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Jointly $\mathcal{A}$-Paranormal Tuples of Operators in Semi-Hilbertian Spaces

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Abstract

This paper introduces the concept of jointly $\mathcal{A}$-paranormal operator tuples acting on semi-Hilbertian spaces. We establish foundational operator inequalities and structural properties governing this new class. Specifically, we prove that joint $\mathcal{A}$-paranormality is preserved under tensor products and stable under taking powers. Furthermore, we demonstrate that every commuting jointly $\mathcal{A}$-hyponormal tuple is jointly $\mathcal{A}$-paranormal. Since every commuting jointly $\mathcal{A}$-paranormal tuple is jointly $\mathcal{A}$-normaloid, we complete the hierarchy of multivariable operator classes within the semi-Hilbertian context.

Keywords: semi-Hilbertian space; jointly $\mathcal{A}$-paranormal tuple; jointly $\mathcal{A}$-hyponormal tuple; jointly $\mathcal{A}$-normaloid tuple; joint operator $\mathcal{A}$-seminorm; operator inequalities; tensor product

MSC: 47B20; 47A13; 46C05; 47A30; 47A63

1. Introduction

The study of multivariable operators on Hilbert spaces has evolved considerably [1–3]. With attention shifting toward specific classes of operator tuples [4–7], the properties of normal and hyponormal tuples have been rigorously established [8–13]. Other notable advancements include the analysis of spherical transforms [14,15] and the transition to joint paranormality [16,17].

While classical spaces rely on standard inner products, extending these multivariable concepts to semi-Hilbertian spaces provides a substantial generalization [18–20]. Employing a positive semi-definite sesquilinear form induced by a bounded positive operator $\mathcal{A}$ establishes a tailored framework for accommodating degenerate inner products. Initial investigations generalized geometric and spectral properties, such as the joint numerical range [21] and the joint spectral radius [22]. Subsequent efforts formalized operator classes within this setting, including jointly $\mathcal{A}$-normal [16], jointly $\mathcal{A}$-hyponormal [8,10], and $(\mathcal{A}, m)$-isometric tuples [23].

To properly situate our contribution, we recall that a bounded linear operator Y on a complex Hilbert space $\mathscr{E}$ admitting an $\mathcal{A}$-adjoint is called $\mathcal{A}$-quasinormal if Y commutes with $Y^{\sharp_{\mathcal{A}}}Y$ in the $\mathcal{A}$-sense (see [24]) this property places it strictly between $\mathcal{A}$-normal and $\mathcal{A}$-hyponormal operators. Since every commuting jointly $\mathcal{A}$-hyponormal tuple is jointly $\mathcal{A}$-paranormal, as we establish in Theorem 2, the jointly $\mathcal{A}$-quasinormal class likewise resides above joint $\mathcal{A}$-paranormality in the hierarchy. It is also worth noting that $(\mathcal{A}, m)$-isometric tuples [23], while sharing the semi-Hilbertian setting, are defined through an entirely

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different mechanism—namely, the vanishing of an alternating sum of $\mathcal{A}$-seminorms—and therefore do not fit directly into the hyponormal–paranormal–normaloid chain.

In the classical setting, an elegant hierarchy bridges the gap between normal and normaloid operators in the single-variable setting [25–27]. When Stanković [17] introduced jointly paranormal tuples, they were positioned as an intermediate class between jointly hyponormal and jointly normaloid tuples. In the semi-Hilbertian framework, recent results [8] established that every commuting jointly $\mathcal{A}$-hyponormal tuple is jointly $\mathcal{A}$-normaloid. However, the specific intermediate class responsible for this implication had not been identified. The principal objective of this paper is to resolve this gap by showing that joint $\mathcal{A}$-paranormality is precisely the missing theoretical bridge.

To this end, we introduce the class of jointly $\mathcal{A}$-paranormal tuples and rigorously establish the following chain of multivariable operator inclusions:

$$\text{jointly } \mathcal{A}\text{-hyponormal} \implies \text{jointly } \mathcal{A}\text{-paranormal} \implies \text{jointly } \mathcal{A}\text{-normaloid}.$$

We emphasize that working in the semi-Hilbertian framework is not a routine adaptation of classical arguments. It introduces genuine analytical obstructions: the semi-inner product may be degenerate, the existence of the $\mathcal{A}$-adjoint requires Douglas's range inclusion theorem [28], and the identity $(Y^{\sharp_{\mathcal{A}}})^{\sharp_{\mathcal{A}}} = \Pi_{\mathcal{A}} Y \Pi_{\mathcal{A}}$, where $\Pi_{\mathcal{A}}$ denotes the orthogonal projection onto the closure of the range of $\mathcal{A}$, reflects a structural degeneracy with no classical counterpart. Moreover, the absence of a Banach or C^*-algebra structure prevents the direct use of standard functional calculus. Establishing the hierarchy above therefore requires new, framework-specific operator inequality techniques. Within this framework, we derive the fundamental operator inequalities and algebraic properties defining this new class, and we show in particular that joint $\mathcal{A}$-paranormality is stable under taking powers and preserved under tensor products.

The paper is organized as follows. Section 2 reviews the foundational concepts of semi-Hilbertian spaces, including the role of Douglas's range inclusion theorem in defining the $\mathcal{A}$-adjoint, and recalls the relevant multivariable operator definitions. Section 3 presents our main results, detailing the fundamental operator inequalities, structural properties, and generalized log-convexity for jointly $\mathcal{A}$-paranormal tuples. It further establishes stability under powers and tensor products and completes the inclusion hierarchy by proving the position of this class between jointly $\mathcal{A}$-hyponormal and jointly $\mathcal{A}$-normaloid tuples.

2. Preliminary Results

Let $(\mathscr{E}, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space equipped with the norm $\| \cdot \|$ induced by its inner product. The C^*-algebra of all bounded linear operators on $\mathscr{E}$ is denoted by $\mathfrak{L}(\mathscr{E})$. For an operator $Y \in \mathfrak{L}(\mathscr{E})$, its range, nullspace, and adjoint are denoted by $\mathrm{Ran}(Y)$, $\mathcal{N}(Y)$, and Y^*, respectively. An operator Y is positive, denoted by $Y \geq \mathbf{0}$, if $\langle Y\xi, \xi \rangle \geq 0$ for all $\xi \in \mathscr{E}$, where $\mathbf{0}$ denotes the zero operator. For operators X, Y, we write $X \geq_{\mathcal{A}} Y$ if $X - Y \geq_{\mathcal{A}} \mathbf{0}$.

Consider a nonzero positive operator $\mathcal{A} \in \mathfrak{L}(\mathscr{E})$. This operator induces a positive semi-definite sesquilinear form, defined as follows:

$$\langle \cdot, \cdot \rangle_{\mathcal{A}} : \mathscr{E} \times \mathscr{E} \to \mathbb{C}, \quad (\xi_1, \xi_2) \mapsto \langle \mathcal{A}\xi_1, \xi_2 \rangle.$$

The pair $(\mathscr{E}, \| \cdot \|_{\mathcal{A}})$ is called a semi-Hilbert space, with the associated seminorm given by $\|\xi\|_{\mathcal{A}} = \sqrt{\langle \xi, \xi \rangle_{\mathcal{A}}}$ for any $\xi \in \mathscr{E}$. We define the unit $\mathcal{A}$-sphere in $\mathscr{E}$ as $\mathbb{S}_{\mathscr{E}}^{\mathcal{A}} := \{\xi \in \mathscr{E} \mid \|\xi\|_{\mathcal{A}} = 1\}$. In the special case where $\mathcal{A} = I_{\mathscr{E}}$, this reduces to the standard unit sphere $\mathbb{S}_{\mathscr{E}}$. It is important to note that $(\mathscr{E}, \| \cdot \|_{\mathcal{A}})$ is generally not a normed or complete space.

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It forms a true Hilbert space if and only if $\mathcal{A}$ is injective and has a closed range; that is, $\mathrm{Ran}(\mathcal{A}) = \overline{\mathrm{Ran}(\mathcal{A})}$.

An operator $Z \in \mathfrak{L}(\mathscr{E})$ is an $\mathcal{A}$-adjoint of $Y \in \mathfrak{L}(\mathscr{E})$ if $\mathcal{A}Z = Y^*\mathcal{A}$. Applying Douglas's theorem [28] reveals that Y admits an $\mathcal{A}$-adjoint if and only if $\mathrm{Ran}(Y^*\mathcal{A}) \subseteq \mathrm{Ran}(\mathcal{A})$. We denote the space of all such operators by $\mathfrak{L}_{\mathcal{A}}(\mathscr{E})$. For any $Y \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})$, the unique reduced solution is $Y^{\sharp_{\mathcal{A}}} = \mathcal{A}^{+}Y^*\mathcal{A}$ [19]. For operators in the broader space of $\mathcal{A}$-bounded operators $\mathfrak{L}_{\sqrt{\mathcal{A}}}(\mathscr{E})$, the $\mathcal{A}$-seminorm is directly defined as [21]:

$$\|Y\|_{\mathcal{A}} = \sup_{\xi \in \mathbb{S}_{\mathscr{E}}^{\mathcal{A}}} \|Y\xi\|_{\mathcal{A}} = \sup_{\xi_1, \xi_2 \in \mathbb{S}_{\mathscr{E}}^{\mathcal{A}}} |\langle Y\xi_1, \xi_2 \rangle_{\mathcal{A}}|.$$

From this identity, we find $\|Y\|_{\mathcal{A}} = \|Y^{\sharp_{\mathcal{A}}}\|_{\mathcal{A}}$ for any $Y \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})$.

