0^3)\mathbf{I}_2 & 3\mathbf{r}_{0,y}^{3T}(\mathbf{q}_0^8 - \mathbf{q}_0^3)\mathbf{I}_2 \\ 0 & 3\mathbf{r}_{0,x}^{3T}(\mathbf{q}_0^7 - \mathbf{q}_0^3)\mathbf{I}_2 & 3\mathbf{r}_{0,y}^{3T}(\mathbf{q}_0^7 - \mathbf{q}_0^3)\mathbf{I}_2 \end{bmatrix} \end{aligned} \right\} \quad (7)$$

In this expression, $\mathbf{r}_{0,x}^i$ and $\mathbf{r}_{0,y}^i$ (for $i = 1, 2, 3$) are the Cartesian gradient coordinates at each element node in the initial configuration; $\mathbf{q}_0^i$ (for $i = 1, 2, \dots, 9$) refers to the initial position vector of the i -th control point; and $\mathbf{I}_2$ denotes the second-order identity matrix.

Similarly, the temperature T is also interpolated using the same set of shape functions. Therefore, it is also necessary to transform the temperature nodal coordinates $\mathbf{e}_T$ in the area coordinates into the Cartesian coordinates $\mathbf{p}_T$ [10].

$$\left. \begin{aligned} \mathbf{e}_T &= \mathbf{T}_{tem} \mathbf{p}_T \\ \mathbf{p}_T &= [T^1 \ T_x^1 \ T_y^1 \ T^2 \ T_x^2 \ T_y^2 \ T^3 \ T_x^3 \ T_y^3]^T \end{aligned} \right\} \quad (8)$$

The transformation matrix $\mathbf{T}_{tem}$ can be expressed as:

$$\left. \begin{aligned} \mathbf{T}_{tem} &= \begin{bmatrix} \mathbf{T}_{tem}^1 & 0 & 0 \\ 0 & \mathbf{T}_{tem}^2 & 0 \\ 0 & 0 & \mathbf{T}_{tem}^3 \end{bmatrix} \\ \mathbf{T}_{tem}^1 &= \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & 3\mathbf{r}_{0,x}^{1T}(\mathbf{q}_0^4 - \mathbf{q}_0^1)\mathbf{I} & 3\mathbf{r}_{0,y}^{1T}(\mathbf{q}_0^4 - \mathbf{q}_0^1)\mathbf{I} \\ 0 & 3\mathbf{r}_{0,x}^{1T}(\mathbf{q}_0^9 - \mathbf{q}_0^1)\mathbf{I} & 3\mathbf{r}_{0,y}^{1T}(\mathbf{q}_0^9 - \mathbf{q}_0^1)\mathbf{I} \end{bmatrix} \\ \mathbf{T}_{tem}^2 &= \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & 3\mathbf{r}_{0,x}^{2T}(\mathbf{q}_0^4 - \mathbf{q}_0^1)\mathbf{I} & 3\mathbf{r}_{0,y}^{2T}(\mathbf{q}_0^4 - \mathbf{q}_0^1)\mathbf{I} \\ 0 & 3\mathbf{r}_{0,x}^{2T}(\mathbf{q}_0^9 - \mathbf{q}_0^1)\mathbf{I} & 3\mathbf{r}_{0,y}^{2T}(\mathbf{q}_0^9 - \mathbf{q}_0^1)\mathbf{I} \end{bmatrix} \\ \mathbf{T}_{tem}^3 &= \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & 3\mathbf{r}_{0,x}^{3T}(\mathbf{q}_0^8 - \mathbf{q}_0^3)\mathbf{I} & 3\mathbf{r}_{0,y}^{3T}(\mathbf{q}_0^8 - \mathbf{q}_0^3)\mathbf{I} \\ 0 & 3\mathbf{r}_{0,x}^{3T}(\mathbf{q}_0^7 - \mathbf{q}_0^3)\mathbf{I} & 3\mathbf{r}_{0,y}^{3T}(\mathbf{q}_0^7 - \mathbf{q}_0^3)\mathbf{I} \end{bmatrix} \end{aligned} \right\} \quad (9)$$

where $\mathbf{I}$ denotes the first-order identity matrix.

2.2. ANCF Heat Transfer Equations and Dynamic Equations

According to the ANCF thermo-mechanical coupling method proposed by Qi et al. [26], the finite element heat transfer equation is expressed in the power balance form:

$$\mathbf{C}\dot{\mathbf{e}}_T + (\mathbf{K}_C + \mathbf{K}_{conv})\mathbf{e}_T = \mathbf{F}_{conv} \quad (10)$$

where $\mathbf{C}$ denotes the heat capacity matrix, $\mathbf{K}_C$ denotes the thermal conductivity matrix, $\mathbf{K}_{conv}$ denotes the convection contribution matrix, and $\mathbf{F}_{conv}$ denotes the thermal load vector due to convection. Their expressions are as follows:

$$\left. \begin{aligned} \mathbf{C} &= \int_V \rho c \mathbf{S}^T \mathbf{S} dV \\ \mathbf{K}_C &= \int_V \lambda \mathbf{B}^T \mathbf{B} dV \\ \mathbf{K}_{conv} &= h \int_{\Gamma} \mathbf{S}^T \mathbf{S} d\Gamma \\ \mathbf{F}_{conv} &= h T_{\infty} \int_{\Gamma} \mathbf{S}^T d\Gamma \end{aligned} \right\} \quad (11)$$

where matrix $\mathbf{B}$ is the Jacobian matrix of the temperature shape function, $\mathbf{S}$ is the temperature shape function, and T_{∞} is the ambient temperature. Their expressions are as follows:

$$\mathbf{B} = \begin{bmatrix} \frac{\partial S_1}{\partial x} & \frac{\partial S_2}{\partial x} & \dots & \frac{\partial S_9}{\partial x} \\ \frac{\partial S_1}{\partial y} & \frac{\partial S_2}{\partial y} & \dots & \frac{\partial S_9}{\partial y} \end{bmatrix} \quad (12)$$

By applying the principle of virtual work, the governing equations of motion for the ANCF triangular planar element can be established as follows [22–24,26]:

$$\mathbf{M}\ddot{\mathbf{p}} = \mathbf{Q}_{ex} - \mathbf{Q}_{in} - \mathbf{Q}_H \quad (13)$$

where $\mathbf{M}$ is the mass matrix:

$$\mathbf{M} = \mathbf{T}^T \left(h \int_{A_0} \rho \mathbf{S}^T \mathbf{S} dA \right) \mathbf{T} \quad (14)$$

In this expression, h stands for the thickness of the element, and A_0 represents the area of the element in its initial configuration. The term $\mathbf{Q}_{ex}$ corresponds to the external forces applied to the element. $\mathbf{Q}_{in}$ refers to the internal elastic forces generated by deformation.

$\mathbf{Q}_H$ is the thermal deformation force induced by temperature change. The formulation for $\mathbf{Q}_{in}$ and $\mathbf{Q}_H$ is given as follows:

$$\left. \begin{aligned} \mathbf{Q}_{in} &= h\mathbf{T}^T \int_{A_0} \left(\frac{\partial \boldsymbol{\varepsilon}}{\partial \mathbf{e}} \right)^T \mathbf{D} \boldsymbol{\varepsilon} dA_0 \\ \mathbf{Q}_H &= h\mathbf{T}^T \int_{A_0} \left(\frac{\partial \boldsymbol{\varepsilon}}{\partial \mathbf{e}} \right)^T \mathbf{D} \boldsymbol{\varepsilon}_H dA_0 \\ \mathbf{D} &= \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \end{aligned} \right\} \quad (15)$$

where E represents Young's modulus and ν denotes Poisson's ratio. The Green–Lagrange strain tensor $\boldsymbol{\varepsilon}$, thermal strain tensor $\boldsymbol{\varepsilon}_H$ and the partial derivative of the Green–Lagrange strain components with respect to the node coordinates $\mathbf{e}$ can be expressed as:

$$\left. \begin{aligned} \boldsymbol{\varepsilon} &= [\varepsilon_{xx} \ \varepsilon_{yy} \ 2\varepsilon_{xy}]^T \\ \boldsymbol{\varepsilon}_H &= [\varepsilon_{xx}^H, \varepsilon_{yy}^H, \varepsilon_{xy}^H]^T = [-\alpha, -\alpha, 0]^T \cdot \Delta T \\ \varepsilon_{ij} &= \frac{1}{2} (\mathbf{r}_i^T \mathbf{r}_j - \mathbf{r}_{i0}^T \mathbf{r}_{j0}) \\ \frac{\partial \varepsilon_{ij}}{\partial \mathbf{e}} &= (\mathbf{S}_i^T \mathbf{r}_j + \mathbf{S}_j^T \mathbf{r}_i)/2 \\ \Delta T &= T - T_0 = \mathbf{S} \mathbf{e}_T - \mathbf{S} \mathbf{e}_{T0} \\ i, j &= x, y \end{aligned} \right\} \quad (16)$$

Here, the subscripts i and j indicate partial differentiation with respect to the spatial coordinates, specifically $i, j = x, y$. The terms $\mathbf{r}_{i0}$ and $\mathbf{r}_{j0}$ denote the partial derivatives of the position vector in the initial configuration with respect to x and y , respectively. α denotes the coefficient of thermal expansion.

