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Article

A Discrete Analog of the Pham–Burr XII Distribution: Properties and Estimation with Medical and Environmental Applications

Heba N. Salem [1,2], Neama T. AL-Sayed [1], Hebatalla H. Mohammad [3], Gannat R. AL-Dayian [1] and Abeer A. EL-Helbawy [1,4,*]

1 Department of Statistics, Faculty of Commerce, Al-Azhar University, (Girls' Branch), Cairo 11675, Egypt
2 Higher Institute of Marketing, Commerce & Information Systems (MCI), Cairo 11511, Egypt
3 Department of Mathematical Sciences, College of Science, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
4 Department of Basic Sciences, Badr Institute of Science and Technology (BIS), Badr 11829, Egypt
* Correspondence: abeer@azhar.edu.eg

Abstract

This article is concerned with proposing a discrete version of a competing risks model, namely, the Pham–Burr XII distribution, via the general approach of discretization. The proposed model's probability mass function displays decreasing, unimodal and decreasing–unimodal patterns, while its hazard rate and alternative hazard rate functions exhibit different and important shapes, which are decreasing, bathtub (Vtub), modified bathtub and unimodal shapes. Through these shapes, the flexibility and diversity in shapes of the characteristic functions of the discrete Pham–Burr XII distribution can be demonstrated. Therefore, the discrete Pham–Burr XII distribution can provide a better fit for several types of discrete data and count data. The main characteristic functions of the discrete Pham–Burr XII distribution are derived, and its properties are studied. Moreover, the parameters and the main characteristic functions of the discrete Pham–Burr XII distribution are estimated via the maximum likelihood method. Also, the asymptotic confidence intervals and percentile bootstrap confidence intervals are considered. Moreover, point and interval estimation of some entropy measures is discussed. Furthermore, a simulation study is achieved to assess the performance of the delivered maximum likelihood estimates. Finally, the applicability of the discrete Pham–Burr XII distribution is examined though different applications.

Keywords: competing risks model; general approach of discretization; hazard rate and alternative hazard rate functions; maximum likelihood estimation; asymptotic confidence intervals; percentile bootstrap

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1. Introduction

Discretization of the competing risks models is an essential way to reach a discrete model with a *hazard rate function* (HRF) accommodating the bathtub or modified version of failure rates. In the statistical literature, there are only two discrete distributions that were introduced by discretizing their continuous competing risks models. The first one is *discrete additive Weibull* (DAW), derived by [1], as a discrete version of the *additive Weibull* (AW) distribution, which was proposed by [2]. Meanwhile, the second is the *discrete additive Perks–Weibull* (DAP-W) distribution, which was introduced by [3], as a discrete analog of the additive Perks–Weibull distribution proposed by [4]. So, in this article, the *Pham–Burr XII* (Ph-BXII) distribution is discretized by applying the general approach of discretization. The reason behind discretizing this distribution is its HRF, which reveals various important shapes of failure. In addition, the main objectives of this article are as follows:

- Presenting a discrete competing risks model, namely the *discrete Ph-BXII* (DPh-BXII) distribution, by applying the general approach of discretization to preserve the reliability characteristics of the continuous Ph-BXII distribution.
- Proposing a discrete model with HRF and its alternative, i.e., the *alternative HRF* (AHRF), which demonstrates important shapes of failure that are commonly used in reliability analysis and data modeling.
- Deriving a discrete model capable of modeling over- and under-dispersed, right- and left-skewed and platykurtic and leptokurtic data.
- Studying the statistical properties of the DPh-BXII distribution and discussing some important entropy measures (Shannon, Rényi, Tsallis) and the extropy with numerical results.
- Deriving the *maximum likelihood* (ML) estimators and the *asymptotic confidence intervals* (ACIs), *percentile bootstrap confidence intervals* (PBCIs) of the parameters, *survival function* (SF), HRF, AHRF and *reversed HRF* (RHRF) for the proposed model. Also, providing ML point and interval estimation of the considered entropies and extropy.
- Constructing a Monte Carlo simulation study to assess the behavior of the derived *ML estimates* (MLEs).
- Presenting a real application on *acquired immunodeficiency syndrome* (AIDS) incidence cases in the *United Kingdom* (UK) and Australia and count data about forest fires in the *United States of America* (USA). These applications cover medical and environmental sectors. These applications are used for assessing the flexibility and applicability of the proposed model in the era of data modeling and examining its superiority over some existing models.

So, the proposed model, namely the DPh-BXII distribution, fills an important gap in the literature by introducing the first discrete analog of the DPh–BXII distribution. In addition, it is a discrete competing risks model. Unlike many existing discrete lifetime models, the DPh-BXII distribution provides a highly flexible framework capable of accommodating decreasing, bathtub, modified bathtub, and unimodal hazard rate behaviors, making it particularly suitable for modeling complex discrete lifetime and count data.

In the last decades, researchers have focused on developing new lifetime distributions; see [5]. Several of these lifetime distributions were continuous ones, although there may be situations when discrete distributions are more suitable for data modeling. Also, the data can be collected in a discrete way. Sometimes, measuring the life length of devices, components, experimental units or individuals on a continuous scale is frequently impossible or inconvenient. In reliability engineering, the lifetime of an on/off turning device, the number of cycles that a machine operates prior to failure, and the life of a printer can be measured as the total number of copies it produces; in computer sciences, the number of breakdowns of a server occurring during an online application submission, and the number of times a software is required to be updated during a specified period of time; and in survival analysis, the survival times for those suffering from diseases such as lung cancer may be recorded as the number of days, weeks, etc., and the period from remission to relapse for a given patient may be recorded in days, weeks, months, etc. Moreover, the count phenomenon happens in a wide range of practical circumstances, including the number of earthquakes recorded through a calendar year, the number of absences, the number of accidents, the number of failures and successes for students, the number of insurance claims, the number of deaths or cases due to the COVID-19 pandemic observed

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over a specified period of time, and so on. In all these circumstances, it is more appropriate to measure these random phenomena on a discrete measure rather than a continuous one.

Deriving discrete analogs (discretization) of a continuous distribution has been a major concern of researchers. In recent years, many research articles relating to discrete distributions derived by discretizing continuous ones have appeared in a scattered manner in the existing statistical literature. There are several ways to derive discrete distribution from continuous ones. In the literature, there are only three articles that deal with surveys of discrete analogs of continuous distributions. The first one is [6]. Following that, Ref. [7] discussed the presented methods of constructing discrete lifetime models by discretization of continuous lifetime models that were presented in the literature and the discrete HRF along with its alternative. Subsequently, a comprehensive survey of different methods of generating discrete probability distributions as analogs of continuous probability distributions along with their applications in the construction of new discrete distributions was presented by [8]. The study classified these methods based on different criteria of discretization. One of these methods is the general approach of discretization, also known as the survival approach, which has been widely used due to its advantages. The main advantage of using the general approach of discretization is that the resulting discrete distribution has the same functional form of the SF as its continuous form. Due to this feature, many of the reliability characteristics of the distribution are unchanged. In addition, this is the easiest discretization approach for the construction of discrete models.

Consider the SF of a continuous *random variable* (RV), X, denoted by $S(x)$, where

$$S(x) = P(X > x), \tag{1}$$

and the corresponding *cumulative distribution function* (CDF) is

$$F(x) = P(X \leq x) = 1 - S(x). \tag{2}$$

If the underlying continuous RV, X, has the SF, $S(x)$, then the discrete RV, $Y = \lfloor X \rfloor$, the greatest integer smaller than or equal to X, will have the following *probability mass function* (PMF):

$$P(Y = y) = P(y \leq Y < y + 1) = S(y) - S(y + 1), \qquad y = 0, 1, 2, \cdots. \tag{3}$$

Alternatively, by considering $Y = \lceil X \rceil$, the smallest integer greater than or equal to X, one can form a discrete version of X with PMF that will preserve the CDF of the continuous distribution as follows:

$$P(Y = y) = P(y - 1 < Y \leq y) = S(y - 1) - S(y), \quad y = 0, 1, 2, \cdots. \tag{4}$$

It can be noted that $\lceil X \rceil = \lfloor Y \rfloor + 1$.

Using the general approach of discretizing, many continuous lifetime distributions have been discretized. Ref. [9] were the first to derive a discrete analogue of a continuous distribution. They presented a discrete version of *Weibull* (W) distribution, namely *discrete W* (DW). Whereas Ref. [10] introduced another discrete version of W distribution by considering different parameterization of W distribution. Ref. [11] developed a discrete version of the normal distribution. While Ref. [12] proposed the discrete Rayleigh distribution. Ref. [13] obtained *discrete Burr XII* (DBXII) distribution. Also, discrete inverse W distribution was proposed by [14]. Ref. [15] introduced a one parameter discrete Lindley distribution. Ref. [16] introduced the discrete Burr III distribution. Ref. [17] introduced the discrete reduced modified W distribution. Ref. [18] introduced the discrete generalized Rayleigh distribution. Ref. [19] proposed the discrete logistic distribution.

Ref. [20] presented the *discrete Chen* (DCh). Ref. [21] derived a discrete version of Marshall-Olkin family of distributions. Ref. [22] introduced the discrete W geometric distribution. Ref. [23] derived the discrete Gompertz distribution. Ref. [24] derived the discrete Gumbel (Type I) extreme value distribution. Moreover, Ref. [25] presented the discrete inverted Kumaraswamy distribution. Also, Ref. [26] derived a discrete version of the alpha power family of distributions.

Recently, Ref. [27] presented the discrete complementary exponentiated-G Poisson family of distributions. Also, Ref. [28] proposed the discrete logistic exponential distribution. Likewise, Ref. [29] discretized the exponentiated-G family of distributions. More recently, Ref. [30] introduced the discrete odd Lindley half-logistic distribution. Also, Ref. [31] derived the discrete power quasi-xgamma distribution. Moreover, Ref. [32] presented two novel discrete counterparts for the Rayleigh–Lindley distribution.

Furthermore, Ref. [33] introduced a discrete analogue of the one and two-parameter half-logistic distribution. Most recently, Ref. [34] introduced the one-parameter discrete analogue distribution of the new xLindley model. Also, Ref. [35] introduced the discrete alpha power W-G family of distributions. Moreover, Ref. [36] presented the discrete analogue of mixture W distribution. Finally, Ref. [37] presented the discrete exponentiated generalized W-G family.

An essential approach for constructing new continuous lifetime distributions is the competing risks approach, which depends on the concept of competing risks. Competing risks appear in many life-testing experiments, where the failure of the tested item may be due to more than one cause or mode of failure. These modes of failure in some sense compete to cause the failure of the tested item. Due to this explanation, in the statistical literature, this is well-known as competing risks. Further, the importance of the competing risks approach stems from the fact that the resulting model has a flexible HRF, which always accommodates as mixture models the bathtub shape or more complex shapes. Based on the competing risks approach, there are many lifetime distributions that have been constructed in the literature, starting from the AW distribution that was introduced by [2] and the additive Burr XII distribution, which was presented by [38], etc. As of 2025, Ref. [39] had proposed the additive Dhillon–Chen distribution, and Ref. [40] had derived the hybrid Weibull–exponential power distribution. Finally, Ref. [41] introduced the Ph-BXII distribution as a competing risks model, by considering two competing risks: the former has the *Pham* (Ph) distribution, and the latter has the *Burr XII* (BXII) distribution. The *probability density function* (PDF) of the Ph-BXII distribution has different shapes, such as decreasing, unimodal, approximately symmetric, decreasing–unimodal and bimodal. Meanwhile, its HRF exhibits the most important shapes: the bathtub and more generalized shapes, i.e., the modified bathtub and modified unimodal shapes. Also, the Ph-BXII distribution is suitable for low, moderate and high variation, left and right skewed and platykurtic and leptokurtic data. The SF, CDF, HRF and PDF of the Ph-BXII distribution are defined, respectively, as follows:

$$S(x;\underline{\theta}) = \exp\left\{1-\beta^{x^{\alpha}} - k\ln(1+x^{c})\right\}, \qquad x>0;\ \alpha,c,k>0,\ \beta>1, \tag{5}$$

$$F(x;\underline{\theta}) = 1-\exp\left\{1-\beta^{x^{\alpha}} - k\ln(1+x^{c})\right\}, \qquad x>0;\ \alpha,c,k>0,\ \beta>1, \tag{6}$$

$$h(x;\underline{\theta}) = \alpha\ln(\beta)x^{\alpha-1}\beta^{x^{\alpha}} + \frac{ckx^{c-1}}{(1+x^{c})}, \qquad x>0;\ \alpha,c,k>0,\ \beta>1, \tag{7}$$

and

$$f(x;\underline{\theta}) = \left[\alpha\ln(\beta)x^{\alpha-1}\beta^{x^{\alpha}} + \frac{ckx^{c-1}}{(1+x^{c})}\right]\exp\left\{1-\beta^{x^{\alpha}} - k\ln(1+x^{c})\right\}, \quad x>0;\ \alpha,c,k>0,\ \beta>1, \tag{8}$$

where $\underline{\theta} = (\alpha,\ \beta,\ c, k)$ represents Ph-BXII's vector of parameters.

The statistical properties of the Ph-BXII distribution were studied, and its parameters, SF, HRF and RHRF, were estimated via the ML method of estimation.

The organization of this paper is as follows: the derivation of the DPh-BXII distribution and its sub-models are presented in Section 2. Also, a graphical explanation of the PMF, HRF and AHRF of the DPh-BXII distribution are given in Section 3. Moreover, some statistical properties of the DPh-BXII distribution are discussed in Section 4. Section 5 considers ML point and interval estimation of the parameters, SF, HRF and AHRF, of the DPh-BXII distribution. In addition, ML point and interval estimation for Shannon, Rényi, Tsallis and the extropy are considered. A simulation study is conducted in Section 6 to evaluate the performance of the MLEs. In Section 7, new applications are considered to demonstrate the applicability of the proposed model and its superiority over some existing distributions. Finally, this paper is completed, with its general conclusion and the potential future work on the proposed model, in Section 8.

2. Model Structure and Sub-Models

In this section, the DPh-BXII distribution is derived by considering the Ph-BXII distribution and by applying the general approach of discretizing continuous distributions. Also, sub-models of the DPh-BXII distribution are discussed.

2.1. Model Structure

Using (3) and (5), the discrete RV, $Y = \lfloor X \rfloor$, where X is a continuous RV with the Ph-BXII distribution, can be derived with the following PMF:

$$P(y;\underline{\theta}) = \left[e^{1-\beta^{y^{\alpha}}-k\ln(1+y^{c})} - e^{1-\beta^{(y+1)^{\alpha}}-k\ln[1+(y+1)^{c}]}\right], \qquad y = 0,\ 1,\ 2,\ \cdots; \alpha, c, k > 0,\ \beta > 1. \tag{9}$$

The SF and CDF of the DPh-BXII distribution are expressed as

$$S(y;\underline{\theta}) = P(Y \geq y) = e^{1-\beta^{y^{\alpha}}-k\ln(1+y^{c})}, \quad y = 0,\ 1,\ 2,\ \cdots; \quad \alpha, c, k > 0,\ \beta > 1, \tag{10}$$

and

$$\begin{aligned} F(y;\underline{\theta}) &= P(Y \leq y) = 1 - S(y;\underline{\theta}) + P(Y = y) \\ &= 1 - e^{1-\beta^{(y+1)^{\alpha}}-k\ln[1+(y+1)^{c}]}, \qquad y = 0,\ 1,\ 2,\ \cdots; \alpha, c, k > 0,\ \beta > 1. \end{aligned} \tag{11}$$

Subsequently, the HRF is expressed as

$$h(y;\underline{\theta}) = \frac{P(y;\underline{\theta})}{S(y;\underline{\theta})} = 1 - \exp\left\{\beta^{y^{\alpha}} - \beta^{(y+1)^{\alpha}} - k\ln\left[\frac{1+(y+1)^{c}}{1+y^{c}}\right]\right\}, \ y = 0,\ 1,\ 2,\ \cdots;\ \alpha, c, k > 0,\ \beta > 1. \tag{12}$$

Studying the HRF, derived in (12), one can reach the following problems:

- It can be interpreted as a conditional probability, so it is always finite and bounded by 1.
- It is not additive for competing risks models, i.e., for $\mathscr{H}$ independent competing risks or system components functioning independently in series, where the HRF of the system is
$$h(y) = 1 - \prod_{\hbar}^{\mathscr{H}}[1 - h_{\hbar}(y)] \neq \sum_{\hbar}^{\mathscr{H}} h_{\hbar}(y),$$
where $h_{\hbar}(y)$ is the $\hbar^{th}$ risk or the $\hbar^{th}$ component.
- The cumulative HRF is
$$H(y) = \sum h(y) \neq -\ln S(y),$$

as for the continuous case. So, in discrete counterpart, the reliability properties defined in terms of the SF will not be equivalent to those defined in terms of the CHRF; see [7].

In order to overcome these problems associated with the HRF in the discrete counterpart, Ref. [42] introduced an alternative definition of the HRF, namely AHRF, defined as

$$ah(y) = \ln\left[\frac{1}{1-h(y)}\right]. \tag{13}$$

Using (13), the AHRF of the DPh-BXII distribution can be derived as

$$ah(y;\underline{\theta}) = \beta^{(y+1)^{\alpha}} - \beta^{y^{\alpha}} - k\ln\left[\frac{1+y^{c}}{1+(y+1)^{c}}\right], \quad y = 0,\ 1,\ 2,\ \cdots;\quad \alpha, c, k > 0,\ \beta > 1, \tag{14}$$

Correspondingly, the RHRF of the DPh-BXII distribution is given as

$$r(y;\underline{\theta}) = \frac{P(y;\underline{\theta})}{F(y;\underline{\theta})} = \frac{\exp\left\{1-\beta^{y^{\alpha}}-k\ln(1+y^{c})\right\}-\exp\left\{1-\beta^{(y+1)^{\alpha}}-k\ln\left[1+(y+1)^{c}\right]\right\}}{1-\exp\left\{1-\beta^{(y+1)^{\alpha}}-k\ln\left[1+(y+1)^{c}\right]\right\}}, \quad y = 0,\ 1,\ 2,\ \cdots; \alpha, c, k > 0,\ \beta > 1. \tag{15}$$

2.2. Sub-Models

There are some discrete distributions that can be derived from the DPh-BXII distribution as sub-models. These sub-models are new and are not present in literature. Table 1 summarizes these sub-models.