For any $Y, Z \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})$ and $\lambda, \gamma \in \mathbb{C}$, we have $\lambda Y + \gamma Z \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})$ and $YZ \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})$, with $(\lambda Y + \gamma Z)^{\sharp_{\mathcal{A}}} = \overline{\lambda} Y^{\sharp_{\mathcal{A}}} + \overline{\gamma} Z^{\sharp_{\mathcal{A}}}$ and $(YZ)^{\sharp_{\mathcal{A}}} = Z^{\sharp_{\mathcal{A}}} Y^{\sharp_{\mathcal{A}}}$ [18,19]. By induction, for any $i \in \mathbb{N}$, we have $Y^i \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})$ and $(Y^i)^{\sharp_{\mathcal{A}}} = (Y^{\sharp_{\mathcal{A}}})^i$. A double application of the $\mathcal{A}$-adjoint yields $(Y^{\sharp_{\mathcal{A}}})^{\sharp_{\mathcal{A}}} = \Pi_{\mathcal{A}} Y \Pi_{\mathcal{A}}$, where $\Pi_{\mathcal{A}}$ is the orthogonal projection onto $\overline{\mathrm{Ran}(\mathcal{A})}$. Due to the generalized C^*-identity, it follows that $\|Y^{\sharp_{\mathcal{A}}} Y\|_{\mathcal{A}} = \|Y Y^{\sharp_{\mathcal{A}}}\|_{\mathcal{A}} = \|Y\|_{\mathcal{A}}^2$.

An operator $Y \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})$ is $\mathcal{A}$-selfadjoint if $\mathcal{A}Y = Y^*\mathcal{A}$. An operator $U \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})$ is an $\mathcal{A}$-isometry if $\|Uv\|_{\mathcal{A}} = \|v\|_{\mathcal{A}}$ for all $v \in \mathscr{E}$, which holds if and only if $U^{\sharp_{\mathcal{A}}} U = \Pi_{\mathcal{A}}$ [18]. Since an $\mathcal{A}$-unitary operator U has both U and $U^{\sharp_{\mathcal{A}}}$ as $\mathcal{A}$-isometries, it preserves the $\mathcal{A}$-seminorm.

Following Guesba et al. [10], a q-tuple $\mathbf{Y} = (Y_1, \ldots, Y_q) \in \mathfrak{L}_{\sqrt{\mathcal{A}}}(\mathscr{E})^q$ is jointly $\mathcal{A}$-hyponormal if the operator matrix $\mathcal{C}_{\mathcal{A}}(\mathbf{Y}) = \left([Y_j^{\sharp_{\mathcal{A}}}, Y_i]\right)_{i,j \in \{1,\ldots,q\}}$ is $\widetilde{\mathcal{A}}$-positive, where $\widetilde{\mathcal{A}} = \mathrm{diag}(\mathcal{A}, \ldots, \mathcal{A})$ on $\mathscr{E}^q$. Equivalently,

$$\sum_{l=1}^{q} \sum_{s=1}^{q} \langle Y_l \xi_s, Y_s \xi_l \rangle_{\mathcal{A}} \geq \left\| \sum_{l=1}^{q} Y_l^{\sharp_{\mathcal{A}}} \xi_l \right\|_{\mathcal{A}}^2 \tag{1}$$

for any choice of vectors $\xi_1, \ldots, \xi_q \in \mathscr{E}$. Note that commutativity is not required. For $\mathcal{A} = I_{\mathscr{E}}$, this reduces to Athavale's definition [29]. We also note that in [10], $\mathcal{C}_{\mathcal{A}}(\mathbf{Y})$ should strictly be $\widetilde{\mathcal{A}}$-positive (not merely $\mathcal{A}$-positive); see [8] for details.

For any operator $Y \in \mathfrak{L}_{\sqrt{\mathcal{A}}}(\mathscr{E})$, the $\mathcal{A}$-spectral radius and the $\mathcal{A}$-numerical radius are defined [20,30] as

$$r_{\mathcal{A}}(Y) := \inf_{i \in \mathbb{N}} \|Y^i\|_{\mathcal{A}}^{\frac{1}{i}} = \lim_{i \to \infty} \|Y^i\|_{\mathcal{A}}^{\frac{1}{i}}, \quad \omega_{\mathcal{A}}(Y) := \sup_{\xi \in \mathbb{S}_{\mathscr{E}}^{\mathcal{A}}} |\langle Y\xi, \xi \rangle_{\mathcal{A}}|.$$

They satisfy the inequalities $r_{\mathcal{A}}(Y) \leq \omega_{\mathcal{A}}(Y) \leq \|Y\|_{\mathcal{A}}$ [30].

Let $\mathbf{Y} = (Y_1, \ldots, Y_q)$ represent a q-tuple of operators in $\mathfrak{L}_{\sqrt{\mathcal{A}}}(\mathscr{E})^q$. The joint operator $\mathcal{A}$-seminorm and the joint $\mathcal{A}$-numerical radius are given by:

$$\|\mathbf{Y}\|_{\mathcal{A}} := \sup_{\xi \in \mathbb{S}_{\mathscr{E}}^{\mathcal{A}}} \sqrt{\sum_{i=1}^{q} \|Y_i \xi\|_{\mathcal{A}}^2} \quad \text{and} \quad \omega_{\mathcal{A}}(\mathbf{Y}) := \sup_{\xi \in \mathbb{S}_{\mathscr{E}}^{\mathcal{A}}} \sqrt{\sum_{i=1}^{q} |\langle Y_i \xi, \xi \rangle_{\mathcal{A}}|^2}.$$

For a commuting q-tuple $\mathbf{Y} = (Y_1, \ldots, Y_q) \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})^q$, the joint $\mathcal{A}$-spectral radius is defined [16] through the limit:

$$r_{\mathcal{A}}(\mathbf{Y}) := \lim_{i \to \infty} \left\| \sum_{\substack{|\nu| = i, \\ \nu \in \mathbb{N}^q}} \frac{i!}{\nu!} \left(\mathbf{Y}^{\sharp_{\mathcal{A}}}\right)^{\nu} \mathbf{Y}^{\nu} \right\|_{\mathcal{A}}^{\frac{1}{2i}},$$

where $\mathbf{Y}^{\sharp_{\mathcal{A}}} = (Y_1^{\sharp_{\mathcal{A}}}, \ldots, Y_q^{\sharp_{\mathcal{A}}})$, and for any multi-index $\nu = (\nu_1, \ldots, \nu_q) \in \mathbb{N}^q$, we use the notations $\mathbf{Y}^{\nu} := \prod_{l=1}^{q} Y_l^{\nu_l}$, $|\nu| := \sum_{i=1}^{q} \nu_i$ and $\nu! := \prod_{i=1}^{q} \nu_i!$.

A commuting q-tuple $\mathbf{Y} = (Y_1, \ldots, Y_q) \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})^q$ is said to be *jointly $\mathcal{A}$-normaloid* [4] if

$$r_{\mathcal{A}}(\mathbf{Y}) = \|\mathbf{Y}\|_{\mathcal{A}}.$$

A useful tool in our analysis is the completely positive linear map $\Theta_{\mathbf{Y},\mathcal{A}} : \mathfrak{L}_{\mathcal{A}}(\mathscr{E}) \to \mathfrak{L}_{\mathcal{A}}(\mathscr{E})$, defined for a tuple $\mathbf{Y} = (Y_1, \ldots, Y_q) \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})^q$ by:

$$Z \mapsto \Theta_{\mathbf{Y},\mathcal{A}}(Z) := \sum_{l=1}^{q} Y_l^{\sharp_{\mathcal{A}}} Z Y_l.$$

Its powers are defined inductively by $\Theta^0_{\mathbf{Y},\mathcal{A}}(Z) = Z$ and $\Theta^i_{\mathbf{Y},\mathcal{A}}(Z) := \Theta_{\mathbf{Y},\mathcal{A}}[\Theta^{i-1}_{\mathbf{Y},\mathcal{A}}(Z)]$ for all $i \in \mathbb{N}$.

Baklouti et al. [16] proved that if $\mathbf{Y} = (Y_1, \ldots, Y_q) \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})^q$ forms a commuting q-tuple, its joint $\mathcal{A}$-spectral radius simplifies to the limit:

$$r_{\mathcal{A}}(\mathbf{Y}) = \lim_{i \to \infty} \|\Theta^i_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\|_{\mathcal{A}}^{\frac{1}{2^i}}. \tag{2}$$

Additionally, an identity established in [16] states that for any tuple $\mathbf{Y} = (Y_1, \ldots, Y_q) \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})^q$, its joint $\mathcal{A}$-seminorm is given by:

$$\|\mathbf{Y}\|_{\mathcal{A}} = \left\| \sum_{l=1}^{q} Y_l^{\sharp_{\mathcal{A}}} Y_l \right\|_{\mathcal{A}}^{\frac{1}{2}} = \|\Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\|_{\mathcal{A}}^{\frac{1}{2}}. \tag{3}$$

Let $\mathbb{U}_q$ represent the open unit ball within $\mathbb{C}^q$, governed by the standard Euclidean norm $\|\tau\|_2 := \sqrt{\sum_{i=1}^{q} |\tau_i|^2}$. Following the framework in [21], we formulate the following quantity for any $\mathbf{Y} = (Y_1, \ldots, Y_q) \in \mathfrak{L}_{\sqrt{\mathcal{A}}}(\mathscr{E})^q$:

$$\|\mathbf{Y}\|_{e,\mathcal{A}} = \sup_{\tau \in \mathbb{U}_q} \left\| \sum_{i=1}^{q} \tau_i Y_i \right\|_{\mathcal{A}}.$$

Notice that for any commuting q-tuple $\mathbf{Y} = (Y_1, \ldots, Y_q) \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})^q$, we have

$$r_{\mathcal{A}}(\mathbf{Y}) \leq \max\left\{ \frac{1}{2\sqrt{q}} \|\mathbf{Y}\|_{\mathcal{A}}, r_{\mathcal{A}}(\mathbf{Y}) \right\} \leq \omega_{\mathcal{A}}(\mathbf{Y}) \leq \|\mathbf{Y}\|_{e,\mathcal{A}} \leq \|\mathbf{Y}\|_{\mathcal{A}} \tag{4}$$

(see [21]). We note that the inequalities in (4) can simultaneously be strict even if $q = 1$.

3. Main Results

We begin by introducing the notion of jointly $\mathcal{A}$-paranormal tuples.

Definition 1. *An operator q-tuple $\mathbf{Y} = (Y_1, \ldots, Y_q) \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})^q$ is called jointly $\mathcal{A}$-paranormal if the inequality*

$$\sum_{l=1}^{q} \sum_{s=1}^{q} \|Y_l Y_s \xi\|_{\mathcal{A}}^2 \geq \left(\sum_{l=1}^{q} \|Y_l \xi\|_{\mathcal{A}}^2 \right)^2$$

holds for all unit vectors $\xi \in \mathbb{S}_{\mathscr{E}}^{\mathcal{A}}$.

