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Kneser-Type Oscillation Criteria for Half-Linear Third-Order Dynamic Equations on Time Scales

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Abstract

This paper establishes new Kneser-type oscillation criteria for a class of third-order half-linear dynamic equations on time scales. The analysis is based on a Riccati-type transformation, which reduces the original equation to an associated dynamic inequality and yields explicit oscillation conditions. The obtained criteria extend existing results from second-order to third-order equations and provide a unified treatment of continuous and discrete cases. To the best of our knowledge, no comparable Kneser-type oscillation criteria have been reported for equations of the form considered here. Several examples illustrate the applicability of the main results.

Keywords: Kneser-type; oscillation criteria; third order; dynamic equations; time scales

MSC: 34K11; 39A10; 39A99; 34N05

1. Introduction

Mathematical models describing oscillatory phenomena arise in many applied fields, including biological processes and diffusion-type problems. For example, chemotaxis equations and porous medium equations are commonly used to model nonlinear interactions and diffusion mechanisms in applied sciences [1–3]. In recent years, considerable attention has been given to half-linear third-order differential and dynamic equations, owing to their role in describing nonlinear processes with higher-order effects and time delays. Such equations arise naturally in models where the evolution depends on interactions between present and deviating states; see [4–7].

Recent developments have extended oscillation theory to dynamic equations on time scales, allowing a unified treatment of continuous and discrete cases. Within this setting, numerous oscillation criteria have been established for second-order dynamic equations; see [8–10]. For third-order dynamic equations, many oscillation and asymptotic results have also been reported, particularly for equations involving delays and mixed nonlinearities [11–14], as well as in [15–17]. Additional qualitative results, including Hille–Nehari-type criteria, can be found in [18–21]. Further studies addressing asymptotic behavior and functional dynamic equations under more general assumptions are presented in [22–26].

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Kneser-type criteria constitute a classical part of oscillation theory and are closely related to Sturm comparison theory and its nonlinear extensions for half-linear equations; see [27–30]. Another line of research concerns Kneser solutions, which belong to a special class of nonoscillatory solutions. A solution x is called a Kneser solution if it satisfies the conditions

$$x(t)x^{\Delta}(t) < 0 \quad \text{and} \quad x(t)x^{\Delta\Delta}(t) > 0$$

for sufficiently large t. Studies in this direction, including [26,31], focus on the existence and qualitative properties of such nonoscillatory solutions for third-order delay dynamic equations on time scales.

In contrast, the present paper is concerned with oscillation criteria of Kneser type. Although third-order dynamic equations have been extensively studied, results of this type remain limited for third-order half-linear dynamic equations with deviating arguments of the general form considered here. The aim of this paper is to derive new Kneser-type oscillation criteria for a general class of third-order half-linear functional dynamic equations with deviating arguments and to establish conditions ensuring that every solution is either oscillatory or convergent.

We investigate the third-order half-linear functional dynamic equation

$$\left\{r_2(t)\phi_{\alpha_2}\left(\left[r_1(t)\phi_{\alpha_1}\left(x^{\Delta}(t)\right)\right]^{\Delta}\right)\right\}^{\Delta} + q(t)\phi_{\alpha}(x(g(t))) = 0 \tag{1}$$

defined on a time scale $\mathbb{T}$ with $\sup \mathbb{T} = \infty$. Here $t \in [t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$, $t_0 \geq 0$, $t_0 \in \mathbb{T}$, and $\phi_{\gamma}(u) := |u|^{\gamma-1}u$ for $\gamma > 0$. The parameters $\alpha_1, \alpha_2 > 0$ satisfy $\alpha = \alpha_1\alpha_2 > 0$.

