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A Baire Category Approach to Rounded Discrete Max Domains of Attraction

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Abstract

We study a topological problem in discrete extreme value theory. Let $\mathcal{P}(\mathbb{N})$ denote the space of probability laws on $\mathbb{N}$, endowed with the total variation metric d_{TV}. Fix a rounding map r, either the ceiling map or the nearest-integer map. We consider the rounded discrete max-domain-of-attraction class $\mathcal{D}_r \subset \mathcal{P}(\mathbb{N})$, consisting of all laws of the form $\mathrm{law}(r(Y))$, where the parent law of Y belongs to a classical max domain of attraction $\mathrm{MDA}(G_\xi)$ for some generalized extreme value shape ξ. The main result of this note is that, for the ceiling and midpoint rounding schemes, each class $\mathcal{D}_r$ is meager in $\mathcal{P}(\mathbb{N})$, while remaining dense; this mirrors the continuous picture of Leonetti and Khorrami Chokami. The proof is based on the Baire category theorem: we construct a comeager subset of $\mathcal{P}(\mathbb{N})$ by forcing infinitely many incompatible dyadic tail ratios on disjoint far-out windows; such behavior cannot occur for any rounded law arising from a single classical max domain of attraction. We also record the corresponding Banach–Mazur game interpretation and explain why the argument applies to the rounded schemes (Types B and C) rather than to the sampled-density discretization (Type A).

Keywords: extreme value theory; discrete distributions; Baire category; max domain of attraction; total variation; Banach–Mazur game; regular variation

MSC: 60G70; 60B10; 54E52; 26A12

1. Introduction

Classical univariate extreme value theory classifies continuous laws according to the possible nondegenerate limits of affine normalizations of maxima. If $Y_1, Y_2, \ldots$ are i.i.d. with a common distribution function F, one asks whether there exist constants $a_n > 0$ and $b_n \in \mathbb{R}$, such that

$$F^n(a_n x + b_n) \longrightarrow G_\xi(x)$$

at continuity points of a generalized extreme value (GEV) law G_ξ. The three possible limit types are Fréchet ($\xi > 0$), Gumbel ($\xi = 0$), and Weibull ($\xi < 0$). For a background on classical max domains of attraction, we refer to Anderson [1,2], de Haan and Ferreira [3], and Resnick [4].

For integer-valued data, the classical continuous convergence picture is often too rigid. Even when the tail is clearly "Fréchet-like" or "Gumbel-like," lattice effects can prevent exact convergence to a continuous limit. One response is to pass from the continuous GEV family to discrete analogs obtained by rounding or binning a continuous GEV variable.

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Another is to define a rounded discrete max domain of attraction by embedding the discrete law into a rounded continuous parent already lying in a classical max domain of attraction.

Such integer-valued data arise in many natural settings: counts of insurance claims, exceedance counts in environmental monitoring, integer-rounded financial tick data, daily rainfall in millimeters, and the number of failures in network reliability studies. In all such cases the underlying physical or economic mechanism may be continuous, but the recorded values are intrinsically discrete. This can arise because of finite measurement precision, currency quantization, or counting itself. Discrete extreme value theory addresses these settings; foundational work goes back to Anderson [1,2]. Anderson's analysis showed in particular that integer-rounded versions of geometric and Poisson-like laws inherit the tail asymptotics of their continuous parents, foreshadowing the survival-function identity formalized in Lemma 1 below. A natural modeling strategy is to view the observed discrete law as the image of a continuous parent under a rounding transform, and ask which properties of the continuous parent—in particular, its max domain of attraction—transfer to the discrete observation. The classes $\mathcal{D}_B$ and $\mathcal{D}_C$ defined below capture exactly this "rounded" structure.

The purpose of this note is topological. We work on the infinite-dimensional simplex $\mathcal{P}(\mathbb{N})$, equipped with total variation distance, and ask how large the rounded class is from the point of view of the Baire category. The surprising answer is that the class is topologically small—it is meager. The continuous analog was established by Leonetti and Khorrami Chokami [5]: the max domain of attraction is dense yet of the first Baire category in the space of Borel probability measures on $\mathbb{R}^k$. The present note proves the discrete counterpart of the meagerness statement: each rounded class $\mathcal{D}_r$ is meager in $\mathcal{P}(\mathbb{N})$.

We also show that each rounded class $\mathcal{D}_r$ is dense in $\mathcal{P}(\mathbb{N})$ (Proposition 2), so the rounded classes are dense yet meager; this mirrors the continuous picture of Leonetti and Khorrami Chokami [5].

The key point is that a law obtained by rounding a continuous parent in a single classical max domain of attraction has one eventual tail regime. In contrast, a generic law on $\mathbb{N}$ can be forced to exhibit infinitely many incompatible local tail patterns. By building a comeager set of such oscillatory laws, we exclude the rounded class.