Note that if $\mathcal{A} = I_{\mathscr{E}}$, this inequality reduces to the classical definition of jointly paranormal tuples, which was recently introduced by Stanković [17].

The following theorem provides equivalent conditions for joint $\mathcal{A}$-paranormality.

Theorem 1. *For any tuple* $\mathbf{Y} = (Y_1, \ldots, Y_q) \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})^q$, *the following are equivalent:*

(i) $\mathbf{Y}$ *is jointly* $\mathcal{A}$*-paranormal;*

(ii) For all $\xi \in \mathbb{S}^{\mathcal{A}}_{\mathscr{E}}$, *the inequality holds:*

$$\sum_{l=1}^{q}\sum_{s=1}^{q} \|Y_l Y_s \xi\|^2_{\mathcal{A}} \geq \left(\sum_{l=1}^{q} \|Y_l \xi\|^2_{\mathcal{A}}\right)^2;$$

(iii) For all $\xi \in \mathbb{S}^{\mathcal{A}}_{\mathscr{E}}$, *we have:*

$$\langle \Theta^2_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi, \xi\rangle_{\mathcal{A}} \geq \langle \Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi, \xi\rangle^2_{\mathcal{A}};$$

(iv) For all $\xi \in \mathscr{E}$, *the inequality holds:*

$$\langle \Theta^2_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi, \xi\rangle_{\mathcal{A}}\langle \xi, \xi\rangle_{\mathcal{A}} \geq \langle \Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi, \xi\rangle^2_{\mathcal{A}};$$

(v) For all $\lambda > 0$, *the following operator inequality holds:*

$$\Theta^2_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}}) - 2\lambda\, \Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}}) + \lambda^2 I_{\mathscr{E}} \geq_{\mathcal{A}} \mathbf{0}. \tag{5}$$

Proof. Since Definition 1 aligns exactly with *(ii)*, the equivalence between *(i)* and *(ii)* is immediate.

From the definition of $\Theta_{\mathbf{Y},\mathcal{A}}$, it follows that

$$\begin{aligned}\langle \Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi, \xi\rangle_{\mathcal{A}} &= \sum_{l=1}^{q} \langle Y_l^{\sharp_{\mathcal{A}}} Y_l \xi, \xi\rangle_{\mathcal{A}} \\ &= \sum_{l=1}^{q} \langle Y_l \xi, Y_l \xi\rangle_{\mathcal{A}} \\ &= \sum_{l=1}^{q} \|Y_l \xi\|^2_{\mathcal{A}}.\end{aligned}$$

Applying $\Theta_{\mathbf{Y},\mathcal{A}}$ twice gives

$$\begin{aligned}\Theta^2_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}}) &= \Theta_{\mathbf{Y},\mathcal{A}}\left(\sum_{s=1}^{q} Y_s^{\sharp_{\mathcal{A}}} Y_s\right) \\ &= \sum_{l=1}^{q} Y_l^{\sharp_{\mathcal{A}}}\left(\sum_{s=1}^{q} Y_s^{\sharp_{\mathcal{A}}} Y_s\right) Y_l \\ &= \sum_{l=1}^{q}\sum_{s=1}^{q} (Y_s Y_l)^{\sharp_{\mathcal{A}}} (Y_s Y_l).\end{aligned}$$

Taking the $\mathcal{A}$-inner product with ξ yields

$$\begin{aligned}\langle \Theta^2_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi, \xi\rangle_{\mathcal{A}} &= \sum_{l=1}^{q}\sum_{s=1}^{q} \langle (Y_s Y_l)^{\sharp_{\mathcal{A}}} Y_s Y_l \xi, \xi\rangle_{\mathcal{A}} \\ &= \sum_{l=1}^{q}\sum_{s=1}^{q} \|Y_s Y_l \xi\|^2_{\mathcal{A}}.\end{aligned}$$

Since the terms match exactly, *(ii)* and *(iii)* are equivalent.

Because $\xi \in \mathbb{S}_{\mathscr{E}}^{\mathcal{A}}$ implies $\langle \xi, \xi \rangle_{\mathcal{A}} = 1$, *(iv)* yields *(iii)*. Conversely, assume *(iii)* holds. Since *(iv)* holds trivially for $\|\xi\|_{\mathcal{A}} = 0$, we consider $\xi \in \mathscr{E}$ with $\|\xi\|_{\mathcal{A}} \neq 0$. Substituting the unit vector $\eta = \frac{\xi}{\|\xi\|_{\mathcal{A}}}$ into *(iii)* produces

$$\left\langle \Theta_{\mathbf{Y},\mathcal{A}}^2(I_{\mathscr{E}}) \frac{\xi}{\|\xi\|_{\mathcal{A}}}, \frac{\xi}{\|\xi\|_{\mathcal{A}}} \right\rangle_{\mathcal{A}} \geq \left\langle \Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}}) \frac{\xi}{\|\xi\|_{\mathcal{A}}}, \frac{\xi}{\|\xi\|_{\mathcal{A}}} \right\rangle_{\mathcal{A}}^2.$$

Multiplying both sides by $\|\xi\|_{\mathcal{A}}^4$ leads to *(iv)*.

For each $\xi \in \mathbb{S}_{\mathscr{E}}^{\mathcal{A}}$, the polynomial

$$p_{\xi}(\lambda) = \lambda^2 - 2\lambda \langle \Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi, \xi \rangle_{\mathcal{A}} + \langle \Theta_{\mathbf{Y},\mathcal{A}}^2(I_{\mathscr{E}})\xi, \xi \rangle_{\mathcal{A}},$$

has the discriminant

$$D_{\xi} = 4\langle \Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi, \xi \rangle_{\mathcal{A}}^2 - 4\langle \Theta_{\mathbf{Y},\mathcal{A}}^2(I_{\mathscr{E}})\xi, \xi \rangle_{\mathcal{A}}.$$

Since $p_{\xi}(\lambda) \geq 0$ for all $\lambda > 0$ if and only if $D_{\xi} \leq 0$, condition *(iii)* ensures $D_{\xi} \leq 0$. Given $D_{\xi} \leq 0$, we have

$$\langle (\Theta_{\mathbf{Y},\mathcal{A}}^2(I_{\mathscr{E}}) - 2\lambda\, \Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}}) + \lambda^2 I_{\mathscr{E}})\xi, \xi \rangle_{\mathcal{A}} \geq 0$$

for all $\xi \in \mathbb{S}_{\mathscr{E}}^{\mathcal{A}}$ and all $\lambda > 0$, which is equivalent to *(v)*. □

The following theorem proves that commuting jointly $\mathcal{A}$-hyponormal tuples are jointly $\mathcal{A}$-paranormal.

Theorem 2. *Let* $\mathbf{Y} = (Y_1, \ldots, Y_q) \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})^q$ *be a jointly* $\mathcal{A}$*-hyponormal* q*-tuple of commuting operators. Then* $\mathbf{Y}$ *is jointly* $\mathcal{A}$*-paranormal.*

Proof. Let $\xi \in \mathbb{S}_{\mathscr{E}}^{\mathcal{A}}$. Substituting $\xi_l = Y_l\xi$ for each $l \in \{1, \ldots, q\}$ into (1) provides a suitable choice of vectors. Since the operators commute, it follows that $Y_l\xi_s = Y_lY_s\xi = Y_sY_l\xi = Y_s\xi_l$. Inserting these vectors into the left-hand side of (1) gives

$$\begin{aligned} \sum_{l=1}^{q}\sum_{s=1}^{q} \langle Y_l\xi_s, Y_s\xi_l \rangle_{\mathcal{A}} &= \sum_{l=1}^{q}\sum_{s=1}^{q} \langle Y_lY_s\xi, Y_sY_l\xi \rangle_{\mathcal{A}} \\ &= \sum_{l=1}^{q}\sum_{s=1}^{q} \|Y_lY_s\xi\|_{\mathcal{A}}^2. \end{aligned} \tag{6}$$

Evaluating the right-hand side of (1) with these vectors yields

$$\left\| \sum_{l=1}^{q} Y_l^{\sharp_{\mathcal{A}}} Y_l\xi \right\|_{\mathcal{A}}^2 = \|\Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi\|_{\mathcal{A}}^2. \tag{7}$$

Applying the Cauchy–Schwarz inequality along with $\|\xi\|_{\mathcal{A}} = 1$ leads to

$$\begin{aligned} \|\Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi\|_{\mathcal{A}}^2 &\geq |\langle \Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi, \xi \rangle_{\mathcal{A}}|^2 \\ &= \left(\sum_{l=1}^{q} \|Y_l\xi\|_{\mathcal{A}}^2 \right)^2. \end{aligned} \tag{8}$$

Combining (6)–(8) with (1) produces

$$\sum_{l=1}^{q}\sum_{s=1}^{q}\|Y_lY_s\xi\|_{\mathcal{A}}^2 \geq \left(\sum_{l=1}^{q}\|Y_l\xi\|_{\mathcal{A}}^2\right)^2.$$

Since this holds for every $\xi \in \mathbb{S}_{\mathscr{E}}^{\mathcal{A}}$, Theorem 1 guarantees that $\mathbf{Y}$ is jointly $\mathcal{A}$-paranormal. □

To concretely illustrate this new class and its strict position in the established hierarchy (jointly $\mathcal{A}$-hyponormal $\Longrightarrow$ jointly $\mathcal{A}$-paranormal $\Longrightarrow$ jointly $\mathcal{A}$-normaloid), we provide two examples showing that neither implication is reversible in general. It is worth noting that several examples in the one-dimensional case ($q = 1$) already exist in the literature; see, for instance, [31].

The following Example presents a jointly $\mathcal{A}$-paranormal tuple that is not jointly $\mathcal{A}$-hyponormal.