The coefficient functions r_1, r_2 and q are assumed to be positive and rd-continuous on $\mathbb{T}$ and satisfy the canonical condition

$$\int_{t_0}^{\infty} \left(\frac{1}{r_i(\tau)}\right)^{1/\alpha_i} \Delta\tau = \infty, \quad i = 1, 2. \tag{2}$$

The deviating function $g : \mathbb{T} \to \mathbb{T}$ satisfies

$$\lim_{\substack{t\to\infty \\ t\in\mathbb{T}}} g(t) = \infty.$$

A function x is said to be a solution of (1) if it is nontrivial, real-valued, and satisfies $x \in C^1_{\mathrm{rd}}[T_x, \infty)_{\mathbb{T}}$ for some $T_x \geq t_0$, together with

$$r_1(t)\phi_{\alpha_1}\left(x^{\Delta}(t)\right), \, r_2(t)\phi_{\alpha_2}\left(\left[r_1(t)\phi_{\alpha_1}\left(x^{\Delta}(t)\right)\right]^{\Delta}\right) \in C^1_{\mathrm{rd}}[T_x, \infty)_{\mathbb{T}}.$$

In addition, $x(t)$ must verify Equation (1) for all $t \in [T_x, \infty)_{\mathbb{T}}$, where C_{rd} is the space of right-dense continuous functions. A solution $x(t)$ of Equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

A time scale $\mathbb{T}$, introduced by Hilger [32], is a nonempty closed subset of $\mathbb{R}$ that unifies continuous and discrete analysis. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

and the Delta derivative $x^{\Delta}(t)$ is understood in the usual sense of time-scale calculus; see [7,33,34].

In particular, when $\mathbb{T} = \mathbb{R}$, Equation (1) reduces to

$$\left(r_2(t)\phi_{\alpha_2}\left(\left[r_1(t)\phi_{\alpha_1}(x'(t))\right]'\right)\right)' + q(t)\phi_{\alpha}(x(g(t))) = 0,$$

which is a third-order half-linear differential equation with deviating arguments, while for $\mathbb{T} = \mathbb{Z}$ it takes the form of a difference equation. Other examples of time scales include $q^{\mathbb{N}_0}$, $h\mathbb{N}$, and $\mathbb{T}_n$.

The development of oscillation theory for equations associated with (1) draws upon classical results for second-order differential equations, where Euler-type equations have long served as a primary tool for understanding oscillatory behavior. A representative case is

$$x''(t) + \frac{\beta}{t^2}x(t) = 0,$$

which is oscillatory precisely when $\beta > \frac{1}{4}$. This benchmark motivates a wide range of comparison arguments that lead to Kneser-type criteria (see [35]) for second-order equations of the form

$$x''(t) + q(t)x(t) = 0,$$

with oscillation ensured whenever

$$\liminf_{t\to\infty} t^2 q(t) > \frac{1}{4}.$$

Extensions of such results have appeared in numerous studies addressing various classes of second-order differential equations, see [36–38].

Within time-scale calculus, analogous developments have been established for dynamic equations. In particular, oscillation criteria have been obtained by Hassan et al. [39], for

$$\left[r(t)\left|x^{\Delta}(t)\right|^{\alpha-1}x^{\Delta}(t)\right]^{\Delta} + q(t)|x(g(t))|^{\alpha-1}x(g(t)) = 0 \tag{3}$$

and established that if

$$\liminf_{t\to\infty} r^{1/\alpha}(t)\mathcal{R}(t)\mathcal{R}^{\alpha}(\Omega(t))q(t) > \frac{1}{\varrho^{\alpha(\alpha+1)}}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1},$$

where

$$\mathcal{R}(t) := \int_{t_0}^{t}\left(\frac{1}{r(\tau)}\right)^{1/\alpha}\Delta\tau,\ \varrho := \liminf_{t\to\infty}\frac{\mathcal{R}(t)}{\mathcal{R}^{\sigma}(t)},$$

and

$$\Omega(t) := \begin{cases} g(t), & g(t) \le t \\ t, & g(t) \ge t, \end{cases}$$

then every solution of (3) is oscillatory.

In view of the above discussion and the related developments reported in [35–39], this paper establishes new Kneser-type oscillation criteria for the third-order dynamic Equation (1) defined on an arbitrary time scale. The contribution generalizes the known oscillation results for second-order dynamic Equation (3) and extends them to the third-order setting, covering both continuous and discrete cases. The analysis is based on a Riccati-type transformation that converts the original equation into an associated Riccati dynamic inequality. This approach provides an effective mechanism for deriving oscillation criteria via comparison arguments and the structural properties of time-scale calculus. It is worth noting that no restrictive assumptions are imposed on the time scale $\mathbb{T}$ beyond being unbounded. Therefore, the obtained results are applicable to a wide class of time scales, including the continuous case $\mathbb{T} = \mathbb{R}$. In the continuous setting, the studied equation reduces to a third-order half-linear differential equation with deviating arguments. To the best of the authors' knowledge, Kneser-type oscillation criteria for such third-order differential equations do not appear to be available in the existing literature. Consequently, the present results provide new contributions even in the classical case.