2. The Ambient Space

Definition 1. *Let*

$$\mathcal{P}(\mathbb{N}) = \Big\{ P = (p_1, p_2, \ldots) : p_k \geq 0, \ \sum_{k \geq 1} p_k = 1 \Big\}.$$

We equip $\mathcal{P}(\mathbb{N})$ with the total variation metric

$$d_{\mathrm{TV}}(P, Q) = \tfrac{1}{2} \sum_{k \geq 1} |p_k - q_k|.$$

For $P \in \mathcal{P}(\mathbb{N})$ and $X \sim P$, we write

$$\overline{P}(n) := P(X > n), \qquad n \in \mathbb{N}_0.$$

The bar over P is a typographical device used solely to denote the survival function and does not refer to a transformed measure; the underlying measure is still P.

Since $\mathcal{P}(\mathbb{N})$ is a closed subset of ℓ^1, it is a complete separable metric space and, hence, a Baire space.

3. Three Discrete GEV Constructions and Scope

We first set up notation for the continuous GEV family, then define the three discrete GEV constructions, and finally define the rounded embedding classes that are the main objects of the meagerness theorem. The reader should carefully distinguish the finite-parameter discrete GEV families from the larger embedding classes.

3.1. Continuous GEV Notation

Let $f_c(\cdot\,;\mu,\sigma,\xi)$ and $F_c(\cdot\,;\mu,\sigma,\xi)$ denote the continuous GEV density and cdf with location $\mu \in \mathbb{R}$, scale $\sigma > 0$, and shape $\xi \in \mathbb{R}$. Define the admissible support set

$$K(\xi) = \{k \in \mathbb{N} : 1 + \xi(k-\mu)/\sigma > 0\}.$$

For $\xi \geq 0$, one has $K(\xi) = \mathbb{N}$; for $\xi < 0$, the set $K(\xi)$ is finite.

3.2. The Three Parametric Families

Definition 2 (Parametric discrete GEV families)**.** *The three parametric discrete GEV families are the following subsets of* $\mathcal{P}(\mathbb{N})$*:*

(a) ***Type A*** *(sampled-density renormalization):*

$$p_A(k;\mu,\sigma,\xi) = C(\mu,\sigma,\xi)\, f_c(k;\mu,\sigma,\xi), \quad k \in K(\xi),$$

where $C(\mu,\sigma,\xi) = \left(\sum_{j\in K(\xi)} f_c(j;\mu,\sigma,\xi)\right)^{-1}$.

(b) ***Type B*** *(ceiling discretization):*

$$p_B(k;\mu,\sigma,\xi) = \frac{F_c(k;\mu,\sigma,\xi) - F_c(k-1;\mu,\sigma,\xi)}{1 - F_c(0;\mu,\sigma,\xi)}, \quad k \in K(\xi),$$

whenever $1 - F_c(0;\mu,\sigma,\xi) > 0$.

(c) ***Type C*** *(midpoint discretization):*

$$p_C(k;\mu,\sigma,\xi) = \frac{F_c(k+\frac{1}{2};\mu,\sigma,\xi) - F_c(k-\frac{1}{2};\mu,\sigma,\xi)}{1 - F_c(\frac{1}{2};\mu,\sigma,\xi)}, \quad k \in K(\xi),$$

whenever $1 - F_c(\frac{1}{2};\mu,\sigma,\xi) > 0$.

We write $\mathcal{A} = \{p_A(\cdot;\mu,\sigma,\xi)\}$, $\mathcal{B} = \{p_B(\cdot;\mu,\sigma,\xi)\}$, *and* $\mathcal{C} = \{p_C(\cdot;\mu,\sigma,\xi)\}$ *for the three parametric families, regarded as subsets of* $\mathcal{P}(\mathbb{N})$.

Types B and C arise from rounding a GEV random variable: if $X \sim \mathrm{GEV}(\mu,\sigma,\xi)$, then Type B is the law of $\lceil X \rceil$ conditional on $X > 0$, while Type C is the law of $\rho_C(X)$ conditional on $X > \frac{1}{2}$.

3.3. The Rounded Embedding Classes

The meagerness theorem concerns not merely the parametric families above, but the much larger classes obtained by rounding any continuous parent law already lying in some classical max domain of attraction.

For $y \in \mathbb{R}$, define the *midpoint rounding map* $\rho_C : \mathbb{R} \to \mathbb{Z}$ by

$$\rho_C(y) \;:=\; \lfloor y + \tfrac{1}{2} \rfloor,$$

i.e., the unique integer n satisfying $n - \frac{1}{2} \leq y < n + \frac{1}{2}$. With this convention ties at half-integers $y \in \frac{1}{2} + \mathbb{Z}$ are broken *upward*: $\rho_C(n + \frac{1}{2}) = n + 1$. Since every parent Y considered

in this paper is absolutely continuous, $\Pr(Y \in \frac{1}{2} + \mathbb{Z}) = 0$ and the tie-breaking convention is immaterial for all distributional statements that follow.

