

Article

The Well-Posedness of Three-Dimensional Hall-Magnetohydrodynamics Model with Partial Viscosity Terms

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Abstract

The paper is concerned with three-dimensional Hall-magnetohydrodynamics for partial viscosity terms. By applying basic inequality and energy estimates, we prove the global existence of classical solution for small initial data. Additionally, the local existence of classical solution is also obtained. The smallness conditions are given by the suitable Sobolev norms. Furthermore, the existence of classical solutions with large initial data can be obtained when the coefficients of viscosity sufficiently large.

Keywords: well-posedness; Hall-magnetohydrodynamics; partial viscosity; classical solution; small initial data

MSC: 35L60; 35Q80

1. Introduction

This paper investigates the following three-dimensional Hall-MHD system (Hall-MHD):

$$\begin{cases} \partial_t b - \Delta b = (b \cdot \nabla)v - \nabla \times ((\nabla \times b) \times b) - (v \cdot \nabla)b, \\ \partial_t v + \nabla(p + \pi) + (v \cdot \nabla)v = \nu_1 v_{x_1 x_1} + \nu_2 v_{x_2 x_2} + \nu_3 v_{x_3 x_3} + (b \cdot \nabla)b, \\ \nabla \cdot b = \nabla \cdot v = 0, \\ v|_{t=0} = v_0, \quad b|_{t=0} = b_0. \end{cases} \quad (1)$$

where b, v represent the magnetic field and velocity vector field, p denotes the scalar pressure, π is a scalar, and $\nu_1, \nu_2, \nu_3 \geq 0$ are the kinematic viscosity. The Hall-MHD is indeed needed for such problems as magnetic reconnection in space plasmas, star formation, neutron stars and geo-dynamo. Compared with the MHD system, the Hall-MHD equations have the Hall term $\nabla \times ((\nabla \times b) \times b)$, which plays an important role in magnetic reconnection, which is happening in the case of large magnetic shear. Hall-MHD is an essential feature in relation to the problem of magnetic reconnection; it corresponds to changes in the topology of magnetic field lines which are ubiquitously observed in space.

There exist many results on incompressible Hall-magnetohydrodynamic equations: In paper [1]’s given derivation of model (1), Chae, Degond and Liu [2] established well-posedness for Hall-magnetohydrodynamics. Chae and Lee [3] established the blow-up criterion and small data global existence for the Hall-magnetohydrodynamics. The papers [4,5] investigated temporal decay and singularity formation for the Hall-magnetohydrodynamic equations. Paper [6] studied the partial regularity for the steady Hall-magnetohydrodynamics system followed by paper [7]. Ye [8] obtained the well-posedness results for the 3D incompressible Hall-MHD equations with fractional dissipa-



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tion. Liu [9] established the global existence to Hall-MHD system. Dai [10] obtained almost certain well-posedness for Hall-MHD.

Dumas and Sueur [11] derived weak solutions to the Maxwell–Landau–Lifshitz equations and to Hall-magnetohydrodynamic equations. Paper [12] established well-posedness for the axisymmetric incompressible viscous Hall-magnetohydrodynamic equations. Homann and Grauer [13] investigated the bifurcation analysis of magnetic reconnection in Hall-MHD systems. Li [14] investigated the local well-posedness of a 3D ideal Hall-MHD system with an azimuthal magnetic field. The global well-posedness of 3D incompressible hyper-dissipative Hall-MHD equations in anisotropic Besov spaces was obtained in [15]. Danchin and Tan [16] established the well-posedness of the Hall-magnetohydrodynamics system in critical spaces.

To inspire this research, we recall related important contributions to incompressible Hall-MHD equations. Fei and Xiang [17] researched Hall-magnetohydrodynamics for partial dissipation, and obtained a blow-up criterion of classical solutions. Du [18] investigated 2.5D Hall-magnetohydrodynamics with partial dissipation and proved the classical solutions that followed in [19]. Two blow-up criteria of smooth solutions and two improved classical solutions with small initial data for the Hall-magnetohydrodynamics with partial viscosity terms were established in paper [20,21]. Additionally, 3D magnetohydrodynamics with partial diffusion have been analysed by Ye and Zhu [22], who obtained the existence of the global classical solutions.

2. Preliminaries

This paper concerns three-dimensional Hall-magnetohydrodynamics (1) with partial viscosity—that is, the case of $\nu_2 = 0, \nu_1, \nu_3 > 0$. We should point out that this case is mainly a mathematical simplification, but it is theoretically significant in mathematics due to the partial viscosity terms leading to no smoothing effect on the vertical derivative. We show the global well-posedness and, more precisely, we prove the global existence and the local existence of classical solutions for system (1) with partial viscosity terms.

In this paper, $\|\cdot\|_2 \triangleq \|\cdot\|_{L^2}$, C represent a generic positive constant, and ∂_k and v_k indicate the corresponding k -th components of ∇, v , respectively.

$$\nabla^\beta := \partial^{|\beta|} / \partial_1^{\beta_1} \partial_2^{\beta_2} \partial_3^{\beta_3},$$

where $\beta = (\beta_1, \beta_2, \beta_3) \in (\mathbb{N} \cup \{0\})^3$ with $|\beta| = \beta_1 + \beta_2 + \beta_3 \leq 3$.

$$\zeta := \min\{\nu_1, \nu_3\} > 0.$$

$$\nabla_p := (\partial_1, 0, \partial_3).$$

3. Results

The paper researches the global well-posedness for model (1) with $\nu_2 = 0, \nu_1, \nu_3 > 0$.

Theorem 1. *Suppose 3D Hall-magnetohydrodynamics (1) with $\nu_2 = 0$, and $\nu_1, \nu_3 > 0$, and assume $\nabla \cdot v_0 = \nabla \cdot b_0 = 0$ and $(b_0, v_0) \in H^3(\mathbb{R}^3)$. Then, there exists $T = T(\|v_0\|_{H^3}, \|b_0\|_{H^3})$ such that there exists a unique solution $(b, v) \in L^\infty(0, T; H^3(\mathbb{R}^3))$.*

Based on the local existence of classical solutions for system (1) with the partial viscosity terms in Theorem 1, we can establish the following global-in-time existence of the classical solution for the system (1) with $\nu_2 = 0, \nu_1, \nu_3 > 0$.

Theorem 2. Consider model (1) with $v_2 = 0$, and $v_1, v_3 > 0$, and assume $(v_0, b_0) \in H^3(\mathbb{R}^3)$ with $\nabla \cdot b_0 = \nabla \cdot v_0 = 0$. Then, there exists $\epsilon_0 > 0$, if

$$\frac{(\|b_0\|_2^2 + \|v_0\|_2^2)(\|v_0\|_{H^2}^2 + \|b_0\|_{H^2}^2)}{\zeta^4} \leq \epsilon_0. \quad (2)$$

System (1) has a unique classical solution (b, v) , satisfying

$$(b, v) \in L^\infty(0, T; H^3(\mathbb{R}^3)), (\nabla_p v, \nabla b) \in L^2(0, T; H^3(\mathbb{R}^3)), \quad (3)$$

Remark 1. Compared to previous results, the existence of classical solutions with large initial data can be obtained when the coefficients of viscosity are sufficiently large. In addition, the initial data $(\|v_0\|_2^2 + \|b_0\|_2^2)(\|v_0\|_{H^2}^2 + \|b_0\|_{H^2}^2)$ instead of $(\|v_0\|_{H^m}^2 + \|b_0\|_{H^m}^2, m > \frac{5}{2})$ in paper [2] are sufficiently small.

Remark 2. It will be interesting if one could obtain the global regularity in terms of only one component of the velocity in system (1), as there is an essential difficulty due to the quadratic Hall term.