Throughout this paper, all limits are taken along the time scale $\mathbb{T}$ and we let

$$x^{[i]} := r_i \phi_{\alpha_i}([x^{[i-1]}]^{\Delta}),\ i = 1, 2, \quad \text{with } x^{[0]} = x, \tag{4}$$

and

$$A := \liminf_{t\to\infty} r_2^{1/\alpha_2}(t) H_1(t, t_0) H_2^{\alpha_2}(\varphi(t), t_0) q(t),$$

with

$$H_1(t,s) := \int_s^t \left(\frac{1}{r_2(\tau)}\right)^{1/\alpha_2} \Delta\tau,$$

$$H_2(t,s) := \left[\int_s^t \left(\frac{H_1(\tau,s)}{r_1(\tau)}\right)^{1/\alpha_1} \Delta\tau\right]^{\alpha_1},$$

and

$$\varphi(t) := \begin{cases} t, & g(t) \geq t, \\ g(t), & g(t) \leq t. \end{cases}$$

2. Main Results

First, we establish the following lemmas, which play a crucial role in proving our main results.

Lemma 1. *Assume that* (2) *holds. If $x(t)$ is an eventually positive solution of Equation* (1)*, then there are only the following two cases:*

(I) $x^{[1]}(t) > 0, x^{[2]}(t) > 0, \left(x^{[2]}(t)\right)^{\Delta} < 0;$

(II) $x^{[1]}(t) < 0, x^{[2]}(t) > 0, \left(x^{[2]}(t)\right)^{\Delta} < 0,$

eventually.

From case (II) of Lemma 2, it follows that $x^{[2]}(t)$ is eventually positive and decreasing, and hence admits a finite limit. Consequently, $x^{[1]}(t)$ and $x(t)$ are monotone for sufficiently large t, which leads to the convergence of the solution, as stated in the following lemma.

Lemma 2. *If $x(t)$ is an eventually positive solution of Equation* (1) *and satisfies (II) of Lemma 1, then $x^{[j]}(t)$, $i = 0, 1, 2$, converge as $t \to \infty$.*

We note that $\phi_\gamma(u) = |u|^{\gamma-1}u$, $\gamma > 0$ is strictly increasing, and hence invertible. Its inverse is given by

$$\phi_\gamma^{-1}(u) = |u|^{1/\gamma-1}u,\ \gamma > 0,$$

which is also strictly increasing. Consequently, the monotonicity of compositions involving ϕ_γ^{-1} follows directly.

Theorem 1. *Assume that* (2) *holds. If*

$$\liminf_{t\to\infty} r_2^{1/\alpha_2}(t) H_1(t, t_0) H_2^{\alpha_2}(\varphi(t), t_0) q(t) = \infty, \tag{5}$$

then every solution of (1) *is either oscillatory or convergent as* $t \to \infty$*.*

Proof. Suppose Equation (1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, assume $x(t) > 0$ and $x(g(t)) > 0$ on $[t_0, \infty)_{\mathbb{T}}$. Then if case (I) of Lemma 1 holds, so that

$$x^{[i]}(t) > 0,\ i = 1, 2, \text{ and } \left[x^{[2]}(t)\right]^{\Delta} < 0 \text{ on } [t_0, \infty)_{\mathbb{T}}.$$

Then for $t \in [t_0, \infty)_{\mathbb{T}}$,

$$\begin{aligned} x^{[1]}(t) &\geq \int_{t_0}^{t} \phi_{\alpha_2}^{-1}\left(x^{[2]}(\tau)\right)\left(\frac{1}{r_2(\tau)}\right)^{1/\alpha_2} \Delta\tau \\ &\geq \phi_{\alpha_2}^{-1}\left(x^{[2]}(t)\right) \int_{t_0}^{t} \left(\frac{1}{r_2(\tau)}\right)^{1/\alpha_2} \Delta\tau \\ &= \phi_{\alpha_2}^{-1}\left(x^{[2]}(t)\right) H_1(t, t_0). \end{aligned} \tag{6}$$