Definition 3 (Rounded embedding classes)**.** *Define two transforms on continuous distribution functions:*

$$T_B(F) = \operatorname{law}\big(\lceil Y \rceil \mid Y > 0\big),$$
$$T_C(F) = \operatorname{law}\big(\rho_C(Y) \mid Y > \tfrac{1}{2}\big),$$

whenever the conditioning events have positive probability and $Y \sim F$. The corresponding rounded discrete max-domain-of-attraction classes are

$$\mathcal{D}_B = \big\{T_B(F) : F \in \mathrm{MDA}(G_\xi) \text{ for some } \xi \in \mathbb{R}\big\},$$
$$\mathcal{D}_C = \big\{T_C(F) : F \in \mathrm{MDA}(G_\xi) \text{ for some } \xi \in \mathbb{R}\big\}.$$

We also write $\mathcal{D} = \mathcal{D}_B \cup \mathcal{D}_C$.

Remark 1 (Well-definedness of T_r)**.** *The transforms T_B and T_C produce elements of $\mathcal{P}(\mathbb{N})$ whenever the relevant conditioning event has positive probability.*

For $r = B$: if $\overline{F}(0) > 0$, then $\lceil Y \rceil \mid Y > 0$ takes values in $\mathbb{N} = \{1, 2, 3, \ldots\}$, with

$$T_B(F)(\{n\}) = \frac{F(n) - F(n-1)}{\overline{F}(0)}, \qquad n \geq 1.$$

Non-negativity follows from monotonicity of F. The values sum to 1:

$$\sum_{n=1}^{\infty} T_B(F)(\{n\}) = \frac{1}{\overline{F}(0)} \sum_{n=1}^{\infty} \big(F(n) - F(n-1)\big) = \frac{1 - F(0)}{\overline{F}(0)} = 1.$$

For $r = C$: if $\overline{F}(\frac{1}{2}) > 0$, then $\rho_C(Y) \mid Y > \frac{1}{2}$ takes values in $\mathbb{N}$, with

$$T_C(F)(\{n\}) = \frac{F(n + \frac{1}{2}) - F(n - \frac{1}{2})}{\overline{F}(\frac{1}{2})}, \qquad n \geq 1.$$

Non-negativity and total mass 1 follow by the same telescoping argument:

$$\sum_{n=1}^{\infty} T_C(F)(\{n\}) = \frac{1}{\overline{F}(\frac{1}{2})} \sum_{n=1}^{\infty} \big(F(n + \tfrac{1}{2}) - F(n - \tfrac{1}{2})\big) = \frac{\overline{F}(\frac{1}{2})}{\overline{F}(\frac{1}{2})} = 1.$$

The conditioning events $\{Y > 0\}$ and $\{Y > \frac{1}{2}\}$ have positive probability for any continuous $F \in \mathrm{MDA}(G_\xi)$ with unbounded right endpoint (Fréchet $\xi > 0$, or Gumbel $\xi = 0$ with $x_F = \infty$), and for finite-endpoint parents whose right endpoint exceeds 0 (resp. $\frac{1}{2}$). When the conditioning event has zero probability, $T_r(F)$ is undefined and is excluded from $\mathcal{D}_r$.

Remark 2 (Scope)**.** *The inclusions*

$$\mathcal{B} \subset \mathcal{D}_B \subset \mathcal{P}(\mathbb{N}) \qquad \text{and} \qquad \mathcal{C} \subset \mathcal{D}_C \subset \mathcal{P}(\mathbb{N})$$

hold. The embedding classes $\mathcal{D}_B$ and $\mathcal{D}_C$ are much larger than the parametric families: they contain the rounded image of every continuous parent law in some classical max domain of attraction, not only the GEV distributions themselves.

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Remark 3 (The Type A family is not addressed). *Sampled-density discretization (Type A) does not arise from a rounding transform, and the meagerness argument below does not extend to it. The reason is structural. For $r \in \{B, C\}$, the rounding construction produces the survival-function identity*

$$\overline{P}(n) \;=\; \frac{\overline{F}(c_r(n))}{\overline{F}(c_r)}, \qquad c_B(n) = n, \;\; c_C(n) = n + \tfrac{1}{2},$$

which expresses the discrete tail as a ratio of continuous tails. Regular and rapid variation of $\overline{F}$ then transfer directly to $\overline{P}$ via this ratio, giving the trichotomy of dyadic regimes used in Lemma 3. By contrast, Type A is defined by $p_A(k) = C\, f_c(k)$, so the discrete survival function is a Riemann-type sum of the continuous density rather than a ratio of continuous survival functions; there is no analogous clean transfer of regular variation, and the dyadic ratio $R_P(N)$ need not have a single eventual limit even when the parent density is regularly varying. Whether $\mathcal{A}$ is meager in $\mathcal{P}(\mathbb{N})$ remains open; see Section 10.