4. Local Existence

Proof of Theorem 1. Applying ∇^β to the first two equations of (1), and taking the inner product of the resulting equations with $\nabla^\beta v$ and $\nabla^\beta b$, respectively, and adding them together, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|v\|_{H^3}^2 + \|b\|_{H^3}^2) + v_1 \|\partial_1 v\|_{H^3}^2 + v_3 \|\partial_3 v\|_{H^3}^2 + \|\Delta b\|_{H^3}^2 \\ &= - \sum_{0 < |\beta| \leq 3} \int_{\mathbb{R}^3} \nabla^\beta (v \cdot \nabla v) \cdot \nabla^\beta v dx + \sum_{0 < |\beta| \leq 3} \int_{\mathbb{R}^3} \nabla^\beta (b \cdot \nabla b) \cdot \nabla^\beta v dx \\ & - \sum_{0 < |\beta| \leq 3} \int_{\mathbb{R}^3} \nabla^\beta (v \cdot \nabla b) \cdot \nabla^\beta b dx + \sum_{0 < |\beta| \leq 3} \int_{\mathbb{R}^3} \nabla^\beta (b \cdot \nabla v) \cdot \nabla^\beta v dx \\ & - \sum_{0 < |\beta| \leq 3} \int_{\mathbb{R}^3} \nabla^\beta (\nabla \times ((\nabla \times b) \times b)) \cdot \nabla^\beta b dx \\ &= Q_1 + Q_2 + Q_3 + Q_4 + Q_5. \end{aligned} \quad (4)$$

Notice that $|\beta| = 0$, and the above equality becomes (11). Using calculus inequality and Sobolev inequality, one obtains

$$\begin{aligned} |Q_1| &= \left| \sum_{0 < |\beta| \leq 3} \int_{\mathbb{R}^3} [\nabla^\beta (v \cdot \nabla v) - (v \cdot \nabla) \nabla^\beta v] \cdot \nabla^\beta v dx \right| \\ &\leq \sum_{0 < |\beta| \leq 3} \|[\nabla^\beta (v \cdot \nabla v) - (v \cdot \nabla) \nabla^\beta v]\|_2 \|v\|_{H^3} \\ &\leq C(\|\nabla v\|_{L^\infty} \|\nabla v\|_{H^2} + \|v\|_{H^3} \|\nabla v\|_{L^\infty}) \|v\|_{H^3} \\ &\leq C\|v\|_{H^3}^3. \end{aligned} \quad (5)$$

Applying Leibnitz formula and Sobolev inequality, we obtain

$$\begin{aligned} |Q_2| &\leq C(\|b\|_{H^3} \|\nabla b\|_{L^\infty} + \|b\|_{L^\infty} \|\nabla b\|_{H^3}) \|v\|_{H^3} \\ &\leq C\|b\|_{H^3}^2 \|v\|_{H^3} + C\|\nabla b\|_{H^3} \|b\|_{H^3} \|v\|_{H^3} \\ &\leq \frac{1}{8} \|\nabla b\|_{H^3}^2 + C\|b\|_{H^3}^2 \|v\|_{H^3}^2. \end{aligned} \quad (6)$$

In a similar manner, we obtain

$$|Q_3| \leq \frac{1}{8} \|\nabla b\|_{H^3}^2 + C \|b\|_{H^3}^2 \|v\|_{H^3}^2. \quad (7)$$

Integrating by parts in Q_4 , and then similarly to the above calculation, one obtains

$$|Q_4| \leq \frac{1}{8} \|\nabla b\|_{H^3}^2 + C \|b\|_{H^3}^2 \|v\|_{H^3}^2. \quad (8)$$

By cancellation property, calculus inequality and Sobolev inequality, one can estimate Q_5 as

$$\begin{aligned} |Q_5| &= \left| \sum_{0 < |\beta| \leq 3} \int_{\mathbb{R}^3} [\nabla^\beta ((\nabla \times b) \times b) - \nabla^\beta (\nabla \times b) \times b] \cdot \nabla^\beta (\nabla \times b) dx \right| \\ &\leq \sum_{0 < |\beta| \leq 3} \|\nabla^\beta ((\nabla \times b) \times b) - \nabla^\beta (\nabla \times b) \times b\|_2 \|\nabla b\|_{H^3} \\ &\leq C (\|\nabla b\|_{L^\infty} \|\nabla b\|_{H^2} + \|b\|_{H^3} \|\nabla b\|_{L^\infty}) \|\nabla b\|_{H^3} \\ &\leq C \|b\|_{H^3}^2 \|\nabla b\|_{H^3} \\ &\leq \frac{1}{8} \|\nabla b\|_{H^3}^2 + C \|b\|_{H^3}^4. \end{aligned} \quad (9)$$

Combining (4)–(9), we obtain

$$\frac{d}{dt} (\|v\|_{H^3}^2 + \|b\|_{H^3}^2) + \zeta \|\nabla_p v\|_{H^3}^2 + \|\nabla b\|_{H^3}^2 \leq C (\|v\|_{H^3}^2 + \|b\|_{H^3}^2)^2. \quad (10)$$

Applying nonlinear Gronwall's inequality to (10), and choosing $T = \frac{1}{2C(\|v_0\|_{H^3}^2 + \|b_0\|_{H^3}^2)}$, we can then obtain

$$\|v\|_{H^3} + \|b\|_{H^3} \leq 2(\|v_0\|_{H^3} + \|b_0\|_{H^3}) \quad \forall t \in (0, T),$$

which completes the proof of Theorem 1. $\square$

5. Priori Estimates

In this section, we require the following useful inequality.

Lemma 1 ([23]). Assume that $h, f, g, \nabla_p f, \partial_3 g, \nabla_p h \in L^2(\mathbb{R}^3)$; then, the following inequality holds:

$$\int_{\mathbb{R}^3} |hgf| dx \leq C \|f\|_2^{\frac{1}{2}} \|g\|_2^{\frac{1}{2}} \|h\|_2^{\frac{1}{2}} \|\nabla_p f\|_2^{\frac{1}{2}} \|\partial_3 g\|_2^{\frac{1}{2}} \|\nabla_p h\|_2^{\frac{1}{2}}.$$

Given a classical solution (v, b) on $(0, T) \times \mathbb{R}^3$ to (1) with $v_2 = 0, v_1, v_3 > 0$, we define

$$G(t) \triangleq \sup_{0 \leq s \leq t} (\|\nabla v(s)\|_2^2 + \|\nabla b(s)\|_2^2 + \|\nabla^2 v(s)\|_2^2 + \|\nabla^2 b(s)\|_2^2), \quad 0 \leq t \leq T.$$

Under $v_2 = 0$, and $v_1, v_3 > 0$, and taking scalar products of the first two equations of model (1) with v and b , respectively, one obtains

$$\frac{1}{2} \frac{d}{dt} (\|v\|_2^2 + \|b\|_2^2) + \zeta (\|\nabla_p v\|_2^2 + \|\nabla b\|_2^2) = 0. \quad (11)$$

Based on the above equality, one can obtain the energy inequality as following.

Lemma 2. Suppose (b, v) solves (1) with the $v_2 = 0, v_1, v_3 > 0$ on $[0, T] \times \mathbb{R}^3$; then,

$$\|v\|_2^2 + \|b\|_2^2 + 2\zeta \int_0^T (\|\nabla_p v\|_2^2 + \|\nabla b\|_2^2) dt \leq \|v_0\|_2^2 + \|b_0\|_2^2. \quad (12)$$

Next, we will structure energy estimates of b, v .