Hence, we conclude that for $t \in (t_0, \infty)_{\mathbb{T}}$,

$$\left(\frac{x^{[1]}(t)}{H_1(t, t_0)}\right)^{\Delta} < 0.$$

Therefore,

$$\begin{aligned} x(t) &\geq \int_{t_0}^{t} \phi_{\alpha_1}^{-1}\left(\frac{x^{[1]}(\tau)}{H_1(\tau, t_0)}\right)\left(\frac{H_1(\tau, t_0)}{r_1(\tau)}\right)^{1/\alpha_1} \Delta\tau \\ &\geq \phi_{\alpha_1}^{-1}\left(\frac{x^{[1]}(t)}{H_1(t, t_0)}\right) \int_{t_0}^{t} \left(\frac{H_1(\tau, t_0)}{r_1(\tau)}\right)^{1/\alpha_1} \Delta\tau \\ &= \phi_{\alpha_1}^{-1}\left(\frac{x^{[1]}(t)}{H_1(t, t_0)}\right) H_2^{1/\alpha_1}(t, t_0). \end{aligned}$$

Hence, there exists a $t_1 \in (t_0, \infty)_{\mathbb{T}}$ such that for $t \in [t_1, \infty)_{\mathbb{T}}$,

$$\left(\frac{x^{[1]}(t)}{H_1(t, t_0)}\right)^{\Delta} < 0, \tag{7}$$

$$\phi_{\alpha_1}(x(t)) \geq \frac{x^{[1]}(t)}{H_1(t, t_0)} H_2(t, t_0), \tag{8}$$

and, from the definition of A, we obtain that for any $\varepsilon > 0$,

$$r_2^{1/\alpha_2}(t) H_1(t, t_0) H_2^{\alpha_2}(\varphi(t), t_0) q(t) \geq A - \varepsilon. \tag{9}$$

Integrating (1) from t_1 to $t \in [t_1, \infty)_{\mathbb{T}}$, we obtain

$$x^{[2]}(t_1) \geq x^{[2]}(t_1) - x^{[2]}(t) = \int_{t_1}^{t} q(\tau)\phi_{\alpha}(x(g(\tau)))\Delta\tau. \tag{10}$$

First, consider the case when $g(t) \leq t$ eventually. Using (7) and (8), we have

$$\phi_{\alpha_1}(x(g(t))) \geq \frac{x^{[1]}(g(t))}{H_1(g(t), t_0)} H_2(g(t), t_0) \geq \frac{x^{[1]}(t)}{H_1(t, t_0)} H_2(g(t), t_0). \tag{11}$$

Next, consider the case when $g(t) \geq t$ eventually. Using the fact that $x(t)$ is strictly increasing and (7), we obtain

$$\phi_{\alpha_1}(x(g(t))) \geq \phi_{\alpha_1}(x(t)) \geq \frac{x^{[1]}(t)}{H_1(t,t_0)} H_2(t,t_0). \tag{12}$$

It follows from (11) and (12) that

$$\phi_\alpha(x(g(t))) \geq \phi_{\alpha_2}\left(x^{[1]}(t)\right)\left(\frac{H_2(\varphi(t),t_0)}{H_1(t,t_0)}\right)^{\alpha_2} \qquad \text{for } t \in [t_1,\infty)_{\mathbb{T}}. \tag{13}$$

Hence, (10) becomes

$$\begin{aligned} x^{[2]}(t_1) &\geq x^{[2]}(t_1) - x^{[2]}(t) \\ &\geq \int_{t_1}^{t}\left(\frac{H_2(\varphi(\tau),t_0)}{H_1(\tau,t_0)}\right)^{\alpha_2} q(\tau)\phi_{\alpha_2}\left(x^{[1]}(\tau)\right)\Delta\tau. \end{aligned}$$

Due to $\left(x^{[1]}\right)^{\Delta} > 0$, then $x^{[1]}(\tau) \geq x^{[1]}(t_1) := c > 0$ for $\tau \geq t_1$, then

$$x^{[2]}(t_1) \geq c^{\alpha_2}\int_{t_1}^{t}\left(\frac{H_2(\varphi(\tau),t_0)}{H_1(\tau,t_0)}\right)^{\alpha_2} q(\tau)\Delta\tau. \tag{14}$$