Table 1. Sub-models of DPh-BXII distribution.

θ	Resulting Distribution	PMF
$c = 1$	Discrete Ph-compound exponential distribution	$P(y;\alpha,\beta,k) = e^{1-\beta^{y^{\alpha}}-k\ln(1+y)} - e^{1-\beta^{(y+1)^{\alpha}}-k\ln(y+2)}$, $y = 0,\ 1,\ 2,\ \cdots;\ \alpha, k > 0,\ \beta > 1.$
$c = 2$	Discrete Ph-compound Rayleigh distribution	$P(y;\alpha,\beta,k) = e^{1-\beta^{y^{\alpha}}-k\ln(1+y^{2})} - e^{1-\beta^{(y+1)^{\alpha}}-k\ln[1+(y+1)^{2}]}$, $y = 0,\ 1,\ 2,\ \cdots;\quad \alpha, k > 0,\ \beta > 1.$
$k = 1$	Discrete Ph-log logistic	$P(y;\alpha,\beta,c) = e^{1-\beta^{y^{\alpha}}-k\ln(1+y^{c})} - e^{1-\beta^{(y+1)^{\alpha}}-k\ln[1+(y+1)^{c}]}$, $y = 0,\ 1,\ 2,\ \cdots;\quad \alpha, c > 0,\ \beta > 1.$
$c = 1$ and $k = 1$	Discrete Ph-standard Lomax	$P(y;\alpha,\beta) = e^{1-\beta^{y^{\alpha}}-\ln(1+y)} - e^{1-\beta^{(y+1)^{\alpha}}-\ln(y+2)}$, $y = 0,\ 1,\ 2,\ \cdots;\quad \alpha > 0,\ \beta > 1.$

3. Graphical Explanation

In this section, graphical explanation of the PMF, HRF and AHRF of the DPh-BXII distribution is presented to show the diversity in shapes and flexibility of the proposed distribution.

Figure 1 displays the PMF of the DPh-BXII distribution for some selected values of the parameters, from which one can observe that the PMF of the DPh-BXII distribution is decreasing, unimodal and decreasing–unimodal. Figure 2 shows the HRF of the DPh-BXII distribution for some selected values of the parameters. While the AHRF of the DPh-BXII distribution for some selected values of the parameters is displayed in Figure 3. The HRF and AHRF of the DPh-BXII distribution exhibit different and important shapes, which are decreasing, bathtub (Vtub), unimodal and modified bathtub shapes. From Figures 1–3, the flexibility and diversity in the shapes of the PMF, HRF and AHRF of the DPh-BXII distribution can be demonstrated. Therefore, the DPh-BXII distribution can provide a convenient fit for several types of discrete data.

Figure 1. Plots of PMF of DPh-BXII distribution at some selected values of the parameters.

To examine the unimodality of the DPh-BXII distribution, the log-concave property can be used. The log-concave property is satisfied for a PMF if the following inequality is achieved:

$$P^2(y) \geq P(y-1)P(y+1). \tag{16}$$

For the DPh-BXII distribution, the log-concave property is assessed numerically through R programming at different combinations of the parameters. Figure 4 provides graphical plots of log-concavity regions for different combinations of the parameters. From Figure 4, for $\alpha > 1$ and as β grows larger than 1, at any values of c and k, the log-concave property of DPh-BXII's PMF is satisfied. This result is consistent with the plots of the PMF given in Figure 1; for more details, see [43,44].

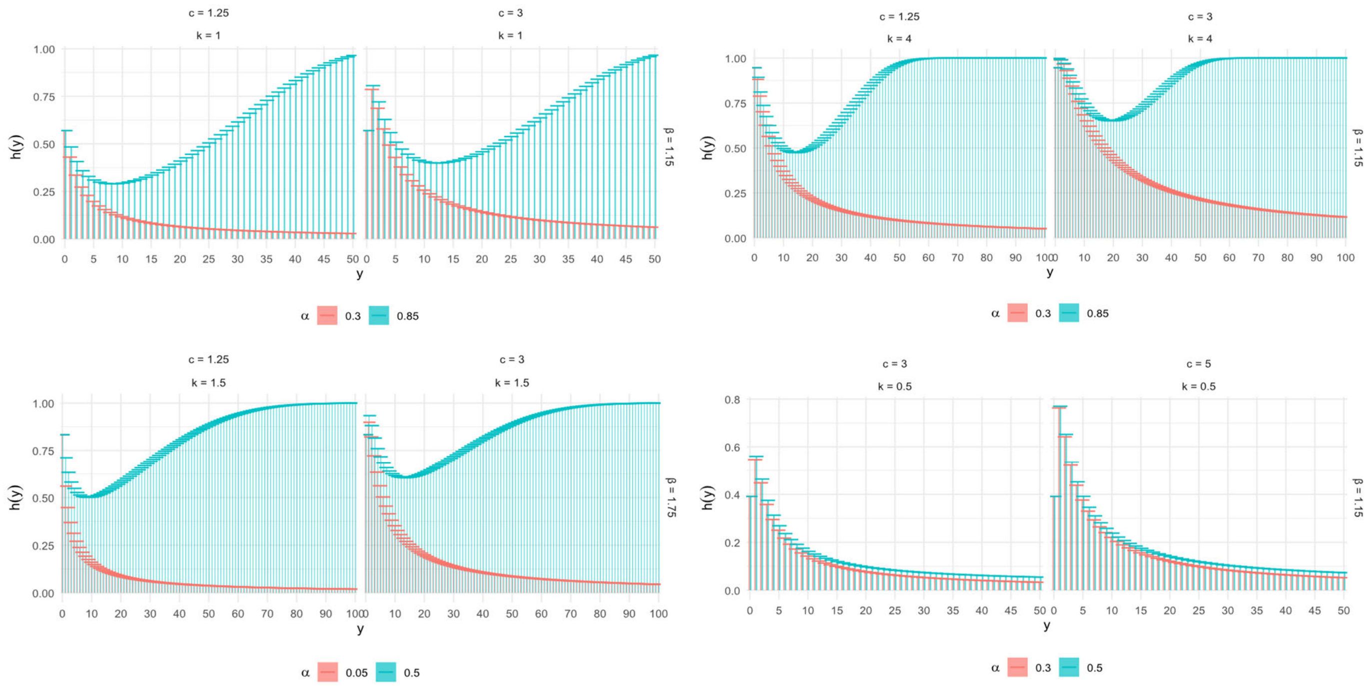

Figure 2. Plots of HRF of DPh-BXII distribution at some selected values of the parameters.

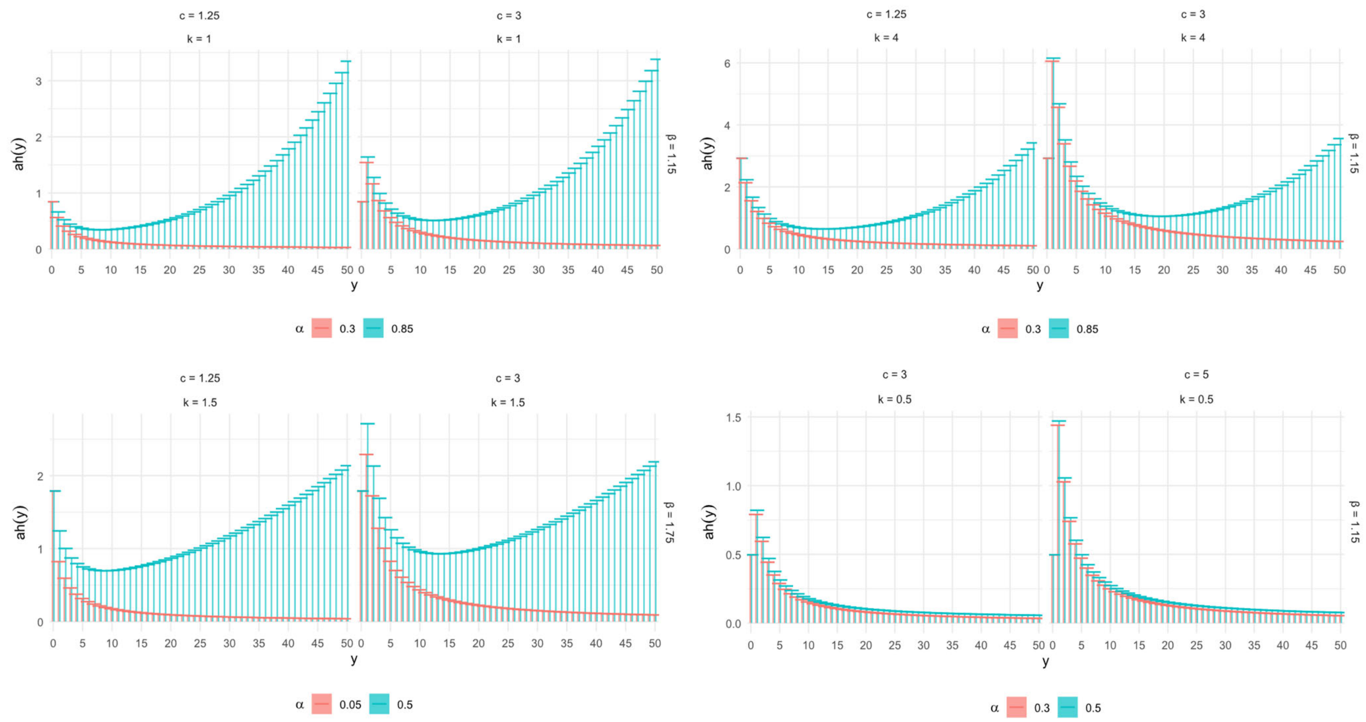

Figure 3. Plots of AHRF of DPh-BXII distribution at some selected values of the parameters.

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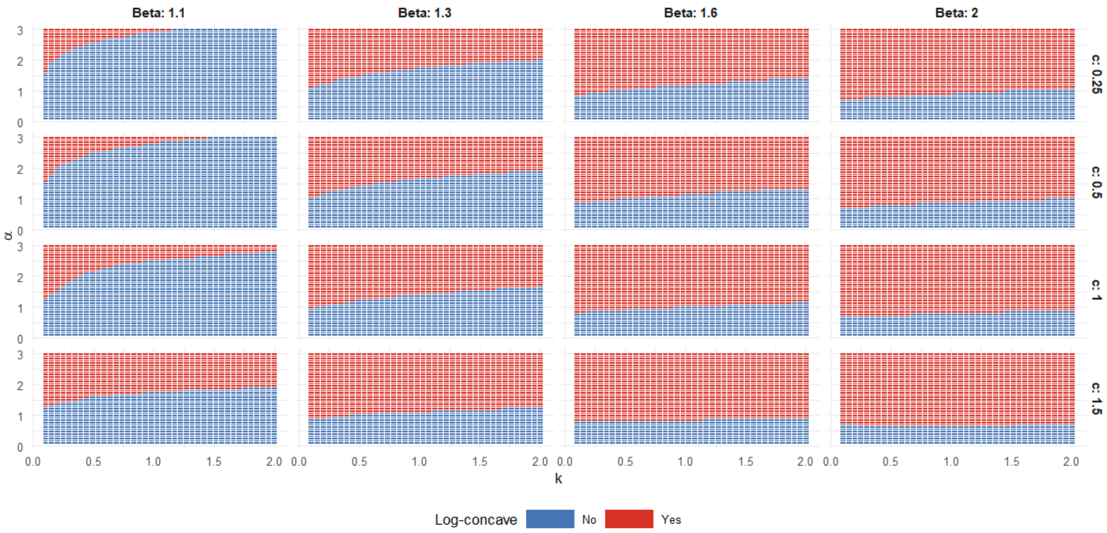


Figure 4. Plot of log-concavity regions of DPh-BXII distribution for different combinations of parameters.

4. Statistical Properties

This section is intended for studying the main properties of the DPh-BXII distribution, including the quantile, central and non-central moments, moment and probability generating functions, and the r^{th} incomplete moment. Also, it is for discussing the *mean residual life* (MRL) and *mean past life* (MPL). Furthermore, this section considers some entropies and extropy for the proposed model. Finally, it explores the order statistics.

4.1. Quantile Function

For a discrete RV with the DPh-BXII distribution, the quantile, $y_{\mathcal{q}}$, can be reached as the flooring value, $\lfloor . \rfloor$, of the root of the following non-linear equation:

$$\beta^{(y_{\mathcal{q}}+1)^{\alpha}} + k\ln\left[1 + (y_{\mathcal{q}} + 1)^{c}\right] = 1 - ln(1 - \mathcal{q}), \tag{17}$$

where $\lfloor . \rfloor$ is the greatest integer smaller than or equal to its argument.

Table 2 presents numerical values of the 1st quartile, median and 3rd quartile, denoted respectively by $y_{0.25}$, $y_{0.5}$ and $y_{0.75}$, besides the mode, y_{mod}, for different combinations of the parameters of the DPh-BXII distribution. From this table, the DPh-BXII distribution can be used for modeling zero inflated data.

Table 2. Some quartiles and the mode of DPh-BXII distribution for different combinations of the parameters.

α	β	c	k	$y_{0.25}$	$y_{0.5}$	$y_{0.75}$	y_{mod}
0.25	1.1	0.5	0.5	0	3	59	0
0.25	1.1	1.5	0.5	0	1	4	0
0.25	1.3	0.5	0.5	0	1	12	0
0.25	1.75	0.5	0.5	0	0	2	0
0.5	1.3	0.85	0.5	0	1	3	0
1.5	1.1	1.5	0.5	0	1	2	0
1.5	1.3	1.5	0.5	0	1	1	0
1.5	1.75	1.5	1.5	0	0	0	0
2.6	1.1	2	0.5	0	1	1	1
2.6	1.3	0.85	0.5	0	1	1	1
2.6	1.3	1.5	1.5	0	0	1	0
2.6	1.3	1.5	1.75	0	0	0	0
3	1.1	0.5	0.5	0	1	1	1
3	1.3	0.5	0.5	0	1	1	1

4.2. Moments

The r^{th} non-central moment of the DPh-BXII distribution is derived as follows:

$$\acute{\mu}_r = \sum_{y=1}^{\infty} \left[y^r - (y-1)^r\right] S(y;\underline{\theta}) = \sum_{y=1}^{\infty} \left[y^r - (y-1)^r\right] e^{1-\beta^{y^\alpha} - k\ln(1+y^c)}, \quad r = 0,1,2,\cdots. \tag{18}$$

Substituting $r = 1$ and 2 into (18), the mean, μ, and the second non-central moment can be obtained. Subsequently, the *variance* (Var) of the DPh-BXII distribution can be obtained as follows:

$$Var(Y) = \mu_2 = \acute{\mu}_2 - \mu^2 = \sum_{y=1}^{\infty} (2y-1) e^{1-\beta^{y^\alpha} - k\ln(1+y^c)} - \left[\sum_{y=1}^{\infty} e^{1-\beta^{y^\alpha} - k\ln(1+y^c)}\right]^2, \tag{19}$$

Similarly, the *coefficient of dispersion* (CD), *coefficient of skewness* (CS) and *coefficient of kurtosis* (CK) are given, respectively, as

$$CD = \frac{\mu_2}{\mu}, \qquad CS = \frac{\acute{\mu}_3 - 3\mu\acute{\mu}_2 + 2\mu^3}{(\acute{\mu}_2 - \mu^2)^{3/2}} \qquad and \qquad Ck = \frac{\acute{\mu}_4 - 4\mu\acute{\mu}_3 + 6\mu^2\acute{\mu}_2 - 3\mu^4}{(\acute{\mu}_2 - \mu^2)^2}, \tag{20}$$

where μ, $\acute{\mu}_2$, $\acute{\mu}_3$ and $\acute{\mu}_4$ can be obtained, respectively, by setting $r = 1,2,3$ and $r = 4$ in (18), and μ_2 is obtained in (19).

Numerical results of the first four non-central moments, Var, CD, CS and CK, for the DPh-BXII distribution at different combinations of the parameters are provided in Table 3.

Table 3. The first four non-central moments, Var, CD, CS and CK, for DPh-BXII distribution for different combinations of the parameters.

α	β	c	k	μ	$\acute{\mu}_2$	$\acute{\mu}_3$	$\acute{\mu}_4$	μ_2	CD	CS	CK
0.25	1.3	0.5	0.5	7.4453	341.5604	21,843.9889	1,604,389.9193	286.1272	38.4304	3.1076	12.9259
0.5	1.3	0.5	0.5	3.8563	65.2788	1741.2129	60,994.0789	50.4078	13.0716	3.0756	15.4654
1.5	1.3	0.5	0.5	0.7714	1.3345	2.6665	5.9595	0.7393	0.9584	0.7808	2.6231
2.6	1.3	0.5	0.5	0.5368	0.5626	0.6144	0.7179	0.2745	0.5114	0.1228	1.6235
3	1.3	0.5	0.5	0.5243	0.5253	0.5273	0.5314	0.2504	0.4776	−0.0855	1.0391
1.5	1.1	1.5	0.5	1.3892	4.1066	15.1466	64.4817	2.1766	1.5668	1.0568	3.5232
1.5	1.3	1.5	0.5	0.7161	1.1454	2.1402	4.5443	0.6326	0.8835	0.8226	2.8716
1.5	1.75	1.5	0.5	0.3447	0.3660	0.4087	0.4941	0.2472	0.7172	0.9122	2.4406
1.5	2	1.5	0.5	0.2613	0.2636	0.2681	0.2773	0.1953	0.7475	1.1265	2.3865
1.5	3	1.5	0.5	0.0957	0.0957	0.0957	0.0957	0.0865	0.9043	2.7487	8.5555
1.5	1.75	0.5	1.75	0.1449	0.1538	0.1717	0.2074	0.1328	0.9167	2.2909	7.1388
1.5	1.75	0.85	1.75	0.1439	0.1508	0.1645	0.1920	0.1301	0.9040	2.2473	6.7870
1.5	1.75	1	1.75	0.1435	0.1496	0.1618	0.1862	0.1290	0.8991	2.2299	6.6442
1.5	1.75	1.5	1.75	0.1424	0.1464	0.1544	0.1703	0.1261	0.8856	2.1790	6.2207
1.5	1.75	2	1.75	0.1417	0.1442	0.1492	0.1592	0.1241	0.8759	2.1404	5.8940
1.5	1.75	1.5	0.5	0.3447	0.3660	0.4087	0.4941	0.2472	0.7172	0.9122	2.4406
1.5	1.75	1.5	1.5	0.1698	0.1754	0.1865	0.2088	0.1465	0.8630	1.9071	5.1219
1.5	1.75	1.5	1.75	0.1424	0.1464	0.1544	0.1703	0.1261	0.8856	2.1790	6.2207
1.5	1.75	1.5	2	0.1195	0.1224	0.1281	0.1395	0.1081	0.9043	2.4655	7.5431
1.5	1.75	1.5	3	0.0594	0.0602	0.0617	0.0646	0.0566	0.9531	3.8100	15.9683

From Table 3, the following is clear:

- For given values of β, c and k, as α increases, the first four non-central moments and Var decrease, the CD varies from over- to under-dispersion, the CS varies from right- (positive) to left (negative)-skewed, and the CK shifts from leptokurtic to platykurtic.
- For given values of α, c and k, as β increases, the first four non-central moments and Var decrease and the CD changes between over- and under-dispersion. The CS is right (positive)-skewed. Also, the CK shifts between leptokurtic and platykurtic.
- For given values of α, β and k, as c increases, the first four non-central moments and Var decrease, the CD is under-dispersed, the CS is right skewed and tends to be approximately symmetric, and the CK can be platykurtic and leptokurtic.