Lemma 3. Suppose (v, b) solves (1) with $v_2 = 0$, and $v_1, v_3 > 0$ on $[0, T] \times \mathbb{R}^3$, if

$$G(T) \leq 2(\|\nabla v_0\|_2^2 + \|\nabla b_0\|_2^2 + \|\nabla^2 v_0\|_2^2 + \|\nabla^2 b_0\|_2^2), \quad (13)$$

and (2) holds, and ϵ_0 is sufficiently small. Then,

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|\nabla b\|_2^2 + \|\nabla v\|_2^2) + 2\zeta \int_0^T (\|\nabla^2 b(t)\|_2^2 + \|\nabla_p \nabla v\|_2^2) dt \\ & \leq \|\nabla b_0\|_2^2 + \|\nabla v_0\|_2^2. \end{aligned} \quad (14)$$

Proof of Lemma 3. Applying ∇ to the first two equations of model (1), taking the inner products with $\nabla b, \nabla v$ and then summing them together results in

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\nabla v\|_2^2 + \|\nabla b\|_2^2) + v_1 \|\nabla \partial_1 v\|_2^2 + v_3 \|\nabla \partial_3 v\|_2^2 + \|\Delta b\|_2^2 \\ & = - \int_{\mathbb{R}^3} \nabla v \cdot \nabla (v \cdot \nabla) v dx - \int_{\mathbb{R}^3} \nabla b \cdot \nabla (v \cdot \nabla) b dx + \int_{\mathbb{R}^3} \nabla v \cdot \nabla (b \cdot \nabla) b dx \\ & + \int_{\mathbb{R}^3} \nabla b \cdot \nabla (b \cdot \nabla) v dx - \int_{\mathbb{R}^3} \nabla b \cdot \nabla [\nabla \times ((\nabla \times b) \times b)] dx \\ & = \sum_{i=1}^5 K_i. \end{aligned} \quad (15)$$

One can split K_1 into three parts:

$$\begin{aligned} K_1 & = - \int_{\mathbb{R}^3} (\nabla_p v \cdot \nabla) v \cdot \nabla_p v dx - \int_{\mathbb{R}^3} (\partial_2 v_p \cdot \nabla_p) v \cdot \partial_2 v dx + \int_{\mathbb{R}^3} (\nabla_p \cdot v_p) \partial_3 v \cdot \partial_2 v dx \\ & = K_{11} + K_{12} + K_{13}. \end{aligned}$$

Applying Hölder inequality, Young's inequality and interpolation result in

$$\begin{aligned} K_{11} & \leq C \|\nabla v\|_2 \|\nabla_p v\|_{L^3} \|\nabla_p v\|_{L^6} \\ & \leq C \|\nabla v\|_2 \|\nabla_p v\|_2^{\frac{1}{2}} \|\nabla \nabla_p v\|_2^{\frac{1}{2}} \|\nabla \nabla_p v\|_2 \\ & \leq C \|\nabla \nabla_p v\|_2^{\frac{3}{2}} \|\nabla v\|_2 \|\nabla_p v\|_2^{\frac{1}{2}} \\ & \leq \frac{C}{\zeta^3} \|\nabla v\|_2^4 \|\nabla_p v\|_2^2 + \frac{\zeta}{6} \|\nabla \nabla_p v\|_2^2. \end{aligned}$$

By Lemma 1, one has

$$\begin{aligned} K_{12} & \leq C \|\partial_2 v\|_2^{\frac{1}{2}} \|\nabla_p v\|_2^{\frac{1}{2}} \|\partial_2 v\|_2^{\frac{1}{2}} \|\partial_2 \nabla_p v\|_2^{\frac{1}{2}} \|\partial_2 \nabla_p v\|_2^{\frac{1}{2}} \|\partial_2 \nabla_p v\|_2^{\frac{1}{2}} \\ & \leq C \|\nabla \nabla_p v\|_2^{\frac{3}{2}} \|\nabla v\|_2 \|\nabla_p v\|_2^{\frac{1}{2}} \\ & \leq \frac{C}{\zeta^3} \|\nabla v\|_2^4 \|\nabla_p v\|_2^2 + \frac{\zeta}{6} \|\nabla \nabla_p v\|_2^2. \end{aligned}$$

In a similar manner, we can obtain

$$K_{13} \leq \frac{C}{\zeta^3} \|\nabla v\|_2^4 \|\nabla_p v\|_2^2 + \frac{\zeta}{6} \|\nabla \nabla_p v\|_2^2.$$

Therefore,

$$K_1 \leq \frac{C}{\zeta^3} \|\nabla v\|_2^4 \|\nabla_p v\|_2^2 + \frac{\zeta}{2} \|\nabla \nabla_p v\|_2^2. \quad (16)$$

By Hölder inequality, we obtain

$$\begin{aligned} |K_2| &= \left| \int_{\mathbb{R}^3} \Delta b \cdot (v \cdot \nabla) b dx \right| \\ &\leq C \|\nabla b\|_{L^3} \|v\|_{L^6} \|\Delta b\|_2 \\ &\leq C \|\Delta b\|_2^{\frac{1}{2}} \|\nabla b\|_2^{\frac{1}{2}} \|\nabla v\|_2 \|\Delta v\|_2 \\ &\leq C \|\Delta b\|_2^{\frac{3}{2}} \|\nabla b\|_2^{\frac{1}{2}} \|\nabla v\|_2 \\ &\leq \frac{C}{\zeta^3} \|\nabla v\|_2^4 \|\nabla b\|_2^2 + \frac{\zeta}{8} \|\Delta b\|_2^2. \end{aligned} \quad (17)$$

Using the cancellation property, we rewrite $K_3 + K_4$ as

$$K_3 + K_4 = \int_{\mathbb{R}^3} (\nabla b \cdot \nabla) b \cdot \nabla v dx + \int_{\mathbb{R}^3} (\nabla b \cdot \nabla) v \cdot \nabla b dx$$

Hence, one can apply the same procedure as K_2 to obtain

$$|K_3 + K_4| \leq \frac{C}{\zeta^3} \|\nabla v\|_2^4 \|\nabla b\|_2^2 + \frac{\zeta}{4} \|\Delta b\|_2^2. \quad (18)$$

Based on cancellation property, one obtains

$$\begin{aligned} |K_5| &= \left| - \int_{\mathbb{R}^3} \nabla (\nabla \times b) \cdot \nabla ((\nabla \times b) \times b) dx \right| \\ &\leq C \|\nabla b\|_4^2 \|\Delta b\|_2 \\ &\leq C \|\nabla^2 b\|_2^{\frac{3}{2}} \|\nabla b\|_2^{\frac{1}{2}} \|\nabla^2 b\|_2 \\ &\leq \frac{C}{\zeta^3} \|\Delta b\|_2^4 \|\nabla b\|_2^2 + \frac{\zeta}{8} \|\Delta b\|_2^2. \end{aligned} \quad (19)$$

Combining (15)–(19) results in

$$\begin{aligned} &\frac{d}{dt} (\|\nabla v\|_2^2 + \|\nabla b\|_2^2) + \zeta (\|\nabla \nabla_p v\|_2^2 + \|\Delta b\|_2^2) \\ &\leq \frac{C}{\zeta^3} (\|\nabla v\|_2^4 + \|\nabla^2 b\|_2^4) (\|\nabla b\|_2^2 + \|\nabla_p v\|_2^2). \end{aligned}$$

Therefore,

$$\begin{aligned} &\sup_{0 \leq t \leq T} (\|\nabla v\|_2^2 + \|\nabla b\|_2^2) + \zeta \int_0^T (\|\nabla \nabla_p v\|_2^2 + \|\Delta b\|_2^2) dt \\ &\leq \frac{C}{\zeta^3} \sup_{0 \leq t \leq T} (\|\nabla v(t)\|_2^4 + \|\nabla^2 b(t)\|_2^4) \int_0^T (\|\nabla b\|_2^2 + \|\nabla_p v\|_2^2) dt \\ &\quad + (\|\nabla b_0\|_2^2 + \|\nabla v_0\|_2^2). \end{aligned} \quad (20)$$