Example 1. *Let $\mathscr{E} = \mathbb{C}^2$ and $\mathcal{A} = I_{\mathscr{E}}$. Consider the tuple $\mathbf{Y} = (Y_1, Y_2)$, where $Y_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $Y_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Since $\mathcal{A} = I_{\mathscr{E}}$, the $\mathcal{A}$-adjoint coincides with the standard adjoint. We first determine $\Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})$:*

$$\Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}}) = Y_1^*Y_1 + Y_2^*Y_2 = I_{\mathscr{E}}.$$

Consequently, we have

$$\Theta_{\mathbf{Y},\mathcal{A}}^2(I_{\mathscr{E}}) = Y_1^*\Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})Y_1 + Y_2^*\Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})Y_2 = Y_1^*I_{\mathscr{E}}Y_1 + Y_2^*I_{\mathscr{E}}Y_2 = I_{\mathscr{E}}.$$

Thus, for each $\lambda > 0$, we obtain

$$\Theta_{\mathbf{Y},\mathcal{A}}^2(I_{\mathscr{E}}) - 2\lambda\Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}}) + \lambda^2 I_{\mathscr{E}} = I_{\mathscr{E}} - 2\lambda I_{\mathscr{E}} + \lambda^2 I_{\mathscr{E}} = (1-\lambda)^2 I_{\mathscr{E}} \geq_{\mathcal{A}} \mathbf{0}.$$

By Theorem 1, this shows that $\mathbf{Y}$ is jointly $\mathcal{A}$-paranormal.

On the other hand, the self-commutator of Y_1 is

$$[Y_1^*, Y_1] = Y_1^*Y_1 - Y_1Y_1^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For $\mathbf{Y}$ to be jointly $\mathcal{A}$-hyponormal, the operator matrix $\mathcal{C}_{\mathcal{A}}(\mathbf{Y})$ must be $\widetilde{\mathcal{A}}$-positive. This implies that its diagonal blocks $[Y_k^*, Y_k]$ must also be positive semi-definite. Since $[Y_1^*, Y_1]$ has a negative eigenvalue (-1), it is not positive semi-definite. Therefore, $\mathbf{Y}$ is not jointly $\mathcal{A}$-hyponormal.

The following Example presents a commuting jointly $\mathcal{A}$-normaloid tuple that is not jointly $\mathcal{A}$-paranormal. More precisely, there exists a commuting jointly $\mathcal{A}$-normaloid tuple (Y_1, Y_2) with $Y_1 \neq Y_2$ which fails to be jointly $\mathcal{A}$-paranormal.

Example 2. *Let $\mathscr{E} = \mathbb{C}^3$ and $\mathcal{A} = I_{\mathscr{E}}$. Let $\mathbf{Y} = (Y_1, Y_2)$ be the tuple defined by*

$$Y_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have $Y_1 \neq Y_2$. *They commute since* $Y_1Y_2 = Y_2Y_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *To compute the joint* $\mathcal{A}$*-seminorm of* $\mathbf{Y}$, *we first determine* $\Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})$:

$$\Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}}) = Y_1^*Y_1 + Y_2^*Y_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The joint $\mathcal{A}$*-seminorm of* $\mathbf{Y}$ *is given by*

$$\|\mathbf{Y}\|_{\mathcal{A}} = \|\Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\|_{\mathcal{A}}^{\frac{1}{2}} = \sqrt{2}.$$

To find the joint $\mathcal{A}$*-spectral radius* $r_{\mathcal{A}}(\mathbf{Y})$, *we compute* $\Theta^i_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})$ *for* $i \geq 2$. *Applying* $\Theta_{\mathbf{Y},\mathcal{A}}$ *to a diagonal matrix* $Z = \operatorname{diag}(z_1, z_2, z_3)$ *yields* $\Theta_{\mathbf{Y},\mathcal{A}}(Z) = \operatorname{diag}(2z_1, 0, z_2)$. *Starting with* $Z_0 = I_{\mathscr{E}}$, *we obtain* $\Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}}) = \operatorname{diag}(2, 0, 1)$ *and* $\Theta^i_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}}) = \operatorname{diag}(2^i, 0, 0)$ *for* $i \geq 2$. *By using* (2), *we deduce that*

$$r_{\mathcal{A}}(\mathbf{Y}) = \lim_{i\to\infty} \|\Theta^i_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\|_{\mathcal{A}}^{\frac{1}{2i}} = \lim_{i\to\infty} (2^i)^{\frac{1}{2i}} = \sqrt{2}.$$

Since $r_{\mathcal{A}}(\mathbf{Y}) = \|\mathbf{Y}\|_{\mathcal{A}} = \sqrt{2}$, *the tuple* $\mathbf{Y}$ *is jointly* $\mathcal{A}$*-normaloid.*

To see that $\mathbf{Y}$ *is not jointly* $\mathcal{A}$*-paranormal, we use Theorem 1 and evaluate the condition for* $\lambda = 1$. *We have* $\Theta^2_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}}) = \operatorname{diag}(4, 0, 0)$. *This yields:*

$$\Theta^2_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}}) - 2(1)\Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}}) + (1)^2 I_{\mathscr{E}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

This resulting matrix has a negative eigenvalue (-1). *Thus, it is not positive semi-definite. Consequently, the condition from Theorem 1 is violated. Therefore,* $\mathbf{Y}$ *is not jointly* $\mathcal{A}$*-paranormal.*

The following theorem shows that joint $\mathcal{A}$-paranormality is preserved under $\mathcal{A}$-unitary equivalence.

Theorem 3. *Let* $\mathbf{Y} = (Y_1, \ldots, Y_q) \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})^q$ *be a jointly* $\mathcal{A}$*-paranormal* q*-tuple and let* $U \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})$ *be an* $\mathcal{A}$*-unitary operator. Then, the tuple*

$$\mathbf{Z} = (UY_1U^{\sharp_{\mathcal{A}}}, \ldots, UY_qU^{\sharp_{\mathcal{A}}}) \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})^q$$

is also jointly $\mathcal{A}$*-paranormal.*

Proof. Since an $\mathcal{A}$-unitary operator U preserves the $\mathcal{A}$-seminorm, we have

$$\|Uv\|_{\mathcal{A}} = \|v\|_{\mathcal{A}}$$

for all $v \in \mathscr{E}$. Applying this property to the $\mathcal{A}$-adjoint further yields $\|U^{\sharp_{\mathcal{A}}}v\|_{\mathcal{A}} = \|v\|_{\mathcal{A}}$.

Since $Y_l \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})$, we have $\operatorname{Ran}(Y_l^*\mathcal{A}) \subseteq \operatorname{Ran}(\mathcal{A})$. This ensures that $\mathcal{N}(\mathcal{A})$ is invariant under Y_l, which implies $\mathcal{A}Y_l\Pi_{\mathcal{A}} = \mathcal{A}Y_l$. Evaluating the $\mathcal{A}$-seminorm gives

$$\begin{aligned} \|Y_l\Pi_{\mathcal{A}}v\|_{\mathcal{A}}^2 &= \langle \mathcal{A}Y_l\Pi_{\mathcal{A}}v, Y_l\Pi_{\mathcal{A}}v\rangle \\ &= \langle \mathcal{A}Y_lv, Y_lv\rangle \\ &= \|Y_lv\|_{\mathcal{A}}^2. \end{aligned}$$

Let $\xi \in \mathbb{S}_{\mathscr{E}}^{\mathcal{A}}$. Setting $\eta = U^{\sharp_{\mathcal{A}}}\xi$ produces $\|\eta\|_{\mathcal{A}} = \|U^{\sharp_{\mathcal{A}}}\xi\|_{\mathcal{A}} = \|\xi\|_{\mathcal{A}} = 1$. Since $\mathbf{Y}$ is jointly $\mathcal{A}$-paranormal, we have

$$\sum_{l=1}^{q}\sum_{s=1}^{q}\|Y_lY_s\eta\|_{\mathcal{A}}^2 \geq \left(\sum_{l=1}^{q}\|Y_l\eta\|_{\mathcal{A}}^2\right)^2. \tag{9}$$

For any $l, s \in \{1, \ldots, q\}$, examining the elements of $\mathbf{Z}$ reveals

$$\begin{aligned} Z_lZ_s\xi &= (UY_lU^{\sharp_{\mathcal{A}}})(UY_sU^{\sharp_{\mathcal{A}}})\xi \\ &= UY_l\Pi_{\mathcal{A}}Y_s\eta. \end{aligned}$$

Applying the norm-preserving properties yields

$$\begin{aligned} \|Z_lZ_s\xi\|_{\mathcal{A}} &= \|UY_l\Pi_{\mathcal{A}}Y_s\eta\|_{\mathcal{A}} \\ &= \|Y_l\Pi_{\mathcal{A}}Y_s\eta\|_{\mathcal{A}} \\ &= \|Y_lY_s\eta\|_{\mathcal{A}}. \end{aligned}$$

Similarly, we find

$$\begin{aligned} \|Z_l\xi\|_{\mathcal{A}} &= \|UY_lU^{\sharp_{\mathcal{A}}}\xi\|_{\mathcal{A}} \\ &= \|UY_l\eta\|_{\mathcal{A}} \\ &= \|Y_l\eta\|_{\mathcal{A}}. \end{aligned}$$

Substituting these expressions into (9) gives

$$\sum_{l=1}^{q}\sum_{s=1}^{q}\|Z_lZ_s\xi\|_{\mathcal{A}}^2 \geq \left(\sum_{l=1}^{q}\|Z_l\xi\|_{\mathcal{A}}^2\right)^2.$$

Since this is true for all $\xi \in \mathbb{S}_{\mathscr{E}}^{\mathcal{A}}$, Theorem 1 establishes that $\mathbf{Z}$ is jointly $\mathcal{A}$-paranormal. □

Remark 1. *Unlike the classical Hilbert space setting, where $\mathcal{A}$-unitary equivalence follows easily from the algebraic identity $(UYU^*)^* = UY^*U^*$, the semi-Hilbertian proof of Theorem 3 must carefully navigate the degeneracies inherent to this framework. The key difficulty is that the expression $(UY_lU^{\sharp_{\mathcal{A}}})^{\sharp_{\mathcal{A}}}$ does not simplify by a direct algebraic manipulation. Instead, the proof relies on two relations that are specific to the semi-Hilbertian structure: the identity $U^{\sharp_{\mathcal{A}}}U = \Pi_{\mathcal{A}}$, which replaces the classical $U^*U = I_{\mathscr{E}}$, and the seminorm-preservation property $\|Y_l\Pi_{\mathcal{A}}v\|_{\mathcal{A}} = \|Y_lv\|_{\mathcal{A}}$, which reflects the fact that $\mathcal{N}(\mathcal{A})$ is invariant under Y_l. Neither of these has a counterpart in the classical setting, and together they constitute the essential mechanism of the argument.*

Remark 2. *It is worth pointing out precisely which results in this paper require commutativity and which do not. The definition of joint $\mathcal{A}$-paranormality (Definition 1), the equivalent characterizations in Theorem 1, the $\mathcal{A}$-unitary invariance in Theorem 3, and the stability under powers in Theorem 8 all hold without any commutativity assumption on the tuple $\mathbf{Y}$. The results that genuinely require commutativity are those involving the joint $\mathcal{A}$-spectral radius, specifically Theorem 6, which establishes that every commuting jointly $\mathcal{A}$-paranormal tuple is jointly $\mathcal{A}$-normaloid, and Corollary 1. Both rely on the spectral radius formula of Baklouti et al. [16], which is defined exclusively for commuting tuples. The core obstruction in the noncommuting case is that the identity*

$$r_{\mathcal{A}}(\mathbf{Y}) = \lim_{n\to\infty}\|\Theta_{\mathbf{Y},\mathcal{A}}^n(I_{\mathscr{E}})\|_{\mathcal{A}}^{\frac{1}{2n}}$$

is no longer available, so the log-convexity argument of Theorem 6 cannot be completed. Whether the normaloid conclusion can be extended to noncommuting tuples, perhaps by replacing classical commutativity with the weaker condition of $\mathcal{A}$-commutativity $\mathcal{A}Y_lY_s = \mathcal{A}Y_sY_l$, is a natural and interesting open problem that we leave for future investigation.