From (9) and (14), we have for $t \in [t_1,\infty)_{\mathbb{T}}$,

$$x^{[2]}(t_1) \geq c^{\alpha_2}(A-\varepsilon)\int_{t_1}^{t}\left(\frac{1}{r_2(\tau)}\right)^{1/\alpha_2}\frac{1}{H_1^{\alpha_2+1}(\tau,t_0)}\Delta\tau.$$

Using the Pötzsche chain rule ([Theorem 1.90] [34]), we obtain

$$\begin{aligned} &\left(\frac{-1}{H_1^{\alpha_2}(\tau,t_0)}\right)^{\Delta} \\ = {}& \alpha_2\int_0^1 \frac{1}{[(1-h)H_1(\tau,t_0)+hH_1^{\sigma}(\tau,t_0)]^{\alpha_2+1}}dh\left(\frac{1}{r_2(\tau)}\right)^{1/\alpha_2} \\ \leq {}& \alpha_2\left(\frac{1}{r_2(\tau)}\right)^{1/\alpha_2}\frac{1}{H_1^{\alpha_2+1}(\tau,t_0)}. \end{aligned} \tag{15}$$

Thus,

$$x^{[2]}(t_1) \geq \frac{1}{\alpha_2}c^{\alpha_2}(A-\varepsilon)\left(\frac{1}{H_1^{\alpha_2}(t_1,t_0)} - \frac{1}{H_1^{\alpha_2}(t,t_0)}\right).$$

Letting $t \to \infty$ and using (2), we obtain a contradiction with (5).

Now if case (II) of Lemma 1 holds, then by Lemma 2, it follows that $x(t)$ is convergent as $t \to \infty$, $t \in \mathbb{T}$, which completes the proof. $\square$

From Theorem 1, we can assume in the next theorem that

$$\liminf_{t\to\infty} r_2^{1/\alpha_2}(t)H_1(t,t_0)H_2^{\alpha_2}(\varphi(t),t_0)q(t) < \infty.$$

Theorem 2. *Assume that* (2) *holds. If*

$$\liminf_{t\to\infty} r_2^{1/\alpha_2}(t)H_1(t,t_0)H_2^{\alpha_2}(\varphi(t),t_0)q(t) > \frac{1}{l^{\alpha_2+1}}\left(\frac{\alpha_2}{\alpha_2+1}\right)^{\alpha_2+1}, \tag{16}$$

where

$$l := \liminf_{t\to\infty}\left[\frac{H_1(t,t_0)}{H_1(\sigma(t),t_0)}\right]^{\alpha_2},$$

then every solution of (1) *is either oscillatory or convergent as* $t \to \infty$.

Proof. Suppose Equation (1) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, assume $x(t) > 0$ and $x(g(t)) > 0$ on $[t_0,\infty)_{\mathbb{T}}$. Then, if case (I) of Lemma 1 holds, so that

$$x^{[i]}(t) > 0,\ i = 1,2, \text{ and } \left[x^{[2]}(t)\right]^{\Delta} < 0 \text{ on } [t_0,\infty)_{\mathbb{T}}.$$

Define

$$z(t) := \frac{x^{[2]}(t)}{(x^{[1]}(t))^{\alpha_2}}. \tag{17}$$

Hence, we obtain

$$\begin{aligned} z^{\Delta}(t) &= \frac{(x^{[2]}(t))^{\Delta}}{(x^{[1]}(t))^{\alpha_2}} + \left(\frac{1}{(x^{[1]}(t))^{\alpha_2}}\right)^{\Delta} x^{[2]}(\sigma(t)) \\ &= \frac{(x^{[2]}(t))^{\Delta}}{(x^{[1]}(t))^{\alpha_2}} - \frac{((x^{[1]}(t))^{\alpha_2})^{\Delta}}{(x^{[1]}(t))^{\alpha_2}(x^{[1]}(\sigma(t)))^{\alpha_2}} x^{[2]}(\sigma(t)). \end{aligned}$$