3. Improved Local Linearization Method

Solving the dynamic equations of ANCF necessitates the evaluation of elastic internal forces at every discrete time step, creating a significant computational bottleneck. Reducing the computational burden associated with these forces is therefore a key factor in accelerating ANCF-based simulations. One common strategy employed to streamline this process is local linearization [18–21], which reduces the complexity of evaluating elastic forces. According to the formulation in Equation (13), the governing dynamic equations of ANCF can be expressed as:

$$\mathbf{M} \ddot{\mathbf{p}} = \mathbf{Q}_{ex} - \mathbf{Q}_{in} - \mathbf{Q}_H \quad (17)$$

Performing a Taylor expansion of the elastic internal force $\mathbf{Q}_{in}$ and thermal deformation force $\mathbf{Q}_H$ about the configuration $\mathbf{p}(t)$ and temperature $\mathbf{p}_T(t)$ yields:

$$\begin{aligned} \mathbf{Q}_{in}(\mathbf{p}(t+t_n)) &= \mathbf{Q}_{in}(\mathbf{p}(t)) + \frac{\partial \mathbf{Q}_{in}}{\partial \mathbf{p}}(\mathbf{p}(t)) \cdot \Delta \mathbf{p}_n + \cdots + \frac{\partial \mathbf{Q}_{in}^n}{n! \partial^n \mathbf{p}}(\mathbf{p}(t)) \cdot [\Delta \mathbf{p}_n]^n + \mathbf{R}_n(\mathbf{p}(t)) \\ \mathbf{Q}_H(\mathbf{p}(t+t_n), \mathbf{p}_T(t+t_n)) &= \mathbf{Q}_H(\mathbf{p}(t), \mathbf{p}_T(t)) + \frac{\partial \mathbf{Q}_H}{\partial \mathbf{p}}(\mathbf{p}(t), \mathbf{p}_T(t)) \cdot \Delta \mathbf{p}_n + \frac{\partial \mathbf{Q}_H}{\partial \mathbf{p}_T}(\mathbf{p}(t), \mathbf{p}_T(t)) \cdot \Delta \mathbf{p}_{T_n} \\ &\quad \cdots + \mathbf{R}_n(\mathbf{p}(t), \mathbf{p}_T(t)) \end{aligned} \quad (18)$$

where t_n denotes a brief time interval, with $\mathbf{p}(t+t_n) = \mathbf{p}(t) + \Delta \mathbf{p}_n$ and $\mathbf{p}_T(t+t_n) = \mathbf{p}_T(t) + \Delta \mathbf{p}_{T_n}$. $\Delta \mathbf{p}_n$ is a small displacement increment and $\Delta \mathbf{p}_{T_n}$ is a small temperature change. This interval t_n may span several time steps. Within the range defined by $\Delta \mathbf{p}_n$ and $\Delta \mathbf{p}_{T_n}$, higher-order terms in the Taylor series are omitted, keeping only the linear component.

Consequently, the elastic internal force and thermal deformation force is approximated as:

$$\left. \begin{aligned} \mathbf{Q}_{in}(\mathbf{p}(t+t_n)) &\approx \mathbf{Q}_{in}(\mathbf{p}(t)) + \frac{\partial \mathbf{Q}_{in}}{\partial \mathbf{p}}(\mathbf{p}(t)) \cdot \Delta \mathbf{p}_n \\ \mathbf{Q}_H(\mathbf{p}(t+t_n), \mathbf{p}_T(t+t_n)) &= \mathbf{Q}_H(\mathbf{p}(t), \mathbf{p}_T(t)) + \frac{\partial \mathbf{Q}_H}{\partial \mathbf{p}}(\mathbf{p}(t), \mathbf{p}_T(t)) \cdot \Delta \mathbf{p}_n + \frac{\partial \mathbf{Q}_H}{\partial \mathbf{p}_T}(\mathbf{p}(t), \mathbf{p}_T(t)) \cdot \Delta \mathbf{p}_T \\ \mathbf{K}_t(\mathbf{p}(t)) &= \frac{\partial \mathbf{Q}_{in}}{\partial \mathbf{p}}(\mathbf{p}(t)) \\ \mathbf{K}_{tem1}(\mathbf{p}(t), \mathbf{p}_T(t)) &= \frac{\partial \mathbf{Q}_H}{\partial \mathbf{p}}(\mathbf{p}(t), \mathbf{p}_T(t)) \\ \mathbf{K}_{tem2}(\mathbf{p}(t), \mathbf{p}_T(t)) &= \frac{\partial \mathbf{Q}_H}{\partial \mathbf{p}_T}(\mathbf{p}(t), \mathbf{p}_T(t)) \end{aligned} \right\} \quad (19)$$

where $\mathbf{K}_t(\mathbf{p}(t))$ is the elastic stiffness matrix of the system at the configuration $\mathbf{p}(t)$, $\mathbf{K}_{tem1}(\mathbf{p}(t), \mathbf{p}_T(t))$ and $\mathbf{K}_{tem2}(\mathbf{p}(t), \mathbf{p}_T(t))$ are the thermal stiffness matrices obtained from the partial derivatives of the thermal deformation force with respect to $\mathbf{p}$ and $\mathbf{p}_T$, respectively. The expression for $\mathbf{K}_t$, $\mathbf{K}_{tem1}$ and $\mathbf{K}_{tem2}$ is as follows:

$$\left. \begin{aligned} \mathbf{K}_T &= \frac{\partial \mathbf{Q}_{in}}{\partial \mathbf{p}} = \mathbf{T} \cdot \frac{\partial \mathbf{Q}_{in}}{\partial \mathbf{e}} \\ \frac{\partial \mathbf{Q}_{in}}{\partial \mathbf{e}} &= h \int_{A_0} \left(\frac{\partial^2 \varepsilon_{xx}}{\partial \mathbf{e}^2} \sigma_{xx} + \frac{\partial^2 \varepsilon_{yy}}{\partial \mathbf{e}^2} \sigma_{yy} + 2 \frac{\partial^2 \varepsilon_{xy}}{\partial \mathbf{e}^2} \sigma_{xy} + \left(\frac{\partial \varepsilon}{\partial \mathbf{e}} \right)^T \mathbf{D} \frac{\partial \varepsilon}{\partial \mathbf{e}} \right) dA_0 \\ \mathbf{K}_{tem1} &= \frac{\partial \mathbf{Q}_H}{\partial \mathbf{p}} = \mathbf{T}^T \cdot \frac{\partial \mathbf{Q}_H}{\partial \mathbf{e}} \cdot \mathbf{T} \\ \mathbf{K}_{tem2} &= \frac{\partial \mathbf{Q}_H}{\partial \mathbf{p}_T} = \mathbf{T}^T \cdot \frac{\partial \mathbf{Q}_H}{\partial \mathbf{e}_T} \cdot \mathbf{T}_{tem} \\ \frac{\partial \mathbf{Q}_H}{\partial \mathbf{e}} &= h \int_{A_0} \left(\frac{\partial^2 \varepsilon_{xx}}{\partial \mathbf{e}^2} \sigma_{Hxx} + \frac{\partial^2 \varepsilon_{yy}}{\partial \mathbf{e}^2} \sigma_{Hyy} + 2 \frac{\partial^2 \varepsilon_{xy}}{\partial \mathbf{e}^2} \sigma_{Hxy} \right) dA_0 \\ \frac{\partial \mathbf{Q}_H}{\partial \mathbf{e}_T} &= h \int_{A_0} \left(\frac{\partial \varepsilon}{\partial \mathbf{e}} \right)^T \mathbf{D} \begin{bmatrix} -\alpha \\ -\alpha \\ 0 \end{bmatrix} \mathbf{S} dA_0 \\ \frac{\partial^2 \varepsilon_{ij}^m}{\partial \mathbf{e}^2} &= (\mathbf{S}_i^T \mathbf{S}_j + \mathbf{S}_j^T \mathbf{S}_i) / 2, i, j = x, y \end{aligned} \right\} \quad (20)$$

where ε_{ij} , $i, j = x, y$ are the elastic strain components, σ_{Hij} , $i, j = x, y$ are the thermal strain components. Within the range defined by $\Delta \mathbf{p}_n$, the tangent stiffness matrix $\mathbf{K}_t(\mathbf{p}(t))$ remains unchanged, allowing the elastic internal force $\mathbf{Q}_{in}(\mathbf{p}(t+t_n))$ to be evaluated directly from this constant matrix. As a result, it is not necessary to recompute the stiffness matrix at every time step over the interval $\Delta \mathbf{p}_n$, which leads to a significant reduction in computational effort.

3.1. Improved Local Linearization Update Rule

In previous model order reduction using local linearization [18], the update rule adopted was:

$$\left. \begin{aligned} \max(\|\Delta \mathbf{p}_n^j\|_2) &\leq \delta, j = 1 \cdots n_e \\ \delta &= \frac{L}{1000} \end{aligned} \right\} \quad (21)$$

where $\Delta \mathbf{p}_n^j$ represents the displacement of the j -th node within the current local linearization range, and n_e denotes the number of finite element nodes in the system. The value of δ is an empirical parameter. In previous studies, δ was determined based on the dimensions of the simulated flexible body, for example, $\delta = L/1000$, where L is the longer side length of the flexible body. However, this update rule has several shortcomings. First, when dealing with objects of different sizes, the selection of δ relies entirely on empirical judgment. When the flexible body itself is small, the chosen δ may become very small. For irregularly shaped objects, it is also difficult to determine δ using the above empirical formula.

Secondly, δ characterizes the displacement of a node within the local linearization time range. Node displacement may result from both rigid-body motion and elastic deformation.

However, as can be seen from Equations (15) and (16), the elastic internal force and thermal deformation force of the flexible body are only related to its current state—that is, its deformation and temperature state—and not to its displacement. In simulation examples where the flexible body undergoes large-scale rigid-body motion (e.g., an elastic sphere falling from a height and colliding with the ground), using displacement as the local linearization update criterion can lead to unnecessary updates during rigid-body motion.