To formalize tuple exponentiation, we introduce a concatenation operation. For arbitrary operator tuples $\mathbf{Y} = (Y_1, \ldots, Y_q) \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})^q$ and $\mathbf{Z} = (Z_1, \ldots, Z_s) \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})^s$, we define their concatenation product as:

$$\mathbf{Y} \circ \mathbf{Z} := (Y_1Z_1, \ldots, Y_1Z_s, \ldots, Y_qZ_1, \ldots, Y_qZ_s).$$

We inductively define the powers of a tuple $\mathbf{Y}$:

$$\mathbf{Y}^1 := \mathbf{Y} \quad \text{and} \quad \mathbf{Y}^{i+1} := \mathbf{Y} \circ \mathbf{Y}^i,$$

for any $i \in \mathbb{N}$.

The next result establishes the linearity and monotonicity of the operator map $\Theta_{\mathbf{Y},\mathcal{A}}$.

Theorem 4. *Let $\mathbf{Y} = (Y_1, \ldots, Y_q) \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})^q$. The operator $\Theta_{\mathbf{Y},\mathcal{A}}$ is linear and monotone; specifically, for any pair of $\mathcal{A}$-selfadjoint operators $Z, W \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})$, we have:*

$$Z \geq_{\mathcal{A}} W \implies \Theta_{\mathbf{Y},\mathcal{A}}(Z) \geq_{\mathcal{A}} \Theta_{\mathbf{Y},\mathcal{A}}(W).$$

Proof. Since the $\mathcal{A}$-adjoint operation and summation are linear, the linearity of $\Theta_{\mathbf{Y},\mathcal{A}}$ is immediate. Assuming $Z \geq_{\mathcal{A}} W$, we deduce $\langle Z\eta, \eta\rangle_{\mathcal{A}} \geq \langle W\eta, \eta\rangle_{\mathcal{A}}$ for all $\eta \in \mathscr{E}$. Substituting $\eta = Y_l\xi$ provides

$$\langle ZY_l\xi, Y_l\xi\rangle_{\mathcal{A}} \geq \langle WY_l\xi, Y_l\xi\rangle_{\mathcal{A}}.$$

The definition of the $\mathcal{A}$-adjoint implies $\langle Y_l^{\sharp_{\mathcal{A}}}ZY_l\xi, \xi\rangle_{\mathcal{A}} \geq \langle Y_l^{\sharp_{\mathcal{A}}}WY_l\xi, \xi\rangle_{\mathcal{A}}$. Summing over $l \in \{1, \ldots, q\}$ leads to

$$\sum_{l=1}^{q} \langle Y_l^{\sharp_{\mathcal{A}}}ZY_l\xi, \xi\rangle_{\mathcal{A}} \geq \sum_{l=1}^{q} \langle Y_l^{\sharp_{\mathcal{A}}}WY_l\xi, \xi\rangle_{\mathcal{A}}.$$

Exploiting the linearity of the $\mathcal{A}$-inner product yields

$$\left\langle \left(\sum_{l=1}^{q} Y_l^{\sharp_{\mathcal{A}}}ZY_l \right)\xi, \xi \right\rangle_{\mathcal{A}} \geq \left\langle \left(\sum_{l=1}^{q} Y_l^{\sharp_{\mathcal{A}}}WY_l \right)\xi, \xi \right\rangle_{\mathcal{A}}.$$

The definition of $\Theta_{\mathbf{Y},\mathcal{A}}$ then implies $\langle \Theta_{\mathbf{Y},\mathcal{A}}(Z)\xi, \xi\rangle_{\mathcal{A}} \geq \langle \Theta_{\mathbf{Y},\mathcal{A}}(W)\xi, \xi\rangle_{\mathcal{A}}$. Since this holds for all $\xi \in \mathscr{E}$, we conclude $\Theta_{\mathbf{Y},\mathcal{A}}(Z) \geq_{\mathcal{A}} \Theta_{\mathbf{Y},\mathcal{A}}(W)$. □

The following theorem provides multiplicative factorization and exponentiation rules for the map Θ.

Theorem 5. *Suppose $\mathbf{Y} = (Y_1, \ldots, Y_q)$, $\mathbf{Z} = (Z_1, \ldots, Z_s)$, and $\mathbf{Y}_1, \ldots, \mathbf{Y}_k$ are operator tuples in $\mathfrak{L}_{\mathcal{A}}(\mathscr{E})$. Then the following properties hold:*

(i) $\Theta_{\mathbf{Y}\circ\mathbf{Z},\mathcal{A}} = \Theta_{\mathbf{Z},\mathcal{A}}\Theta_{\mathbf{Y},\mathcal{A}}$;

(ii) $\Theta_{\mathbf{Y}_1\circ\cdots\circ\mathbf{Y}_k,\mathcal{A}} = \Theta_{\mathbf{Y}_k,\mathcal{A}} \ldots \Theta_{\mathbf{Y}_1,\mathcal{A}}$;

(iii) $\Theta^l_{\mathbf{Y}^i,\mathcal{A}} = \Theta^{il}_{\mathbf{Y},\mathcal{A}}$ *for all integers $i, l \in \mathbb{N}$.*

Proof. For *(i)*, the definition of $\Theta_{\mathbf{Y}\circ\mathbf{Z},\mathcal{A}}$ dictates that

$$\Theta_{\mathbf{Y}\circ\mathbf{Z},\mathcal{A}}(X) = \sum_{l=1}^{q}\sum_{m=1}^{s}(Y_lZ_m)^{\sharp_{\mathcal{A}}}X(Y_lZ_m).$$

Applying $(Y_lZ_m)^{\sharp_{\mathcal{A}}} = Z_m^{\sharp_{\mathcal{A}}}Y_l^{\sharp_{\mathcal{A}}}$ yields

$$\Theta_{\mathbf{Y}\circ\mathbf{Z},\mathcal{A}}(X) = \sum_{l=1}^{q}\sum_{m=1}^{s}Z_m^{\sharp_{\mathcal{A}}}(Y_l^{\sharp_{\mathcal{A}}}XY_l)Z_m.$$

Rearranging the summation gives

$$\begin{aligned}\Theta_{\mathbf{Y}\circ\mathbf{Z},\mathcal{A}}(X) &= \sum_{m=1}^{s}Z_m^{\sharp_{\mathcal{A}}}\left(\sum_{l=1}^{q}Y_l^{\sharp_{\mathcal{A}}}XY_l\right)Z_m\\ &= \Theta_{\mathbf{Z},\mathcal{A}}(\Theta_{\mathbf{Y},\mathcal{A}}(X)).\end{aligned}$$

Since X is arbitrary, it follows that $\Theta_{\mathbf{Y}\circ\mathbf{Z},\mathcal{A}} = \Theta_{\mathbf{Z},\mathcal{A}}\Theta_{\mathbf{Y},\mathcal{A}}$.

For *(ii)*, since *(i)* establishes the base case for $k = 2$, assume *(ii)* holds for $k = n$. Setting $\mathbf{W} = \mathbf{Y}_1 \circ \cdots \circ \mathbf{Y}_n$, we observe

$$\Theta_{\mathbf{W}\circ\mathbf{Y}_{n+1},\mathcal{A}} = \Theta_{\mathbf{Y}_{n+1},\mathcal{A}}\Theta_{\mathbf{W},\mathcal{A}}.$$

The induction hypothesis then yields

$$\Theta_{\mathbf{W}\circ\mathbf{Y}_{n+1},\mathcal{A}} = \Theta_{\mathbf{Y}_{n+1},\mathcal{A}}(\Theta_{\mathbf{Y}_n,\mathcal{A}}\ldots\Theta_{\mathbf{Y}_1,\mathcal{A}}).$$

For *(iii)*, since $\mathbf{Y}^i$ is the i-fold concatenation of $\mathbf{Y}$, applying *(ii)* to its l-fold concatenation yields $\Theta^l_{\mathbf{Y}^i,\mathcal{A}} = \Theta_{\mathbf{Y}^{il},\mathcal{A}}$. A further application of *(ii)* produces $\Theta_{\mathbf{Y}^{il},\mathcal{A}} = \Theta^{il}_{\mathbf{Y},\mathcal{A}}$. □

Next, we establish the fact that every commuting jointly $\mathcal{A}$-paranormal tuple is jointly $\mathcal{A}$-normaloid.

Theorem 6. *Let* $\mathbf{Y} = (Y_1, \ldots, Y_q) \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})^q$ *be a commuting jointly* $\mathcal{A}$*-paranormal* q*-tuple. Then,* $\mathbf{Y}$ *is jointly* $\mathcal{A}$*-normaloid.*

Proof. Since (5) provides the base case $n = 1$, assume (10) holds for some $n \in \mathbb{N}$:

$$\Theta^{n+1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}}) - 2\lambda\,\Theta^{n}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}}) + \lambda^2\Theta^{n-1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}}) \geq_{\mathcal{A}} \mathbf{0}. \tag{10}$$

The monotonicity of $\Theta_{\mathbf{Y},\mathcal{A}}$ yields

$$\Theta_{\mathbf{Y},\mathcal{A}}\big(\Theta^{n+1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}}) - 2\lambda\,\Theta^{n}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}}) + \lambda^2\Theta^{n-1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\big) \geq_{\mathcal{A}} \mathbf{0}.$$

Simplifying this expression gives

$$\Theta^{n+2}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}}) - 2\lambda\,\Theta^{n+1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}}) + \lambda^2\Theta^{n}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}}) \geq_{\mathcal{A}} \mathbf{0}.$$

Mathematical induction establishes (10) for all $n \in \mathbb{N}$.