- For given values of α, β and c, as k increases, the first four non-central moments and Var decrease, the CD is under-dispersed and tends to be less dispersed, the CS is right skewed with an increasing right tail, and the CK shifts from platykurtic to leptokurtic.

Summarizing, the four parameters of the DPh-BXII distribution provide a significant influence on the location, Var, dispersion, skewness and kurtosis of the distribution. This reflects the flexibility of the proposed model in accommodating diverse distributional shapes.

4.3. Moment and Probability Generating Functions

The moment generating function, denoted by $M_Y(t)$, of a discrete RV, Y, with the DPh-BXII distribution can be obtained as follows:

$$M_Y(t) = E\left(e^{ty}\right)f = \sum\nolimits_{r=0}^{\infty} \frac{t^r}{r!} \sum\nolimits_{y=0}^{\infty} y^r P(y;\underline{\theta}) = \sum\nolimits_{r=0}^{\infty} \frac{t^r}{r!} \acute{\mu}_r. \tag{21}$$

Therefore, the moment generating function can be obtained by substituting (18) into (21) as

$$M_Y(t) = \sum\nolimits_{r=0}^{\infty} \sum\nolimits_{y=1}^{\infty} \frac{t^r\left[y^r - (y-1)^r\right]}{r!} e^{1-\beta^{y^\alpha} - k\ln(1+y^c)}, \qquad r = 0,\ 1,\ 2,\ \cdots. \tag{22}$$

Meanwhile, the probability generating function, denoted by $G_Y(t)$, of DPh-BXII's RV, Y, is as follows:

$$G_Y(t) = E\left(t^y\right) = \sum\nolimits_{y=0}^{\infty} t^y P(y;\underline{\theta}) = \sum\nolimits_{y=0}^{\infty} t^y \left[e^{1-\beta^{y^\alpha} - k\ln(1+y^c)} - e^{1-\beta^{(y+1)^\alpha} - k\ln[1+(y+1)^c]}\right]. \tag{23}$$

4.4. Incomplete Moments

The r^{th} incomplete moment of a discrete RV, Y, with the DPh-BXII distribution is

$$\mu_r(t) = \sum\nolimits_{y=0}^{t} y^r P(y;\underline{\theta}) = \sum\nolimits_{y=0}^{t} y^r \left[e^{1-\beta^{y^\alpha} - k\ln(1+y^c)} - e^{1-\beta^{(y+1)^\alpha} - k\ln[1+(y+1)^c]}\right], \qquad r = 0, 1, 2, \cdots. \tag{24}$$

Substituting $r = 1$ into (24), the first incomplete moment, $\mu(t)$, can be obtained.

4.5. Mean Residual Life and Mean Past Life

In reliability and survival analysis, there are several measures that quantify the ageing behavior of a component or a system of components. MRL and MPL are helpful tools that model and analyze the burn-in and maintenance strategies. The MRL measures the expected value of the remaining lifetime $Y - t$ given that Y has survived at least t units of time.

For a discrete RV, Y, with the DPh-BXII distribution, the MRL is defined as

$$\varphi(t) = E(Y - t|Y > t) = \frac{1}{S(t;\underline{\theta})} \sum\nolimits_{y=t+1}^{\infty} S(y;\underline{\theta}) = \frac{\sum_{y=t+1}^{\infty} e^{-\beta^{y^\alpha} - k\ln(1+y^c)}}{e^{-\beta^{t^\alpha} - k\ln(1+t^c)}}, \quad y, t = 0,\ 1,\ 2,\ \cdots. \tag{25}$$

When $t = 0$, the MRL reaches the mean of the DPh-BXII distribution, μ.

Meanwhile, the MPL, also referred to as the mean inactivity time, mean waiting time, or mean reversed residual life function, is denoted by $\psi(t)$. It quantifies the expected elapsed time since the failure of a component or a system of components, conditional on the failure having occurred sometimes before t. For a discrete RV with the DPh-BXII distribution, the MPL is defined as

$$\psi(t) = E[t - Y|Y < t] = \frac{\sum_{y=1}^{t} F(y-1;\underline{\theta})}{F(t-1;\underline{\theta})} = \frac{\sum_{y=1}^{t}\left\{1 - e^{1-\beta^{(y-1)^\alpha} - k\ln[1+(y-1)^c]}\right\}}{1 - e^{1-\beta^{(t-1)^\alpha} - k\ln[1+(t-1)^c]}}, \quad y = 0,\ 1,\ 2,\ \cdots,\ t = 1,\ 2,\ 3,\ \cdots, \tag{26}$$

where $\psi(0) = 0$. For more details about the MRL and MPL in discrete circumstances, see [45].

4.6. Entropy and Extropy Measures

Entropy measures the amount of uncertainty, randomness or variation that is associated with a given RV. It is also used in information theory to measure the amount of information lost by a given system. In this subsection, some important entropies are derived for a DPh-BXII RV. Additionally, extropy is a complementary dual concept of entropy firstly introduced by [46]. It is, like entropy, used as a measure of uncertainty associated with given RVs, and it can be used as a measure of information.

Ref. [47] presented the well-known Shannon entropy. For DPh-BXII's RV, Shannon entropy is derived as

$$\begin{aligned} H(Y;\underline{\theta}) &= -E(\ln[P(y)]) = -\textstyle\sum_y P(y)\ln[P(y)] \\ &= -\textstyle\sum_{y=0}^{\infty}\left[e^{1-\beta^{y^\alpha}-k\ln(1+y^c)} - e^{1-\beta^{(y+1)^\alpha}-k\ln[1+(y+1)^c]}\right]\ln\left\{e^{1-\beta^{y^\alpha}-k\ln(1+y^c)} - e^{1-\beta^{(y+1)^\alpha}-k\ln[1+(y+1)^c]}\right\}, \end{aligned} \tag{27}$$

As a generalization of Shannon entropy, Rényi entropy was introduced by [48]. For the DPh-BXII distribution, Rényi entropy is derived as

$$I_1(Y;\delta,\underline{\theta}) = \tfrac{1}{1-\delta}\ln\left[\textstyle\sum_y P^\delta(y)\right] = \tfrac{1}{1-\delta}\ln\left\{\textstyle\sum_{y=0}^{\infty}\left[e^{1-\beta^{y^\alpha}-k\ln(1+y^c)} - e^{1-\beta^{(y+1)^\alpha}-k\ln[1+(y+1)^c]}\right]^\delta\right\}, \quad \delta \neq 1, \delta > 0. \tag{28}$$

As $\delta \to 1$, Rényi entropy tends to Shannon entropy.

As another entropy measure, Tsallis entropy (also called q-entropy) was presented by [49]. Tsallis entropy for a discrete RV with the DPh-BXII distribution is given by

$$I_2(Y;\delta,\underline{\theta}) = \tfrac{1}{\delta-1}\left[1 - \textstyle\sum_y P^\delta(y)\right] = \tfrac{1}{\delta-1}\left\{1 - \textstyle\sum_{y=0}^{\infty}\left[e^{1-\beta^{y^\alpha}-k\ln(1+y^c)} - e^{1-\beta^{(y+1)^\alpha}-k\ln[1+(y+1)^c]}\right]^\delta\right\}, \ \delta \neq 1, \delta > 0. \tag{29}$$

The Boltizmann–Gibbs statistic can be obtained from Tsallis entropy as $\delta \to 1$.

On the other hand, extropy for DPh-BXII's RV is given by

$$\begin{aligned} J(Y;\underline{\theta}) = -\textstyle\sum_y [1-P(y)]\ln[1-P(y)] = -\textstyle\sum_{y=0}^{\infty}&\left[e^{1-\beta^{(y+1)^\alpha}-k\ln[1+(y+1)^c]} - e^{1-\beta^{y^\alpha}-k\ln(1+y^c)} + 1\right]\times \\ &\ln\left\{e^{1-\beta^{(y+1)^\alpha}-k\ln[1+(y+1)^c]} - e^{1-\beta^{y^\alpha}-k\ln(1+y^c)} + 1\right\}, \end{aligned} \tag{30}$$

Numerical values of the considered entropies (Shannon, Rényi, Tsallis) and the extropy of the discrete RV with the DPh-BXII distribution at different values of the parameters, $\underline{\theta} = (\alpha,\ \beta,\ c,\ k)$, are presented in Table 4. Rényi and Tsallis entropies are computed at different values of the entropy parameter, $\delta = (0.5,\ 1.5,\ 2)$.

Table 4. Numerical results of Shannon, Rényi, Tsallis, and the extropy of DPh-BXII's RV for various combinations of the distribution parameters and at different values of Rényi and Tsallis parameter.

α	β	c	k	$H(Y;\underline{\theta})$	$I_1(Y;\delta,\underline{\theta})$			$I_2(Y;\delta,\underline{\theta})$			$J(Y;\underline{\theta})$
					$\delta = 0.5$	$\delta = 1.5$	$\delta = 2$	$\delta = 0.5$	$\delta = 1.5$	$\delta = 2$	
0.25	1.3	0.5	0.5	3.0346	5.4404	1.8765	1.4316	28.3661	1.2174	0.7611	0.8562
0.5	1.3	0.5	0.5	2.2012	3.0816	1.6726	1.3767	7.3365	1.1334	0.7476	0.8493
1.5	1.3	0.5	0.5	1.1408	1.2560	1.0761	1.0314	1.7478	0.8322	0.6435	0.7907
3	1.3	0.5	0.5	0.6960	0.7235	0.6929	0.6919	0.8717	0.5856	0.4994	0.6924
1.5	1.1	1.5	0.5	1.5933	1.7706	1.4881	1.4158	2.8474	1.0496	0.7573	0.8644
1.5	1.75	1.5	0.5	0.6842	0.7950	0.6336	0.6011	0.9763	0.5431	0.4518	0.6412
1.5	2	1.5	0.5	0.5805	0.6770	0.5273	0.4870	0.8057	0.4635	0.3855	0.5735
1.5	3	1.5	0.5	0.3155	0.4627	0.2341	0.1900	0.5206	0.2209	0.1731	0.3155
1.5	1.75	0.5	1.75	0.4255	0.6190	0.3313	0.2779	0.7255	0.3053	0.2426	0.4064
1.5	1.75	0.85	1.75	0.4219	0.6090	0.3301	0.2776	0.7119	0.3043	0.2424	0.4063
1.5	1.75	1	1.75	0.4205	0.6048	0.3297	0.2774	0.7062	0.3040	0.2423	0.4062
1.5	1.75	1.5	1.75	0.4162	0.5911	0.3285	0.2770	0.6878	0.3029	0.2420	0.4061
1.5	1.75	1.5	1.5	0.4653	0.6307	0.3800	0.3273	0.7415	0.3461	0.2791	0.4516
1.5	1.75	1.5	2	0.3708	0.5529	0.2829	0.2340	0.6369	0.2638	0.2086	0.3633
1.5	1.75	1.5	3	0.2266	0.4166	0.1517	0.1178	0.4631	0.1461	0.1112	0.2244

From Table 4, the following is clear:

- Shannon, Rényi, Tsallis, and the extropy exhibit a decreasing pattern with increasing values of the parameter $\alpha,\ \beta,\ c$ or k while the other parameters are at given values. Thus, increases in any of the model parameters are associated with lower values of these considered entropies and extropy measures.
- Rényi and Tsallis entropies present a decreasing pattern as the entropy parameter δ increases.

Overall, the results indicate that larger values of the model parameters are associated with less uncertainty and reduced randomness in the DPh-BXII' RV, as reflected by the decrease in the considered entropy and extropy measures at different combinations of the parameters.

4.7. Order Statistics

Let $Y_1,\ Y_2,\ \ldots,\ Y_n$ be independently and identically distributed RVs with the DPh-BXII distribution, where $Y_{(1)} \leq\ Y_{(2)} \leq\ \ldots \leq\ Y_{(n)}$ are their corresponding order statistics. Then, the CDF and the PMF of the s^{th} order statistic are as follows:

$$F_{s:n}(y;\underline{\theta}) = \sum_{\iota=s}^{n} \sum_{j=0}^{\iota} \binom{n}{\iota}\binom{\iota}{j}(-1)^j e^{(n+j-\iota)[1-\beta^{y_{(s)}^{\alpha}} - k\ln(1+y_{(s)}^{c})]},\quad y_{(s)} = 0,\ 1,\ 2,\ \cdots, \tag{31}$$

and

$$f_{s:n}(y;\underline{\theta}) = \sum_{\iota=s}^{n} \binom{n}{\iota}\Bigg\{\sum_{j=0}^{\iota} \binom{\iota}{j}(-1)^j e^{(n+j-\iota)[1-\beta^{y_{(s)}^{\alpha}} - k\ln(1+y_{(s)}^{c})]} - \sum_{j=0}^{\iota} \binom{\iota}{j}(-1)^j e^{(n+j-\iota)[1-\beta^{(y_{(s)}-1)^{\alpha}} - k\ln[1+(y_{(s)}-1)^{c}]]}\Bigg\},\quad y_{(s)} = 0,\ 1,\ 2,\ \cdots, \tag{32}$$

As special cases of (32), the PMF of the first-order statistics and the PMF of the last-order statistic can be obtained by setting $s = 1$ and $s = n$, respectively. For more details, see [50].

5. Maximum Likelihood Estimation

In this section, the ML estimation of DPh-BXII's parameters, SF, HRF, AHRF and RHRF, are discussed. Also, the assumption of an approximate normality distribution of the ML estimators is considered to derive the ACIs of the parameters, SF, HRF, AHRF and RHRF. Moreover, the parametric *percentile bootstrap* (PB) method is considered. Additionally, point and interval estimation of the considered entropy measures and the dual entropy is discussed.

Let $y_{(1)},\ y_{(2)},\ \ldots,\ y_{(n)}$ be a discrete random sample from the DPh-BXII distribution with parameter vector $\underline{\theta} = (\alpha, \beta, c, k)$, and then the likelihood function is defined as

$$L\left(\underline{\theta};\underline{y}\right) = n!\prod_{i=1}^{n}\left[e^{1-\beta^{y_{(i)}^{\alpha}} - kln(1+y_{(i)}^{c})} - e^{1-\beta^{(y_{(i)}+1)^{\alpha}} - kln[1+(y_{(i)}+1)^{c}]}\right], \tag{33}$$

5.1. Point Estimation

Point estimators for the parameters of the DPh-BXII distribution can be obtained using the log-likelihood function, given by

$$\ell = \ln(n!) + \sum_{i=1}^{n}\ln\left[e^{1-\beta^{y_{(i)}^{\alpha}} - k\ln(1+y_{(i)}^{c})} - e^{1-\beta^{(y_{(i)}+1)^{\alpha}} - k\ln[1+(y_{(i)}+1)^{c}]}\right], \tag{34}$$

The derivatives of the log-likelihood function with respect to α, β, k and c are given below:

$$\frac{\partial \ell}{\partial \alpha} = \sum_{i=1}^{n} \frac{1}{e^{1-\beta^{y_{(i)}^{\alpha}} - k\ln\left(1+y_{(i)}^{c}\right)} - e^{1-\beta^{(y_{(i)}+1)^{\alpha}} - k\ln\left[1+(y_{(i)}+1)^{c}\right]}} \left\{ \beta^{(y_{(i)}+1)^{\alpha}} \left(y_{(i)}+1\right)^{\alpha} \ln\left(y_{(i)}+\beta+1\right) \times e^{1-\beta^{(y_{(i)}+1)^{\alpha}} - k\ln\left[1+(y_{(i)}+1)^{c}\right]} - \beta^{y_{(i)}^{\alpha}} y_{(i)}^{\alpha} \ln\left(y_{(i)}+\beta\right) e^{1-\beta^{y_{(i)}^{\alpha}} - k\ln\left(1+y_{(i)}^{c}\right)} \right\}, \tag{35}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{1}{\beta} \sum_{i=1}^{n} \frac{1}{e^{1-\beta^{y_{(i)}^{\alpha}} - k\ln\left(1+y_{(i)}^{c}\right)} - e^{1-\beta^{(y_{(i)}+1)^{\alpha}} - k\ln\left[1+(y_{(i)}+1)^{c}\right]}} \left\{ \beta^{(y_{(i)}+1)^{\alpha}} \left(y_{(i)}+1\right)^{\alpha} e^{1-\beta^{(y_{(i)}+1)^{\alpha}} - k\ln\left[1+(y_{(i)}+1)^{c}\right]} - \beta^{y_{(i)}^{\alpha}} y_{(i)}^{\alpha} e^{1-\beta^{y_{(i)}^{\alpha}} - k\ln\left(1+y_{(i)}^{c}\right)} \right\}, \tag{36}$$

$$\frac{\partial \ell}{\partial c} = k \sum_{i=1}^{n} \frac{1}{e^{1-\beta^{y_{(i)}^{\alpha}} - k\ln\left(1+y_{(i)}^{c}\right)} - e^{1-\beta^{(y_{(i)}+1)^{\alpha}} - k\ln\left[1+(y_{(i)}+1)^{c}\right]}} \left\{ \frac{\left(y_{(i)}+1\right)^{c} \ln\left(y_{(i)}+1\right)}{1+\left(y_{(i)}+1\right)^{c}} e^{1-\beta^{(y_{(i)}+1)^{\alpha}} - k\ln\left[1+(y_{(i)}+1)^{c}\right]} - \frac{y_{(i)}^{c} \ln\left(y_{(i)}\right)}{1+y_{(i)}^{c}} e^{1-\beta^{y_{(i)}^{\alpha}} - k\ln\left(1+y_{(i)}^{c}\right)} \right\}, \tag{37}$$

and

$$\frac{\partial \ell}{\partial k} = \sum_{i=1}^{n} \frac{1}{e^{1-\beta^{y_{(i)}^{\alpha}} - k\ln\left(1+y_{(i)}^{c}\right)} - e^{1-\beta^{(y_{(i)}+1)^{\alpha}} - k\ln\left[1+(y_{(i)}+1)^{c}\right]}} \left\{ \ln\left[1+\left(y_{(i)}+1\right)^{c}\right] e^{1-\beta^{(y_{(i)}+1)^{\alpha}} - k\ln\left[1+(y_{(i)}+1)^{c}\right]} - \ln\left(1+y_{(i)}^{c}\right) e^{1-\beta^{y_{(i)}^{\alpha}} - k\ln\left(1+y_{(i)}^{c}\right)} \right\}, \tag{38}$$

The MLEs of DPh-BXII's parameters are obtained by equating Equations (35) and (36) to zero and using a numerical algorithm.