For example, as shown in Figure 2, a sphere moves under gravity from time t_n to time t_{n+1} , disregarding air resistance. Since no external forces act on the elastic sphere except the uniform gravitational field, and its temperature remains constant while the sphere is not in contact with other bodies, the state of the elastic sphere remains unchanged from t_n to t_{n+1} . If the update rule in Equation (21) is adopted, several stiffness matrix updates would occur during the displacement from t_n to t_{n+1} . However, because the state of the elastic sphere does not change during the displacement, its elastic internal forces and thermal deformation force remain constant, and thus, no updates to the elastic and thermal stiffness matrix are necessary. Therefore, using Equation (21) as the update rule in scenarios involving large-scale displacement leads to unnecessary stiffness updates.

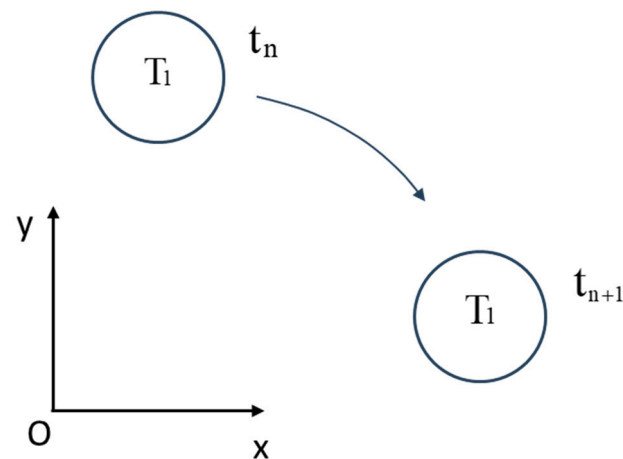


Figure 2. Schematic diagram of the displacement of an elastic sphere under the influence of gravity.

To avoid the aforementioned shortcomings, this paper proposes a new update rule. We take the maximum value of the total strain differences at each node across different times as the criterion for determining whether to update the stiffness matrix. Its expression is as follows:

$$\left. \begin{aligned} \max(\|\Delta \epsilon_n^j\|_2) &\leq \delta, j = 1 \cdots n_e \\ \Delta \epsilon_n^j &= \epsilon_{t+t_n}^j - \epsilon_t^j \\ \epsilon_t^j &= \begin{bmatrix} \epsilon_{xxt}^j \\ \epsilon_{yyt}^j \\ 2\epsilon_{xyt}^j \end{bmatrix} = \begin{bmatrix} \frac{1}{2}[(\mathbf{r}_x^j(t))^T \mathbf{r}_x^j(t) - \mathbf{I}] \\ \frac{1}{2}[(\mathbf{r}_y^j(t))^T \mathbf{r}_y^j(t) - \mathbf{I}] \\ (\mathbf{r}_x^j(t))^T \mathbf{r}_y^j(t) \end{bmatrix} \\ \delta &= 0.001 \sim 0.005 \end{aligned} \right\} \quad (22)$$

where $\Delta \epsilon_n^j = \epsilon_t^j - \epsilon_{t_n}^j$ represents the difference between the total strain $\epsilon_{t+t_n}^j$ at the current time step and the total strain ϵ_t^j at the moment when the tangent stiffness matrix was last updated for the j -th node. We stipulate that the stiffness matrix should be updated when the change in total strain exceeds 0.1% to 0.5% for any node of the flexible body. In ANCF, the Green–Lagrange strain used is numerically approximate to the Cauchy strain under small deformations. That is, we consider that the stiffness matrix needs to be updated

when the flexible body undergoes elastic deformation equivalent to 0.1% to 0.5% of its own dimensions.

Adopting total strain as the update criterion effectively addresses the issues present when using displacement as the rule. Strain is dimensionless, thus eliminating the need to consider the size of the flexible body when setting the update threshold. Furthermore, strain is independent of displacement. Regardless of whether the flexible body undergoes large-scale displacement, strain can accurately capture the magnitude of elastic and thermal deformation, enabling more effective updates of the stiffness matrix.

3.2. Incorporating Full Elastic Force Calculation Within the Local Linearization Cycle

In previous model reduction using local linearization, the Taylor expansions of the elastic internal force and the thermal deformation force retain only the first-order linear terms, which introduces an error of order $O((\Delta p)^2)$. Since the calculation of the elastic and thermal deformation force for the subsequent step is based on the previously obtained force containing this error, error accumulation occurs. During long-duration simulations, this accumulated error causes a mismatch between the elastic and thermal deformation forces of the flexible body and its state, thereby reducing the computational accuracy of the simplified model. To extend the feasible simulation duration of the simplified model, we propose a method to mitigate error accumulation.

During the simulation, we periodically perform a full calculation of the elastic or thermal deformation forces. This is not the force obtained through simplified computation via local linearization, but rather the accurate elastic or thermal deformation force that matches the current flexible body state, derived from Equations (15) and (16). Subsequent local linearization iterations are then based on this accurately obtained force. This approach corrects the mismatch between the forces and the flexible body state caused by prolonged local linearization calculations, ensuring that the next step of elastic or thermal deformation force computation iterates from a correct foundation, thereby mitigating the error accumulation associated with long-term local linearization. In the three subsequent case studies, we calculate the accurate elastic or thermal deformation force using Equation (15) every 1000 time steps, and the time step size of 1×10^{-3} s was adopted for three examples.

3.3. I-LL Model Simplification Procedure

In our tests, we found that applying local linearization to both the elastic internal force and the thermal deformation force can effectively reduce computational time. However, this approach suffers from rapid error accumulation and can only maintain accuracy over a relatively short time. Therefore, in the I-LL method, we apply local linearization to only one of these forces—either the elastic internal force or the thermal deformation force. The I-LL method is able to maintain simulation accuracy over a longer simulation time while effectively reducing computational cost. The flowchart for the improved local linearization-simplified computation process is shown in Figure 3:

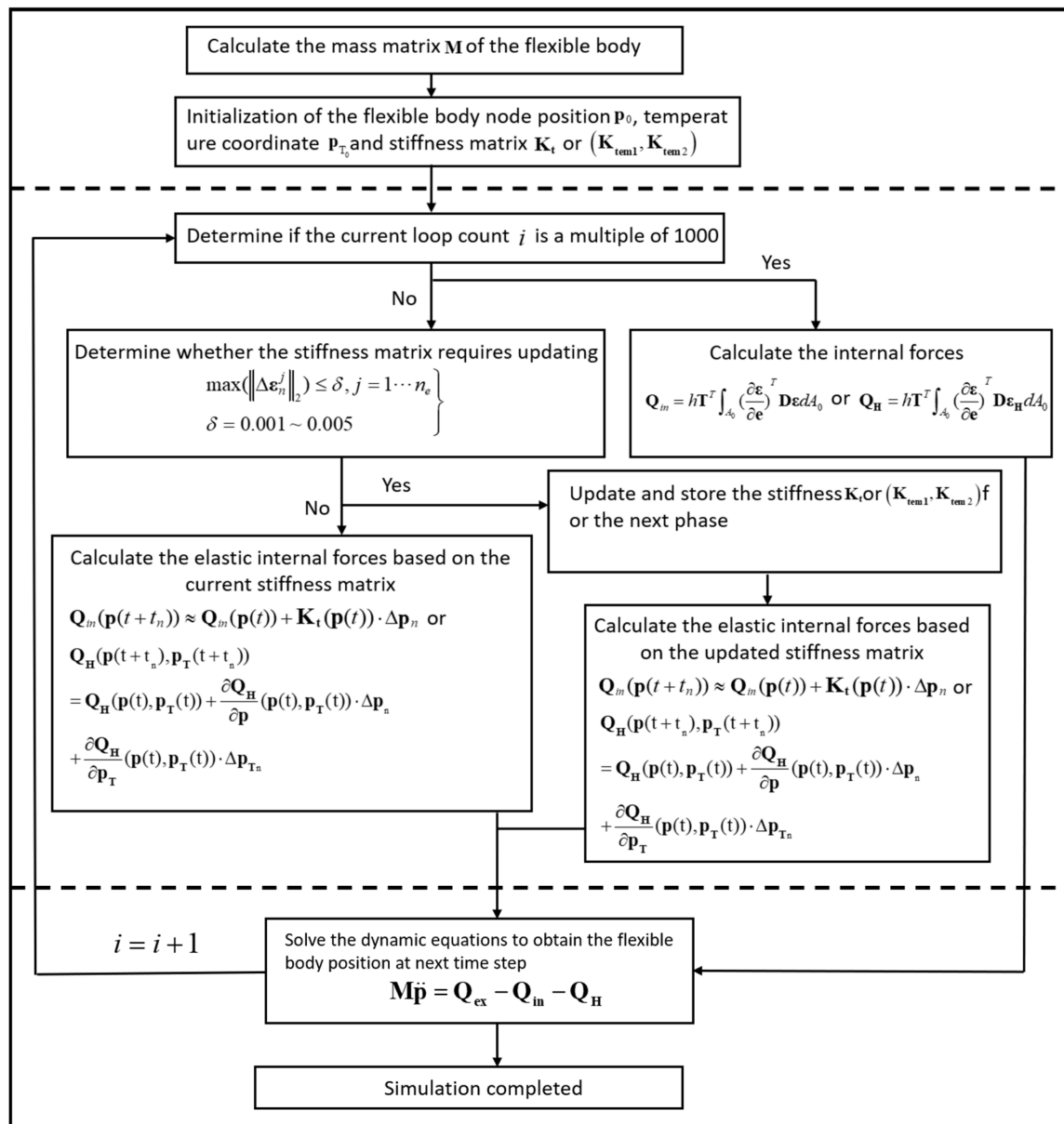


Figure 3. Improved local linearization process diagram.