Collecting Lemma 2, (13), (20), one obtains

$$\begin{aligned} &\sup_{0 \leq t \leq T} (\|\nabla b\|_2^2 + \|\nabla v\|_2^2) + \zeta \int_0^T (\|\nabla^2 b\|_2^2 + \|\nabla \nabla_p v\|_2^2) dt \\ &\leq C \frac{(\|v_0\|_2^2 + \|b_0\|_2^2) (\|\nabla v_0\|_2^2 + \|\nabla b_0\|_2^2 + \|\nabla^2 v_0\|_2^2 + \|\nabla^2 b_0\|_2^2)}{\zeta^4} \\ &\quad \times \sup_{0 \leq t \leq T} (\|\nabla v\|_2^2 + \|\nabla b\|_2^2) + (\|\nabla v_0\|_2^2 + \|\nabla b_0\|_2^2), \end{aligned}$$

which, together with (2), and upon choosing a sufficiently small ϵ_0 , yields (14). $\square$

Lemma 4. Under the same conditions of Lemma 3,

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|\nabla^2 v\|_2^2 + \|\nabla^2 b\|_2^2) + 2\zeta \int_0^T (\|\nabla^2 \nabla_p v\|_2^2 + \|\nabla^3 b\|_2^2) dt \\ & \leq \|\nabla^2 v_0\|_2^2 + \|\nabla^2 b_0\|_2^2. \end{aligned} \quad (21)$$

Proof of Lemma 4. Similarly to the derivation of (15), one has

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\nabla^2 b\|_2^2 + \|\nabla^2 v\|_2^2) + \nu_1 \|\nabla^2 \partial_1 v\|_2^2 + \nu_3 \|\nabla^2 \partial_3 v\|_2^2 + \|\nabla^3 b\|_2^2 \\ & = - \int \Delta(v \cdot \nabla)v \cdot \Delta v dx - \int \Delta \nabla^2 b \cdot (v \cdot \nabla) b dx + \int \Delta v \cdot \Delta(b \cdot \nabla) b dx \\ & \quad + \int \Delta(b \cdot \nabla)v \cdot \Delta b dx - \int \Delta b \cdot \Delta[\nabla \times ((\nabla \times b) \times b)] dx \\ & = T_1 + T_2 + T_3 + T_4 + T_5. \end{aligned} \quad (22)$$

T_1 can be rewritten as

$$\begin{aligned} T_1 & = - \int_{\mathbb{R}^3} (\nabla^2 v \cdot \nabla)v \cdot \nabla^2 v dx - 2 \int_{\mathbb{R}^3} (\nabla v \cdot \nabla) \nabla v \cdot \nabla^2 v dx \\ & = T_{11} + T_{12}. \end{aligned}$$

Using the same procedure as H^2 estimates in [19], we can split T_{11} and T_{12} into three parts, respectively.

$$\begin{aligned} T_{11} & = - \int_{\mathbb{R}^3} (\nabla \nabla_p v \cdot \nabla)v \cdot \nabla \nabla_p v dx \\ & \quad - \int_{\mathbb{R}^3} (\partial_2^2 v_p \cdot \nabla_p)v \cdot \partial_2^2 v dx + \int_{\mathbb{R}^3} (\partial_2 \nabla_p \cdot v_p) \partial_2 v \cdot \partial_2^2 v dx \\ & = T_{111} + T_{112} + T_{113}, \end{aligned}$$

and

$$\begin{aligned} T_{12} & = -2 \int_{\mathbb{R}^3} (\nabla v \cdot \nabla) \nabla_p v \cdot \nabla \nabla_p v dx \\ & \quad - 2 \int_{\mathbb{R}^3} (\partial_2 v_p \cdot \nabla_p) \partial_2 v \cdot \partial_2^2 v dx + 2 \int_{\mathbb{R}^3} (\nabla_p \cdot v_p) \partial_2^2 v \cdot \partial_2^2 v dx \\ & = T_{121} + T_{122} + T_{123}. \end{aligned}$$

By Hölder and Young's inequality, we obtain

$$\begin{aligned} T_{111} & \leq C \|\nabla v\|_{L^3} \|\nabla \nabla_p v\|_{L^3}^2 \\ & \leq C \|\nabla v\|_2^{\frac{1}{2}} \|\nabla^2 v\|_2^{\frac{1}{2}} \|\nabla^2 \nabla_p v\|_2^{\frac{3}{2}} \|\nabla_p v\|_2^{\frac{1}{2}} \\ & \leq C \|\nabla^2 \nabla_p v\|_2^{\frac{3}{2}} (\|\nabla v\|_2 + \|\nabla^2 v\|_2) \|\nabla_p v\|_2^{\frac{1}{2}} \\ & \leq \frac{C}{\zeta^3} (\|\nabla v\|_2^4 + \|\nabla^2 v\|_2^4) \|\nabla_p v\|_2^2 + \frac{\zeta}{12} \|\nabla^2 \nabla_p v\|_2^2. \end{aligned}$$

Applying Lemma 1, we estimate T_{112} , T_{113} as follows.

$$\begin{aligned} T_{112} & \leq C \|\partial_2^2 v\|_2^{\frac{1}{2}} \|\nabla_p v\|_2^{\frac{1}{2}} \|\partial_2^2 v\|_2^{\frac{1}{2}} \|\partial_2^2 \nabla_p v\|_2^{\frac{1}{2}} \|\partial_3 \nabla_p v\|_2^{\frac{1}{2}} \|\partial_2^2 \nabla_p v\|_2^{\frac{1}{2}} \\ & \leq C \|\nabla^2 \nabla_p v\|_2 \|\nabla \nabla_p v\|_2^{\frac{1}{2}} \|\nabla^2 v\|_2 \|\nabla_p v\|_2^{\frac{1}{2}} \\ & \leq C \|\nabla^2 \nabla_p v\|_2^{\frac{3}{2}} \|\nabla^2 v\|_2 \|\nabla_p v\|_2^{\frac{1}{2}} \\ & \leq \frac{C}{\zeta^3} \|\nabla^2 v\|_2^4 \|\nabla_p v\|_2^2 + \frac{\zeta}{12} \|\nabla^2 \nabla_p v\|_2^2, \end{aligned}$$

and

$$\begin{aligned}
 T_{113} &\leq C \|\partial_2 \nabla_p v\|_2^{\frac{1}{2}} \|\partial_2 v\|_2^{\frac{1}{2}} \|\partial_2^2 v\|_2^{\frac{1}{2}} \|\partial_2^2 \nabla_p v\|_2^{\frac{1}{2}} \|\partial_2 \nabla_p v\|_2^{\frac{1}{2}} \|\partial_2^2 \nabla_p v\|_2^{\frac{1}{2}} \\
 &\leq C \|\nabla^2 \nabla_p v\|_2 \|\nabla \nabla_p v\|_2 \|\nabla^2 v\|_2^{\frac{1}{2}} \|\nabla v\|_2^{\frac{1}{2}} \\
 &\leq C \|\nabla^2 \nabla_p v\|_2^{\frac{3}{2}} \|\nabla^2 v\|_2^{\frac{1}{2}} \|\nabla v\|_2^{\frac{1}{2}} \|\nabla_p v\|_2^{\frac{1}{2}} \\
 &\leq C \|\nabla^2 \nabla_p v\|_2^{\frac{3}{2}} (\|\nabla^2 v\|_2 + \|\nabla v\|_2) \|\nabla_p v\|_2^{\frac{1}{2}} \\
 &\leq \frac{C}{\zeta^3} (\|\nabla^2 v\|_2^4 + \|\nabla v\|_2^4) \|\nabla_p v\|_2^2 + \frac{\zeta}{12} \|\nabla^2 \nabla_p v\|_2^2.
 \end{aligned}$$