When applying the invariance property of the ML estimators, the ML estimators of the SF, HRF, AHRF and RHRF can be obtained. Also, the ML estimators of the entropy measures, Shannon, Rényi and Tsallis, and the dual entropy (extropy) can be derived by applying the invariance property of the ML estimators.

5.2. Interval Estimation

Since the ML estimators, $\underline{\hat{\theta}} = \left(\hat{\alpha}, \hat{\beta}, \hat{k}, \hat{c}\right)$, cannot be achieved in closed form and a numerical algorithm is needed, the exact distribution of the ML estimators cannot be determined, and *confidence intervals* (CIs) of the parameters of the DPh-BXII distribution cannot be obtained. So, in this subsection, ACIs of the parameters, SF, HRF, AHRF and RHRF, are derived by applying the asymptotic normality of the ML estimators. Also, the ACIs of the considered entropy and extropy measures are given. Moreover, the parametric PB method for computing CIs is considered.

5.2.1. Asymptotic Confidence Interval

When applying the property of asymptotic normality of the ML estimators, the ML estimators, $\underline{\hat{\theta}} = \left(\hat{\alpha}, \hat{\beta}, \hat{k}, \hat{c}\right)$, of DPh-BXII's parameters are asymptotically distributed with means (α, β, c, k), and Vars can be obtained from the asymptotic Var-covariance matrix. So, the ACIs of the parameters, $\underline{\theta} = (\alpha, \beta, c, k)$, can be achieved rather than the exact CIs. The asymptotic Var-covariance matrix can be derived as the inverse of the asymptotic Fisher information matrix, defined as

$$\widetilde{I}(\underline{\theta}) \simeq -\left[\frac{\partial^2 \ell}{\partial\theta_{\imath}\,\partial\theta_{\jmath}}\right], \qquad \imath, \jmath = 1,\ 2,\ 3,\ 4. \tag{39}$$

Subsequently, the $(1-\omega)100\%$ bounds of the ACIs of the parameters $\underline{\theta} = (\alpha, \beta, k, c)$ are given by

$$\hat{\theta}_{\imath} \pm Z_{(1-\frac{\omega}{2})}\sqrt{\widetilde{var}(\hat{\theta}_{\imath})}, \qquad \imath = 1,\ 2,\ 3,\ 4, \tag{40}$$

where $\hat{\theta}_1 = \alpha$, $\hat{\theta}_2 = \beta$, $\hat{\theta}_3 = c$, $\hat{\theta}_4 = k$, $\widetilde{var}(\hat{\theta}_{\imath})$ is the asymptotic Var of $\hat{\theta}_{\imath}$ and $Z_{(1-\frac{\omega}{2})}$ is the $(1-\omega)100\%$ standard normal percentage point.

To derive ACIs for the SF, HRF, AHRF, RHRF and the derived entropy and extropy measures of the DPh-BXII distribution, Vars of their ML estimators must be obtained first. Therefore, the delta method discussed in [51] and applied by [52–54] can be used to derive the asymptotic Vars. Therefore, for a given function, $\phi(y;\underline{\theta})$, the asymptotic Vars, denoted by $\widetilde{var}(\hat{\phi}(y;\hat{\underline{\theta}}))$, can be obtained as

$$\widetilde{var}(\hat{\phi}(y;\hat{\underline{\theta}})) = [\nabla\phi(y;\underline{\theta})]^T\, \widetilde{I}^{-1}(\underline{\theta})[\nabla\phi(y;\underline{\theta})]\Big|_{\hat{\underline{\theta}}}, \tag{41}$$

where $[\nabla\phi(y;\underline{\theta})]$ is the gradient of $\phi(y;\underline{\theta})$ with respect to α, β, c and k.

Thus, the $(1-\omega)100\%$ bound of the ACIs of $\phi(y;\underline{\theta})$ is

$$\hat{\phi}(y;\hat{\underline{\theta}}) \pm Z_{(1-\frac{\omega}{2})}\sqrt{\widetilde{var}(\hat{\phi}(y;\hat{\underline{\theta}}))}. \tag{42}$$

The ACIs of the SF, HRF, AHRF and RHRF as well as Shannon, R*é*nyi, Tsallis and extropy for the DPh-BXII distribution are numerically computed based on the delta method.

5.2.2. Bootstrap Confidence Interval

A parametric bootstrap CI delivers more information about the population parameter than its point estimator. Also, in small sample sizes, the performance of the ACI is unsatisfactory. So, the parametric PB method is used to provide CIs of DPh-BXII's parameters, SF, HRF, AHRF and RHRF. Moreover, CIs of the considered entropy and extropy measures are provided. The PBCI is constructed based on the idea of [55] using the following algorithm:

- Based on a generated sample, $\underline{y} = \left(y_{(1)},\ y_{(2)},\ \ldots,\ y_{(n)}\right)$, compute the MLEs of the parameters, $\hat{\underline{\theta}} = \left(\hat{\alpha}, \hat{\beta}, \hat{k}, \hat{c}\right)$.
- Repeat the previous two steps N-boot times to obtain $\hat{\theta}_{w}, w = 1,\ 2,\ \cdots,\ N-boot$ for each parameter of the distribution.
- Arrange $\hat{\theta}_{w}$, $w = 1, 2, \cdots, N-boot$ and $\hat{\phi}_{w}(y;\hat{\underline{\theta}}), w = 1, 2, \cdots, N-boot$ in ascending order and find $\hat{\theta}_{(1)}, \hat{\theta}_{(2)}, \cdots, \hat{\theta}_{(N-boot)}$.
- Let $G_1(q) = P(\hat{\theta} \le q)$ be the CDF of $\hat{\theta}$, and define $\hat{\theta}_{boot} = G^{-1}(q)$ for a given q. The approximate PB $(1-\omega)100\%$ CI's *lower limit* (LL) and *upper limit* (UL) of θ are, respectively,

$$LL = \hat{\theta}_{boot}\left(\frac{\omega}{2}\right) \qquad and \qquad UL = \hat{\theta}_{boot}\left(1-\frac{\omega}{2}\right). \tag{43}$$

6. Simulation Study

In this section, the performance of the MLEs of the parameters, SF, HRF, AHRF and RHRF, as well as Shannon, Rényi, Tsallis and extropy of the DPh-BXII distribution, is examined throughout the conduction of a simulation study using the following steps:

1. Set initial values of the parameters:
   I. $(\alpha = 1.5,\ \beta = 1.15,\ c = 0.85,\ k = 0.15)$;
   II. $(\alpha = 0.5,\ \beta = 1.15,\ c = 1.25,\ k = 0.5)$;
   III. $(\alpha = 0.15,\ \beta = 1.3,\ c = 1.5,\ k = 0.25)$.
2 Specify different sample sizes $(n = 30,\ 60,\ 100)$ to generate random samples from a uniform $(0, 1)$ distribution.
3. Define the quantile function of the DPh-BXII distribution given in (17) and the log-likelihood function presented in (34) in R programming language.
4. Using a number of replications NR = 10,000, the MLE of each parameter can be obtained as the average of all MLEs calculated from each simulated sample, that is, $\hat{\theta} = \frac{1}{NR}\sum_{w=1}^{NR} \hat{\theta}_w$.
5. For given time point $(y_0 = 1, 2, 5)$, apply the invariance property of the MLEs calculated from each simulated sample to obtain MLEs of the SF, HRF, AHRF and RHRF using Equations (10), (12), (14) and (15).
6. For given entropy parameter $(\delta = 0.5, 1.5, 2)$, apply the invariance property of the MLEs calculated at each simulated sample to achieve the MLEs of entropy and extropy measures given in Equations (27)–(30).
7. From each simulated sample, derive the asymptotic Var-covariance matrix for the parameters, and calculate the UL and LL of the 95% ACIs and their width. Also, compute the PBCI limits using $N - boot = NR$, by applying the 0.025 and 0.975 quartiles of the computed MLEs.
8. Compute the 95% ACIs of SF, HRF, AHRF and RHRF by applying the delta method from each simulated sample, then compute their means. Also, compute the PBCI limits by computing the 0.025 and 0.975 quartiles of the MLEs derived at every replication.
9. Calculate the *mean square error* (MSE), *relative mean square error* (RMSE), *bias* (Bias), and *relative absolute bias* (RAB), where the MSE, RMSE, Bias and RAB are defined as

$$MSE(\hat{\theta}) = \frac{\sum_{w=1}^{NR} (\hat{\theta}_w - \theta)^2}{NR}, \quad RMSE(\hat{\theta}) = \frac{MSE(\hat{\theta})}{\theta}, \quad Bias(\hat{\theta}) = \hat{\theta}_w - \theta$$

   and $RAB(\hat{\theta}) = \frac{|Bias(\hat{\theta})|}{\theta}$.
10. Compute the UL and LL of the 95% ACI of the parameters, SF, HRF, RHRF, entropy and extropy measures, with their widths as the means of their calculated values from each simulated sample.

The simulation results are tabulated in Tables 5–14. Tables 5–7 present the MLEs of DPh-BXII's parameters and the considered criteria for evaluating the simulation process. Also, the UL and LL of the 95% ACI and the PBCI along with their width are displayed.

Table 5. MLEs of DPh-BXII's parameters, Biases, RABs, MSEs, RMSEs, the 95% ACI limits with their widths, and the 95% PBCI limits with their widths for different sample sizes at ($\alpha = 1.5,\ \beta = 1.15,\ c = 0.85,\ k = 0.15$).

n	θ	MLE	Bias	RAB	MSE	RMSE	95% ACI			95% PBCI		
							LL	UL	Width	LL	UL	Width
30	α	2.2701	0.7701	0.5134	1.4784	0.9856	0.5457	3.9945	3.4487	1.0731	3.8503	2.7772
	β	1.0942	−0.0558	0.0485	0.0122	0.0106	1.0000	1.2470	0.2470	1.0028	1.2959	0.2931
	c	1.3295	0.4795	0.5641	0.8344	0.9817	0.0000	3.6323	3.6323	0.2332	2.9870	2.7538
	k	0.2143	0.0643	0.4287	0.0465	0.3097	0.0000	0.4765	0.4765	0.0000	0.6345	0.6345
60	α	2.1039	0.6039	0.4026	1.0164	0.6776	0.5097	3.6981	3.1884	1.0803	3.6652	2.5849
	β	1.1019	−0.0481	0.0418	0.0115	0.0100	1.0000	1.2537	0.2537	1.0036	1.2972	0.2936
	c	1.3221	0.4721	0.5554	0.6525	0.7677	0.0000	3.3399	3.3399	0.2734	2.7092	2.4358
	k	0.2122	0.0622	0.4150	0.0358	0.2389	0.0000	0.4485	0.4485	0.0000	0.5401	0.5401
100	α	2.0121	0.5121	0.3414	0.7557	0.5038	0.6027	3.4215	2.8188	1.1078	3.5084	2.4006
	β	1.1034	−0.0466	0.0405	0.0104	0.0091	1.0000	1.2564	0.2564	1.0050	1.2873	0.2822
	c	1.3400	0.4900	0.5764	0.6212	0.7308	0.0000	3.3397	3.3397	0.2847	2.5857	2.3009
	k	0.2113	0.0613	0.4087	0.0293	0.1956	0.0000	0.4520	0.4520	0.0000	0.4706	0.4706

Table 6. MLEs of DPh-BXII's parameters, Biases, RABs, MSEs, RMSEs, the 95% ACI limits with their widths, and the 95% PBCI limits with their widths for different sample sizes at ($\alpha = 1.5,\ \beta = 1.15,\ c = 0.85,\ k = 0.15$).

n	θ	MLE	Bias	RAB	MSE	RMSE	95% ACI			95% PBCI		
							LL	UL	Width	LL	UL	Width
30	α	1.0125	0.5125	1.0251	0.5537	1.1073	0.5099	1.5152	1.0053	0.1729	1.7118	1.5390
	β	1.1239	−0.0261	0.0227	0.0394	0.0342	1.0000	1.4586	0.4586	1.0000	1.6395	0.6395
	c	1.6904	0.4404	0.3523	3.6989	2.9591	0.0000	6.1558	6.1558	0.4673	10.0000	9.5327
	k	0.5324	0.0324	0.0648	0.0909	0.1818	0.0000	1.2333	1.2333	0.0417	1.1245	1.0828
60	α	0.9454	0.4454	0.8908	0.4361	0.8722	0.4342	1.4566	1.0224	0.2396	1.6282	1.3886
	β	1.1270	−0.0230	0.0200	0.0335	0.0292	1.0000	1.4618	0.4618	1.0007	1.5661	0.5655
	c	1.5746	0.3246	0.2596	2.2689	1.8152	0.0000	5.1464	5.1464	0.6562	6.7091	6.0529
	k	0.5312	0.0312	0.0624	0.0693	0.1386	0.0000	1.1520	1.1520	0.0485	0.9757	0.9271
100	α	0.8580	0.3580	0.7159	0.3348	0.6696	0.2695	1.4465	1.1770	0.2754	1.5731	1.2977
	β	1.1316	−0.0184	0.0160	0.0271	0.0236	1.0000	1.5064	0.5064	1.0009	1.4958	0.4949
	c	1.5463	0.2963	0.2371	1.8442	1.4753	0.0000	4.4501	4.4501	0.7755	5.1217	4.3462
	k	0.5266	0.0266	0.0531	0.0604	0.1209	0.0000	1.1570	1.1570	0.0600	0.9130	0.8530

Table 7. MLEs of DPh-BXII's parameters, Biases, RABs, MSEs, RMSEs, the 95% ACI limits with their widths and the 95% PBCI limits with their widths for different sample sizes at ($\alpha = 0.15,\ \beta = 1.3,\ c = 1.5,\ k = 0.25$).

n	θ	MLE	Bias	RAB	MSE	RMSE	95% ACI			95% PBCI		
							LL	UL	Width	LL	UL	Width
30	α	0.2745	0.1245	0.8302	0.0737	0.4916	0.1505	0.3986	0.2481	0.1157	0.9557	0.8400
	β	1.4202	0.1202	0.0925	0.0662	0.0509	1.0544	1.7861	0.7317	1.0010	1.8143	0.8133
	c	3.8622	2.3622	1.5748	18.6660	12.4440	0.0000	25.5674	25.5674	0.3928	10.0000	9.6072
	k	0.1336	−0.1164	0.4654	0.0577	0.2307	0.0000	0.4780	0.4780	0.0000	0.7585	0.7585
60	α	0.2529	0.1029	0.6857	0.0526	0.3503	0.1768	0.3289	0.1521	0.1402	0.8992	0.7591
	β	1.4170	0.1170	0.0900	0.0453	0.0348	1.1524	1.6816	0.5291	1.0013	1.6813	0.6800
	c	3.8056	2.3056	1.5371	17.7342	11.8228	0.0000	21.1415	21.1415	0.4636	10.0000	9.5364
	k	0.1391	−0.1109	0.4435	0.0573	0.2291	0.0000	0.4222	0.4222	0.0061	0.7146	0.7085
100	α	0.2518	0.1018	0.6784	0.0495	0.3297	0.1752	0.3283	0.1531	0.1463	0.8882	0.7419
	β	1.4028	0.1028	0.0791	0.0367	0.0282	1.1885	1.6171	0.4286	1.0014	1.5995	0.5981
	c	3.6050	2.1050	1.4033	13.8060	9.2040	0.0000	16.6407	16.6407	0.5667	10.0000	9.4333
	k	0.1412	−0.1088	0.4352	0.0574	0.2295	0.0000	0.4130	0.4130	0.0098	0.7339	0.7241

Table 8. MLEs of DPh-BXII's SF, HRF, AHRF and RHRF, RABs, MSEs, RMSEs, the 95% ACI limits with their widths, and the 95% PBCI limits with their widths for different sample sizes at ($\alpha = 1.5,\ \beta = 1.15,\ c = 0.85,\ k = 0.15$) and different y_0.