Let $\xi \in \mathscr{E}$. Since the quadratic polynomial in λ given by

$$\langle\Theta^{n+1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}} - 2\lambda\langle\Theta^{n}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}} + \lambda^2\langle\Theta^{n-1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}} \geq 0$$

is non-negative for all $\lambda > 0$ and achieves its minimum at a non-negative value, its discriminant must be non-positive. This discriminant condition implies

$$\langle \Theta^{n+1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}\langle \Theta^{n-1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}} \geq \langle \Theta^{n}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle^2_{\mathcal{A}}. \tag{11}$$

We now prove $\langle \Theta^{n}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle^{\frac{1}{n}}_{\mathcal{A}} \leq \langle \Theta^{n+1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle^{\frac{1}{n+1}}_{\mathcal{A}}$. Taking $n = 1$ as the base case, assume the statement holds for a given n. Applying (11) alongside the induction hypothesis provides

$$\begin{aligned}\langle \Theta^{n+1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle^2_{\mathcal{A}} &\leq \langle \Theta^{n+2}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}\,\langle \Theta^{n}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}\\ &\leq \langle \Theta^{n+2}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}\,\langle \Theta^{n+1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle^{\frac{n}{n+1}}_{\mathcal{A}}.\end{aligned}$$

Since the inequality holds trivially when $\langle \Theta^{n+1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}} = 0$, we may assume this quantity is strictly positive. Rearranging algebraically yields

$$\langle \Theta^{n+1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle^{2-\frac{n}{n+1}}_{\mathcal{A}} \leq \langle \Theta^{n+2}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}.$$

Simplifying the fractions provides

$$\langle \Theta^{n+1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle^{\frac{n+2}{n+1}}_{\mathcal{A}} \leq \langle \Theta^{n+2}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}.$$

Raising to the power of $\frac{1}{n+2}$ yields

$$\langle \Theta^{n+1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle^{\frac{1}{n+1}}_{\mathcal{A}} \leq \langle \Theta^{n+2}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle^{\frac{1}{n+2}}_{\mathcal{A}}.$$

Mathematical induction therefore establishes the fundamental chain:

$$\langle \Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}} \leq \langle \Theta^{2}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle^{\frac{1}{2}}_{\mathcal{A}} \leq \ldots \leq \langle \Theta^{n}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle^{\frac{1}{n}}_{\mathcal{A}}. \tag{12}$$

Since the inequality $\|\Theta^{n}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\|_{\mathcal{A}} \leq \|\Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\|^n_{\mathcal{A}}$ is known to hold, we proceed to establish the reverse inequality. Taking the supremum over all unit vectors $\xi \in \mathbb{S}^{\mathcal{A}}_{\mathscr{E}}$ on the fundamental chain provides

$$\|\Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\|_{\mathcal{A}} \leq \|\Theta^{n}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\|^{\frac{1}{n}}_{\mathcal{A}}.$$

Extracting the square root and invoking (3) gives

$$\|\mathbf{Y}\|_{\mathcal{A}} \leq \|\Theta^{n}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\|^{\frac{1}{2n}}_{\mathcal{A}}.$$

Taking the limit as $n \to \infty$ and utilizing (2) produces

$$\|\mathbf{Y}\|_{\mathcal{A}} \leq \lim_{n\to\infty} \|\Theta^{n}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\|^{\frac{1}{2n}}_{\mathcal{A}} = r_{\mathcal{A}}(\mathbf{Y}).$$

Since $r_{\mathcal{A}}(\mathbf{Y}) \leq \|\mathbf{Y}\|_{\mathcal{A}}$ always holds, we conclude $r_{\mathcal{A}}(\mathbf{Y}) = \|\mathbf{Y}\|_{\mathcal{A}}$. Thus, $\mathbf{Y}$ is jointly $\mathcal{A}$-normaloid. □

Remark 3. *It is instructive to contrast the proof of Theorem 6 with its classical counterpart. In the standard Hilbert space setting, the fact that a paranormal operator is normaloid follows from elementary norm estimates. In the semi-Hilbertian framework, the argument is considerably more involved and proceeds through three distinct steps: First, constructing the completely positive map $\Theta_{\mathbf{Y},\mathcal{A}}$ and establishing its monotonicity (Theorem 4). Second, propagating this monotonicity*

inductively to derive the log-convexity inequality (11) for the sequence $\langle \Theta^n_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}$. *Third, connecting the resulting growth estimates to the joint* $\mathcal{A}$*-spectral radius via the limit formula (2) of Baklouti et al. [16]. Each of these steps is specific to the semi-Hilbertian structure and has no direct classical analog, which illustrates why the transition to this framework demands new techniques rather than a straightforward adaptation of known arguments.*

Corollary 1. *Let* $\mathbf{Y} = (Y_1,\ldots,Y_q) \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})^q$ *be a commuting jointly* $\mathcal{A}$*-paranormal q-tuple. Then, the following equalities hold:*

$$r_{\mathcal{A}}(\mathbf{Y}) = \omega_{\mathcal{A}}(\mathbf{Y}) = \|\mathbf{Y}\|_{e,\mathcal{A}} = \|\mathbf{Y}\|_{\mathcal{A}}.$$

Proof. Since $\mathbf{Y}$ is a commuting jointly $\mathcal{A}$-paranormal tuple, it follows that $r_{\mathcal{A}}(\mathbf{Y}) = \|\mathbf{Y}\|_{\mathcal{A}}$. From (4), we deduce

$$r_{\mathcal{A}}(\mathbf{Y}) \leq \omega_{\mathcal{A}}(\mathbf{Y}) \leq \|\mathbf{Y}\|_{e,\mathcal{A}} \leq \|\mathbf{Y}\|_{\mathcal{A}}.$$

Since the bounds are equal, we conclude $r_{\mathcal{A}}(\mathbf{Y}) = \omega_{\mathcal{A}}(\mathbf{Y}) = \|\mathbf{Y}\|_{e,\mathcal{A}} = \|\mathbf{Y}\|_{\mathcal{A}}$. □

Evaluating the squared $\mathcal{A}$-seminorm of $\mathbf{Y}^n$ acting on ξ yields:

$$\|\mathbf{Y}^n\xi\|^2_{\mathcal{A}} = \langle \Theta^n_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}.$$

Since this holds for every $n \in \mathbb{N}$ and every $\xi \in \mathscr{E}$, extracting the square roots in (12) generates the following chain:

$$\|\mathbf{Y}\xi\|_{\mathcal{A}} \leq \|\mathbf{Y}^2\xi\|^{1/2}_{\mathcal{A}} \leq \cdots \leq \|\mathbf{Y}^n\xi\|^{1/n}_{\mathcal{A}} \leq \cdots,$$

for all $\xi \in \mathbb{S}^{\mathcal{A}}_{\mathscr{E}}$.

Our next theorem establishes the invariance of the $\mathcal{A}$-kernel under taking powers of the tuple.

Theorem 7. *Let* $\mathbf{Y} = (Y_1,\ldots,Y_q) \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})^q$ *be a jointly* $\mathcal{A}$*-paranormal q-tuple. Then,*

$$\mathcal{N}_{\mathcal{A}}(\mathbf{Y}) = \mathcal{N}_{\mathcal{A}}(\mathbf{Y}^n) \tag{13}$$

for all $n \in \mathbb{N}$, *where* $\mathcal{N}_{\mathcal{A}}(\mathbf{Y}) := \{\xi \in \mathscr{E} : \|Y_l\xi\|_{\mathcal{A}} = 0,\ l = 1,\ldots,q\}$.

Proof. For an arbitrary $n \in \mathbb{N}$, Theorem 5*(iii)* implies:

$$\begin{aligned}\mathcal{N}_{\mathcal{A}}(\Theta^n_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})) &= \mathcal{N}_{\mathcal{A}}(\Theta_{\mathbf{Y}^n,\mathcal{A}}(I_{\mathscr{E}}))\\ &= \mathcal{N}_{\mathcal{A}}(\mathbf{Y}^n).\end{aligned}$$

Since the log-convexity inequality (11) holds for all $n \in \mathbb{N}$, it produces:

$$\mathcal{N}_{\mathcal{A}}(\mathbf{Y}^{n-1}) \subseteq \mathcal{N}_{\mathcal{A}}(\mathbf{Y}^n) \supseteq \mathcal{N}_{\mathcal{A}}(\mathbf{Y}^{n+1}),$$

where $\mathbf{Y}^0 := \mathbf{I}$. Applying these inclusions inductively confirms (13). □

The theorem below proves that the powers of a jointly $\mathcal{A}$-paranormal tuple remain jointly $\mathcal{A}$-paranormal.

Theorem 8. *Let* $\mathbf{Y} = (Y_1,\ldots,Y_q) \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})^q$ *be a jointly* $\mathcal{A}$*-paranormal q-tuple. Then,* $\mathbf{Y}^n$ *is jointly* $\mathcal{A}$*-paranormal for all* $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$ and $\xi \in \mathbb{S}^{\mathcal{A}}_{\mathscr{E}}$. Since $\xi \in \mathcal{N}_{\mathcal{A}}(\mathbf{Y})$ implies $\xi \in \mathcal{N}_{\mathcal{A}}(\mathbf{Y}^k)$ by Theorem 7, we have $\langle \Theta^k_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}} = 0$. When $\langle \Theta^k_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}} = 0$, the inequality holds trivially.