4. A Correction for the Type A Normalizing Constant

Proposition 1 (Type A is well defined for all admissible parameters). *Fix $\mu \in \mathbb{R}$, $\sigma > 0$, and $\xi \in \mathbb{R}$, and assume $K(\xi) \neq \varnothing$. Then,*

$$0 < \sum_{j \in K(\xi)} f_c(j; \mu, \sigma, \xi) < \infty.$$

Consequently the Type A normalizing constant $C(\mu, \sigma, \xi)$ exists and is finite for every admissible (μ, σ, ξ).

Proof. If $\xi < 0$, then $K(\xi)$ is finite, so the claim is immediate. If $\xi = 0$, then the Gumbel density satisfies $f_c(x; \mu, \sigma, 0) \leq \sigma^{-1} \exp(-(x-\mu)/\sigma)$, so $\sum_{j \geq 1} f_c(j; \mu, \sigma, 0)$ converges by comparison with a geometric series. If $\xi > 0$, then $f_c(x; \mu, \sigma, \xi) \sim C_{\mu,\sigma,\xi}\, x^{-1/\xi - 1}$ as $x \to \infty$, so the sampled series is comparable to a p-series with exponent $1 + 1/\xi > 1$ and therefore converges. □

Remark 4. *The threshold $\xi < 1$ is the condition for the existence of the mean of a Fréchet-type law; it is not the condition for the Type A normalizing constant to be finite.*

5. Tail Behavior: Regular and Rapid Variation

Fact 1 (Regular variation; de Haan and Ferreira [3]). *If $F \in \mathrm{MDA}(G_\xi)$ with $\xi > 0$, then $\overline{F}(x) := 1 - F(x) = x^{-\alpha} L(x)$ for some $\alpha = 1/\xi > 0$ and a slowly varying function L. In particular, for every $c > 0$,*

$$\frac{\overline{F}(cx)}{\overline{F}(x)} \;\longrightarrow\; c^{-\alpha} \qquad (x \to \infty).$$

Fact 2 (Rapid variation; Embrechts et al. [6]). *If $F \in \mathrm{MDA}(G_0)$ and F has an infinite right endpoint, then $\overline{F}$ is rapidly varying of index $-\infty$:*

$$\frac{\overline{F}(cx)}{\overline{F}(x)} \;\longrightarrow\; 0 \qquad (x \to \infty) \quad \textit{for every } c > 1.$$

We briefly recall the proof of Fact 2. The standard characterization of $\mathrm{MDA}(G_0)$ states that $F \in \mathrm{MDA}(G_0)$ if and only if there exists a positive measurable auxiliary function $a(t)$, such that

$$\lim_{t \to x_F} \frac{\overline{F}\big(t + x\, a(t)\big)}{\overline{F}(t)} = e^{-x} \qquad \text{for all } x \in \mathbb{R}; \tag{1}$$

see de Haan and Ferreira [3] (Theorem 1.2.5). When $x_F = \infty$, the auxiliary function satisfies $a(t)/t \to 0$ as $t \to \infty$ [3] (Corollary 1.2.10). Fix $c > 1$ and write, for large t,

$$\frac{\overline{F}(ct)}{\overline{F}(t)} = \frac{\overline{F}\big(t + x_t\, a(t)\big)}{\overline{F}(t)}, \qquad x_t := \frac{(c-1)t}{a(t)}.$$

Since $a(t)/t \to 0$ we have $x_t \to +\infty$. For any fixed $x_0 > 0$,

$$\frac{\overline{F}(ct)}{\overline{F}(t)} \;\le\; \frac{\overline{F}\big(t + x_0\, a(t)\big)}{\overline{F}(t)} \;\longrightarrow\; e^{-x_0}$$

for all sufficiently large t. Since x_0 is arbitrary, $\overline{F}(ct)/\overline{F}(t) \to 0$.

Fact 3 (Finite endpoint cases). *If $F \in \mathrm{MDA}(G_\xi)$ with $\xi < 0$, then F has a finite right endpoint. If $F \in \mathrm{MDA}(G_0)$ and F has a finite right endpoint, then the rounded law $T_r(F)$ has finite support for $r \in \{B, C\}$.*

Together, Facts 1–3 yield a trichotomy for any rounded law $P \in \mathcal{D}_r$: the survival function is eventually regularly varying, eventually rapidly varying, or has finite support.

The following lemma transfers these continuous tail asymptotics to the rounded discrete survival function $\overline{P}$. It is the technical backbone of the dyadic-ratio analysis in Section 6.