n	$\phi(y_0;\underline{\theta})$	MLE	Bias	RAB	MSE	RMSE	95% ACI			95% PBCI		
							LL	UL	Width	LL	UL	Width
30	$S(1;\underline{\theta})$	0.7861	0.0104	0.0134	0.0052	0.0067	0.6515	0.9207	0.2692	0.6328	0.9034	0.2707
	$S(2;\underline{\theta})$	0.5557	0.0281	0.0532	0.0068	0.0129	0.3947	0.7167	0.3220	0.3971	0.7052	0.3081
	$S(5;\underline{\theta})$	0.0292	0.0111	0.6101	0.0005	0.0248	0.0000	0.0771	0.0771	0.0014	0.0709	0.0695
	$h(1;\underline{\theta})$	0.2981	−0.0217	0.0679	0.0041	0.0129	0.1574	0.4390	0.2817	0.1819	0.4266	0.2447
	$h(2;\underline{\theta})$	0.3964	−0.0644	0.1397	0.0067	0.0145	0.2600	0.5327	0.2727	0.2940	0.4777	0.1838
	$h(5;\underline{\theta})$	0.9320	−0.0205	0.0216	0.0059	0.0062	0.7551	1.1090	0.3539	0.7391	1.0000	0.2609
	$ah(1;\underline{\theta})$	0.3578	−0.0276	0.0716	0.0086	0.0223	0.1546	0.5612	0.4066	0.2008	0.5562	0.3554
	$ah(2;\underline{\theta})$	0.5083	−0.1093	0.1770	0.0188	0.0305	0.2807	0.7358	0.4551	0.3481	0.6496	0.3015
	$ah(5;\underline{\theta})$	6.9181	3.8708	1.2703	205.4687	67.4272	0.0000	24.3159	24.3159	1.3435	30.2806	28.9371
	$r(1;\underline{\theta})$	0.5308	0.0056	0.0107	0.0099	0.0189	0.3258	0.7357	0.4100	0.2967	0.6981	0.4014
	$r(2;\underline{\theta})$	0.3345	−0.0052	0.0154	0.0049	0.0144	0.2027	0.4660	0.2633	0.2053	0.4769	0.2716
	$r(5;\underline{\theta})$	0.0264	0.0092	0.5305	0.0003	0.0186	0.0000	0.0667	0.0667	0.0014	0.0596	0.0582
60	$S(1;\underline{\theta})$	0.7812	0.0055	0.0070	0.0027	0.0034	0.6830	0.8792	0.1962	0.6692	0.8687	0.1995
	$S(2;\underline{\theta})$	0.5388	0.0112	0.0213	0.0033	0.0063	0.4223	0.6553	0.2330	0.4246	0.6502	0.2256
	$S(5;\underline{\theta})$	0.0259	0.0078	0.4302	0.0002	0.0116	0.0000	0.0596	0.0596	0.0038	0.0538	0.0500
	$h(1;\underline{\theta})$	0.3115	−0.0083	0.0260	0.0021	0.0067	0.2054	0.4178	0.2124	0.2227	0.4033	0.1807
	$h(2;\underline{\theta})$	0.4122	−0.0486	0.1055	0.0039	0.0085	0.3058	0.5184	0.2125	0.3340	0.4794	0.1454
	$h(5;\underline{\theta})$	0.9403	−0.0122	0.0128	0.0042	0.0044	0.8133	1.0675	0.2542	0.7828	1.0000	0.2172
	$ah(1;\underline{\theta})$	0.3755	−0.0099	0.0258	0.0045	0.0118	0.2204	0.5307	0.3102	0.2519	0.5164	0.2645
	$ah(2;\underline{\theta})$	0.5335	−0.0841	0.1362	0.0115	0.0187	0.3522	0.7147	0.3625	0.4064	0.6527	0.2463
	$ah(5;\underline{\theta})$	4.8022	1.7549	0.5759	26.0277	8.5413	0.0000	12.4965	12.4965	1.5269	19.6454	18.1186
	$r(1;\underline{\theta})$	0.5299	0.0047	0.0089	0.0053	0.0100	0.3789	0.6808	0.3019	0.3670	0.6498	0.2827
	$r(2;\underline{\theta})$	0.3265	−0.0133	0.0391	0.0027	0.0079	0.2273	0.4254	0.1981	0.2304	0.4279	0.1975
	$r(5;\underline{\theta})$	0.0239	0.0067	0.3849	0.0002	0.0088	0.0000	0.0530	0.0530	0.0038	0.0474	0.0436
100	$S(1;\underline{\theta})$	0.7802	0.0045	0.0058	0.0016	0.0021	0.7028	0.8575	0.1547	0.6998	0.8503	0.1505
	$S(2;\underline{\theta})$	0.5323	0.0047	0.0090	0.0020	0.0038	0.4404	0.6241	0.1837	0.4441	0.6184	0.1743
	$S(5;\underline{\theta})$	0.0241	0.0060	0.3323	0.0001	0.0069	0.0000	0.0497	0.0497	0.0083	0.0458	0.0375
	$h(1;\underline{\theta})$	0.3154	−0.0045	0.0140	0.0014	0.0044	0.2290	0.4018	0.1727	0.2393	0.3860	0.1467
	$h(2;\underline{\theta})$	0.4202	−0.0406	0.0881	0.0029	0.0063	0.3325	0.5077	0.1753	0.3496	0.4800	0.1305
	$h(5;\underline{\theta})$	0.9449	−0.0077	0.0080	0.0029	0.0030	0.8355	1.0546	0.2192	0.8229	1.0000	0.1771
	$ah(1;\underline{\theta})$	0.3804	−0.0050	0.0131	0.0030	0.0079	0.2539	0.5069	0.2530	0.2735	0.4877	0.2142
	$ah(2;\underline{\theta})$	0.5469	−0.0707	0.1145	0.0087	0.0140	0.3954	0.6981	0.3027	0.4301	0.6540	0.2239
	$ah(5;\underline{\theta})$	4.1902	1.1429	0.3751	10.0957	3.3130	0.0000	9.3808	9.3808	1.7310	11.8153	10.0843
	$r(1;\underline{\theta})$	0.5299	0.0047	0.0089	0.0034	0.0065	0.4075	0.6520	0.2445	0.4015	0.6302	0.2287
	$r(2;\underline{\theta})$	0.3242	−0.0156	0.0458	0.0018	0.0052	0.2432	0.4049	0.1617	0.2486	0.4000	0.1514
	$r(5;\underline{\theta})$	0.0225	0.0052	0.3038	0.0001	0.0053	0.0003	0.0447	0.0444	0.0083	0.0408	0.0325

Table 9. MLEs of DPh-BXII's SF, HRF, AHRF and RHRF, Biases, RABs, MSEs, RMSEs, the 95% ACI limits with their widths, and the 95% percentile bootstrap CI limits with their widths for different sample sizes at ($\alpha = 0.5,\ \beta = 1.15,\ c = 1.25,\ k = 0.5$) and different y_0.

n	$\phi(y_0;\underline{\theta})$	MLE	Bias	RAB	MSE	RMSE	95% ACI			95% PBCI		
							LL	UL	Width	LL	UL	Width
30	$S(1;\underline{\theta})$	0.6239	0.0152	0.0251	0.0081	0.0133	0.4521	0.7957	0.3436	0.4472	0.7908	0.3436
	$S(2;\underline{\theta})$	0.4571	0.0198	0.0453	0.0061	0.0140	0.3072	0.6065	0.2993	0.3072	0.6053	0.2981
	$S(5;\underline{\theta})$	0.2717	0.0337	0.1416	0.0053	0.0224	0.1341	0.4090	0.2749	0.1513	0.4056	0.2543
	$h(1;\underline{\theta})$	0.2583	−0.0232	0.0826	0.0041	0.0145	0.1398	0.3775	0.2378	0.1544	0.3947	0.2403
	$h(2;\underline{\theta})$	0.1967	−0.0215	0.0987	0.0027	0.0122	0.0999	0.2940	0.1940	0.1199	0.3008	0.1809
	$h(5;\underline{\theta})$	0.1150	−0.0184	0.1376	0.0009	0.0064	0.0615	0.1686	0.1071	0.0749	0.1632	0.0883
	$ah(1;\underline{\theta})$	0.3022	−0.0285	0.0862	0.0077	0.0232	0.1396	0.4658	0.3262	0.1678	0.5020	0.3343
	$ah(2;\underline{\theta})$	0.2207	−0.0254	0.1033	0.0042	0.0171	0.0987	0.3434	0.2447	0.1277	0.3578	0.2301
	$ah(5;\underline{\theta})$	0.1225	−0.0206	0.1441	0.0011	0.0076	0.0617	0.1834	0.1216	0.0779	0.1782	0.1003
	$r(1;\underline{\theta})$	0.3066	0.0021	0.0069	0.0093	0.0306	0.1249	0.4887	0.3638	0.1498	0.5260	0.3762
	$r(2;\underline{\theta})$	0.1421	−0.0029	0.0198	0.0014	0.0093	0.0708	0.2135	0.1427	0.0806	0.2241	0.1435
	$r(5;\underline{\theta})$	0.0403	0.0003	0.0074	0.0001	0.0019	0.0230	0.0574	0.0343	0.0240	0.0579	0.0340
60	$S(1;\underline{\theta})$	0.6129	0.0043	0.0071	0.0041	0.0067	0.4909	0.7350	0.2441	0.4835	0.7324	0.2489
	$S(2;\underline{\theta})$	0.4486	0.0113	0.0259	0.0030	0.0068	0.3427	0.5543	0.2117	0.3436	0.5511	0.2076
	$S(5;\underline{\theta})$	0.2591	0.0211	0.0887	0.0026	0.0107	0.1623	0.3557	0.1934	0.1718	0.3513	0.1796
	$h(1;\underline{\theta})$	0.2678	−0.0138	0.0488	0.0022	0.0077	0.1805	0.3559	0.1755	0.1895	0.3672	0.1777
	$h(2;\underline{\theta})$	0.2041	−0.0141	0.0648	0.0014	0.0065	0.1336	0.2749	0.1413	0.1414	0.2782	0.1368
	$h(5;\underline{\theta})$	0.1196	−0.0137	0.1030	0.0005	0.0035	0.0805	0.1588	0.0783	0.0881	0.1515	0.0634
	$ah(1;\underline{\theta})$	0.3136	−0.0171	0.0516	0.0041	0.0125	0.1932	0.4350	0.2418	0.2101	0.4576	0.2475
	$ah(2;\underline{\theta})$	0.2292	−0.0170	0.0689	0.0023	0.0092	0.1401	0.3188	0.1788	0.1525	0.3261	0.1736
	$ah(5;\underline{\theta})$	0.1276	−0.0156	0.1087	0.0006	0.0042	0.0831	0.1722	0.0892	0.0922	0.1643	0.0721
	$r(1;\underline{\theta})$	0.3009	−0.0036	0.0117	0.0046	0.0150	0.1747	0.4273	0.2526	0.1867	0.4479	0.2611
	$r(2;\underline{\theta})$	0.1422	−0.0027	0.0189	0.0007	0.0046	0.0919	0.1926	0.1007	0.0961	0.1944	0.0983
	$r(5;\underline{\theta})$	0.0397	−0.0003	0.0074	3.6518×10^{-5}	0.0009	0.0274	0.0519	0.0245	0.0283	0.0520	0.0237
100	$S(1;\underline{\theta})$	0.6101	0.0015	0.0024	0.0024	0.0040	0.5153	0.7049	0.1896	0.5126	0.7018	0.1891
	$S(2;\underline{\theta})$	0.4447	0.0075	0.0171	0.0018	0.0041	0.3626	0.5269	0.1643	0.3607	0.5260	0.1653
	$S(5;\underline{\theta})$	0.2521	0.0141	0.0594	0.0016	0.0065	0.1771	0.3271	0.1500	0.1799	0.3262	0.1463
	$h(1;\underline{\theta})$	0.2733	−0.0083	0.0295	0.0013	0.0046	0.2035	0.3435	0.1400	0.2045	0.3444	0.1399
	$h(2;\underline{\theta})$	0.2092	−0.0090	0.0411	0.0008	0.0038	0.1531	0.2655	0.1125	0.1582	0.2665	0.1083
	$h(5;\underline{\theta})$	0.1230	−0.0104	0.0783	0.0003	0.0022	0.0918	0.1541	0.0624	0.0978	0.1515	0.0537
	$ah(1;\underline{\theta})$	0.3203	−0.0103	0.0312	0.0024	0.0074	0.2240	0.4175	0.1935	0.2287	0.4222	0.1934
	$ah(2;\underline{\theta})$	0.2354	−0.0108	0.0438	0.0013	0.0054	0.1641	0.3068	0.1427	0.1722	0.3099	0.1377
	$ah(5;\underline{\theta})$	0.1313	−0.0119	0.0828	0.0004	0.0027	0.0957	0.1669	0.0712	0.1029	0.1643	0.0614
	$r(1;\underline{\theta})$	0.3014	−0.0031	0.0103	0.0027	0.0090	0.2032	0.3999	0.1967	0.2075	0.4082	0.2007
	$r(2;\underline{\theta})$	0.1434	−0.0015	0.0105	0.0004	0.0028	0.1044	0.1825	0.0782	0.1053	0.1836	0.0782
	$r(5;\underline{\theta})$	0.0395	−0.0005	0.0127	2.3903×10^{-5}	0.0006	0.0297	0.0492	0.0196	0.0304	0.0492	0.0189

For the reliability measures, SF, HRF, AHRF and RHRF, Tables 8–10 exhibit the MLEs of the SF, HRF, AHRF and RHRF for the DPh-BXII distribution at different time points and for the considered combinations of the parameters (initial values), I, II and III. Also, the UL and LL of the 95% ACI and the PBCI along with their widths as well are achieved. Furthermore, Table 11 presents the MLEs, Biases, RAB, MSEs, RMSEs of Shannon and extropy measures for the predetermined combinations of initial values of the parameters, I, II and III. Also, Table 11 displays the 95% UL and LL of the ACI and the PBCI along with their width. In addition, Tables 12–14 exhibit MLEs, Biases, RAB, MSEs, and RMSEs of Rènyi and Tsallis at different values for the entropy parameter. Moreover, Tables 12–14 present interval estimation of Rènyi and Tsallis measures through the 95% ACI and PBCI, the UL, and the LL along with their width.

Table 10. MLEs of DPh-BXII's SF, HRF, AHRF and RHRF, Biases, RABs, MSEs, RMSEs, the 95% ACI limits with their widths, and the 95% PBCI limits with their widths for different sample sizes at ($\alpha = 0.15,\ \beta = 1.3,\ c = 1.5,\ k = 0.25$) and different y_0.

n	$\phi(y_0;\underline{\theta})$	MLE	Bias	RAB	MSE	RMSE	95% ACI			95% PBCI		
							LL	UL	Width	LL	UL	Width
30	$S(1;\underline{\theta})$	0.6106	−0.0124	0.0199	0.0075	0.0121	0.4389	0.7822	0.3433	0.4388	0.7695	0.3308
	$S(2;\underline{\theta})$	0.5288	0.0189	0.0371	0.0056	0.0110	0.3686	0.6887	0.3201	0.3862	0.6672	0.2810
	$S(5;\underline{\theta})$	0.4195	0.0594	0.1650	0.0076	0.0210	0.2652	0.5737	0.3085	0.2959	0.5403	0.2444
	$h(1;\underline{\theta})$	0.1339	−0.0475	0.2619	0.0030	0.0163	0.0558	0.2122	0.1564	0.0913	0.1917	0.1004
	$h(2;\underline{\theta})$	0.0940	−0.0410	0.3034	0.0020	0.0150	0.0391	0.1491	0.1099	0.0632	0.1322	0.0691
	$h(5;\underline{\theta})$	0.0498	−0.0238	0.3230	0.0006	0.0086	0.0225	0.0771	0.0546	0.0352	0.0660	0.0307
	$ah(1;\underline{\theta})$	0.1443	−0.0560	0.2795	0.0041	0.0204	0.0533	0.2353	0.1820	0.0957	0.2128	0.1170
	$ah(2;\underline{\theta})$	0.0990	−0.0460	0.3175	0.0025	0.0175	0.0381	0.1600	0.1219	0.0653	0.1418	0.0766
	$ah(5;\underline{\theta})$	0.0511	−0.0253	0.3310	0.0007	0.0093	0.0223	0.0799	0.0576	0.0359	0.0682	0.0324
	$r(1;\underline{\theta})$	0.1818	−0.0489	0.2119	0.0068	0.0294	0.0477	0.3159	0.2682	0.0847	0.3166	0.2319
	$r(2;\underline{\theta})$	0.0982	−0.0250	0.2027	0.0016	0.0128	0.0343	0.1621	0.1278	0.0500	0.1662	0.1162
	$r(5;\underline{\theta})$	0.0352	−0.0046	0.1151	0.0001	0.0026	0.0167	0.0535	0.0368	0.0199	0.0542	0.0342
60	$S(1;\underline{\theta})$	0.6095	−0.0135	0.0217	0.0037	0.0059	0.4866	0.7323	0.2457	0.5028	0.7226	0.2198
	$S(2;\underline{\theta})$	0.5285	0.0186	0.0364	0.0028	0.0056	0.4149	0.6421	0.2272	0.4314	0.6349	0.2035
	$S(5;\underline{\theta})$	0.4090	0.0490	0.1360	0.0046	0.0129	0.3000	0.5180	0.2180	0.3091	0.4871	0.1780
	$h(1;\underline{\theta})$	0.1393	−0.0422	0.2325	0.0022	0.0119	0.0789	0.1997	0.1208	0.1061	0.1766	0.0705
	$h(2;\underline{\theta})$	0.0961	−0.0389	0.2881	0.0017	0.0126	0.0556	0.1366	0.0810	0.0683	0.1192	0.0509
	$h(5;\underline{\theta})$	0.0517	−0.0219	0.2978	0.0005	0.0070	0.0314	0.0720	0.0407	0.0398	0.0626	0.0227
	$ah(1;\underline{\theta})$	0.1502	−0.0500	0.2497	0.0030	0.0150	0.0798	0.2207	0.1409	0.1121	0.1943	0.0822
	$ah(2;\underline{\theta})$	0.1012	−0.0439	0.3024	0.0022	0.0148	0.0563	0.1460	0.0898	0.0707	0.1270	0.0562
	$ah(5;\underline{\theta})$	0.0531	−0.0234	0.3056	0.0006	0.0077	0.0317	0.0746	0.0429	0.0407	0.0646	0.0239
	$r(1;\underline{\theta})$	0.1822	−0.0484	0.2100	0.0043	0.0188	0.0839	0.2806	0.1967	0.1111	0.2746	0.1635
	$r(2;\underline{\theta})$	0.0989	−0.0242	0.1966	0.0011	0.0093	0.0524	0.1455	0.0930	0.0619	0.1457	0.0838
	$r(5;\underline{\theta})$	0.0347	−0.0050	0.1266	0.0001	0.0016	0.0218	0.0476	0.0258	0.0230	0.0473	0.0242
100	$S(1;\underline{\theta})$	0.6196	−0.0034	0.0054	0.0019	0.0030	0.5249	0.7142	0.1894	0.5560	0.6957	0.1397
	$S(2;\underline{\theta})$	0.5282	0.0183	0.0359	0.0019	0.0038	0.4378	0.6147	0.1769	0.4460	0.5985	0.1525
	$S(5;\underline{\theta})$	0.4087	0.0486	0.1351	0.0039	0.0109	0.3248	0.4925	0.1677	0.3344	0.4855	0.1511
	$h(1;\underline{\theta})$	0.1405	−0.0409	0.2256	0.0020	0.0110	0.0941	0.1869	0.0928	0.1009	0.1696	0.0687
	$h(2;\underline{\theta})$	0.0988	−0.0362	0.2679	0.0015	0.0108	0.0664	0.1320	0.0656	0.0758	0.1206	0.0448
	$h(5;\underline{\theta})$	0.0542	−0.0194	0.2633	0.0004	0.0053	0.0381	0.0703	0.0322	0.0472	0.0611	0.0140
	$ah(1;\underline{\theta})$	0.1517	−0.0486	0.2426	0.0028	0.0140	0.0975	0.2059	0.1084	0.1064	0.1859	0.0795
	$ah(2;\underline{\theta})$	0.1042	−0.0409	0.2818	0.0019	0.0128	0.0681	0.1411	0.0729	0.0788	0.1285	0.0497
	$ah(5;\underline{\theta})$	0.0557	−0.0207	0.2707	0.0004	0.0058	0.0387	0.0728	0.0341	0.0483	0.0631	0.0148
	$r(1;\underline{\theta})$	0.1880	−0.0426	0.1848	0.0030	0.0131	0.1102	0.2658	0.1556	0.1270	0.2460	0.1190
	$r(2;\underline{\theta})$	0.1000	−0.0231	0.1878	0.0008	0.0063	0.0629	0.1365	0.0736	0.0753	0.1240	0.0487
	$r(5;\underline{\theta})$	0.0363	−0.0035	0.0877	0.0000	0.0009	0.0261	0.0464	0.0203	0.0265	0.0459	0.0195

Table 11. MLEs of DPh-BXII's Shannon and extropy, Biases, RABs, MSEs, RMSEs, the 95% ACI limits along with their widths, and the 95% PBCI limits with their widths for different sample sizes and combinations of the parameters I, II, and III.