For $\xi \notin \mathcal{N}_{\mathcal{A}}(\mathbf{Y})$, we know $\langle \Theta^k_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}} > 0$ for all $k \in \mathbb{N}$; applying (11) then yields:

$$\langle \Theta^{k+1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}} \langle \Theta^{k-1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}} \geq \langle \Theta^{k}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle^2_{\mathcal{A}}.$$

Algebraic rearrangement leads to:

$$\frac{\langle \Theta^{k+1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}{\langle \Theta^{k}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}} \geq \frac{\langle \Theta^{k}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}{\langle \Theta^{k-1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}.$$

Since this relation holds for every $k \in \mathbb{N}$, iterative expansion yields the following descending chain:

$$\begin{aligned}\frac{\langle \Theta^{2^n}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}{\langle \Theta^{2^n-1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}} &\geq \cdots \geq \frac{\langle \Theta^{n+1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}{\langle \Theta^{n}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}} \\ &\geq \frac{\langle \Theta^{n}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}{\langle \Theta^{n-1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}} \geq \cdots \geq \frac{\langle \Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}{\langle \Theta^{0}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}.\end{aligned}$$

Pairing terms symmetrically from the upper and lower halves yields:

$$\frac{\langle \Theta^{n+j}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}{\langle \Theta^{n+j-1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}} \geq \frac{\langle \Theta^{j}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}{\langle \Theta^{j-1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}$$

for each $j \in \{1,\ldots,n\}$. Taking the product of these inequalities produces:

$$\prod_{j=1}^{n} \frac{\langle \Theta^{n+j}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}{\langle \Theta^{n+j-1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}} \geq \prod_{j=1}^{n} \frac{\langle \Theta^{j}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}{\langle \Theta^{j-1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}.$$

Recognizing that these products are telescoping, simplifying them yields:

$$\frac{\langle \Theta^{2^n}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}{\langle \Theta^{n}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}} \geq \frac{\langle \Theta^{n}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}{\langle \Theta^{0}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}.$$

Since $\langle \Theta^0_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}} = \langle I_{\mathscr{E}}\xi,\xi\rangle_{\mathcal{A}} = 1$, we obtain:

$$\langle \Theta^{2^n}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}} \geq \langle \Theta^{n}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle^2_{\mathcal{A}}.$$

Theorem 5*(iii)* dictates that $\Theta^{2^n}_{\mathbf{Y},\mathcal{A}} = \Theta^2_{\mathbf{Y}^n,\mathcal{A}}$ and $\Theta^n_{\mathbf{Y},\mathcal{A}} = \Theta_{\mathbf{Y}^n,\mathcal{A}}$. Thus, it follows that:

$$\langle \Theta^{2}_{\mathbf{Y}^n,\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}} \geq \langle \Theta_{\mathbf{Y}^n,\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle^2_{\mathcal{A}}.$$

Since this inequality remains valid for any $\xi \in \mathbb{S}^{\mathcal{A}}_{\mathscr{E}}$, Theorem 1 ensures that $\mathbf{Y}^n$ is jointly $\mathcal{A}$-paranormal. □

Remark 4. *The classical analog of Theorem 8 assumes the algebraic convenience of the standard adjoint identity $(Y^n)^* = (Y^*)^n$. In our setting, properly applying the $\mathcal{A}$-adjoint to the map $\Theta_{\mathbf{Y},\mathcal{A}}$ relies intrinsically on the non-trivial factorization $\Theta^l_{\mathbf{Y}^i,\mathcal{A}} = \Theta^{il}_{\mathbf{Y},\mathcal{A}}$ (established in Theorem 5), without which the structural stability under powers cannot be verified.*

The following corollary generalizes the log-convexity property for tuples.

Corollary 2. *Let* $\mathbf{Y} = (Y_1, \ldots, Y_q) \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E})^q$ *be a jointly* $\mathcal{A}$*-paranormal* q*-tuple, and let* $\xi \in \mathscr{E}$ *be arbitrary. Then,*

$$\langle \Theta^n_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}} \langle \Theta^m_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}} \geq \langle \Theta^{\frac{n+m}{2}}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle^2_{\mathcal{A}}, \tag{14}$$

holds for each $n \in \mathbb{N}$ *and each* $0 \leq m \leq n$*, such that* $n \equiv_2 m$*.*

Proof. While the statement is trivially true for $\xi \in \mathcal{N}_{\mathcal{A}}(\mathbf{Y})$, we assume $\xi \notin \mathcal{N}_{\mathcal{A}}(\mathbf{Y})$. Since $n - \left(\frac{n+m}{2} + 1\right) = \frac{n+m}{2} - (m+1)$, the descending chain of ratio inequalities established in Theorem 8 leads to:

$$\begin{aligned}
\frac{\langle \Theta^n_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}{\langle \Theta^{\frac{n+m}{2}}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}} &= \frac{\langle \Theta^n_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}{\langle \Theta^{n-1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}} \cdots \frac{\langle \Theta^{\frac{n+m}{2}+1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}{\langle \Theta^{\frac{n+m}{2}}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}} \\
&\geq \frac{\langle \Theta^{\frac{n+m}{2}}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}{\langle \Theta^{\frac{n+m}{2}-1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}} \cdots \frac{\langle \Theta^{m+1}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}{\langle \Theta^{m}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}} \\
&= \frac{\langle \Theta^{\frac{n+m}{2}}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}{\langle \Theta^{m}_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}})\xi,\xi\rangle_{\mathcal{A}}}.
\end{aligned}$$

Multiplying appropriately on both sides produces (14). □

In the sequel, we establish the behavior of the map Θ on tensor products of operator tuples. Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be complex Hilbert spaces equipped with positive operators $\mathcal{A} \in \mathfrak{L}(\mathscr{E}_1)^+$ and $\mathcal{B} \in \mathfrak{L}(\mathscr{E}_2)^+$. The completed Hilbert tensor product space $\mathscr{E}_1 \overline{\otimes} \mathscr{E}_2$ naturally becomes a semi-Hilbertian space with respect to the positive tensor operator $\mathcal{A}\overline{\otimes}\mathcal{B}$ [18,20].

For tuples $\mathbf{Y} = (Y_1, \ldots, Y_q) \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E}_1)^q$ and $\mathbf{Z} = (Z_1, \ldots, Z_s) \in \mathfrak{L}_{\mathcal{B}}(\mathscr{E}_2)^s$, we consider the tensor extension tuples. We shall write $I_{\mathscr{E}_1}$ and $I_{\mathscr{E}_2}$ for the identity operators on $\mathscr{E}_1$ and $\mathscr{E}_2$, respectively. Let $\mathbf{I}_1 = (I_{\mathscr{E}_1}, \ldots, I_{\mathscr{E}_1})$ and $\mathbf{I}_2 = (I_{\mathscr{E}_2}, \ldots, I_{\mathscr{E}_2})$ be the corresponding identity tuples. Let $\mathbf{Y}\overline{\otimes}\mathbf{I}_2 := (Y_1\overline{\otimes}I_{\mathscr{E}_2}, \ldots, Y_q\overline{\otimes}I_{\mathscr{E}_2})$.

Theorem 9. *Let* $\mathbf{Y} = (Y_1, \ldots, Y_q) \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E}_1)^q$*. Then, for all* $n \in \mathbb{N}$ *and all* $X \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E}_1)$*, we have:*

$$\Theta^n_{\mathbf{Y}\overline{\otimes}\mathbf{I}_2,\mathcal{A}\overline{\otimes}\mathcal{B}}(X\overline{\otimes}I_{\mathscr{E}_2}) = \Theta^n_{\mathbf{Y},\mathcal{A}}(X)\overline{\otimes}\Pi_{\mathcal{B}},$$

where $\Pi_{\mathcal{B}}$ *is the orthogonal projection onto* $\overline{\mathrm{Ran}(\mathcal{B})}$*. By symmetry, an analogous identity holds for* $\mathbf{I}_1\overline{\otimes}\mathbf{Z}$*.*

Proof. The properties of the $\mathcal{A}\overline{\otimes}\mathcal{B}$-adjoint for $n = 1$ assert that $(Y_l\overline{\otimes}I_{\mathscr{E}_2})^{\sharp_{\mathcal{A}\overline{\otimes}\mathcal{B}}} = Y_l^{\sharp_{\mathcal{A}}}\overline{\otimes}\Pi_{\mathcal{B}}$. Substituting this identity yields

$$\begin{aligned}
\Theta_{\mathbf{Y}\overline{\otimes}\mathbf{I}_2,\mathcal{A}\overline{\otimes}\mathcal{B}}(X\overline{\otimes}I_{\mathscr{E}_2}) &= \sum_{l=1}^{q} (Y_l\overline{\otimes}I_{\mathscr{E}_2})^{\sharp_{\mathcal{A}\overline{\otimes}\mathcal{B}}}(X\overline{\otimes}I_{\mathscr{E}_2})(Y_l\overline{\otimes}I_{\mathscr{E}_2}) \\
&= \sum_{l=1}^{q} (Y_l^{\sharp_{\mathcal{A}}}\overline{\otimes}\Pi_{\mathcal{B}})(X\overline{\otimes}I_{\mathscr{E}_2})(Y_l\overline{\otimes}I_{\mathscr{E}_2}) \\
&= \sum_{l=1}^{q} (Y_l^{\sharp_{\mathcal{A}}}XY_l)\overline{\otimes}(\Pi_{\mathcal{B}}I_{\mathscr{E}_2}I_{\mathscr{E}_2}) \\
&= \left(\sum_{l=1}^{q} Y_l^{\sharp_{\mathcal{A}}}XY_l\right)\overline{\otimes}\Pi_{\mathcal{B}} \\
&= \Theta_{\mathbf{Y},\mathcal{A}}(X)\overline{\otimes}\Pi_{\mathcal{B}}.
\end{aligned}$$

Having shown the base case $n = 1$, assume the identity holds for some $n \in \mathbb{N}$. Since $\Pi_{\mathcal{B}}$ is an orthogonal projection, it satisfies $\Pi_{\mathcal{B}}^2 = \Pi_{\mathcal{B}}$. Applying the map Θ once more alongside this projection property gives

$$\begin{aligned}
\Theta^{n+1}_{\mathbf{Y}\overline{\otimes}\mathbf{I}_2,\mathcal{A}\overline{\otimes}\mathcal{B}}(X\overline{\otimes}I_{\mathscr{E}_2}) &= \Theta_{\mathbf{Y}\overline{\otimes}\mathbf{I}_2,\mathcal{A}\overline{\otimes}\mathcal{B}}\big(\Theta^n_{\mathbf{Y},\mathcal{A}}(X)\overline{\otimes}\Pi_{\mathcal{B}}\big) \\
&= \sum_{l=1}^{q}(Y_l^{\sharp_{\mathcal{A}}}\overline{\otimes}\Pi_{\mathcal{B}})\big(\Theta^n_{\mathbf{Y},\mathcal{A}}(X)\overline{\otimes}\Pi_{\mathcal{B}}\big)(Y_l\overline{\otimes}I_{\mathscr{E}_2}) \\
&= \sum_{l=1}^{q}\Big(Y_l^{\sharp_{\mathcal{A}}}\Theta^n_{\mathbf{Y},\mathcal{A}}(X)Y_l\Big)\overline{\otimes}(\Pi_{\mathcal{B}}\Pi_{\mathcal{B}}I_{\mathscr{E}_2}) \\
&= \Theta_{\mathbf{Y},\mathcal{A}}\big(\Theta^n_{\mathbf{Y},\mathcal{A}}(X)\big)\overline{\otimes}\Pi_{\mathcal{B}} \\
&= \Theta^{n+1}_{\mathbf{Y},\mathcal{A}}(X)\overline{\otimes}\Pi_{\mathcal{B}}.
\end{aligned}$$

Mathematical induction establishes the desired identity. □

We finish the paper by showing that joint $\mathcal{A}$-paranormality is preserved under tensor products.