4. Case Studies

In order to validate the precision of the enhanced local linearization algorithm for extended-duration simulations, three numerical cases are investigated herein. Since collision cases are involved, this study adopts an explicit time integration method to solve the dynamic equations. The simulations were carried out using MATLAB 2024a (MathWorks, Natick, MA, USA) on a personal computer equipped (Honor Device Co., Ltd., Shenzhen, Guangdong, China) with an Intel(R) Core(TM) Ultra 5 225H processor (1.70 GHz) and 32 GB of RAM.

4.1. Flexible Sphere Collision

To verify the simplification capability of the I-LL method for purely dynamic models, a dynamics simulation of a two-dimensional flexible sphere falling under a uniform gravitational field and colliding with the ground was conducted. The initial state of the flexible sphere is shown in Figure 4, and the relevant parameters for the dynamics simulation are listed in Table 1.

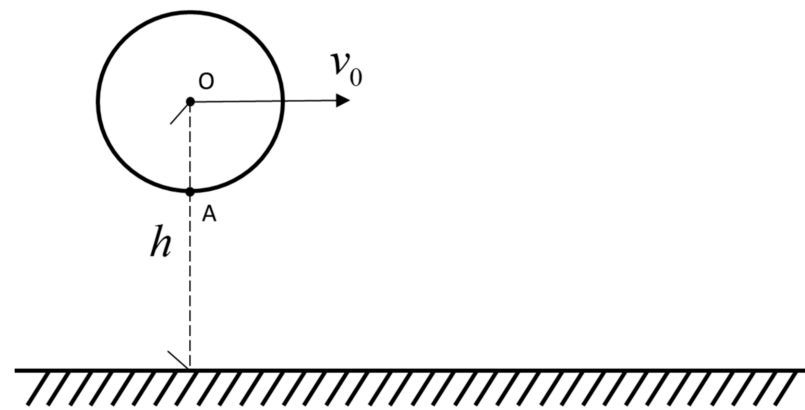


Figure 4. Illustration of the initial configuration of the flexible sphere.

Table 1. Flexible sphere structure parameters.

Structural Parameters		Value
Sphere radius R		5 m
The distance from the center of the sphere to the ground h		15 m
Young's modulus E		9×10^7 Pa
Poisson's ratio ν		0.3
Element thickness t		0.01 m
Gravitational acceleration g		9.81
Horizontal initial velocity v_0		2 m/s
Ground friction coefficient μ		0.6
Normal contact stiffness		1×10^6 N/m
Normal contact damping		1000 N · s/m
Constraint tolerance		1×10^4

The flexible ball is discretized into 235 ANCF planar triangular elements, with a total of 834 degrees of freedom. This study employs a spring–damper model to describe the collision between the ball and the ground and a Coulomb friction model to describe the friction between them. During the collision between the flexible sphere and the ground, significant instantaneous impact loads are generated, imposing high demands on the simulation accuracy of the simplified model. The spring–damper contact model and the Coulomb friction model were employed to simulate the collision and friction, and the dynamic response of the flexible sphere over 20 s was analyzed. Elastic internal force cloud maps of the flexible sphere at its current position were generated every 2 s, as shown in Figure 5a. The trajectory of point A at the sphere center is illustrated in Figure 5b,c.

Simplified computations were performed using both the traditional local linearization method (LL) and the improved local linearization method (I-LL). LL and I-LL use the same mesh and solver. In the traditional local linearization method, the update criterion is $\max(\|\Delta \mathbf{p}_n^j\|_2) \leq 0.001$; for the improved local linearization method proposed in this paper, the update criterion is $\max(\|\Delta \boldsymbol{\varepsilon}_n^j\|_2) \leq 0.003$, and an accurate calculation of the elastic internal forces is performed every 1000 time steps. As shown in Equation (22), we consider that the tangent stiffness matrix can be updated when the total strain difference is in $0.001 \sim 0.005$. The reason why this example uses $\max(\|\Delta \boldsymbol{\varepsilon}_n^j\|_2) \leq 0.003$ is that the total strain is relatively large during the sphere collision, so the update criterion can be appropriately raised to achieve a balance between accuracy and computational speed. The computational results are shown in Figure 6:

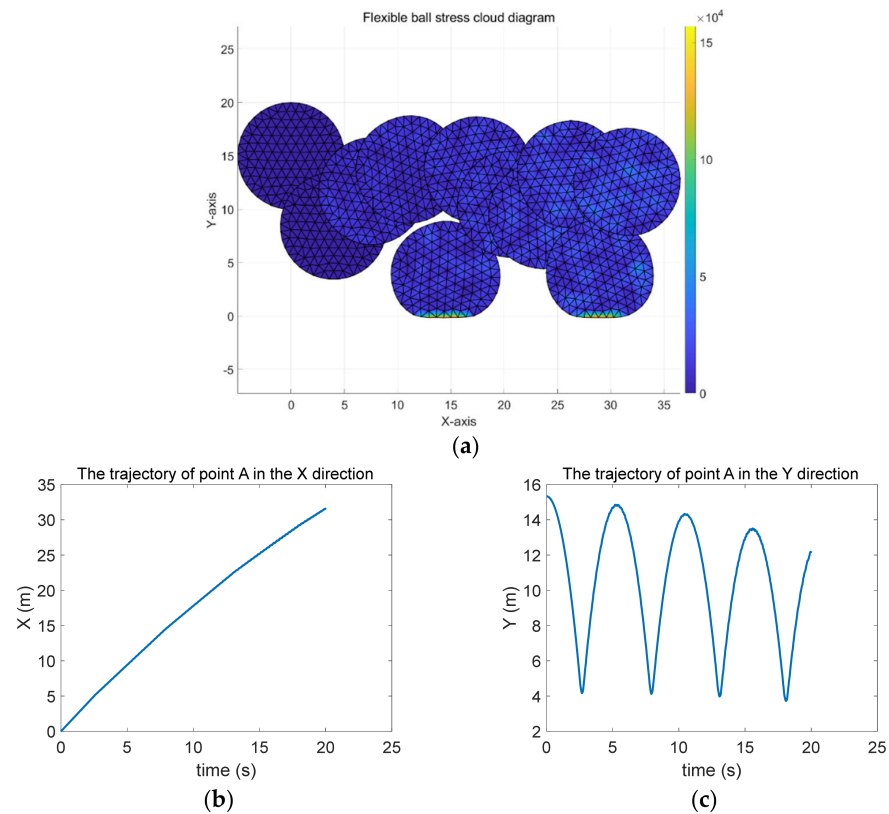


Figure 5. (a) Stress cloud map of the flexible sphere. (b) X-direction trajectory of point A. (c) Y-direction trajectory of point A.

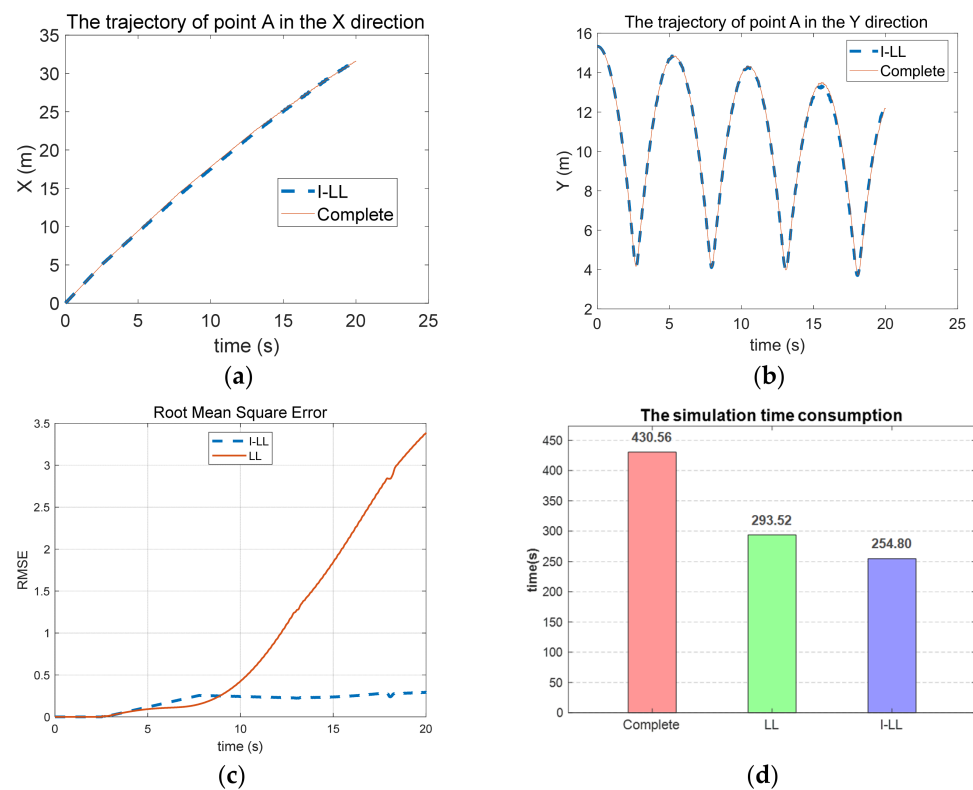


Figure 6. (a) Comparative results for the X-direction trajectory of point A as predicted by the I-LL and complete models. (b) Comparative results for the Y-direction trajectory of point A as predicted by the I-LL and complete models. (c) Comparison of the RMSE between the LL model and the I-LL model. (d) Comparison of the simulation time for the complete model, LL model, and I-LL model.

As illustrated in Figure 6a,b, the trajectories obtained from the I-LL model agree well with those from the complete model. Figure 6c shows that the root mean square error (RMSE) between the I-LL model and the complete model is maintained below 0.3. In contrast, the traditional local linearization model maintains good accuracy only up to 10 s, after which the RMSE begins to increase rapidly. By the 20 s of simulation, the RMSE of the traditional local linearization model has expanded to ten times that of the I-LL model.