$\underline{\theta}$	n	$\phi(y;\underline{\theta})$	MLE	Bias	RAB	MSE	RMSE	95% ACI			95% PBCI		
								LL	UL	Width	LL	UL	Width
I	30	$H(Y;\underline{\theta})$	1.6436	0.0236	0.0146	0.0038	0.0023	1.4746	1.8128	0.3383	1.5349	1.7534	0.2185
		$J(Y;\underline{\theta})$	0.8859	0.0006	0.0007	0.0001	0.0001	0.8596	0.9121	0.0524	0.8616	0.9005	0.0389
	60	$H(Y;\underline{\theta})$	1.6429	0.0230	0.0142	0.0018	0.0011	1.5261	1.7597	0.2336	1.5792	1.7200	0.1409
		$J(Y;\underline{\theta})$	0.8871	0.0018	0.0020	3.8530×10^{-5}	4.3523×10^{-5}	0.8698	0.9043	0.0344	0.8737	0.8976	0.0239
	100	$H(Y;\underline{\theta})$	1.6421	0.0222	0.0137	0.0013	0.0008	1.5523	1.7319	0.1796	1.5962	1.7024	0.1062
		$J(Y;\underline{\theta})$	0.8875	0.0022	0.0025	2.3594×10^{-5}	2.6652×10^{-5}	0.8746	0.9003	0.0256	0.8782	0.8958	0.0176
II	30	$H(Y;\underline{\theta})$	2.5196	0.1433	0.0603	0.1054	0.0443	1.8739	3.1652	1.2913	1.9305	3.0766	1.1462
		$J(Y;\underline{\theta})$	0.8855	−0.0007	0.0008	0.0015	0.0017	0.8094	0.9616	0.1522	0.7895	0.9433	0.1538
	60	$H(Y;\underline{\theta})$	2.4714	0.0951	0.0400	0.0500	0.0211	2.0167	2.9261	0.9093	2.0625	2.8530	0.7905
		$J(Y;\underline{\theta})$	0.8863	0.0001	0.0001	0.0008	0.0009	0.8331	0.9394	0.1063	0.8217	0.9289	0.1072
	100	$H(Y;\underline{\theta})$	2.4524	0.0761	0.0320	0.0321	0.0135	2.1028	2.8019	0.6991	2.1199	2.7586	0.6387
		$J(Y;\underline{\theta})$	0.8861	−0.0001	0.0001	0.0005	0.0005	0.8449	0.9272	0.0823	0.8386	0.9212	0.0825
III	30	$H(Y;\underline{\theta})$	3.5571	0.3801	0.1197	0.2894	0.0911	2.5697	4.5446	1.9749	2.8294	4.2538	1.4244
		$J(Y;\underline{\theta})$	0.8967	−0.0057	0.0063	0.0017	0.0019	0.8107	0.9826	0.1719	0.8006	0.9575	0.1569
	60	$H(Y;\underline{\theta})$	3.5497	0.3727	0.1173	0.2218	0.0698	2.8517	4.2477	1.3960	2.9508	3.9927	1.0418
		$J(Y;\underline{\theta})$	0.8986	−0.0037	0.0041	0.0011	0.0012	0.8382	0.9590	0.1209	0.8222	0.9460	0.1238
	100	$H(Y;\underline{\theta})$	3.5441	0.3671	0.1156	0.1966	0.0619	3.0077	4.0805	1.0728	3.0475	3.9467	0.8992
		$J(Y;\underline{\theta})$	0.9029	0.0005	0.0006	0.0006	0.0006	0.8573	0.9484	0.0911	0.8470	0.9373	0.0903

Table 12. MLEs of DPh-BXII's Rènyi and Tsallis, Biases, RABs, MSEs, RMSEs, the 95% ACI limits with their widths, and the 95% PBCI limits with their widths for different sample sizes and combinations of the parameters I, II, and III at $\delta = 0.5$.

$\underline{\theta}$	n	$\phi(y;\underline{\theta})$	MLE	Bias	RAB	MSE	RMSE	95% ACI			95% PBCI		
								LL	UL	Width	LL	UL	Width
I	30	$I_1(Y;\delta,\underline{\theta})$	1.7279	0.0237	0.0139	0.0044	0.0026	1.5574	1.8985	0.3411	1.6071	1.8526	0.2455
		$I_2(Y;\delta,\underline{\theta})$	2.7472	0.0582	0.0217	0.0249	0.0093	2.3415	3.1533	0.8118	2.4670	3.0503	0.5834
	60	$I_1(Y;\delta,\underline{\theta})$	1.7249	0.0208	0.0122	0.0023	0.0014	1.6026	1.8472	0.2446	1.6420	1.8210	0.1790
		$I_2(Y;\delta,\underline{\theta})$	2.7391	0.0502	0.0187	0.0133	0.0049	2.4488	3.0295	0.5806	2.5456	2.9712	0.4256
	100	$I_1(Y;\delta,\underline{\theta})$	1.7219	0.0178	0.0104	0.0015	0.0009	1.6260	1.8179	0.1919	1.6648	1.7939	0.1291
		$I_2(Y;\delta,\underline{\theta})$	2.7316	0.0426	0.0158	0.0083	0.0031	2.5043	2.9588	0.4545	2.5977	2.9041	0.3064
II	30	$I_1(Y;\delta,\underline{\theta})$	3.7299	0.2061	0.0585	0.1120	0.0318	2.8722	4.5879	1.7157	3.2992	4.3385	1.0393
		$I_2(Y;\delta,\underline{\theta})$	11.0268	1.3799	0.1430	5.0959	0.5282	5.3068	16.7496	11.4428	8.4097	15.5036	7.0939
	60	$I_1(Y;\delta,\underline{\theta})$	3.6874	0.1636	0.0464	0.0660	0.0187	3.0405	4.3339	1.2934	3.3706	4.1224	0.7518
		$I_2(Y;\delta,\underline{\theta})$	10.7029	1.0560	0.1095	2.7988	0.2901	6.5281	14.8746	8.3465	8.7882	13.7106	4.9224
	100	$I_1(Y;\delta,\underline{\theta})$	3.6578	0.1340	0.0380	0.0422	0.0120	3.1542	4.1613	1.0072	3.3971	3.9975	0.6004
		$I_2(Y;\delta,\underline{\theta})$	10.4922	0.8454	0.0876	1.7065	0.1769	7.3177	13.6665	6.3488	8.9321	12.7599	3.8277
III	30	$I_1(Y;\delta,\underline{\theta})$	5.8608	0.3484	0.0632	0.1509	0.0274	5.2329	6.4882	1.2553	5.5374	6.1752	0.6378
		$I_2(Y;\delta,\underline{\theta})$	35.6089	6.1281	0.2079	48.0032	1.6283	23.8754	47.3319	23.4565	29.8763	41.8498	11.9735
	60	$I_1(Y;\delta,\underline{\theta})$	5.8541	0.3417	0.0620	0.1317	0.0239	5.4074	6.2985	0.8911	5.5805	6.0587	0.4782
		$I_2(Y;\delta,\underline{\theta})$	35.4151	5.9343	0.2013	40.3785	1.3697	27.0873	43.6982	16.6110	30.5702	39.3676	8.7974
	100	$I_1(Y;\delta,\underline{\theta})$	5.8406	0.3282	0.0595	0.1215	0.0220	5.4995	6.1818	0.6823	5.6200	6.0613	0.4412
		$I_2(Y;\delta,\underline{\theta})$	35.1584	5.6776	0.1926	37.0000	1.2551	28.8376	41.4793	12.6418	31.2222	39.4210	8.1988

Table 13. MLEs of DPh-BXII's Rènyi and Tsallis, Biases, RABs, MSEs, RMSEs, the 95% ACI limits with their widths, and the 95% PBCI limits with their widths for different sample sizes and combinations of the parameters I, II, and III at $\delta = 1.5$.

$\underline{\theta}$	n	$\phi(y;\underline{\theta})$	MLE	Bias	RAB	MSE	RMSE	95% ACI			95% PBCI		
								LL	UL	Width	LL	UL	Width
I	30	$I_1(Y;\delta,\underline{\theta})$	1.5973	0.0183	0.0116	0.0041	0.0026	1.4143	1.7804	0.3661	1.4688	1.7134	0.2447
		$I_2(Y;\delta,\underline{\theta})$	1.0997	0.0078	0.0072	0.0008	0.0008	1.0171	1.1824	0.1653	1.0404	1.1509	0.1105
	60	$I_1(Y;\delta,\underline{\theta})$	1.6005	0.0215	0.0136	0.0021	0.0013	1.4755	1.7255	0.2500	1.5191	1.6847	0.1655
		$I_2(Y;\delta,\underline{\theta})$	1.1014	0.0095	0.0087	0.0004	0.0004	1.0451	1.1576	0.1125	1.0643	1.1386	0.0743
	100	$I_1(Y;\delta,\underline{\theta})$	1.5995	0.0205	0.0130	0.0013	0.0009	1.5044	1.6946	0.1902	1.5412	1.6650	0.1238
		$I_2(Y;\delta,\underline{\theta})$	1.1010	0.0092	0.0084	0.0003	0.0002	1.0582	1.1438	0.0856	1.0745	1.1301	0.0556
II	30	$I_1(Y;\delta,\underline{\theta})$	1.9506	0.0846	0.0453	0.0932	0.0499	1.3480	2.5532	1.2052	1.3627	2.5052	1.1424
		$I_2(Y;\delta,\underline{\theta})$	1.2377	0.0244	0.0201	0.0134	0.0110	1.0083	1.4670	0.4588	0.9882	1.4285	0.4403
	60	$I_1(Y;\delta,\underline{\theta})$	1.9163	0.0503	0.0270	0.0459	0.0246	1.4943	2.3378	0.8435	1.4986	2.3117	0.8131
		$I_2(Y;\delta,\underline{\theta})$	1.2286	0.0154	0.0127	0.0068	0.0056	1.0659	1.3911	0.3253	1.0546	1.3704	0.3158
	100	$I_1(Y;\delta,\underline{\theta})$	1.8986	0.0326	0.0175	0.0278	0.0149	1.5741	2.2228	0.6487	1.5739	2.2061	0.6322
		$I_2(Y;\delta,\underline{\theta})$	1.2234	0.0101	0.0083	0.0042	0.0034	1.0974	1.3492	0.2518	1.0895	1.3363	0.2467
III	30	$I_1(Y;\delta,\underline{\theta})$	2.3075	0.1308	0.0601	0.1908	0.0877	1.4325	3.1825	1.7501	1.5849	3.1899	1.6050
		$I_2(Y;\delta,\underline{\theta})$	1.3590	0.0325	0.0245	0.0176	0.0133	1.0816	1.6363	0.5547	1.0852	1.5746	0.4894
	60	$I_1(Y;\delta,\underline{\theta})$	2.2979	0.1212	0.0557	0.1146	0.0527	1.6813	2.9144	1.2331	1.6635	2.8729	1.2093
		$I_2(Y;\delta,\underline{\theta})$	1.3581	0.0316	0.0238	0.0115	0.0086	1.1614	1.5547	0.3934	1.1294	1.5244	0.3950
	100	$I_1(Y;\delta,\underline{\theta})$	2.2825	0.1059	0.0486	0.0687	0.0316	1.8052	2.7598	0.9546	1.8063	2.6231	0.8168
		$I_2(Y;\delta,\underline{\theta})$	1.3566	0.0302	0.0228	0.0067	0.0050	1.2037	1.5096	0.3059	1.1894	1.4612	0.2718

Table 14. MLEs of DPh-BXII's Rènyi and Tsallis, Biases, RABs, MSEs, RMSEs, the 95% ACI limits with their widths, and the 95% PBCI limits with their widths for different sample sizes and combinations of the parameters I, II, and III at $\delta = 2$.

$\underline{\theta}$	n	$\phi(y;\underline{\theta})$	MLE	Bias	RAB	MSE	RMSE	95% ACI			95% PBCI		
								LL	UL	Width	LL	UL	Width
I	30	$I_1(Y;\delta,\underline{\theta})$	1.5658	0.0121	0.0078	0.0050	0.0032	1.3678	1.7640	0.3962	1.4054	1.6940	0.2887
		$I_2(Y;\delta,\underline{\theta})$	0.7906	0.0020	0.0026	0.0002	0.0003	0.7486	0.8326	0.0839	0.7547	0.8162	0.0615
	60	$I_1(Y;\delta,\underline{\theta})$	1.5717	0.0180	0.0116	0.0023	0.0015	1.4370	1.7064	0.2695	1.4728	1.6564	0.1836
		$I_2(Y;\delta,\underline{\theta})$	0.7921	0.0036	0.0045	0.0001	0.0001	0.7640	0.8203	0.0563	0.7707	0.8092	0.0385
	100	$I_1(Y;\delta,\underline{\theta})$	1.5714	0.0177	0.0114	0.0015	0.0010	1.4688	1.6740	0.2052	1.4967	1.6394	0.1426
		$I_2(Y;\delta,\underline{\theta})$	0.7921	0.0036	0.0045	0.0001	0.0001	0.7707	0.8135	0.0428	0.7761	0.8059	0.0298
II	30	$I_1(Y;\delta,\underline{\theta})$	1.6814	0.0733	0.0456	0.0894	0.0556	1.0974	2.2651	1.1677	1.1082	2.2286	1.1203
		$I_2(Y;\delta,\underline{\theta})$	0.8058	0.0061	0.0076	0.0035	0.0043	0.6930	0.9185	0.2255	0.6699	0.8923	0.2225
	60	$I_1(Y;\delta,\underline{\theta})$	1.6386	0.0306	0.0190	0.0439	0.0273	1.2281	2.0488	0.8207	1.2184	2.0333	0.8148
		$I_2(Y;\delta,\underline{\theta})$	0.8015	0.0018	0.0022	0.0018	0.0022	0.7202	0.8828	0.1626	0.7043	0.8691	0.1648
	100	$I_1(Y;\delta,\underline{\theta})$	1.6238	0.0157	0.0098	0.0260	0.0162	1.3082	1.9391	0.6309	1.3042	1.9293	0.6251
		$I_2(Y;\delta,\underline{\theta})$	0.8003	0.0006	0.0007	0.0010	0.0013	0.7373	0.8633	0.1260	0.7286	0.8548	0.1261
III	30	$I_1(Y;\delta,\underline{\theta})$	1.8025	0.0375	0.0212	0.1378	0.0781	1.0305	2.5745	1.5440	1.1268	2.5725	1.4457
		$I_2(Y;\delta,\underline{\theta})$	0.8239	−0.0049	0.0060	0.0041	0.0049	0.6924	0.9553	0.2629	0.6759	0.9237	0.2477
	60	$I_1(Y;\delta,\underline{\theta})$	1.7972	0.0321	0.0182	0.0708	0.0401	1.2501	2.3443	1.0942	1.3313	2.3613	1.0300
		$I_2(Y;\delta,\underline{\theta})$	0.8285	−0.0003	0.0004	0.0020	0.0024	0.7362	0.9208	0.1845	0.7359	0.9057	0.1698
	100	$I_1(Y;\delta,\underline{\theta})$	1.8027	0.0377	0.0213	0.0420	0.0238	1.3771	2.2283	0.8512	1.4480	2.1636	0.7156
		$I_2(Y;\delta,\underline{\theta})$	0.8318	0.0030	0.0036	0.0012	0.0014	0.7609	0.9026	0.1417	0.7648	0.8851	0.1203

6.1. Concluding Remarks

Based on the simulation tables, the following is observed:

- For the parameters, reliability characteristics and entropy and extropy measures of the DPh-BXII distribution, as the sample size increases, the MSE and RMSE always decrease. Also, as the sample size increases, the Bias, RAB and SE decrease in most cases.
- The MLEs of the SF and RHRF decrease as the considered time point, y_0, increases for all the predetermined combinations of initial values of the parameters, I, II and III. Meanwhile, the MLEs of the HRF and AHRF for the first set of parameters, I, increase as the time point increases, reflecting an increasing failure pattern, whereas for the second and third sets of parameters, II and III, the MLEs of the HRF and AHRF decrease as the time point, y_0, increases, displaying a decreasing failure pattern.
- The MLEs of Shannon, Rènyi and Tsallis entropy measures always decrease as the sample size increases, while the MLEs of the extropy increase mostly as the sample size increases. Also, the MSEs of Rènyi and Tsallis entropy measures increase as the entropy parameter δ increases.
- The 95% ACI and PBCI for DPh-BXII's parameters, reliability characteristics, entropy and extropy measures display a narrower trend as the sample size increases and their widths consistently decrease.
- The PB method provides a narrower 95% CI than the 95% ACI in most cases.