Theorem 10. *Let $\mathbf{Y} = (Y_1, \ldots, Y_q) \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E}_1)^q$ be a jointly $\mathcal{A}$-paranormal q-tuple. Then, the tensor tuple $\mathbf{Y}\overline{\otimes}\mathbf{I}_2$ is jointly $(\mathcal{A}\overline{\otimes}\mathcal{B})$-paranormal on $\mathscr{E}_1\overline{\otimes}\mathscr{E}_2$.*

Proof. Let $\lambda > 0$. We must show that the operator $\mathcal{W}_\lambda$ defined by

$$\mathcal{W}_\lambda := \Theta^2_{\mathbf{Y}\overline{\otimes}\mathbf{I}_2,\mathcal{A}\overline{\otimes}\mathcal{B}}(I_{\mathscr{E}_1}\overline{\otimes}I_{\mathscr{E}_2}) - 2\lambda\Theta_{\mathbf{Y}\overline{\otimes}\mathbf{I}_2,\mathcal{A}\overline{\otimes}\mathcal{B}}(I_{\mathscr{E}_1}\overline{\otimes}I_{\mathscr{E}_2}) + \lambda^2(I_{\mathscr{E}_1}\overline{\otimes}I_{\mathscr{E}_2})$$

is $(\mathcal{A}\overline{\otimes}\mathcal{B})$-positive. Theorem 9 provides

$$\mathcal{W}_\lambda = \Theta^2_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}_1})\overline{\otimes}\Pi_{\mathcal{B}} - 2\lambda\Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}_1})\overline{\otimes}\Pi_{\mathcal{B}} + \lambda^2 I_{\mathscr{E}_1}\overline{\otimes}I_{\mathscr{E}_2}.$$

Let $\zeta \in \mathscr{E}_1\overline{\otimes}\mathscr{E}_2$. For any operator $W \in \mathfrak{L}_{\mathcal{A}}(\mathscr{E}_1)$, evaluating $W\overline{\otimes}\Pi_{\mathcal{B}}$ within the $\mathcal{A}\overline{\otimes}\mathcal{B}$ form indicates $\langle(W\overline{\otimes}\Pi_{\mathcal{B}})\zeta,\zeta\rangle_{\mathcal{A}\overline{\otimes}\mathcal{B}} = \langle(\mathcal{A}W\overline{\otimes}\mathcal{B}\Pi_{\mathcal{B}})\zeta,\zeta\rangle$. Since $\Pi_{\mathcal{B}}$ is the orthogonal projection onto $\overline{\mathrm{Ran}(\mathcal{B})}$, it follows that $\mathcal{B}\Pi_{\mathcal{B}} = \mathcal{B}$. Utilizing this identity, we obtain

$$\begin{aligned}
\langle(\mathcal{A}W\overline{\otimes}\mathcal{B})\zeta,\zeta\rangle &= \langle(\mathcal{A}\overline{\otimes}\mathcal{B})(W\overline{\otimes}I_{\mathscr{E}_2})\zeta,\zeta\rangle \\
&= \langle(W\overline{\otimes}I_{\mathscr{E}_2})\zeta,\zeta\rangle_{\mathcal{A}\overline{\otimes}\mathcal{B}}.
\end{aligned}$$

Applying this equivalence to the first two terms of $\mathcal{W}_\lambda$ produces

$$\begin{aligned}
\langle\mathcal{W}_\lambda\zeta,\zeta\rangle_{\mathcal{A}\overline{\otimes}\mathcal{B}} &= \langle(\Theta^2_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}_1})\overline{\otimes}I_{\mathscr{E}_2} - 2\lambda\Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}_1})\overline{\otimes}I_{\mathscr{E}_2} + \lambda^2 I_{\mathscr{E}_1}\overline{\otimes}I_{\mathscr{E}_2})\zeta,\zeta\rangle_{\mathcal{A}\overline{\otimes}\mathcal{B}} \\
&= \langle([\Theta^2_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}_1}) - 2\lambda\Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}_1}) + \lambda^2 I_{\mathscr{E}_1}]\overline{\otimes}I_{\mathscr{E}_2})\zeta,\zeta\rangle_{\mathcal{A}\overline{\otimes}\mathcal{B}}.
\end{aligned}$$

Since $\mathbf{Y}$ is jointly $\mathcal{A}$-paranormal, Theorem 1 implies

$$P_\lambda := \Theta^2_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}_1}) - 2\lambda\Theta_{\mathbf{Y},\mathcal{A}}(I_{\mathscr{E}_1}) + \lambda^2 I_{\mathscr{E}_1} \geq_{\mathcal{A}} \mathbf{0}.$$

Since the tensor product of an $\mathcal{A}$-positive operator (P_λ) and a $\mathcal{B}$-positive operator ($I_{\mathscr{E}_2}$) is inherently $(\mathcal{A}\overline{\otimes}\mathcal{B})$-positive, it follows that $\langle\mathcal{W}_\lambda\zeta,\zeta\rangle_{\mathcal{A}\overline{\otimes}\mathcal{B}} \geq 0$. Since this holds for all $\lambda > 0$, it guarantees that $\mathbf{Y}\overline{\otimes}\mathbf{I}_2$ is jointly $(\mathcal{A}\overline{\otimes}\mathcal{B})$-paranormal. □

Remark 5. *The tensor product stability established in Theorem 10 diverges significantly from classical results. Because the $(\mathcal{A}\overline{\otimes}\mathcal{B})$-adjoint inevitably introduces the orthogonal projection $\Pi_{\mathcal{B}}$ via $(Y_l\overline{\otimes}I_{\mathscr{E}_2})^{\sharp_{\mathcal{A}\overline{\otimes}\mathcal{B}}} = Y_l^{\sharp_{\mathcal{A}}}\overline{\otimes}\Pi_{\mathcal{B}}$, the proof must systematically track the propagation of this projection*

through all iterates of Θ*. The subsequent reliance on the identity* $\mathcal{B}\Pi_{\mathcal{B}} = \mathcal{B}$ *to finalize the positivity argument is an obstruction unique to the semi-Hilbertian framework and is entirely absent from any classical treatment.*

Remark 6. *It is worth noting that even in the classical Hilbert space setting* ($\mathcal{A} = I_{\mathscr{E}_1}, \mathcal{B} = I_{\mathscr{E}_2}$)*, if* $\mathbf{Y}$ *and* $\mathbf{Z}$ *are jointly paranormal tuples, their full tensor product* $\mathbf{Y}\overline{\otimes}\mathbf{Z}$ *may not be jointly paranormal, even for single operators* ($q = 1$)*, as demonstrated by Ando [25].*

4. Conclusions and Open Problems

In this paper, we introduced the class of jointly $\mathcal{A}$-paranormal tuples of operators on semi-Hilbertian spaces and established its natural position within the operator hierarchy

$$\text{jointly } \mathcal{A}\text{-hyponormal} \implies \text{jointly } \mathcal{A}\text{-paranormal} \implies \text{jointly } \mathcal{A}\text{-normaloid},$$

filling a gap that previously existed in the multivariable semi-Hilbertian theory. Along the way, we derived equivalent characterizations of joint $\mathcal{A}$-paranormality via operator inequalities; the completely positive map $\Theta_{\mathbf{Y},\mathcal{A}}$ established a generalized log-convexity property for its iterates, proved stability under $\mathcal{A}$-unitary equivalence and under taking powers, and showed that the class is preserved under tensor product extensions. We regard this work as a natural starting point for a broader research program in multivariable semi-Hilbertian operator theory, and we now outline what we believe are the most promising directions for future investigation. A natural and immediate next step is the introduction and systematic study of jointly $\mathcal{A}$-quasinormal tuples in semi-Hilbertian spaces, where each component Y_l commutes with $Y_l^{\sharp_{\mathcal{A}}} Y_l$ in the $\mathcal{A}$-sense; this class would sit between jointly $\mathcal{A}$-normal and jointly $\mathcal{A}$-hyponormal tuples, further refining the hierarchy and demonstrating that the tools developed in the present paper—particularly the monotonicity of $\Theta_{\mathbf{Y},\mathcal{A}}$ established in Theorem 4—are well suited to its analysis. Several of our results rely on classical commutativity of the tuple components, and a compelling open question is whether these can be extended to the weaker and more natural condition of $\mathcal{A}$-commutativity, namely, $\mathcal{A}Y_lY_s = \mathcal{A}Y_sY_l$ for all l, s; this is a strictly weaker requirement whenever $\mathcal{A}$ has a nontrivial null space, and resolving it would require, in particular, developing a joint $\mathcal{A}$-spectral radius theory under this relaxed hypothesis. Finally, exploring connections with $(\mathcal{A}, m)$-isometric tuples [23] and investigating the spectral consequences of joint $\mathcal{A}$-paranormality—such as the structure of the joint $\mathcal{A}$-spectrum and the existence of common invariant subspaces—are further directions that we believe deserve attention, and we hope the framework developed here will serve as a useful foundation for these and related future investigations.

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