From the definition of $z(t)$ and (1), we see that

$$z^{\Delta}(t) = -\frac{\phi_{\alpha}(x(g(t)))}{\phi_{\alpha_2}(x^{[1]}(t))}q(t) - \frac{((x^{[1]}(t))^{\alpha_2})^{\Delta}}{(x^{[1]}(t))^{\alpha_2}} z^{\sigma}(t).$$

As shown in the proof of Theorem 1, there exists a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $g(t) > t_0$ and

$$\frac{\phi_{\alpha}(x(g(t)))}{\phi_{\alpha_2}(x^{[1]}(t))} \geq \left(\frac{H_2(\varphi(t),t_0)}{H_1(t,t_0)}\right)^{\alpha_2} \qquad \text{for } t \in [t_1,\infty)_{\mathbb{T}}.$$

Hence, we conclude that for $t \in [t_1,\infty)_{\mathbb{T}}$,

$$z^{\Delta}(t) \leq -\left(\frac{H_2(\varphi(t),t_0)}{H_1(t,t_0)}\right)^{\alpha_2} q(t) - \frac{((x^{[1]}(t))^{\alpha_2})^{\Delta}}{(x^{[1]}(t))^{\alpha_2}} z^{\sigma}(t).$$

By the Pötzsche chain rule, we obtain

$$((x^{[1]}(t))^{\alpha_2})^{\Delta} \geq \begin{cases} \alpha_2\left[x^{[1]}(\sigma(t))\right]^{\alpha_2-1} (x^{[1]}(t))^{\Delta}, & 0 < \alpha_2 \leq 1 \\ \alpha_2\left[x^{[1]}(t)\right]^{\alpha_2-1} (x^{[1]}(t))^{\Delta}, & \alpha_2 \geq 1. \end{cases}$$

If $0 < \alpha_2 \leq 1$, we have

$$z^{\Delta}(t) \leq -\left(\frac{H_2(\varphi(t),t_0)}{H_1(t,t_0)}\right)^{\alpha_2} q(t) - \alpha_2 \frac{(x^{[1]}(t))^{\Delta}}{x^{[1]}(\sigma(t))}\left(\frac{x^{[1]}(\sigma(t))}{x^{[1]}(t)}\right)^{\alpha_2} z^{\sigma}(t); \tag{18}$$

and if $\alpha_2 \geq 1$, we obtain

$$z^{\Delta}(t) \leq -\left(\frac{H_2(\varphi(t),t_0)}{H_1(t,t_0)}\right)^{\alpha_2} q(t) - \alpha_2 \frac{(x^{[1]}(t))^{\Delta}}{x^{[1]}(\sigma(t))} \frac{x^{[1]}(\sigma(t))}{x^{[1]}(t)} z^{\sigma}(t). \tag{19}$$

Due to $x^{[1]}(t)$ is strictly increasing, we have for $\alpha_2 > 0$,

$$z^{\Delta}(t) \leq -\left(\frac{H_2(\varphi(t),t_0)}{H_1(t,t_0)}\right)^{\alpha_2} q(t) - \alpha_2 \left(\frac{1}{r_2(t)}\right)^{1/\alpha_2} \frac{\phi_{\alpha_2}^{-1}\left(x^{[2]}(t)\right)}{x^{[1]}(\sigma(t))} z^{\sigma}(t). \tag{20}$$

Also, since $x^{[2]}(t)$ is strictly decreasing, we obtain

$$\begin{aligned} z^{\Delta}(t) &\leq -\left(\frac{H_2(\varphi(t),t_0)}{H_1(t,t_0)}\right)^{\alpha_2} q(t) - \alpha_2 \left(\frac{1}{r_2(t)}\right)^{1/\alpha_2} \left[\left(\frac{x^{[2]}(t)}{(x^{[1]}(t))^{\alpha_2}}\right)^{\sigma}\right]^{1/\alpha_2} z^{\sigma}(t) \\ &= -\left(\frac{H_2(\varphi(t),t_0)}{H_1(t,t_0)}\right)^{\alpha_2} q(t) - \alpha_2 \left(\frac{1}{r_2(t)}\right)^{1/\alpha_2} (z^{\sigma}(t))^{1+1/\alpha_2}. \end{aligned} \tag{21}$$