Lemma 1 (Rounding preserves tail regimes). *Let $r \in \{B, C\}$ with $c_B = 0$ and $c_C = \frac{1}{2}$. Let $F \in \mathrm{MDA}(G_\xi)$ be a continuous parent with $\overline{F}(c_r) > 0$, and let $P = T_r(F) \in \mathcal{D}_r$.*

(Survival identity.) *For every $n \in \mathbb{N}_0$,*

$$\overline{P}(n) \;=\; \frac{\overline{F}(n + c_r)}{\overline{F}(c_r)}.$$

(i) *If $\xi > 0$, then for every $\lambda > 0$,*

$$\frac{\overline{F}(\lambda(n + c_r))}{\overline{F}(n + c_r)} \;\longrightarrow\; \lambda^{-\alpha}, \qquad n \to \infty,$$

where $\alpha = 1/\xi$, uniformly for λ in compact subsets of $(0, \infty)$.

(ii) *If $\xi = 0$ and F has infinite right endpoint, then for every $\lambda > 1$,*

$$\frac{\overline{F}(\lambda(n + c_r))}{\overline{F}(n + c_r)} \;\longrightarrow\; 0, \qquad n \to \infty.$$

(iii) *If F has finite right endpoint, then P has finite support.*

Proof. *Survival identity.* By the definition of T_r, X is distributed as $\rho_r(Y)$ conditional on $Y > c_r$, where $Y \sim F$. Since $\rho_r(Y) > n$ if and only if $Y > n + c_r$ (with endpoint events of probability zero under continuity), and $\{Y > n + c_r\} \subseteq \{Y > c_r\}$ for $n \ge 0$,

$$\overline{P}(n) = P(\rho_r(Y) > n \mid Y > c_r) = \frac{P(Y > n + c_r)}{\overline{F}(c_r)} = \frac{\overline{F}(n + c_r)}{\overline{F}(c_r)}.$$

Case (i). By Fact 1, $\overline{F}$ is regularly varying with index $-\alpha$. Setting $t_n = n + c_r \to \infty$, the result follows directly from Karamata's uniform convergence theorem (de Haan and Ferreira [3]): $\overline{F}(\lambda t)/\overline{F}(t) \to \lambda^{-\alpha}$ uniformly for λ in any compact subset of $(0, \infty)$.

Case (ii). By Fact 2, $\overline{F}(\mu t)/\overline{F}(t) \to 0$ for every fixed $\mu > 1$. Setting $t_n = n + c_r$, $\overline{F}(\lambda t_n)/\overline{F}(t_n) \to 0$ for every $\lambda > 1$ as $n \to \infty$.

Case (iii). If F has finite right endpoint x_F, then $\overline{F}(n + c_r) = 0$ for all $n + c_r \geq x_F$, hence $\overline{P}(n) = 0$ for all sufficiently large n. □

6. The Dyadic Block Ratio

Definition 4. *For $P \in \mathcal{P}(\mathbb{N})$ and $N \in \mathbb{N}$, define*

$$A_P(N) = P(N < X \leq 2N) = \overline{P}(N) - \overline{P}(2N),$$

$$B_P(N) = P(2N < X \leq 4N) = \overline{P}(2N) - \overline{P}(4N).$$

Whenever $A_P(N) > 0$, define the dyadic block ratio

$$R_P(N) = \frac{B_P(N)}{A_P(N)}.$$

Lemma 2 (Stability under far-tail surgery)**.** *If $P, Q \in \mathcal{P}(\mathbb{N})$ agree on $\{1, \ldots, 4N\}$, then $R_P(N) = R_Q(N)$ whenever these ratios are defined.*

Proof. Both $A_P(N)$ and $B_P(N)$ depend only on masses at indices in $\{N + 1, \ldots, 4N\}$. □

Lemma 3 (Single eventual ratio regime)**.** *Fix $r \in \{B, C\}$ and let $P \in \mathcal{D}_r$.*

(i) *If the parent lies in* $\mathrm{MDA}(G_\xi)$ *with $\xi > 0$, then $R_P(N) \to 2^{-\alpha} \in (0, 1)$ for some $\alpha > 0$.*
(ii) *If the parent lies in* $\mathrm{MDA}(G_0)$ *with an infinite right endpoint, then $R_P(N) \to 0$.*
(iii) *If the parent lies in* $\mathrm{MDA}(G_\xi)$ *with $\xi < 0$, or in* $\mathrm{MDA}(G_0)$ *with a finite right endpoint, then P has finite support.*

Proof. By Lemma 1, $\overline{P}$ inherits the tail regime of $\overline{F}$, with $\overline{P}(n) = \overline{F}(n + c_r)/\overline{F}(c_r)$. Substituting into the dyadic ratio,

$$R_P(N) = \frac{\overline{P}(2N) - \overline{P}(4N)}{\overline{P}(N) - \overline{P}(2N)} = \frac{\overline{F}(2N + c_r) - \overline{F}(4N + c_r)}{\overline{F}(N + c_r) - \overline{F}(2N + c_r)}.$$

Case (i): $\xi > 0$. By Lemma 1(i), for $\lambda \in \{2, 4\}$,

$$\frac{\overline{F}(\lambda(N + c_r))}{\overline{F}(N + c_r)} \longrightarrow \lambda^{-\alpha}.$$