Therefore,

$$T_{11} \leq \frac{C}{\zeta^3} (\|\nabla^2 v\|_2^4 + \|\nabla v\|_2^4) \|\nabla_p v\|_2^2 + \frac{\zeta}{4} \|\nabla^2 \nabla_p v\|_2^2.$$

Clearly, T_{121} , T_{122} , and T_{123} can be estimated as T_{111} , T_{113} , and T_{112} ; thus, one has

$$T_{12} \leq \frac{C}{\zeta^3} (\|\nabla^2 v\|_2^4 + \|\nabla v\|_2^4) \|\nabla_p v\|_2^2 + \frac{\zeta}{4} \|\nabla^2 \nabla_p v\|_2^2.$$

Hence, we can obtain

$$T_1 \leq \frac{C}{\zeta^3} (\|\nabla^2 v\|_2^4 + \|\nabla v\|_2^4) \|\nabla_p v\|_2^2 + \frac{\zeta}{2} \|\nabla^2 \nabla_p v\|_2^2. \quad (23)$$

One decomposes T_2 into two terms:

$$\begin{aligned}
 T_2 &= - \int_{\mathbb{R}^3} (\nabla^2 v \cdot \nabla) b \cdot \nabla^2 b dx - 2 \int_{\mathbb{R}^3} (\nabla v \cdot \nabla) \nabla b \cdot \nabla^2 b dx \\
 &= T_{21} + T_{22}.
 \end{aligned}$$

Using interpolation, one obtains

$$\begin{aligned}
 T_{21} &\leq C \|\nabla^2 v\|_2 \|\nabla b\|_{L^3} \|\nabla^2 b\|_{L^6} \\
 &\leq C \|\nabla^2 v\|_2 \|\nabla b\|_2^{\frac{1}{2}} \|\nabla^2 b\|_2^{\frac{1}{2}} \|\nabla^3 b\|_2^{\frac{1}{2}} \\
 &\leq C \|\nabla^3 b\|_2^{\frac{3}{2}} \|\nabla^2 v\|_2 \|\nabla b\|_2^{\frac{1}{2}} \\
 &\leq \frac{C}{\zeta^3} \|\nabla^2 v\|_2^4 \|\nabla b\|_2^2 + \frac{\zeta}{14} \|\nabla^3 b\|_2^2,
 \end{aligned}$$

and

$$\begin{aligned}
 T_{22} &\leq C \|\nabla v\|_{L^3} \|\nabla^2 b\|_{L^3}^2 \\
 &\leq C \|\nabla^2 v\|_2^{\frac{1}{2}} \|\nabla v\|_2^{\frac{1}{2}} \|\nabla b\|_2^{\frac{1}{2}} \|\nabla^3 b\|_2^{\frac{3}{2}} \\
 &\leq C \|\nabla^3 b\|_2^{\frac{3}{2}} (\|\nabla^2 v\|_2 + \|\nabla v\|_2) \|\nabla b\|_2^{\frac{1}{2}} \\
 &\leq \frac{C}{\zeta^3} (\|\nabla^2 v\|_2^4 + \|\nabla v\|_2^4) \|\nabla b\|_2^2 + \frac{\zeta}{14} \|\nabla^3 b\|_2^2.
 \end{aligned}$$

Thus, one obtains

$$T_2 \leq \frac{C}{\zeta^3} (\|\nabla^2 v\|_2^4 + \|\nabla v\|_2^4) \|\nabla b\|_2^2 + \frac{\zeta}{7} \|\nabla^3 b\|_2^2. \quad (24)$$

Based on cancellation property, one can write $T_3 + T_4$ into four terms:

$$\begin{aligned} & \int_{\mathbb{R}^3} (\nabla^2 b \cdot \nabla) b \cdot \nabla^2 v dx + \int_{\mathbb{R}^3} (\nabla^2 b \cdot \nabla) v \cdot \nabla^2 b dx \\ & + 2 \int_{\mathbb{R}^3} (\nabla b \cdot \nabla) \nabla b \cdot \nabla^2 v dx + 2 \int_{\mathbb{R}^3} (\nabla b \cdot \nabla) \nabla v \cdot \nabla^2 b dx \\ & = T_{341} + T_{342} + T_{343} + T_{344}. \end{aligned}$$

$(T_{341}, T_{343}, T_{344})$, and T_{342} can be estimated as T_{21} and T_{22} ; hence, one has

$$|T_3 + T_4| \leq \frac{C}{\zeta^3} (\|\nabla^2 v\|_2^4 + \|\nabla v\|_2^4) \|\nabla b\|_2^2 + \frac{2\zeta}{7} \|\nabla^3 b\|_2^2. \quad (25)$$

Similarly to (19), we obtain

$$\begin{aligned} |T_5| &= \left| \int_{\mathbb{R}^3} \Delta((\nabla \times b) \times b) \cdot \Delta(\nabla \times b) dx \right| \\ &\leq C \|\nabla^3 b\|_2 \|\nabla b\|_{L^6} \|\nabla^2 b\|_{L^3} \\ &\leq C \|\nabla^3 b\|_2 \|\nabla^2 b\|_2 \|\nabla^2 b\|_2^{\frac{1}{2}} \|\nabla^3 b\|_2^{\frac{1}{2}} \\ &\leq C \|\nabla^3 b\|_2^{\frac{3}{2}} \|\nabla^2 b\|_2^{\frac{3}{2}} \\ &\leq \frac{C}{\zeta^3} \|\nabla^2 b\|_2^4 \|\nabla^2 b\|_2^2 + \frac{\zeta}{14} \|\nabla^3 b\|_2^2. \end{aligned} \quad (26)$$

Putting (22)–(26) together results in

$$\begin{aligned} & \frac{d}{dt} (\|\nabla^2 v(t)\|_2^2 + \|\nabla^2 b(t)\|_2^2) + \zeta (\|\nabla^2 \nabla_p v\|_2^2 + \|\nabla^3 b\|_2^2) \\ & \leq \frac{C}{\zeta^3} (\|\nabla^2 v\|_2^4 + \|\nabla^2 b\|_2^4 + \|\nabla v\|_2^4) (\|\nabla b\|_2^2 + \|\nabla_p v\|_2^2). \end{aligned}$$

Hence, one obtains

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|\Delta b\|_2^2 + \|\nabla^2 v\|_2^2) + \zeta \int_0^T (\|\nabla^2 \nabla_p v(t)\|_2^2 + \|\nabla^3 b(t)\|_2^2) dt \\ & \leq \frac{C}{\zeta^3} \sup_{0 \leq t \leq T} (\|\nabla^2 v(t)\|_2^4 + \|\nabla^2 b(t)\|_2^4 + \|\nabla v\|_2^4) \\ & \quad \times \int_0^T (\|\nabla_p v\|_2^2 + \|\nabla b\|_2^2) dt + (\|\nabla^2 b_0\|_2^2 + \|\nabla^2 v_0\|_2^2). \end{aligned} \quad (27)$$

Combining Lemma 2, (13) and (27), we obtain

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|\nabla^2 v\|_2^2 + \|\nabla^2 b\|_2^2) + \zeta \int_0^T (\|\Delta \nabla_p v\|_2^2 + \|\nabla^3 b\|_2^2) dt \\ & \leq C \frac{(\|v_0\|_2^2 + \|b_0\|_2^2)(\|v_0\|_{H^2}^2 + \|b_0\|_{H^2}^2)}{\zeta^4} \\ & \quad \times \sup_{0 \leq t \leq T} (\|\nabla^2 v\|_2^2 + \|\nabla^2 b\|_2^2 + \|\nabla v\|_2^2) + (\|\nabla^2 v_0\|_2^2 + \|\nabla^2 b_0\|_2^2), \end{aligned}$$

which, together with (2), and taking ϵ_0 as sufficiently small, results in (21). $\square$