6.2. Computational Aspects of the Maximum Likelihood Estimation

The ML estimators of the DPh-BXII distribution parameters cannot be derived in closed-form. Therefore, a numerical solution is needed by maximizing the log-likelihood function through software programming. Despite the four-parameter structure of the model, the optimization procedure was found to be stable across the considered simulation study. The invariance property of MLEs enables straightforward estimation of the considered reliability characteristics, entropy and extropy measures without requiring additional optimization procedures.

Besides the invariance property of the MLEs, the ML estimation method has some advantages that are satisfied for large sample sizes: its asymptotic consistency, asymptotic efficiency, and asymptotic normality, which facilitate the construction of CIs for the model parameters. In addition, the percentile bootstrap procedure delivers more information about the population parameter than its point estimator. Also, in small sample sizes, the performance of the ACI is unsatisfactory.

A limitation of the ML method is that numerical optimization is required, which may increase computational time compared with models possessing closed-form estimators. Furthermore, the quality of the estimates may depend on the choice of initial values of the parameters and the convergence behavior of the optimization algorithm, particularly for small sample sizes. A final limitation is that of applying the asymptotic normality based on small sample sizes.

7. Applications

The applicability and flexibility of the DPh-BXII distribution for data modeling are approved through different datasets. Moreover, these datasets are used to prove the superiority of the DPh-BXII distribution over some competitor discrete models that are introduced in the literature: DAW, DAP-W, DBXII proposed by [13], DW presented by [9] and DCh derived by [20].

The comparison models were selected based on their methodological relevance to the proposed distribution. The DAW and DAP-W distributions were included because they are the only discrete competing risks models currently available in the literature and therefore represent the most direct competitors. The DBXII distribution was also considered as one of the marginal components of the proposed model. Furthermore, the DW and DCh distributions were selected because the proposed model can be viewed as a flexible extension of the Weibull family. Consequently, these models provide appropriate and meaningful benchmarks for evaluating the performance of the proposed distribution.

The MLEs of the parameters and their SEs, the *Kolmogorov–Smirnov* (K-S) statistic and its corresponding p-value are calculated for the considered datasets. For count data, the χ^2 statistic and its p-value are obtained. The *log-likelihood statistic* (ℓ), *Akaike information criterion* (AIC), *Bayesian information criterion* (BIC) and *corrected Akaike information criterion* (CAIC) are used to compare the fit of the competitor distributions, where

$$AIC = 2m - 2\ell, \quad BIC = m\ln(n) - 2\ell, \text{ and } CAIC = AIC + 2\left(\frac{m(m+1)}{n-m-1}\right),$$

where ℓ is the natural logarithm of the likelihood function evaluated at the MLEs, n is the number of observations and m is the number of estimated parameters.

The best distribution corresponds to the largest value of ℓ and the lowest values of AIC, BIC and CAIC. Also, the most appropriate distribution will have the lowest K–S and χ^2 statistics and the highest p-value.

7.1. Application 1

In this application, discrete data generated from the *Organisation for Economic Cooperation and Development* (OECD), 2025, https://data-explorer.oecd.org, accessed on 11 November 2025 [56], are used to examine the applicability of the DPh-BXII distribution in data modeling. This application contains time-series data on AIDS incidence cases at the year of diagnosis in some countries that are members in the OECD: UK and Australia for the period from 1981 to 2022.

7.1.1. United Kingdom Data

This section addresses the AIDS number of incidence cases per year of diagnosis for the UK during the period of 1981–2022. Table 15 provides some descriptive measures for this data. A graphical description is displayed in Figure 5.

Table 15. Descriptive measures of AIDS incidence cases for UK.

Min	$y_{0.25}$	y_m	$y_{0.75}$	Max	$\bar{y}$	Var	CD	CS	CK
4	256	734	1011	1879	720	274,225.9	380.8442	0.5778	2.4804

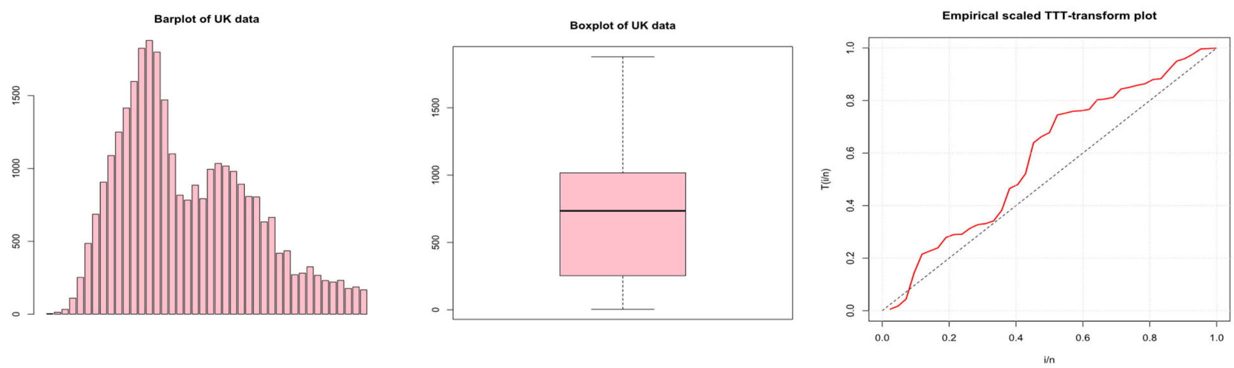


Figure 5. Bar plot, boxplot and empirical scaled TTT-transform plot of UK's number of AIDS incidence cases.

From Table 15, the CD is 380.8442, which indicates over-dispersed data. The CS and CK are 0.5778 and 2.4804, respectively, suggesting that these data have a slightly right skewed tail with a platykurtic distribution. These results are further supported by the graphical description given in Figure 5. The bar plot exhibits that the number of AIDS incidence cases in the UK are concentrated around moderate numbers, with frequencies decreasing toward larger numbers of AIDS cases, indicating a right skewed PMF. The boxplot also displays a longer upper tail than lower tail, which is consistent with the right-skewness and substantial variability in the number of AIDS cases in the UK. Furthermore, the empirical scaled TTT-transform plot is first convex and then initially concave, so the failure of these data exhibits a decreasing pattern followed by an increasing one. Therefore, the empirical scaled TTT-transform plot implies that the data follows a bathtub HRF. This graphical description supports the flexibility of the DPh-BXII distribution for modeling the number of AIDS cases in UK data with such HRF behavior.

Figure 6 exhibits a graphical valuation of how well the DPh-BXII distribution fits the UK's number of AIDS incidence cases. In the Q–Q and P–P plots, the red straight line represents the theoretical quantiles or probabilities of the assumed distribution, whereas the plotted black points correspond to the empirical value computed from the data. For the empirical–theoretical CDF comparison, the smooth curve denotes the theoretical CDF and represents the empirical CDF. From Figure 6 it is indicated that the proposed distribution provides a good fit for the present data. Additionally, Table 16 presents K–S statistics and their corresponding p-values for the DPh-BXII distribution and its competitors along with some information criteria including AIC, BIC and CAIC. From Table 16, the DPh-BXII distribution provides the best fit for the UK's number of AIDS incidence cases as compared to the other models, since it has the lowest K–S statistic and the highest p-value. Also, the DPh-BXII distribution has the lowest values of information criteria. Moreover, Table 17 presents MLEs and their *standard errors* (SEs) of DPh-BXII's parameters.

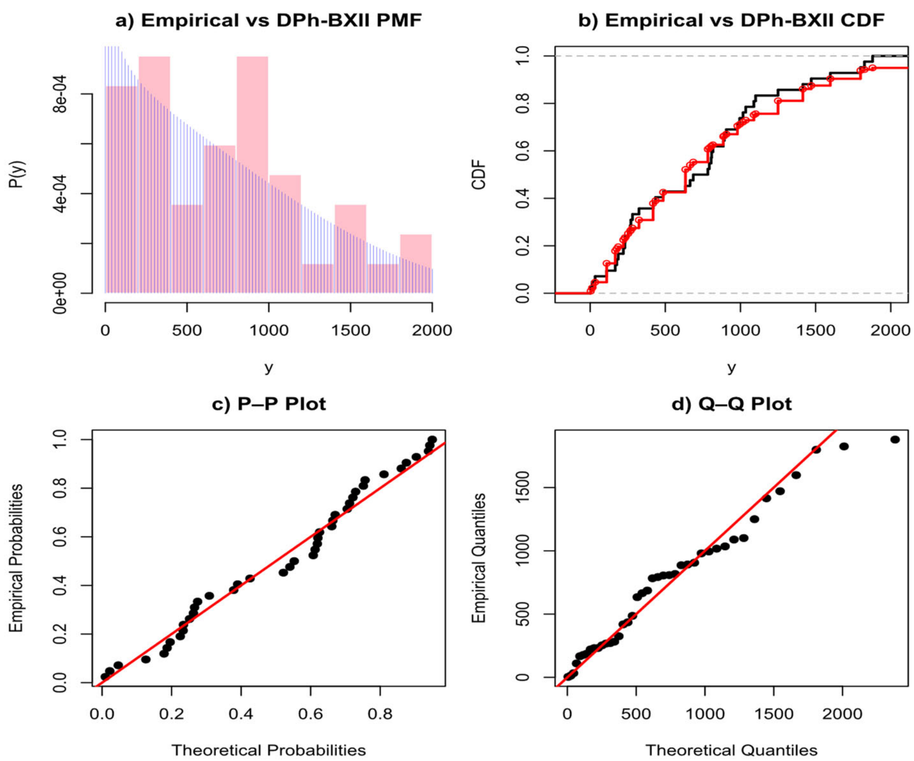


Figure 6. (**a**) Empirical and DPh-BXII PMFs, (**b**) empirical and DPh-BXII CDFs, (**c**) P–P plot and (**d**) Q–Q plot for UK's number of AIDS incidence cases.

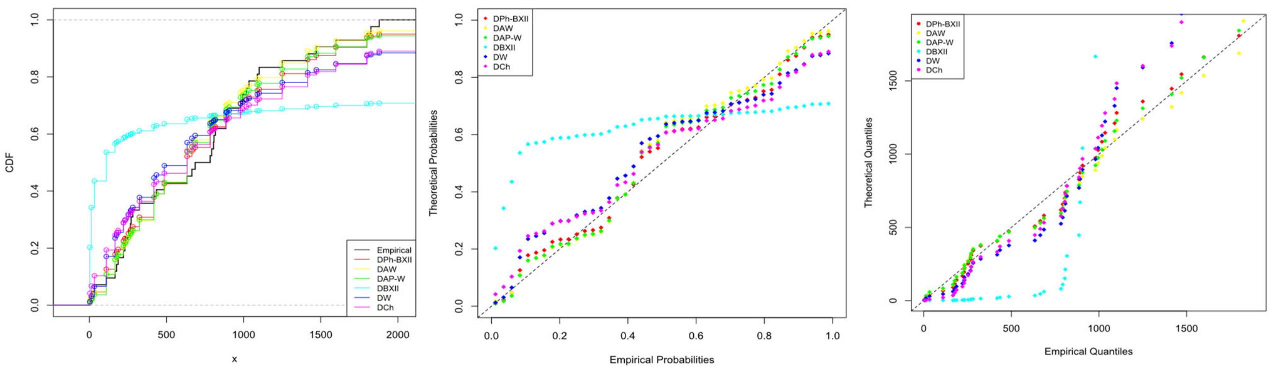
Table 16. K–S statistics and their corresponding p-values for DPh-BXII distribution and its competitors along with the AIC, BIC and CAIC for UK's number of AIDS incidence cases.

Model	K–S	p-value	ℓ	AIC	BIC	CAIC
DPh-BXII	0.1074	0.6776	−315.6811	639.3623	646.3130	640.4434
DAW	0.1336	0.4058	−315.8610	639.7221	646.6728	640.8032
DAP-W	0.1290	0.4490	−316.5201	641.0401	647.9908	642.1212
DBXII	0.4714	4.391×10^{-5}	−375.2806	754.5612	758.0366	754.8689
DW	0.1414	0.3380	−320.6599	645.3198	648.7951	645.6275
DCh	0.1503	0.2709	−320.0896	644.1791	647.6544	644.4868

Table 17. MLEs along with their SEs of the parameters of DPh-BXII distribution for UK's number of AIDS incidence cases.

$\underline{\theta}$	MLE	SE
α	0.8472	0.0112
β	1.0023	0.1267
c	0.2497	0.1717
k	0.0003	0.0375

Additionally, Figure 7 exhibits a comparative visualization of the DPh-BXII distribution and its competitor distributions. Empirical and theoretical CDFs are exhibited in Figure 7a, P–P plots are presented in Figure 7b and Q–Q plots are shown in Figure 7c. These figures indicate that the DPh-BXII distribution offers the best fit of the UK's number of AIDS incidence cases as compared to its competitors.



(**a**) (**b**) (**c**)

Figure 7. (**a**) Empirical and theoretical CDFs, (**b**) P–P plots, and (**c**) Q–Q plots of DPh-BXII distribution and its competitor distributions for UK's number of AIDS incidence cases.

7.1.2. Australia Data

In this application, the AIDS number of cases per year of diagnosis in Australia are addressed during 1982–2012. Table 18 presents some descriptive measures for this data. A graphical description is exhibited in Figure 8.

Table 18. Descriptive measures of AIDS incidence cases for Australia.

Min	$y_{0.25}$	y_m	$y_{0.75}$	Max	$\overline{y}$	Var	CD	CS	CK
1.0	126	232.0	575.0	953.0	346.1	79,713.38	230.2996	0.7764	2.2491

Figure 8. Bar plot, boxplot and empirical scaled TTT-transform plot of Australia's number of AIDS incidence cases.

From Table 18, the CD is 230.2996, indicating it is over-dispersed. The CS and CK are 0.7764 and 2.2491, implying slightly right skewed tail data with a platykurtic distribution. Figure 6 provides graphical support for the results of Table 18. The bar plot shows that the number of AIDS incidence cases in Australia is concentrated around moderate numbers, with frequencies decreasing toward larger numbers of AIDS cases, indicating a right skewed PMF. The boxplot also displays a longer upper tail than the lower one, which is consistent with the positive-skewness and substantial variability in the number of AIDS cases in Australia. Moreover, the empirical scaled TTT-transform plot is first convex and then concave, and finally convex, so the failure of these data displays an increasing, followed by decreasing and then increasing pattern. Therefore, the empirical scaled TTT-transform plot indicates that the data follows a modified bathtub HRF. This graphical description supports the flexibility of the DPh-BXII distribution for modeling the number of AIDS cases in Australian data with such complicated HRF behavior.

Figure 9 provides a graphical assessment of how well the DPh-BXII distribution fits the data of the number of AIDS incidence cases in Australia. In the Q–Q and P–P plots, the red straight line represents the theoretical quantiles or probabilities of the assumed distribution, whereas the plotted black points correspond to the empirical value computed from the data. For the empirical–theoretical CDF comparison, the smooth curve denotes the theoretical CDF and represents the empirical CDF. From these figures, it is evident that the proposed distribution fits the data well. Moreover, Table 19 presents K–S statistics and their corresponding p-values for the DPh-BXII distribution and its competitors, DAW, DAP-W, DBXII, DW and DCh distributions, along with some information criteria, including AIC, BIC and CAIC. In Table 19, the DPh-BXII distribution provides the best fit for Australia's number of AIDS incidence cases as compared to the other models, since it has the lowest K–S statistic and the highest p-value. Also, the DPh-BXII distribution has the lowest information criteria. Table 20 provides MLEs and their SEs of DPh-BXII's parameters.

Table 19. K–S statistics and their corresponding p-values for DPh-BXII distribution and its competitors along with the AIC, BIC and CAIC for Australia's number of AIDS incidence cases.