Since $(2N + c_r)/(N + c_r) \to 2$ and $(4N + c_r)/(N + c_r) \to 4$ as $N \to \infty$, regular variation of $\overline{F}$ with index $-\alpha$ gives $\overline{F}(2N + c_r)/\overline{F}(N + c_r) \to 2^{-\alpha}$ and $\overline{F}(4N + c_r)/\overline{F}(N + c_r) \to 4^{-\alpha} = (2^{-\alpha})^2$. Dividing numerator and denominator by $\overline{F}(N + c_r)$,

$$R_P(N) \longrightarrow \frac{2^{-\alpha} - 4^{-\alpha}}{1 - 2^{-\alpha}} = \frac{2^{-\alpha}(1 - 2^{-\alpha})}{1 - 2^{-\alpha}} = 2^{-\alpha}.$$

Case (ii): $\xi = 0$, infinite right endpoint. By Lemma 1(ii), each ratio in the preceding display tends to 0, so $R_P(N) \to 0$.

Case (iii): finite right endpoint. By Lemma 1(iii), P has finite support and $R_P(N)$ is eventually undefined. □

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7. Open Dense Witness Sets

Fix two open intervals $I_-, I_+ \subset (0,1)$ with disjoint closures (we take $I_- = (\frac{1}{20}, \frac{1}{10})$ and $I_+ = (\frac{1}{4}, \frac{1}{3})$ for concreteness).

For each $m \in \mathbb{N}$, define

$$U_m^- = \bigcup_{N \geq m} \{P \in \mathcal{P}(\mathbb{N}) : A_P(N) > 0,\ R_P(N) \in I_-\},$$

$$U_m^+ = \bigcup_{N \geq m} \{P \in \mathcal{P}(\mathbb{N}) : A_P(N) > 0,\ R_P(N) \in I_+\}.$$

Lemma 4 (Openness). *For each $m \in \mathbb{N}$, the sets U_m^- and U_m^+ are open in $(\mathcal{P}(\mathbb{N}), d_{\mathrm{TV}})$.*

Proof. The maps $P \mapsto A_P(N)$ and $P \mapsto B_P(N)$ are continuous linear functionals on ℓ^1. On the open set $\{P : A_P(N) > 0\}$, the ratio $R_P(N)$ is continuous, so $\{P : A_P(N) > 0,\ R_P(N) \in I_\pm\}$ is open. Taking the union over $N \geq m$ gives the result. □

Lemma 5 (Density). *For each $m \in \mathbb{N}$, the sets U_m^- and U_m^+ are dense in $(\mathcal{P}(\mathbb{N}), d_{\mathrm{TV}})$.*

Proof. We prove the claim for U_m^-; the proof for U_m^+ is identical.

Fix $P \in \mathcal{P}(\mathbb{N})$, $\varepsilon > 0$, $m \in \mathbb{N}$, and choose $\eta \in (0, \varepsilon/4)$. Since $\overline{P}(N) \to 0$, choose $N \geq m$ with

$$\tau := P(N < X \leq 4N) < \eta \qquad \text{and} \qquad P(X \leq N) > \eta.$$

Fix $c \in I_-$ and set $a = (\tau + \eta)/(1 + c)$, $b = c(\tau + \eta)/(1 + c)$, so $b/a = c$ and $a + b = \tau + \eta$. Since $P(X \leq N) > \eta$, choose $\delta_1, \ldots, \delta_N$ with $0 \leq \delta_k \leq p_k$ and $\sum \delta_k = \eta$. Define Q using

$$q_k = \begin{cases} p_k - \delta_k, & 1 \leq k \leq N, \\ a, & k = N+1, \\ b, & k = 2N+1, \\ 0, & N < k \leq 4N,\ k \notin \{N+1, 2N+1\}, \\ p_k, & k > 4N. \end{cases}$$

Verification that $Q \in \mathcal{P}(\mathbb{N})$. Non-negativity: for $1 \leq k \leq N$, $q_k = p_k - \delta_k \geq 0$ by the choice $0 \leq \delta_k \leq p_k$. By construction, $q_{N+1} = a > 0$ and $q_{2N+1} = b > 0$. The remaining indices in $(N, 4N]$ are assigned $q_k = 0$, and $q_k = p_k \geq 0$ for $k > 4N$.

Total mass:

$$\begin{aligned} \sum_{k \geq 1} q_k &= \sum_{k=1}^{N} (p_k - \delta_k) + a + b + \sum_{k > 4N} p_k \\ &= \sum_{k=1}^{N} p_k - \eta + (\tau + \eta) + \sum_{k > 4N} p_k \\ &= \sum_{k=1}^{N} p_k + \tau + \sum_{k > 4N} p_k \\ &= \sum_{k=1}^{N} p_k + \sum_{k=N+1}^{4N} p_k + \sum_{k > 4N} p_k = \sum_{k \geq 1} p_k = 1, \end{aligned}$$

using $\sum_{k=1}^{N} \delta_k = \eta$ and $\tau = P(N < X \leq 4N) = \sum_{k=N+1}^{4N} p_k$.