Proposition 1. Suppose (v, b) solves the system (1) with case $v_2 = 0$, and $v_1, v_3 > 0$, and there exists $\epsilon_0 > 0$; if (2) and (13) hold, then

$$G(T) \leq (\|\nabla v_0\|_2^2 + \|\nabla b_0\|_2^2 + \|\nabla^2 v_0\|_2^2 + \|\nabla^2 b_0\|_2^2). \quad (28)$$

Proof of Proposition 1. Based on the following Lemmas 3 and 4, adding (14) and (21) results in

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|\nabla v\|_2^2 + \|\nabla b\|_2^2 \|\nabla^2 v\|_2^2 + \|\nabla^2 b\|_2^2) \\ & + 2\zeta \int_0^T (\|\nabla \nabla_p v\|_2^2 + \|\nabla^2 b\|_2^2 + \|\nabla^2 \nabla_p v\|_2^2 + \|\nabla^3 b\|_2^2) dt \\ & \leq \|\nabla v_0\|_2^2 + \|\nabla b_0\|_2^2 \|\nabla^2 v_0\|_2^2 + \|\nabla^2 b_0\|_2^2. \end{aligned}$$

Therefore,

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|\nabla v\|_2^2 + \|\nabla b\|_2^2 \|\nabla^2 v\|_2^2 + \|\nabla^2 b\|_2^2) \\ & \leq (\|\nabla v_0\|_2^2 + \|\nabla b_0\|_2^2 \|\nabla^2 v_0\|_2^2 + \|\nabla^2 b_0\|_2^2). \end{aligned}$$

This completes the proof of Proposition 1. $\square$

From Proposition 1, one deduces energy estimates as follows.

Proposition 2. Under the same conditions as in Proposition 1,

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|\nabla^3 v(t)\|_2^2 + \|\nabla^3 b(t)\|_2^2) \\ & + 2\zeta \int_0^T (\|\nabla^3 \nabla_p v\|_2^2 + \|\nabla^4 b\|_2^2) dt \\ & \leq C(\|\nabla^3 v_0\|_2^2 + \|\nabla^3 b_0\|_2^2). \end{aligned} \quad (29)$$

Proof of Proposition 2. Similarly to (15), one obtains

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\nabla^3 v(t)\|_2^2 + \|\nabla^3 b(t)\|_2^2) \\ & + \nu_1 \|\nabla^3 \partial_1 v\|_2^2 + \nu_3 \|\nabla^3 \partial_2 v\|_2^2 + \|\nabla^4 b\|_2^2 \\ & = - \int_{\mathbb{R}^3} \nabla^3 (v \cdot \nabla) v \cdot \nabla^3 v dx - \int_{\mathbb{R}^3} \nabla^3 (v \cdot \nabla) b \cdot \nabla^3 b dx \\ & + \int_{\mathbb{R}^3} \nabla^3 (b \cdot \nabla) b \cdot \nabla^3 v dx + \int_{\mathbb{R}^3} \nabla^3 (b \cdot \nabla) v \cdot \nabla^3 b dx \\ & - \int_{\mathbb{R}^3} \nabla^3 [\nabla \times ((\nabla \times b) \times b)] \cdot \nabla^3 b dx \\ & = X_1 + X_2 + X_3 + X_4 + X_5. \end{aligned} \quad (30)$$

One can rewrite X_1 into three parts:

$$\begin{aligned} & - \int_{\mathbb{R}^3} (\nabla^3 v \cdot \nabla) v \cdot \nabla^3 v dx - 3 \int_{\mathbb{R}^3} (\nabla^2 v \cdot \nabla) \nabla v \cdot \nabla^3 v dx \\ & - 3 \int_{\mathbb{R}^3} (\nabla v \cdot \nabla) \nabla^2 v \cdot \nabla^3 v dx = X_{11} + X_{12} + X_{13}. \end{aligned}$$

Similarly to the H^3 estimates in paper [19], X_{11} and X_{12} can be decomposed into three parts, respectively.

$$\begin{aligned} & - \int_{\mathbb{R}^3} (\nabla^2 \nabla_p v \cdot \nabla) v \cdot \nabla^2 \nabla_p v dx - \int_{\mathbb{R}^3} (\partial_2^3 v_p \cdot \nabla_p) v \cdot \partial_2^3 v dx \\ & + \int_{\mathbb{R}^3} (\partial_2^2 \nabla_p \cdot v_p) \partial_2 v \cdot \partial_2^3 v dx = X_{111} + X_{112} + X_{113}, \end{aligned}$$

and

$$\begin{aligned} & - 3 \int_{\mathbb{R}^3} (\nabla \nabla_p v \cdot \nabla) \nabla v \cdot \nabla^2 \nabla_p v dx - 3 \int_{\mathbb{R}^3} (\partial_2^2 v_p \cdot \nabla_p) \partial_2 v \cdot \partial_2^2 v dx \\ & + 3 \int_{\mathbb{R}^3} (\partial_2 \nabla_p \cdot v_p) \partial_2^2 v \cdot \partial_2^2 v dx = X_{121} + X_{122} + X_{123}. \end{aligned}$$

Using Hölder inequality results in

$$\begin{aligned}
 X_{111} &\leq C \|\nabla^2 \nabla_p v\|_2 \|\nabla v\|_{L^6} \|\nabla^2 \nabla_p v\|_{L^3} \\
 &\leq C \|\nabla^2 \nabla_p v\|_2 \|\nabla^2 v\|_2 \|\nabla^3 \nabla_p v\|_2^{\frac{1}{2}} \|\nabla^2 \nabla_p v\|_2^{\frac{1}{2}} \\
 &\leq C \|\nabla^3 \nabla_p v\|_2^{\frac{1}{2}} \|\nabla^2 \nabla_p v\|_2^{\frac{3}{2}} \|\nabla^2 v\|_2 \\
 &\leq C \|\nabla^3 \nabla_p v\|_2^{\frac{1}{2}} \|\nabla^3 v\|_2^{\frac{3}{2}} \|\nabla^2 v\|_2 \\
 &\leq C \|\nabla^2 v\|_2^{\frac{4}{3}} \|\nabla^3 v\|_2^2 + \frac{\zeta}{18} \|\nabla^3 \nabla_p v\|_2^2.
 \end{aligned}$$

By Lemma 1, one can estimate X_{112} , X_{113} as follows:

$$\begin{aligned}
 X_{112} &\leq C \|\partial_2^3 v\|_2^{\frac{1}{2}} \|\nabla_p v\|_2^{\frac{1}{2}} \|\partial_2^3 u\|_2^{\frac{1}{2}} \|\partial_2^3 \nabla_p v\|_2^{\frac{1}{2}} \|\partial_2 \nabla_p v\|_2^{\frac{1}{2}} \|\partial_2^3 \nabla_p v\|_2^{\frac{1}{2}} \\
 &\leq C \|\nabla^3 \nabla_p v\|_2 \|\nabla^3 v\|_2 \|\nabla \nabla_p v\|_2^{\frac{1}{2}} \|\nabla v\|_2^{\frac{1}{2}} \\
 &\leq C \|\nabla^3 \nabla_p v\|_2 \|\nabla^3 v\|_2 (\|\nabla \nabla_p v\|_2 + \|\nabla v\|_2) \\
 &\leq C (\|\nabla \nabla_p v\|_2^2 + \|\nabla v\|_2^2) \|\nabla^3 v\|_2^2 + \frac{\zeta}{18} \|\nabla^3 \nabla_p v\|_2^2,
 \end{aligned}$$

and

$$\begin{aligned}
 X_{113} &\leq C \|\partial_2^2 \nabla_p v\|_2^{\frac{1}{2}} \|\partial_2 v\|_2^{\frac{1}{2}} \|\partial_2^3 v\|_2^{\frac{1}{2}} \|\partial_2^3 \nabla_p v\|_2^{\frac{1}{2}} \|\partial_2 \nabla_p v\|_2^{\frac{1}{2}} \|\partial_2^3 \nabla_p v\|_2^{\frac{1}{2}} \\
 &\leq C \|\nabla^3 \nabla_p v\|_2 \|\nabla^3 v\|_2^{\frac{1}{2}} \|\nabla^2 \nabla_p v\|_2^{\frac{1}{2}} \|\nabla \nabla_p v\|_2^{\frac{1}{2}} \|\nabla v\|_2^{\frac{1}{2}} \\
 &\leq C \|\nabla^3 \nabla_p v\|_2 \|\nabla^3 v\|_2 (\|\nabla \nabla_p v\|_2 + \|\nabla v\|_2) \\
 &\leq C (\|\nabla \nabla_p v\|_2^2 + \|\nabla v\|_2^2) \|\nabla^3 v\|_2^2 + \frac{\zeta}{18} \|\nabla^3 \nabla_p v\|_2^2.
 \end{aligned}$$