As illustrated in Figure 6d and Table 2, the simulation time of the I-LL model is reduced by 40.82% compared to the complete model and by 13.19% compared to the traditional local linearization model. The number of tangent stiffness matrix updates for I-LL is reduced by 12.36% relative to the traditional local linearization approach. These simulation results demonstrate that, compared with the traditional local linearization model, the I-LL model reduces simulation time while maintaining simulation accuracy. The strategy of inserting accurate elastic internal force calculations during the local linearization process effectively mitigates error accumulation, while the adoption of strain as the update criterion reduces the frequency of tangent stiffness matrix updates and more accurately reflects the deformation state of the flexible body.

Table 2. Comparison of the times of updates in the tangent stiffness matrix in the complete model, the LL model, and the I-LL model.

	Complete Model	Local Linearization	Improved Local Linearization
Times of updates in the tangent stiffness matrix	20,000	10,000	8764

To investigate the reasonableness of selecting the threshold value of $\max(\|\Delta \epsilon_n^j\|_2) \leq 0.003$ for the total-strain-based local linearization criterion in the I-LL method, we adjusted the threshold and tested the RMSE and simulation time of the I-LL model. The computed results are shown in Figure 7:

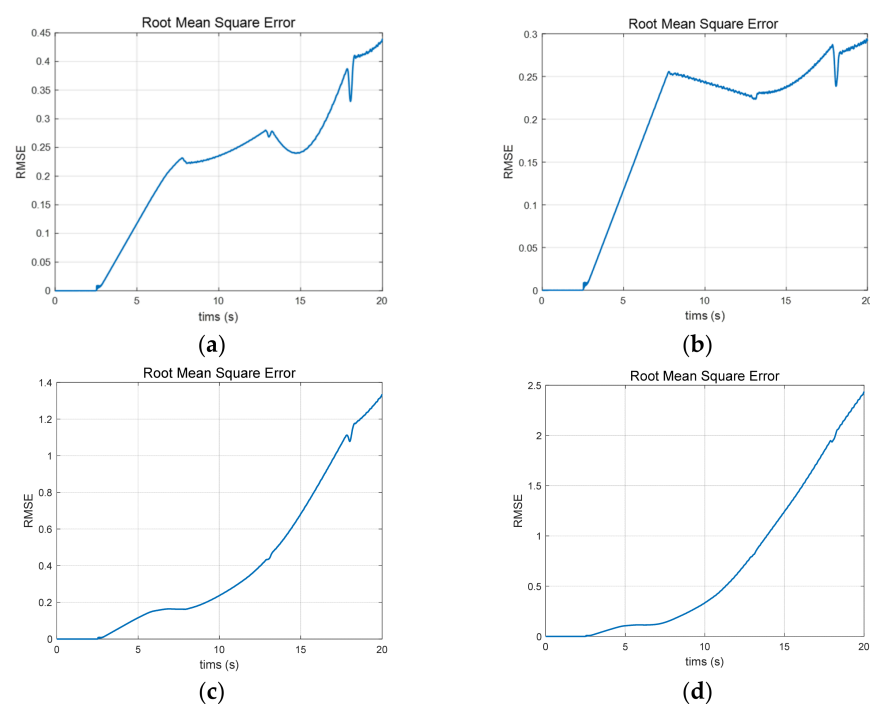


Figure 7. (a) The RMSE when the ILL threshold is 0.005. (b) The RMSE when the ILL threshold is 0.001. (c) The RMSE when the ILL threshold is 0.007. (d) The RMSE when the ILL threshold is 0.01.

Figure 7a shows the RMSE between the I-LL-simplified model and the full model when the threshold is 0.005, with a computation time of 257.19 s. Figure 7b shows the RMSE between the I-LL-simplified model and the full model when the threshold is 0.001, with a computation time of 263.08 s. The computation time of the I-LL model with a threshold of 0.005 is slightly lower than that with a threshold of 0.001, but the error of the model with a threshold of 0.005 is slightly higher than that with a threshold of 0.001. Therefore, we consider that the difference in the I-LL model's performance is small when the threshold is between 0.001 and 0.005, and a balance between error and computational speed can be achieved by adjusting the threshold. To demonstrate the rationality of staying within this threshold range of 0.001–0.005, we further tested the I-LL computation time and error at thresholds of 0.007 and 0.01. At a threshold of 0.007, the computation time is 248.63 s, and the RMSE is shown in Figure 7c. At a threshold of 0.01, the computation time is 227.18 s, and the RMSE is shown in Figure 7d. As can be seen from Figure 7c,d, when the threshold exceeds 0.005, the peak RMSE value rises from 0.45 to 1.2 and even above 2, which leads to considerable errors. Therefore, although a threshold larger than 0.005 can further reduce the simulation time, we still select the threshold within the range of 0.001 to 0.005 in order to maintain accuracy.

4.2. Bimetallic Cantilever

To further evaluate the performance of the I-LL method in thermo-mechanical coupling cases, we analyze a bimetal flexible cantilever beam. This flexible metal cantilever beam consists of two layers, a and b, of metals with different thermal expansion coefficients. The initial temperature of the bimetallic strip is 0 °C, and the ambient temperature is 20 °C. The left side of the flexible cantilever beam is clamped, and the temperature varies between 175 °C and 225 °C. The remaining three edges of the bimetallic cantilever beam exchange heat with the ambient temperature. When the ambient temperature is higher than that of the bimetallic cantilever beam, heat is transferred from the environment to the beam. When the temperature of the bimetallic cantilever beam is higher than the ambient temperature, the beam dissipates heat to the environment. The schematic diagram of the bimetal flexible cantilever beam structure and the temperature variation curve at the left end are shown in Figure 8, and the structural parameters are listed in Table 3.

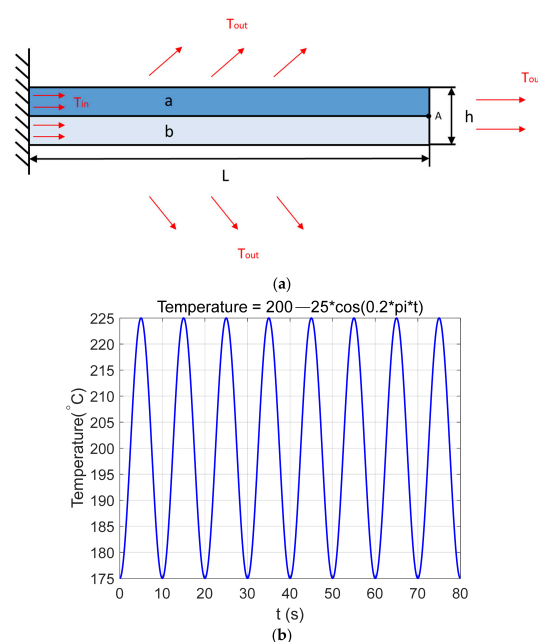


Figure 8. (a) Schematic diagram of the bimetallic flexible cantilever beam structure. (b) The left-end boundary temperature.

Table 3. Structure parameters.

Structural Parameters	Value
Beam length L	20 m
Beam width h	2 m
Young's modulus E	1×10^7 Pa
Poisson's ratio ν	0.3
Density ρ	4800 kg/m^3
Element thickness t	0.01 m
Coefficient of thermal expansion of layer a	$5 \times 10^{-4}/^\circ\text{C}$
Coefficient of thermal expansion of layer b	$1 \times 10^{-5}/^\circ\text{C}$
Specific heat capacity c	$50 \text{ J/(kg}\cdot^\circ\text{C)}$
Thermal conductivity coefficient λ	$500 \text{ W/(m}\cdot^\circ\text{C)}$
Convective heat transfer coefficient	$0.5 \text{ W/(m}^2\cdot^\circ\text{C)}$

The bimetallic cantilever beam is discretized into 80 planar triangular ANCF elements, yielding a total of 378 degrees of freedom. When the temperature from the left side transfers into the bimetal cantilever beam, the difference in the thermal expansion coefficients of the two layers a and b causes the cantilever beam to bend. We analyze the dynamic response of the bimetal cantilever beam over 80 s. The trajectory and temperature at point A at the free end of the cantilever beam are shown in Figure 9 as follows:

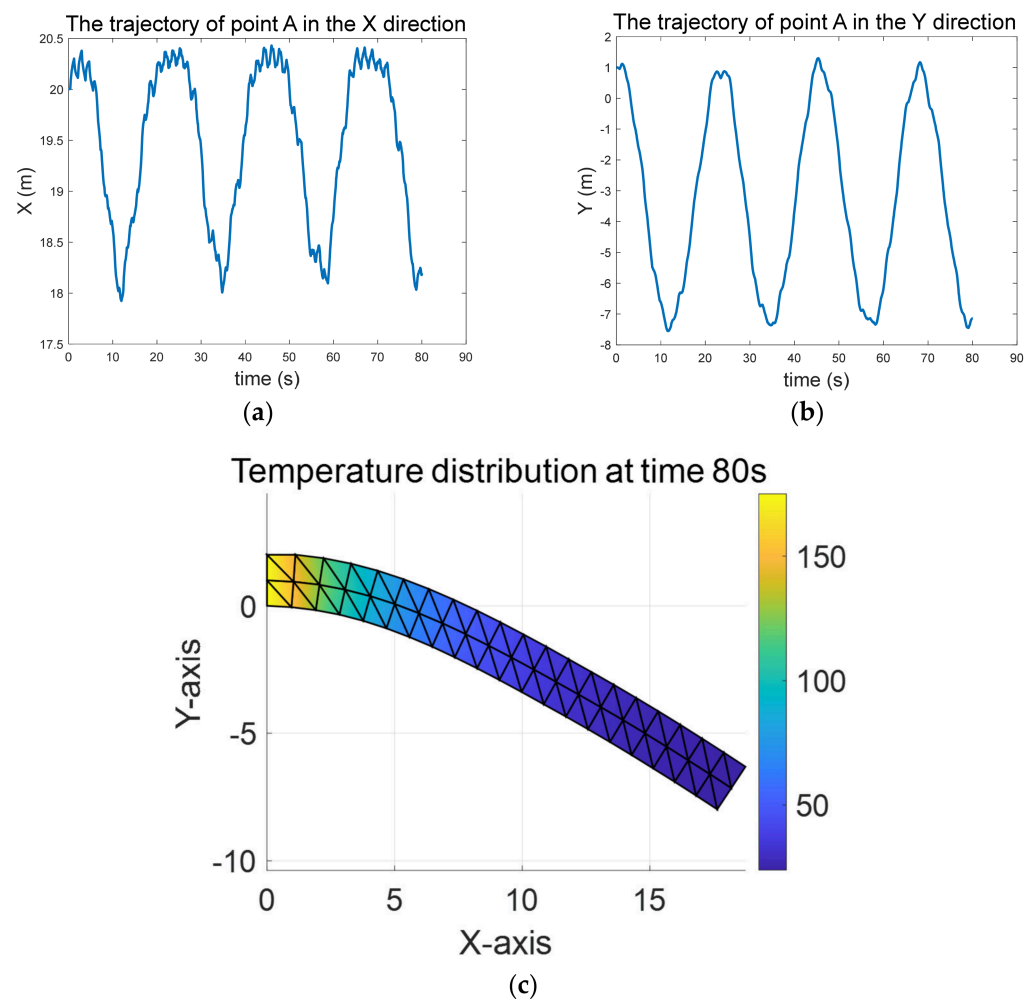


Figure 9. (a) The trajectory of point A in the X direction. (b) The trajectory of point A in the Y direction. (c) Temperature distribution map of the bimetallic cantilever beam at 80s.

We adopt the conventional local linearization method (LL) and the improved local linearization method (I-LL) to simplify the model, respectively. LL and I-LL use the same mesh and solver. To reduce error accumulation, local linearization is applied only to the thermal deformation force $\mathbf{Q}_H$, while the elastic internal force is not locally linearized. In the conventional local linearization method, the update criterion is $\max(\|\Delta \mathbf{p}_n^j\|_2) \leq 0.001$; whereas the update criterion of the improved local linearization method proposed in this paper is $\max(\|\Delta \boldsymbol{\varepsilon}_n^j\|_2) \leq 0.005$, and the thermal deformation force is computed exactly every 1000 time steps. As shown in Equation (22), we consider that the thermal stiffness matrix can be updated when the total strain difference exceeds 0.3%. The computed results are presented in Figure 10 as follows:

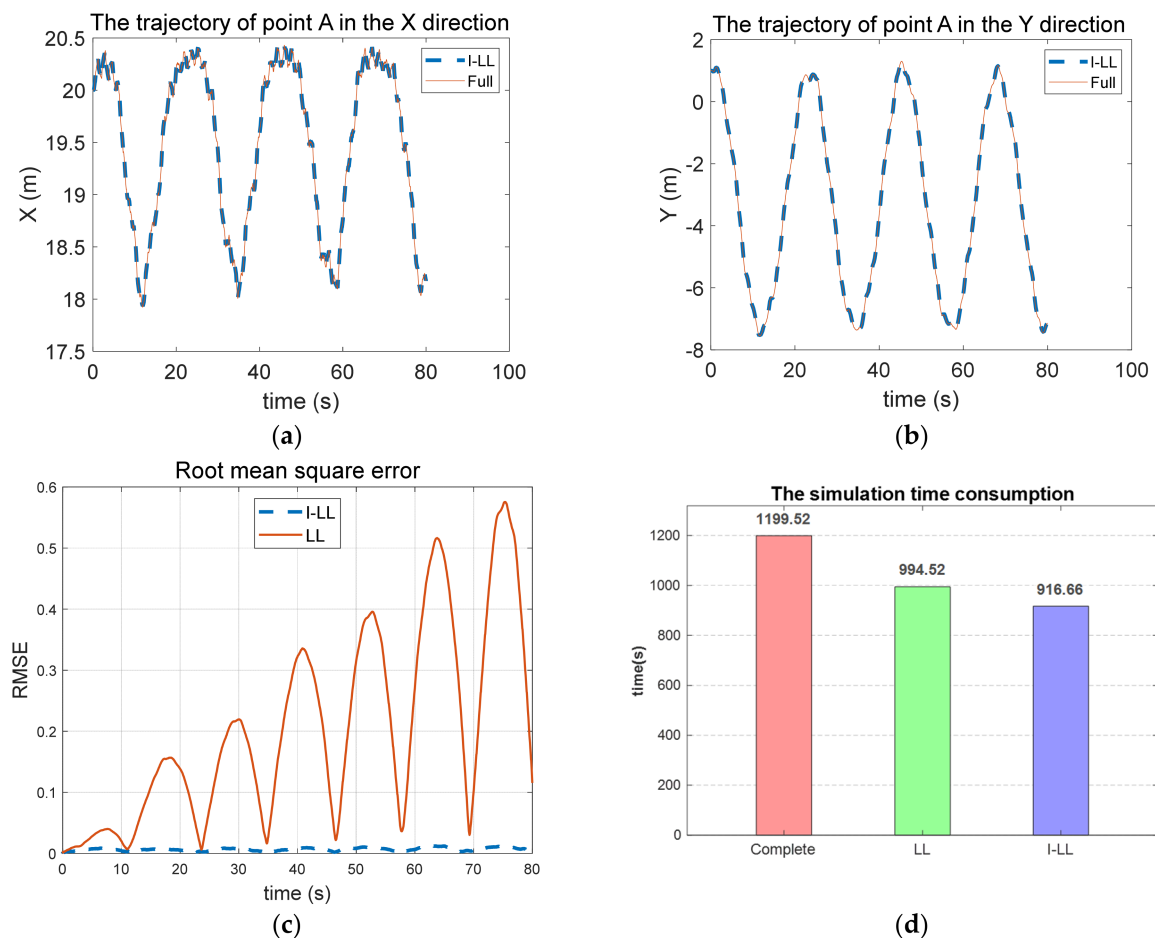


Figure 10. (a) Comparative results for the X-direction trajectory of point A as predicted by the I-LL and complete models. (b) Comparative results for the Y-direction trajectory of point A as predicted by the I-LL and complete models. (c) Comparison of the RMSE between the LL model and the I-LL model. (d) Comparison of the simulation time for the complete model, LL model, and I-LL model.

As shown in Figure 10a,b, the trajectory obtained by the I-LL model closely matches that obtained by the full model. Figure 10c shows that the RMSE between the I-LL model and the full model remains below 0.05. In contrast, the conventional LL model exhibits rapid error accumulation, and its RMSE is much higher than that of the I-LL model. The peak RMSE of the LL model is more than 10 times that of the I-LL model.

As shown in Figure 10d and Table 4, compared with the full model, the simulation time of the I-LL model is reduced by 23.58%, and by 7.82% compared with the traditional local linearization model. Compared with the traditional local linearization method, the number of thermal stiffness matrix updates in the I-LL model is reduced by 32.82%. These

simulation results demonstrate that in the solid–thermal coupling model, the I-LL model significantly reduces the number of thermal stiffness matrix updates, effectively decreasing the simulation time while ensuring simulation accuracy. The strategy of inserting exact thermal deformation force calculations during the local linearization of the thermal deformation force effectively reduces error accumulation, while the adoption of strain as the update criterion greatly reduces the update frequency of the thermal stiffness matrix and more accurately describes the deformation state of the flexible body.

Table 4. Comparison of the times of updates in the tangent stiffness matrix in the complete model, the LL model, and the I-LL model.

	Complete Model	Local Linearization	Improved Local Linearization
Times of updates in the tangent stiffness matrix	80,000	29,836	20,043

4.3. Solid–Thermal Coupled Slider–Crank Mechanism

To verify the accuracy and effectiveness of the I-LL method in a solid–thermal coupled multi-flexible-body system, we perform a dynamic analysis of a flexible slider–crank mechanism. The slider–crank mechanism moves under the action of a torque M applied at point O. The revolute joint B is connected to a 20 kg mass, and damping exists on the guide rail of slider B. It is assumed that during the rotation of revolute joint A of this flexible slider–crank mechanism, improper lubrication causes joint A temperature variation, and heat is transferred to the flexible links on both sides, resulting in thermal deformation of the flexible links. The crank–slider mechanism does not dissipate heat to the external environment, and heat transfer occurs only within the mechanism itself. The structural diagram of the slider–crank mechanism and the temperature variation curve at point A are shown in Figure 11a,b, and the structural parameters are listed in Table 5.