Therefore $Q \in \mathcal{P}(\mathbb{N})$. By construction $A_Q(N) = a > 0$ and $B_Q(N) = b$, so $R_Q(N) = b/a = c \in I_-$, hence $Q \in U_m^-$.

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The total variation distance satisfies

$$2\,d_{\mathrm{TV}}(P,Q) = \sum_{k=1}^{N} \delta_k + \sum_{k=N+1}^{4N} |p_k - q_k| \le \eta + (\tau + \eta + \tau) = 2\eta + 2\tau < 4\eta < \varepsilon. \quad \square$$

8. The Meagerness Theorem

Recall from Definition 3 that T_B is the rounded-law transform built from ceiling rounding (Type B) and T_C from nearest-integer (midpoint) rounding (Type C); the classes $\mathcal{D}_B$ and $\mathcal{D}_C$ are the corresponding rounded discrete max-domain-of-attraction classes generated by these two schemes.

Theorem 1 (Rounded discrete MDA classes are meager)**.** *The classes $\mathcal{D}_B$ and $\mathcal{D}_C$ are meager subsets of $(\mathcal{P}(\mathbb{N}), d_{\mathrm{TV}})$. Consequently, $\mathcal{D} = \mathcal{D}_B \cup \mathcal{D}_C$ is meager as well.*

Proof. Define

$$\mathcal{G} = \bigcap_{m=1}^{\infty} (U_m^- \cap U_m^+).$$

By Lemmas 4 and 5, each $U_m^{\pm}$ is open and dense. Since $(\mathcal{P}(\mathbb{N}), d_{\mathrm{TV}})$ is a Baire space, $\mathcal{G}$ is comeager.

Let $P \in \mathcal{G}$. For every m, there exist $N_m, M_m \ge m$ with $R_P(N_m) \in I_-$ and $R_P(M_m) \in I_+$. Since $\overline{I_-} \cap \overline{I_+} = \varnothing$, the sequence $\{R_P(N)\}_N$ admits no single eventual limit. Moreover, the membership $P \in U_m^- \cap U_m^+$ for every m requires $A_P(N) > 0$ at arbitrarily large scales N, so P cannot have finite support. By Lemma 3, no law in $\mathcal{D}_B$ or $\mathcal{D}_C$ exhibits this behavior: every element of $\mathcal{D}_r$ either has $R_P(N)$ converging to a single limit (cases (i)–(ii)) or has finite support (case (iii)).

Therefore, $\mathcal{G} \cap \mathcal{D}_B = \varnothing$ and $\mathcal{G} \cap \mathcal{D}_C = \varnothing$. Since $\mathcal{G}$ is comeager, its complement is meager, so $\mathcal{D}_B$, $\mathcal{D}_C$, and $\mathcal{D}$ are all meager. $\square$

Corollary 1 (Parametric families are meager)**.** *The parametric families $\mathcal{B}$ and $\mathcal{C}$ are meager subsets of $\mathcal{P}(\mathbb{N})$.*

Proof. By Remark 2, $\mathcal{B} \subset \mathcal{D}_B$ and $\mathcal{C} \subset \mathcal{D}_C$. Apply Theorem 1. $\square$

Proposition 2 (Density via tail grafting)**.** *For $r \in \{B, C\}$, the rounded embedding class $\mathcal{D}_r$ is dense in $(\mathcal{P}(\mathbb{N}), d_{\mathrm{TV}})$. Consequently $\mathcal{D} = \mathcal{D}_B \cup \mathcal{D}_C$ is dense in $(\mathcal{P}(\mathbb{N}), d_{\mathrm{TV}})$.*

Proof. Fix $p \in \mathcal{P}(\mathbb{N})$ and $\varepsilon > 0$. We construct $p' \in \mathcal{D}_r$ such that $d_{\mathrm{TV}}(p, p') < \varepsilon$.

Choose $N \ge 2$ large enough that

$$\beta := p(\{n \ge N\}) < \varepsilon.$$

For $r \in \{B, C\}$, let $I_n^{(r)}$ denote the rounding bin corresponding to $n \in \mathbb{N}$:

$$I_n^{(B)} = (n-1, n], \qquad I_n^{(C)} = [n - \tfrac{1}{2}, n + \tfrac{1}{2}).$$

Let

$$T^{(r)} := \bigcup_{n \ge N} I_n^{(r)},$$

so $T^{(B)} = (N-1, \infty)$ and $T^{(C)} = [N - \frac{1}{2}, \infty)$. Write y_0 for the left endpoint of $T^{(r)}$, namely $y_0 = N - 1$ for $r = B$ and $y_0 = N - \frac{1}{2}$ for $r = C$. By the choice $N \ge 2$, we have $y_0 > 0$.