Therefore,

$$X_{11} \leq C (\|\nabla \nabla_p v\|_2^2 + \|\nabla v\|_2^2 + \|\nabla^2 v\|_2^2) \|\nabla^3 v\|_2^2 + \frac{\zeta}{6} \|\nabla^3 \nabla_p v\|_2^2.$$

Via applying a similar process as with X_{111} , one obtains

$$\begin{aligned}
 X_{121} &\leq C \|\nabla \nabla_p v\|_{L^6} \|\nabla^2 v\|_2 \|\nabla^2 \nabla_p v\|_{L^3} \\
 &\leq C \|\nabla^2 \nabla_p v\|_2 \|\nabla^2 v\|_2 \|\nabla^3 \nabla_p v\|_2^{\frac{1}{2}} \|\nabla^2 \nabla_p v\|_2^{\frac{1}{2}} \\
 &\leq C \|\nabla^3 \nabla_p v\|_2^{\frac{1}{2}} \|\nabla^2 \nabla_p v\|_2^{\frac{3}{2}} \|\nabla^2 v\|_2 \\
 &\leq C \|\nabla^3 \nabla_p v\|_2^{\frac{1}{2}} \|\nabla^3 v\|_2^{\frac{3}{2}} \|\nabla^2 v\|_2 \\
 &\leq C \|\nabla^2 v\|_2^{\frac{4}{3}} \|\nabla^3 v\|_2^2 + \frac{\zeta}{18} \|\nabla^3 \nabla_p v\|_2^2.
 \end{aligned}$$

Based on Lemma 1, we estimate X_{122} as follows:

$$\begin{aligned}
 X_{122} &\leq C \|\partial_2^2 v\|_2^{\frac{1}{2}} \|\partial_2 \nabla_p v\|_2^{\frac{1}{2}} \|\partial_2^3 v\|_2^{\frac{1}{2}} \|\partial_2^3 \nabla_p v\|_2^{\frac{1}{2}} \|\partial_2^2 \nabla_p v\|_2^{\frac{1}{2}} \|\partial_2^3 \nabla_p v\|_2^{\frac{1}{2}} \\
 &\leq C \|\nabla^3 \nabla_p v\|_2^{\frac{1}{2}} \|\nabla^2 \nabla_p v\|_2 \|\nabla^3 v\|_2^{\frac{1}{2}} \|\nabla \nabla_p v\|_2^{\frac{1}{2}} \|\nabla^2 v\|_2^{\frac{1}{2}} \\
 &\leq C \|\nabla^3 \nabla_p v\|_2^{\frac{1}{2}} \|\nabla^3 v\|_2^{\frac{3}{2}} \|\nabla^2 v\|_2 \\
 &\leq C \|\nabla^2 v\|_2^{\frac{4}{3}} \|\nabla^3 v\|_2^2 + \frac{\zeta}{18} \|\nabla^3 \nabla_p v\|_2^2.
 \end{aligned}$$

One can apply same procedure as X_{122} to obtain

$$X_{123} \leq C \|\nabla^2 v\|_2^{\frac{4}{3}} \|\nabla^3 v\|_2^2 + \frac{\zeta}{18} \|\nabla^3 \nabla_p v\|_2^2.$$

Thus,

$$X_{12} \leq C \|\nabla^2 v\|_2^{\frac{4}{3}} \|\nabla^3 v\|_2^2 + \frac{\zeta}{6} \|\nabla^3 \nabla_p v\|_2^2.$$

Obviously, X_{13} can be estimated as X_{11} ; hence,

$$X_{13} \leq C(\|\nabla \nabla_p v\|_2^2 + \|\nabla v\|_2^2 + \|\nabla^2 v\|_2^2) \|\nabla^3 v\|_2^2 + \frac{\zeta}{6} \|\nabla^3 \nabla_p v\|_2^2.$$

Collecting these above estimates, we obtain

$$X_1 \leq C(\|\nabla \nabla_p v\|_2^2 + \|\nabla^2 v\|_2^2) \|\nabla^3 v\|_2^2 + \frac{\zeta}{2} \|\nabla^3 \nabla_p v\|_2^2. \quad (31)$$

One can write X_2 into three parts:

$$\begin{aligned} & -3 \int_{\mathbb{R}^3} (\nabla v \cdot \nabla) \nabla^2 b \cdot \nabla^3 b dx - 3 \int_{\mathbb{R}^3} (\nabla^2 v \cdot \nabla) \nabla b \cdot \nabla^3 b dx \\ & - \int_{\mathbb{R}^3} (\nabla^3 v \cdot \nabla) v \cdot \nabla^3 v dx = X_{21} + X_{22} + X_{23}. \end{aligned}$$

Applying Young's inequality and interpolation results in

$$\begin{aligned} X_{21} & \leq C \|\nabla v\|_{L^6} \|\nabla^3 b\|_2 \|\nabla^3 b\|_{L^3} \\ & \leq C \|\nabla^2 v\|_2 \|\nabla^3 b\|_2 \|\nabla^3 b\|_2^{\frac{1}{2}} \|\nabla^4 b\|_2^{\frac{1}{2}} \\ & \leq C \|\nabla^2 v\|_2 \|\nabla^3 b\|_2^{\frac{3}{2}} \|\nabla^4 b\|_2^{\frac{1}{2}} \\ & \leq C \|\nabla^2 v\|_2^{\frac{4}{3}} \|\nabla^3 b\|_2^2 + \frac{\zeta}{22} \|\nabla^4 b\|_2^2. \end{aligned}$$

In a similar manner, one can obtain

$$\begin{aligned} X_{22} & \leq C \|\nabla^2 v\|_2 \|\nabla^2 b\|_{L^6} \|\nabla^3 b\|_{L^3} \\ & \leq C \|\nabla^2 v\|_2 \|\nabla^3 b\|_2 \|\nabla^3 b\|_2^{\frac{1}{2}} \|\nabla^4 b\|_2^{\frac{1}{2}} \\ & \leq C \|\nabla^2 v\|_2 \|\nabla^3 b\|_2^{\frac{3}{2}} \|\nabla^4 b\|_2^{\frac{1}{2}} \\ & \leq C \|\nabla^2 v\|_2^{\frac{4}{3}} \|\nabla^3 b\|_2^2 + \frac{\zeta}{22} \|\nabla^4 b\|_2^2. \end{aligned}$$

and

$$\begin{aligned} X_{23} & \leq C \|\nabla^3 v\|_2 \|\nabla b\|_{L^3} \|\nabla^3 b\|_6 \\ & \leq C \|\nabla^3 v\|_2 \|\nabla b\|_2^{\frac{1}{2}} \|\nabla^2 b\|_2^{\frac{1}{2}} \|\Delta^2 b\|_2 \\ & \leq C \|\nabla^3 v\|_2 (\|\nabla b\|_2 + \|\Delta b\|_2) \|\Delta^2 b\|_2 \\ & \leq \frac{\zeta}{22} \|\Delta^2 b\|_2^2 + C(\|\Delta b\|_2^2 + \|\nabla b\|_2^2) \|\nabla^3 v\|_2^2. \end{aligned}$$