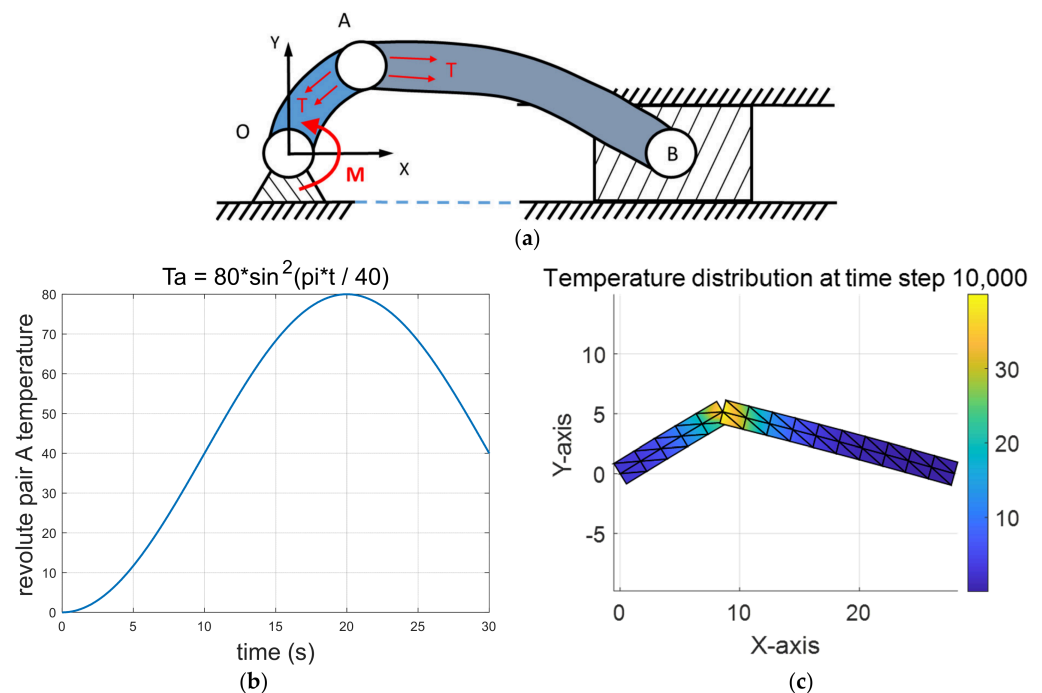


Figure 11. (a) Schematic diagram of flexible crank–slider structure. (b) Schematic diagram of temperature variation at revolute joint A. (c) Temperature distribution of the crank–slider mechanism at 10,000 time steps

Table 5. Structure parameters.

Structural Parameters	Value
AO Beam length	10 m
AB Beam length	10 m
Beam width h	2 m
Young's modulus E	1×10^7 Pa
Poisson's ratio ν	0.3
Density ρ	4800 kg/m ³
Element thickness t	0.01 m
Coefficient of thermal expansion α	$1 \times 10^{-4}/^{\circ}\text{C}$
Specific heat capacity c	200 J/(kg $\cdot^{\circ}\text{C}$)
Thermal conductivity coefficient λ	100 W/(m $\cdot^{\circ}\text{C}$)
Mass block	20 kg
Torque $\mathbf{M}$	1800 N $\cdot\text{m}$
Rail damping	25 N $\cdot\text{s}/\text{m}$

The crank–slider mechanism is discretized into 60 planar triangular ANCF elements, resulting in 306 degrees of freedom. The dynamic response of the crank–slider mechanism over 30 s is analyzed. The kinematic trajectory of revolute joint A is shown in Figure 12a–c, and the temperature variation of revolute joint B is shown in Figure 12d.

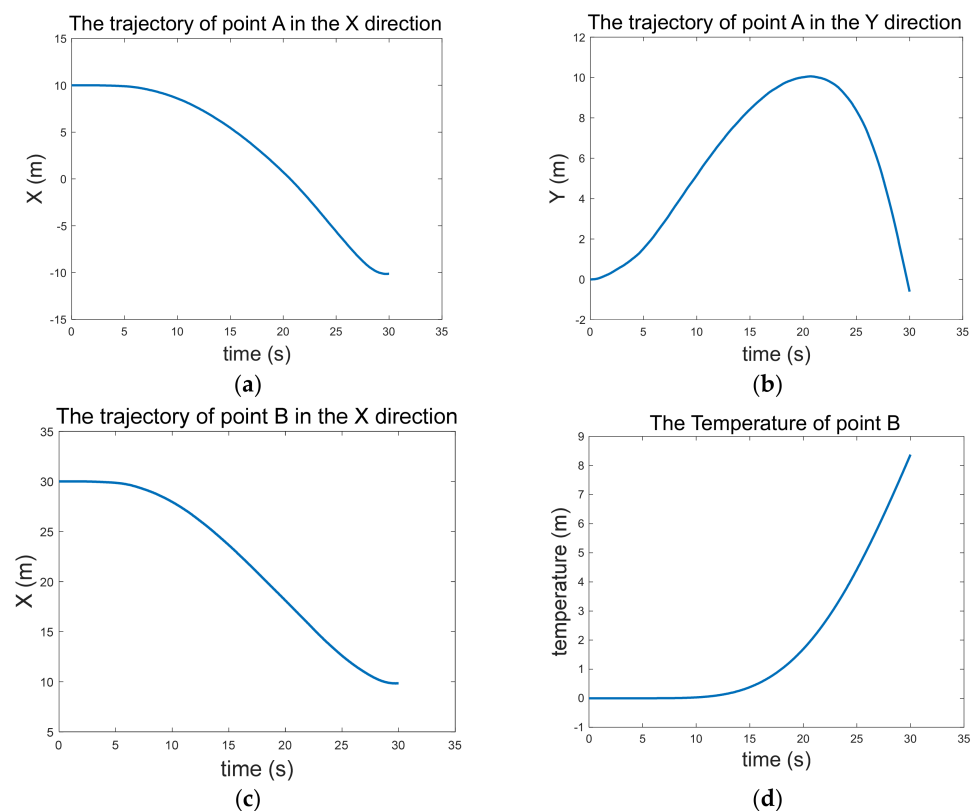


Figure 12. (a) The trajectory of point A in the X direction. (b) The trajectory of point A in the Y direction. (c) The trajectory of point B in the X direction. (d) The temperature of point B.

The I-LL method and the traditional local linearization method are used to simplify the model, respectively. LL and I-LL use the same mesh and solver. To reduce error accumulation, only the thermal deformation force $\mathbf{Q}_H$ is locally linearized, while the elastic internal force is not. In the traditional local linearization method, the update criterion is $\max(\|\Delta \mathbf{p}_n^j\|_2) \leq 0.003$, whereas the update criterion of the improved local linearization

method proposed in this paper is $\max(\|\Delta \varepsilon_n^j\|_2) \leq 0.003$, and the thermal deformation force is calculated exactly every 1000 time steps. As shown in Equation (22), it is considered that the thermal stiffness matrix can be updated when the total strain difference exceeds 0.3%. The calculation results are shown in Figure 13.

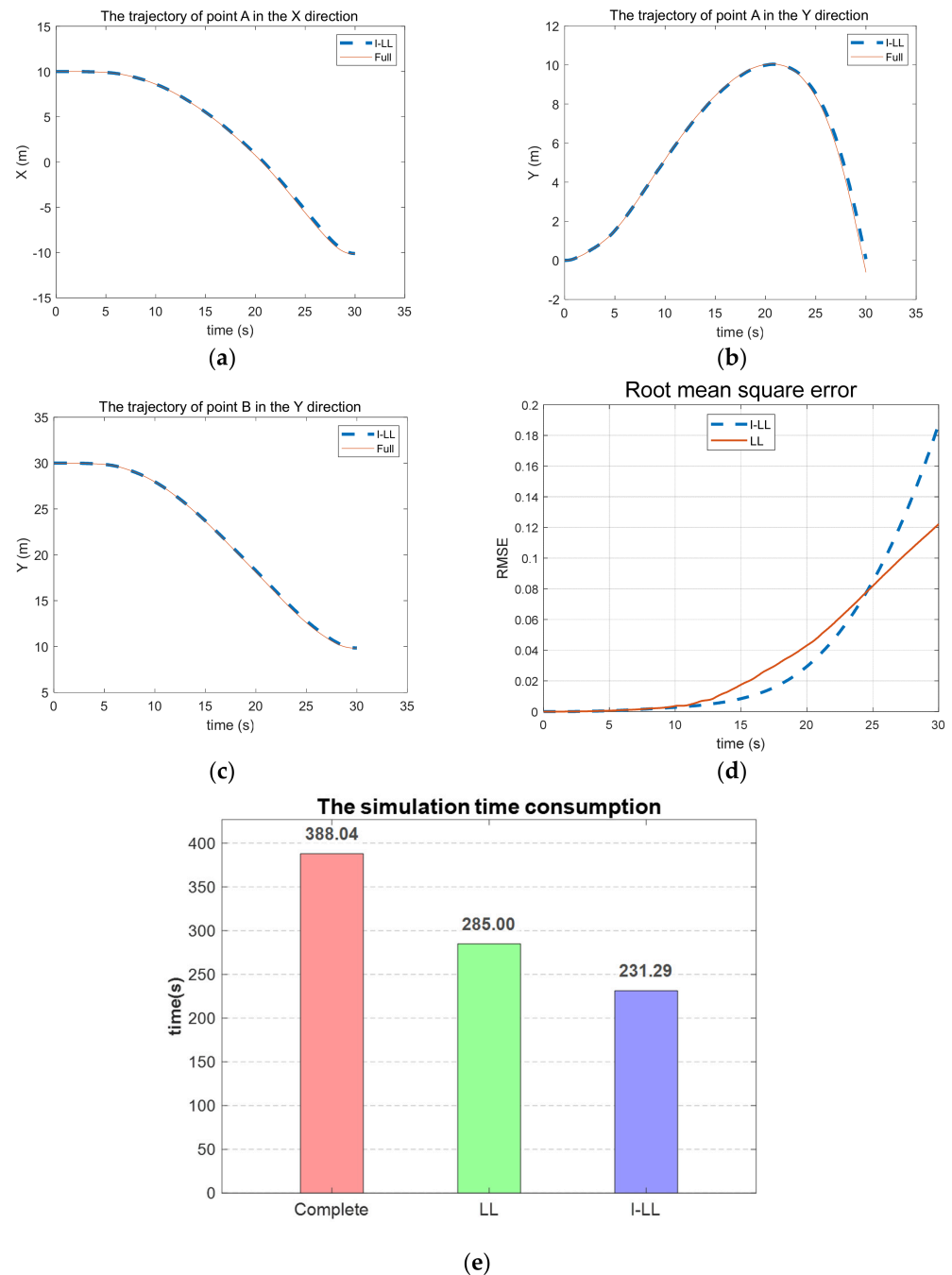


Figure 13. (a) Comparative results for the X-direction trajectory of point A as predicted by the I-LL and complete models. (b) Comparative results for the Y-direction trajectory of point A as predicted by the I-LL and complete models. (c) Comparative results for the X-direction trajectory of point B as predicted by the I-LL and complete models. (d) Comparison of the RMSE between the LL model and the I-LL model. (e) Comparison of the simulation time for the complete model, LL model, and I-LL model.