Define a density g on $\mathbb{R}$ as follows. Fix $\alpha > 0$. For $1 \le n < N$, set

$$g(y) = p(\{n\}), \qquad y \in I_n^{(r)}.$$

On the tail half-line $T^{(r)}$, set

$$g(y) = \beta\,\alpha\,y_0^{\alpha}\,y^{-\alpha-1}, \qquad y \in T^{(r)}.$$

Set $g(y) = 0$ elsewhere.

Since each bin $I_n^{(r)}$ has length one,

$$\int_{I_n^{(r)}} g(y)\,dy = p(\{n\}), \qquad 1 \le n < N.$$

A direct computation gives

$$\int_{T^{(r)}} \beta\,\alpha\,y_0^{\alpha}\,y^{-\alpha-1}\,dy = \beta,$$

so

$$\int_{\mathbb{R}} g(y)\,dy = \sum_{n=1}^{N-1} p(\{n\}) + \beta = 1.$$

Let Y' have density g and let $F_{Y'}$ denote its cdf. For $y \ge y_0$,

$$\overline{F}_{Y'}(y) = \int_y^{\infty} g(u)\,du = \beta\left(\frac{y}{y_0}\right)^{-\alpha}.$$

Hence $\overline{F}_{Y'}$ is regularly varying at infinity with index $-\alpha$, so $F_{Y'} \in \mathrm{MDA}(G_\xi)$ with $\xi = 1/\alpha > 0$ (Fact 1).

Since g is supported on $(0,\infty)$ when $r = B$ and on $[\frac{1}{2},\infty)$ when $r = C$, the conditioning event in the definition of T_r has probability one. In particular, $p' := \mathrm{law}(\rho_r(Y'))$ satisfies $p' = T_r(F_{Y'})$, so $p' \in \mathcal{D}_r$.

For $1 \le n < N$,

$$p'(\{n\}) = \int_{I_n^{(r)}} g(y)\,dy = p(\{n\}),$$

so p and p' agree on $\{1,\dots,N-1\}$. Hence

$$2\,d_{\mathrm{TV}}(p,p') = \sum_{k\ge1} |p_k - p'_k| = \sum_{k\ge N} |p_k - p'_k| \le \sum_{k\ge N} p_k + \sum_{k\ge N} p'_k = 2\beta,$$

so $d_{\mathrm{TV}}(p,p') \le \beta < \varepsilon$. Since p and ε were arbitrary, $\mathcal{D}_r$ is dense in $(\mathcal{P}(\mathbb{N}), d_{\mathrm{TV}})$. □

9. Banach–Mazur Game Interpretation

The proof of Theorem 1 admits a natural reading via the Banach–Mazur game [7–9]. Since $\mathcal{G}$ is comeager and disjoint from each rounded class, one may interpret the construction as follows: at stage m, Player II forces the play into the open dense set $U_m^- \cap U_m^+$ beyond the current scale via the tail surgery of Lemma 5, without disturbing earlier witnesses (Lemma 2). Any limit point lies in $\mathcal{G} \subset \mathcal{D}^c$. This is a heuristic reformulation; the rigorous content is in Sections 7 and 8.

10. Discussion and Open Directions

The presented theorem shows that rounded discrete MDA classes are topologically small in the sense of Baire category, even though they are dense in $\mathcal{P}(\mathbb{N})$ (Proposition 2). Density alone is therefore not the right notion of size in this setting: every neighborhood of a generic law contains rounded laws, yet the collection of all such rounded laws is itself

nowhere dense in a suitable comeager sense. The Baire category distinguishes these classes from generic laws on $\mathbb{N}$ and mirrors the continuous picture of Leonetti and Khorrami Chokami [5], where the multivariate max-domain-of-attraction is shown to be dense yet of the first category.

Type A. The argument does not address the sampled-density family $\mathcal{A}$. As explained in Remark 3, this family lacks the survival-function identity $\overline{P}(n) = \overline{F}(\,\cdot\,)/\overline{F}(\,\cdot\,)$ that drives the trichotomy lemma; consequently a generic Type A law could in principle exhibit oscillating dyadic ratios while still arising from a regularly varying parent density. A meagerness result for $\mathcal{A}$ would therefore require either a substitute for the trichotomy—perhaps via a careful study of the discrete normalizing sums $\sum_j f_c(j)$ and their dyadic differences—or a different witness construction altogether. We expect $\mathcal{A}$ to be meager but leave this as an open problem.

Parametric closures and boundaries. Characterizing the closures and topological boundaries of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ within $\mathcal{P}(\mathbb{N})$ would be of independent interest.

Alternative topologies. One may ask whether analogous meagerness phenomena persist under Wasserstein metrics restricted to moment classes.

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