Hence, we obtain

$$X_2 \leq C(\|\nabla^2 v\|_2^{\frac{4}{3}} + \|\nabla^2 b\|_2^2) (\|\nabla^3 v\|_2^2 + \|\nabla^3 b\|_2^2) + \frac{3\zeta}{22} \|\nabla^4 b\|_2^2. \quad (32)$$

Applying cancellation property, one writes $X_3 + X_4$ into six parts:

$$\begin{aligned} & \int_{\mathbb{R}^3} (\nabla^3 b \cdot \nabla) b \cdot \nabla^3 v dx + 3 \int_{\mathbb{R}^3} (\nabla^2 b \cdot \nabla) \nabla^3 v \cdot \nabla b dx \\ & + 3 \int_{\mathbb{R}^3} (\nabla b \cdot \nabla) \nabla^2 b \cdot \nabla^3 v dx + \int_{\mathbb{R}^3} (\nabla^3 b \cdot \nabla) v \cdot \nabla^3 b dx \\ & + 3 \int_{\mathbb{R}^3} (\nabla b \cdot \nabla) \nabla^3 b \cdot \nabla^2 v dx + 3 \int_{\mathbb{R}^3} (\nabla^2 b \cdot \nabla) \nabla^3 b \cdot \nabla v dx \\ & = X_{341} + X_{342} + X_{343} + X_{344} + X_{345} + X_{346}. \end{aligned}$$

X_{342} can be further divided into two terms:

$$-3 \int_{\mathbb{R}^3} (\nabla^3 b \cdot \nabla) \nabla b \cdot \nabla^2 v dx - 3 \int_{\mathbb{R}^3} (\nabla^2 b \cdot \nabla) \nabla^2 b \cdot \nabla^2 v dx.$$

$(X_{341}, X_{343}, X_{346}), (X_{342}, X_{345}),$ and X_{344} can be estimated as $X_{21}, X_{22},$ and $X_{23};$ therefore,

$$|X_3 + X_4| \leq C(\|\nabla^2 v\|_2^{\frac{4}{3}} + \|\nabla^2 b\|_2^2)(\|\nabla^3 v\|_2^2 + \|\nabla^3 b\|_2^2) + \frac{7\zeta}{22} \|\nabla^4 b\|_2^2. \quad (33)$$

One can apply a similar process as in (13) to deduce that

$$\begin{aligned} |X_5| &= \left| - \int_{\mathbb{R}^3} \nabla^3 (\nabla \times b) \cdot \nabla^3 [(\nabla \times b) \times b] dx \right| \\ &\leq C \|\nabla^4 b\|_2 \|\nabla b\|_{L^6} \|\nabla^3 b\|_{L^3} \\ &\leq C \|\nabla^4 b\|_2 \|\nabla^2 b\|_2 \|\nabla^3 b\|_2^{\frac{1}{2}} \|\nabla^4 b\|_2^{\frac{1}{2}} \\ &\leq C \|\nabla^4 b\|_2^{\frac{3}{2}} \|\nabla^2 b\|_2 \|\nabla^3 b\|_2^{\frac{1}{2}} \\ &\leq C \|\nabla^2 b\|_2^4 \|\nabla^3 b\|_2^2 + \frac{\zeta}{22} \|\nabla^4 b\|_2^2. \end{aligned} \quad (34)$$

Combining (30)–(34) results in

$$\begin{aligned} &\frac{d}{dt} (\|\nabla^3 b\|_2^2 + \|\nabla^3 v\|_2^2) + \zeta (\|\nabla^3 \nabla_p v\|_2^2 + \|\nabla^4 b\|_2^2) \\ &\leq C(\|\nabla^2 v\|_2^2 + \|\nabla^2 b\|_2^4 + \|\nabla b\|_2^2)(\|\nabla^3 v\|_2^2 + \|\nabla^3 b\|_2^2). \end{aligned}$$

Based on Gronwall's inequality, we obtain

$$\begin{aligned} &\sup_{0 \leq t \leq T} (\|\nabla^3 v\|_2^2 + \|\nabla^3 b\|_2^2) + \zeta \int_0^T (\|\nabla^3 \nabla_p v\|_2^2 + \|\nabla^4 b\|_2^2) dt \\ &\leq (\|\nabla^3 v_0\|_2^2 + \|\nabla^3 b_0\|_2^2) [C \int_0^T (\|\nabla b\|_2^2 + \|\Delta b\|_2^4 + \|\Delta v\|_2^2) dt] \\ &\times \exp(C \int_0^T (\|\nabla^2 v\|_2^2 + \|\nabla^2 b\|_2^4 + \|\nabla b\|_2^2) dt). \end{aligned}$$

The above inequality, together with Lemmas 3 and 4, results in (29). $\square$

6. Global Existence

Proof of Theorem 2. Based on the Theorem 1 and Propositions 1 and 2, one can imply the following:

$$\begin{aligned} &\frac{d}{dt} (\|b\|_{H^3}^2 + \|v\|_{H^3}^2) + \zeta (\|\nabla_p v\|_{H^3}^2 + \|\nabla b\|_{H^3}^2) \\ &\leq C(\|v\|_{H^2}^2 + \|b\|_{H^2}^4 + \|b\|_{H^1}^2)(\|v\|_{H^3}^2 + \|b\|_{H^3}^2). \end{aligned}$$

Applying Gronwall's inequality results in

$$\begin{aligned} &\sup_{0 \leq t \leq T} (\|v\|_{H^3}^2 + \|b\|_{H^3}^2) + \zeta \int_0^T (\|\nabla_p v\|_{H^3}^2 + \|\nabla b\|_{H^3}^2) dt \\ &\leq C \frac{(\|v_0\|_2^2 + \|b_0\|_2^2)(\|v_0\|_{H^2}^2 + \|b_0\|_{H^2}^2)}{\zeta^4} \\ &\times \sup_{0 \leq t \leq T} (\|v\|_{H^3}^2 + \|b\|_{H^3}^2 + \|v\|_{H^1}^2) + (\|v_0\|_{H^3}^2 + \|b_0\|_{H^3}^2). \end{aligned}$$

We can observe that $(\|v_0\|_2^2 + \|b_0\|_2^2)(\|v_0\|_{H^2}^2 + \|b_0\|_{H^2}^2)/\zeta^4 \leq \epsilon_0$; therefore, if $\epsilon_0 < \frac{1}{2C}$, then

$$\begin{aligned} & \|v\|_{H^3}^2 + \|b\|_{H^3}^2 + 2\zeta \int_0^T (\|\nabla_p v\|_{H^3}^2 + \|\nabla b\|_{H^3}^2) dt \\ & \leq 2(\|v_0\|_{H^3}^2 + \|b_0\|_{H^3}^2), \end{aligned}$$

which ends the existence of classical solutions that can be obtained; one can use the L^2 -method to prove the uniqueness of classical solution. $\square$

7. Conclusions

In this paper, we establish the global existence and the local existence of classical solutions to the incompressible resistive Hall-magnetohydrodynamic equations with partial viscosity terms. There exist two main difficulties: the first one is due to the partial viscosity terms that mean there is no smoothing effect on the vertical derivative; the other one arises from the Hall term, which is quadratic in the magnetic field and involves second-order derivatives. To overcome these two difficulties, we apply delicate estimates for the energy method. It is very important to establish the global regularity for three-dimensional Hall-magnetohydrodynamic equations with partial magnetic diffusion, which need new technical means to overcome the difficulties that arise from the Hall term.

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Abbreviations

The following abbreviation is used in this manuscript:

Hall-MHD equations Hall-magnetohydrodynamics

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