Model	K–S	p-value	ℓ	AIC	BIC	CAIC
DPh-BXII	0.1080	0.8252	−210.0030	429.9878	435.7237	431.7553
DAW	0.1501	0.4446	−212.8595	433.7190	439.4549	435.2574
DAP-W	0.1759	0.2609	−211.4220	430.8440	436.5800	432.3825
DBXII	0.4623	1.386×10^{-6}	−248.1678	500.3356	503.2036	500.7642
DW	0.2223	0.0795	−214.6018	433.2035	436.0715	433.6321
DCh	0.2302	0.0631	−218.5823	441.1647	444.0327	441.5932

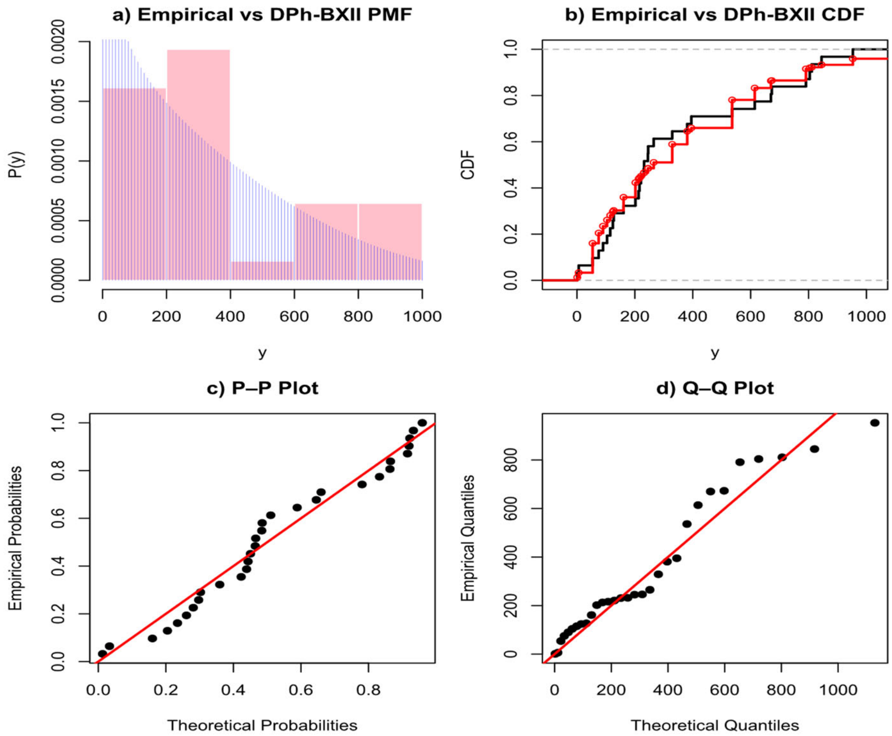
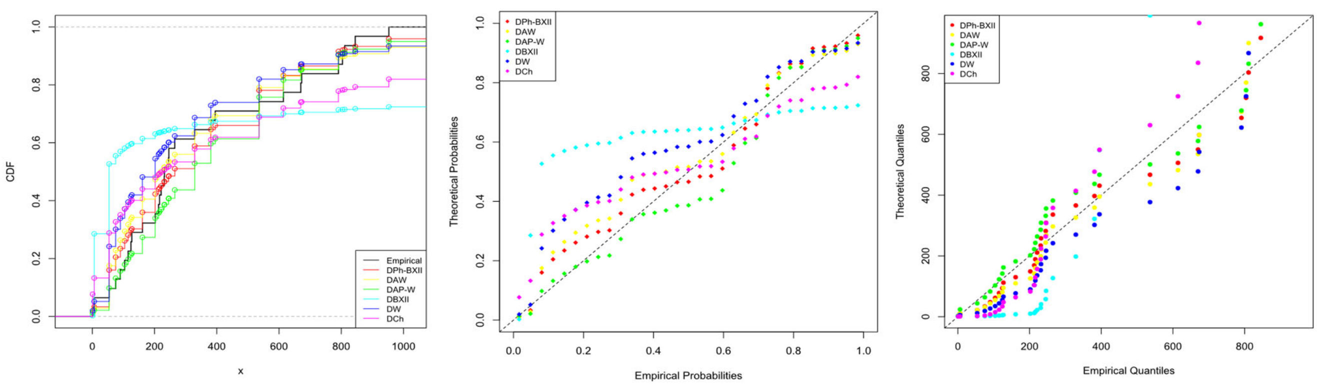


Figure 9. (**a**) Empirical and DPh-BXII PMFs, (**b**) empirical and DPh-BXII CDFs, (**c**) P–P plot and (**d**) Q–Q plot for Australia's number of AIDS incidence cases.

Table 20. MLEs along with their SEs of the parameters of DPh-BXII distribution for Australia's number of AIDS incidence cases.

$\boldsymbol{\theta}$	MLE	SE
α	0.7667	0.0081
β	1.0075	0.0626
c	0.3677	0.1226
k	1.3284×10^{-5}	0.0106

Furthermore, Figure 10 displays a graphical comparison of the DPh-BXII distribution and its competitor distributions.



(**a**) (**b**) (**c**)

Figure 10. (**a**) Empirical and theoretical CDFs, (**b**) P–P plots, and (**c**) Q–Q plots of DPh-BXII distribution and its competitor distributions for Australia's number of AIDS incidence cases.

Empirical and theoretical CDFs are exhibited in Figure 10a, P–P plots are presented in Figure 10b and Q–Q plots are given in Figure 10c. These figures indicate that the DPh-

BXII distribution provides the best fit of Australia's number of AIDS incidence cases as compared to its competitors.

7.2. Application 2

In this application, discrete data generated from the Global Forest Watch Platform, https://www.globalforestwatch.org/dashboards/country/USA/5/?category=fires, accessed on 14 January 2026 [57], are used to examine the applicability of the DPh-BXII distribution in modeling count data. This application contains time-series data on the number of forest fire alerts that occurred in the USA from the 35th week in 2025 to the 3rd week in 2026. A summary of descriptive measures for the number of forest fire alerts in the USA is delivered in Table 21. In Table 21, the CD indicates over-dispersed data. Also, the CS and CK indicate right skewed and platykurtic data.

Table 21. Descriptive measures for number of forest fire alerts in the USA.

Min	$y_{0.25}$	y_m	$y_{0.75}$	Max	$\bar{y}$	Var	CD	CS	CK
2	2.75	6	18	31	10.19	85.36	8.38	0.91	2.58

Also, a graphical description is presented in Figure 11, which gives a bar plot, boxplot and empirical scaled TTT-transform plot of the number of forest fire alerts in the USA. These plots provide graphical support for the results from Table 21. The bar plot indicates that the number of forest fire alerts in the USA has substantial variation across recorded numbers. The small count values of forest fire alerts exhibit notably high alert frequencies, while most count values record relatively low numbers of fire alerts, suggesting a right skewed distribution. Moreover, the boxplot displays a longer upper tail than the lower one, which is consistent with the positive CS and substantial variability in the number of forest fire alerts in the USA. Additionally, the empirical scaled TTT-transform plot reflects a modified bathtub HRF. The DPh-BXII distribution is suitable for modeling such data since it presents diverse patterns of dispersion, skewness and kurtosis. Also, the HRF of the DPh-BXII distribution accommodates different and complicated shapes of failure; therefore, this model will provide a reasonable fit for these data.

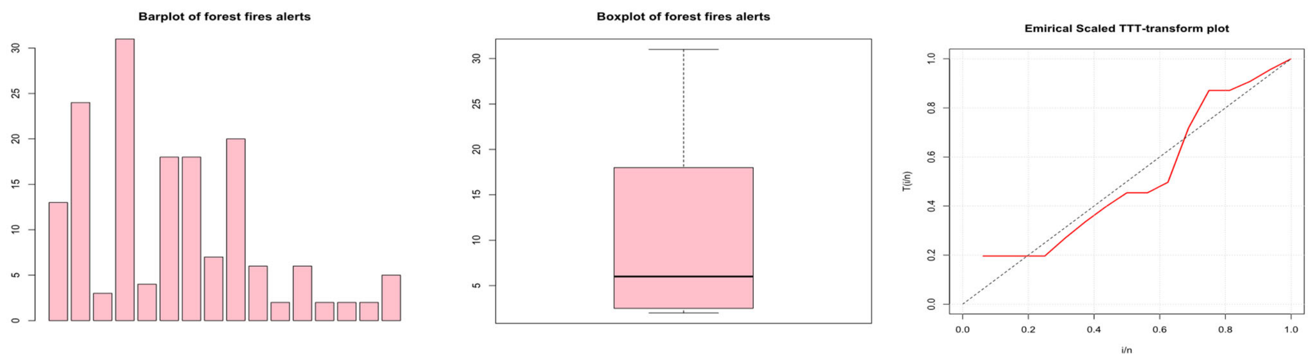


Figure 11. Bar plot, boxplot and empirical scaled TTT-transform plot of number of forest fire alerts in the USA.

The observed and expected frequencies of the number of forest fire alerts in the USA are displayed in Table 22 when fitted to the DPh-BXII distribution. Also, Figure 12 provides a graphical evaluation of how well the DPh-BXII distribution fits the number of forest fire alerts in the USA. From these figures, it is evidenced that the proposed distribution fits the data well. Moreover, Table 23 delivers K–S, χ^2 statistics, *degrees of freedom* (DF) and their corresponding p-values for the DPh-BXII distribution and its competitor distributions, along with some information criteria, including AIC, BIC and CAIC. In Table 23, the DPh-BXII distribution offers the best fit for the number of forest fire alerts in the USA as compared to the other models, since it has the lowest K–S and χ^2 statistics and the highest p-value. Also, the DPh-BXII distribution has the lowest information criteria. Meanwhile, Table 24 presents MLEs and their SEs of DPh-BXII's parameters.

Table 22. Observed and expected frequencies of number of forest fire alerts in the USA.

Y	2	3	4	5	6	7	13	18	20	24	31	Total
Frequency	4	1	1	1	2	1	1	2	1	1	1	16
Expected Frequency	3.3491	2.4528	2.0008	1.7329	1.5572	1.4333	1.0651	0.8547	0.7632	0.5641	0.2269	16

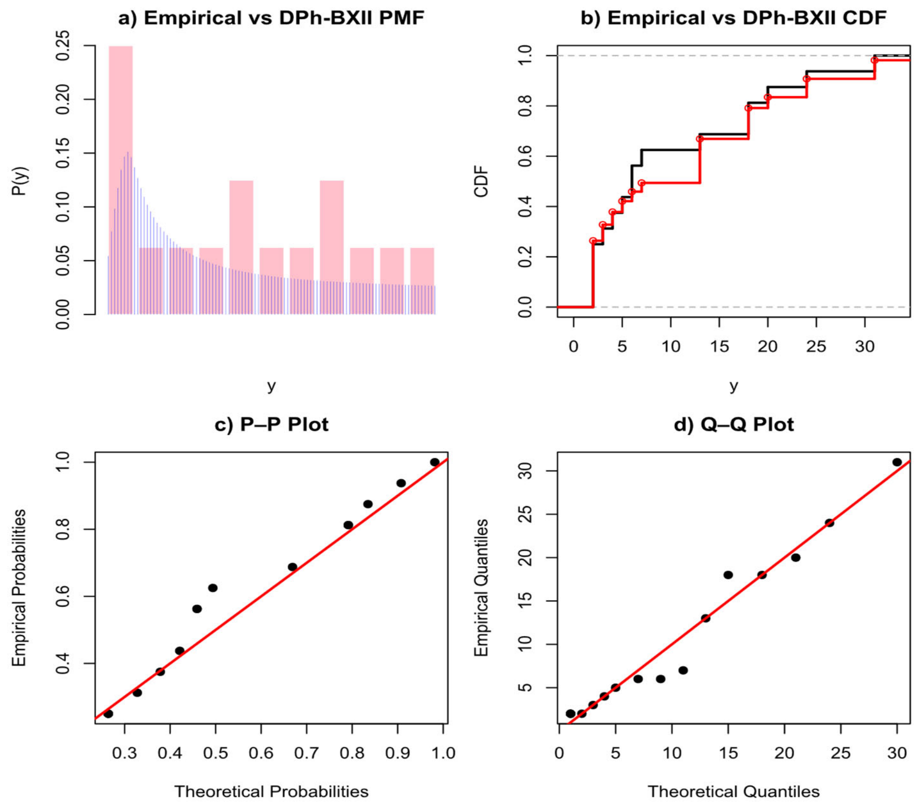


Figure 12. (**a**) Empirical and DPh-BXII PMFs, (**b**) empirical and DPh-BXII CDFs, (**c**) P–P plot and (**d**) Q–Q plot for number of forest fire alerts in the USA.

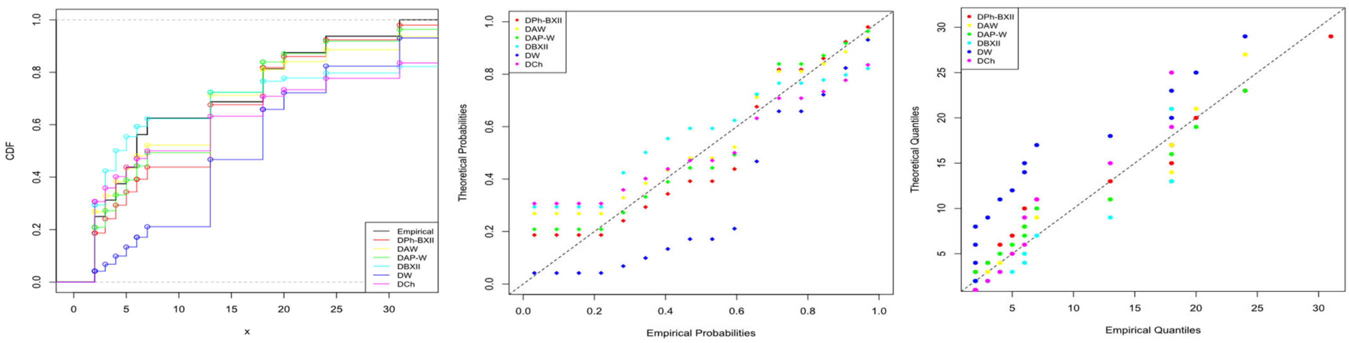
Table 23. $\mathrm{K}-\mathrm{S}$, χ^2 statistics, DFs and the corresponding p-values for DPh-BXII distribution and its competitors along with the AIC, BIC and CAIC for the number of forest fire alerts in the USA.

Model	$K-S$	p-value	χ^2	DF	p-value	ℓ	AIC	BIC	CAIC
DPh-BXII	0.1870	0.6308	5.9529	6	0.4285	−52.5838	113.1676	116.2580	116.8040
DAW	0.2686	0.1985	9.4068	6	0.1520	−54.7888	117.5775	120.6679	121.2139
DAP-W	0.2085	0.4897	11.4258	6	0.0761	−53.7491	115.4981	118.5885	119.1345
DBXII	0.2940	0.1257	28.8403	8	0.0003	−58.9385	121.8770	123.4222	122.1847
DW	0.4140	0.0083	10.9370	8	0.2053	−57.4193	118.8387	120.3838	119.1464
DCh	0.3068	0.0983	9.4823	8	0.3033	−56.9155	117.8311	119.3763	118.7542

Table 24. MLEs along with their SEs of the parameters of DPh-BXII distribution for the number of forest fire alerts in the USA.

$\boldsymbol{\theta}$	MLE	SE
α	1.6403	0.0214
β	1.0048	0.0845
c	10.0079	0.2322
k	0.0252	0.0392

Furthermore, Figure 13 presents a graphical comparison of the DPh-BXII distribution and its competitors. Empirical and theoretical CDFs are displayed in Figure 13a, P–P plots are given in Figure 13b and Q–Q plots are exhibited in Figure 13c. In the Q–Q and P–P plots, the red straight line represents the theoretical quantiles or probabilities of the assumed distribution, whereas the plotted black points correspond to the empirical value computed from the data. For the empirical–theoretical CDF comparison, the smooth curve denotes the theoretical CDF and represents the empirical CDF. These figures indicate that the DPh-BXII distribution provides the best fit for the number of forest fire alerts in the USA as compared to its competitors.



(**a**) (**b**) (**c**)

Figure 13. (**a**) Empirical and theoretical CDFs, (**b**) P–P plots, and (**c**) Q–Q plots of DPh-BXII distribution and its competitor distributions for number of forest fire alerts in the USA.

7.3. Concluding Remarks

From Tables 15–24 and Figures 5–13, the following can be concluded:

- When analyzing the number of AIDS incidence cases, there is high variation in the number of AIDS incidence cases from year to another, with a bathtub failure rate for the UK and a modified bathtub failure rate for Australia. Also, the mean number of incidence cases in the UK is higher than in Australia. Moreover, the maximum observed incidence cases in the UK total 1879 which is more than the maximum number of incidence cases observed in Australia (346.1).
- When analyzing the number of forest fire alerts in the USA as discrete and count data, the average number of forest fire alerts in the USA is 10.19 $\simeq$ 10 alerts with an over-CD. Also, the failure rate is modified bathtub. The minimum and maximum numbers of forest fire alerts are, respectively, 2 and 31.
- When applying the DPh-BXII distribution as a model for these data in order to assess the ability of DPh-BXII in modeling discrete data, it is observed that it provides the best fit in terms of the lowest K–S and χ^2 statistics and the highest p-values compared to DAW, DAP-W, DBXII, DW and DCh distributions. Also, when using information criteria such as AIC, BIC and CAIC, the DPh-BXII distribution presents the lowest value and the largest values of ℓ as compared to the competitor distributions.

- The superior fit of the DPh-BXII model is not only reflected by goodness-of-fit statistics and their corresponding *p*-values but also supported by its ability to present the key empirical characteristics of the considered medical and environmental datasets, including dispersion, skewness, kurtosis and failure behavior.
- The MLEs of the parameters for the presented applications have small SEs, indicating the efficiency of these estimates.

8. Conclusions and Potential Future Work

In this paper, a discrete analog of the Ph-BXII competing risks model is introduced by applying the general approach of discretization. The main characteristic functions of the proposed model are derived: PMF, SF, CDF, HRF, AHRF, and RHRF. A graphical description of the DPh-BXII distribution is presented for the PMF, HRF and AHRF. The PMF displays decreasing, unimodal and decreasing–unimodal patterns, while its HRF and AHRF exhibit different and important shapes: decreasing, bathtub (Vtub), modified bathtub and unimodal shapes. Through these shapes, the flexibility and diversity in the shapes of the DPh-BXII distribution is demonstrated. Therefore, the DPh-BXII distribution can provide a good fit for several types of discrete data and count data. The main statistical properties of the DPh-BXII distribution are studied. Moreover, the parameters and the main characteristic functions of the DPh-BXII distribution are estimated through the ML method. Furthermore, the 95% ACIs and PBCIs are considered. Likewise, point and interval estimation of the presented entropy and extropy measures is discussed. Additionally, a simulation study is presented to assess the performance of the delivered MLEs. Finally, the applicability of the DPh-BXII distribution is examined though medical and environmental applications. From these applications, it is demonstrated that the DPh-BXII distribution is suitable for data modeling and for modeling count data. Also, it provides the best fit compared to its competitor distributions.

Future research could investigate new discrete analogs of the Ph-BXII distribution by applying other discretization methods, classified in [8]. Alternatively, estimation problems of the parameters, SF, HRF, AHRF and RHRF, of the DPh-BXII distribution could be studied using different techniques. Also, Bayesian and E-Bayesian approaches could be considered for estimation purposes. Moreover, future work could explore the estimation of the parameters, SF, HRF and AHRF, based on censored, hybrid or progressive censored samples. Furthermore, prediction of future observations from the DPh-BXII distribution could be considered for future research. Finally, new discrete competing risks models can be derived by discretizing different competing risks distributions from literature.

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Data Availability Statement: The data presented in this study are openly available with the Organisation for Economic Cooperation and Development (OECD) at https://data-explorer.oecd.org, accessed on 11 November 2025 and the Global Forest Watch Platform at https://www.globalforestwatch.org/dashboards/country/USA/5/?category=fires, accessed on 14 January 2026.

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