As shown in Figure 13a–c, the trajectory obtained by the I-LL model closely matches that obtained by the full model. Figure 13d shows that the root mean square error (RMSE)

between the I-LL model and the full model remains below 0.2. In contrast, the RMSE of the traditional local linearization model begins to increase rapidly after 15 s. By 30 s into the simulation, the RMSE of the traditional local linearization model has grown to more than five times that of the I-LL model.

As shown in Figure 13e and Table 6, compared with the full model, the simulation time of the I-LL model is reduced by 40.36%, and by 18.84% compared with the traditional local linearization model. Compared with the traditional local linearization method, the number of thermal stiffness matrix updates in the I-LL model is reduced by 98.50%. These simulation results demonstrate that in the solid–thermal coupled model, the I-LL model significantly reduces the number of thermal stiffness matrix updates, effectively decreasing the simulation time while ensuring simulation accuracy. The strategy of inserting exact thermal deformation force calculations during the local linearization of the thermal deformation force effectively reduces error accumulation, while the adoption of strain as the update criterion greatly reduces the update frequency of the thermal stiffness matrix and more accurately describes the deformation state of the flexible body.

Table 6. Comparison of the times of updates in the tangent stiffness matrix in the complete model, the LL model, and the I-LL model.

	Complete Model	Local Linearization	Improved Local Linearization
Times of updates in the tangent stiffness matrix	30,000	6058	91

To investigate the differences caused by varying the time intervals for performing exact nonlinear internal force calculations, we evaluate the performance of the exact nonlinear internal force calculation for the flexible crank–slider mechanism at different time intervals. Specifically, the exact thermal deformation force calculation is performed once every 500, 1000, and 2000 time steps, and the results are shown in Figure 14:

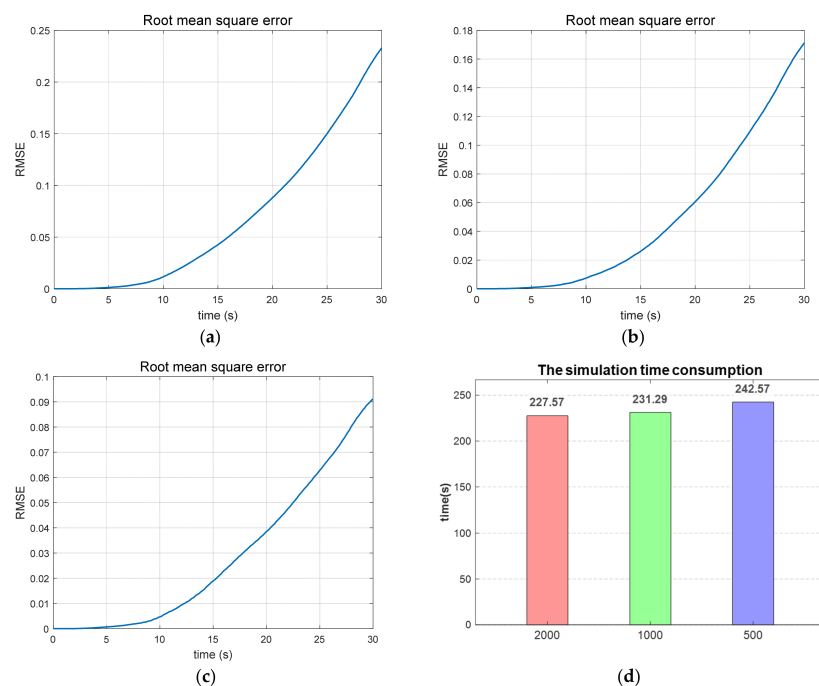


Figure 14. (a) I-LL model RMSE (2000-step interval). (b) I-LL model RMSE (1000-step interval). (c) I-LL model RMSE (500-step interval). (d) Comparison of simulation time for the I-LL model at different time intervals.

Figure 14a–c present the RMSE of the dynamic response obtained when the exact nonlinear internal force calculation is performed once every 500, 1000, and 2000 time steps, respectively, for the I-LL model. It can be observed from the figures that as the interval for the exact nonlinear internal force calculation is reduced, the accuracy of the dynamic response improves. The peak RMSE of the I-LL model exceeds 0.2 at an interval of 2000 time steps, exceeds 0.16 at an interval of 1000 time steps, and drops to 0.009 at an interval of 500 time steps. Meanwhile, the improvement in computational accuracy also brings an increase in computational time. As shown in Figure 14d, the shorter the interval for the exact nonlinear internal force calculation, the slightly higher the computational time. Therefore, the simulation results indicate that the time step interval for the exact nonlinear internal force calculation affects both the computational time and the model accuracy of the I-LL model, but the sensitivity is not particularly high. In this study, the exact nonlinear internal force calculation is performed once every 1000 time steps, i.e., at an interval of 1 s. For a 30 s dynamic simulation, 30 exact nonlinear internal force calculations do not impose an excessive computational burden while effectively mitigating error accumulation. This interval is chosen as a trade-off that balances simulation speed and model accuracy.

5. Conclusions

To address the issues of error accumulation in traditional local linearization methods and the adaptability of the tangent stiffness matrix update criterion, this paper proposes an improved local linearization method (I-LL). The I-LL method enhances the conventional LL approach in two main aspects. First, to mitigate the rate of error accumulation, I-LL periodically inserts an accurate calculation of the elastic or thermal deformation forces at specified time intervals and uses the obtained forces as the basis for the subsequent local linearization iteration. Second, the update criterion for the stiffness matrix is modified from displacement-based to total strain-based. Total strain accurately captures the extent of deformation of the flexible body and provides a more effective criterion for determining whether the stiffness matrix needs to be updated, particularly in scenarios involving large rigid-body displacements.

The accuracy and effectiveness of the I-LL method are demonstrated through three numerical cases. Compared with the traditional local linearization method, I-LL exhibits significant advantages in mitigating error accumulation and maintaining long-term simulation accuracy. Furthermore, in these examples, I-LL requires fewer stiffness matrix updates and less computational time than the conventional LL approach.

In the three numerical examples, different benchmark thresholds are chosen for the local linearization in the conventional LL model. In the ball impact example, the conventional local linearization method (LL) must cope with the instantaneous impact of the ball colliding with the ground, which requires a relatively small benchmark threshold to ensure simulation accuracy for the impact deformation. In this example, we also compared the variation in simulation accuracy of the I-LL model when other benchmark thresholds are selected. In the bimetallic cantilever beam example, the deformation of the bimetallic strip caused by temperature variation is relatively large. If a larger benchmark threshold were chosen for the LL model, its simulation accuracy would differ too much from that of the I-LL model, making the comparison meaningless. From this example we can also observe that even though the LL model has already adopted a small threshold of 0.001, the simulation error remains considerable. Therefore, we increased the benchmark threshold for the LL model and selected a higher value of 0.003. In the crank–slider example, the crank–slider undergoes large rigid-body motion, which allows a larger benchmark threshold to be chosen while still achieving good simulation accuracy. Hence, we adopted a benchmark threshold of 0.003 for both the LL and I-LL models.

The differences in the reduction of stiffness matrix update counts among the three cases do not indicate a failure of the I-LL method in impact problems; rather, they are determined by the physical deformation characteristics of each case. The core advantage of the I-LL method is that it eliminates unnecessary stiffness updates caused by rigid-body motion in the conventional LL method, while fully retaining the necessary updates induced by genuine deformation. There is a significant difference in the ratio of “necessary updates” to “unnecessary updates” among these three cases. In the bimetallic cantilever beam example, the conventional LL adopts a displacement-based update criterion: even when the bimetallic cantilever beam undergoes only slight bending, the nodal displacements accumulate continuously and trigger updates. In contrast, the I-LL employs a strain-based criterion and updates only when the maximum nodal strain change exceeds the preset threshold, thereby almost completely eliminating unnecessary updates. Similarly, in the flexible crank–slider mechanism, rigid-body motion dominates the rigid–flexible coupling movement, and the thermal strain varies slowly only near the joints. The conventional LL is frequently triggered to update by the continuous displacements of the connecting rod and slider, whereas the I-LL can accurately identify the elastic deformation and updates only when the accumulated total strain exceeds the threshold. In the ball impact example, purely rigid-body motion occurs during the free fall before the first impact. When the ball collides with the ground, relatively severe elastic deformation occurs. Even after the ball rebounds from the ground, repeated elastic oscillations are induced inside the ball due to the impact. Therefore, the ball continues to undergo elastic deformation after leaving the ground. The I-LL method completely eliminates unnecessary updates during the free-fall phase before the first impact. However, after the collision with the ground, the elastic deformation is relatively severe, necessitating continuous updates of the stiffness matrix to ensure computational accuracy. Therefore, in the ball impact example, the number of stiffness matrix updates saved by the I-LL method is smaller than in the other two cases.

However, the I-LL method has a drawback: in solid–thermal coupling cases, only one of the two—either the thermal deformation force or the elastic internal force—can be simplified. If both the thermal deformation force and the elastic internal force are simplified simultaneously, the error accumulates rapidly. In future work, we will further investigate methods for simultaneously simplifying multiple nonlinear internal forces and apply the I-LL method to more complex examples to achieve long-duration, high-accuracy simulations. Meanwhile, we will develop more effective computational strategies for simplifying nonlinear internal forces to further reduce error accumulation.

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