Integrating from $\sigma(t)$ to v, we obtain

$$\begin{aligned} -z^{\sigma}(t) &\leq z(v) - z^{\sigma}(t) \\ &\leq -\int_{\sigma(t)}^{v} \left(\frac{H_2(\varphi(\tau),t_0)}{H_1(\tau,t_0)}\right)^{\alpha_2} q(\tau)\Delta\tau - \alpha_2 \int_{\sigma(t)}^{v} \left(\frac{1}{r_2(\tau)}\right)^{1/\alpha_2} (z^{\sigma}(\tau))^{1+1/\alpha_2}\Delta\tau. \end{aligned} \tag{22}$$

From the definitions of l and A, we obtain that, for any $\varepsilon > 0$, there exists a $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that for $t \in [t_2,\infty)_{\mathbb{T}}$,

$$\left[\frac{H_1(t,t_0)}{H_1(\sigma(t),t_0)}\right]^{\alpha_2} \geq l - \varepsilon \quad \text{and} \quad r_2^{1/\alpha_2}(t) H_1(t,t_0) H_2^{\alpha_2}(\varphi(t),t_0) q(t) \geq A - \varepsilon, \tag{23}$$

and

$$H_1^{\alpha_2}(t,t_0) z^{\sigma}(t) \geq \Gamma - \varepsilon, \tag{24}$$

where

$$\Gamma := \liminf_{t\to\infty} H_1^{\alpha_2}(t,t_0) z^{\sigma}(t). \tag{25}$$

We note that $0 \leq \Gamma \leq 1$ due to (6) and (17). Using (23) and (24) in (22), we obtain

$$\begin{aligned} -z^{\sigma}(t) &\leq -(A-\varepsilon)\int_{\sigma(t)}^{v} \left(\frac{1}{r_2(\tau)}\right)^{1/\alpha_2} \frac{1}{H_1^{\alpha_2+1}(\tau,t_0)}\Delta\tau \\ &\quad - \alpha_2(\Gamma-\varepsilon)^{1+1/\alpha_2} \int_{\sigma(t)}^{v} \left(\frac{1}{r_2(\tau)}\right)^{1/\alpha_2} \frac{1}{H_1^{\alpha_2+1}(\tau,t_0)}\Delta\tau \\ &= -\left[\frac{A-\varepsilon}{\alpha_2} + (\Gamma-\varepsilon)^{1+1/\alpha_2}\right] \int_{\sigma(t)}^{v} \frac{\alpha_2}{H_1^{\alpha_2+1}(\tau,t_0)} \left(\frac{1}{r_2(\tau)}\right)^{1/\alpha_2}\Delta\tau. \end{aligned} \tag{26}$$

Substituting (15) into (26), we have

$$\begin{aligned} -z^{\sigma}(t) &\leq -\left[\frac{A-\varepsilon}{\alpha_2} + (\Gamma-\varepsilon)^{1+1/\alpha_2}\right] \int_{\sigma(t)}^{v} \left(\frac{-1}{H_1^{\alpha_2}(\tau,t_0)}\right)^{\Delta}\Delta\tau \\ &= -\left[\frac{A-\varepsilon}{\alpha_2} + (\Gamma-\varepsilon)^{1+1/\alpha_2}\right] \left(\frac{1}{H_1^{\alpha_2}(\sigma(t),t_0)} - \frac{1}{H_1^{\alpha_2}(v,t_0)}\right). \end{aligned}$$

Letting $v \to \infty$, we see

$$-z^{\sigma}(t) \leq -\left[\frac{A-\varepsilon}{\alpha_2} + (\Gamma-\varepsilon)^{1+1/\alpha_2}\right] \left(\frac{1}{H_1^{\alpha_2}(\sigma(t),t_0)}\right).$$

Therefore,
$$A - \varepsilon \leq \alpha_2 H_1^{\alpha_2}(\sigma(t), t_0) z^{\sigma}(t) - \alpha_2(\Gamma - \varepsilon)^{1+1/\alpha_2}.$$
Using (23), we achieve
$$A - \varepsilon \leq \frac{\alpha_2}{l - \varepsilon} H_1^{\alpha_2}(t, t_0) z^{\sigma}(t) - \alpha_2(\Gamma - \varepsilon)^{1+1/\alpha_2}. \tag{27}$$
Taking $\liminf_{t\to\infty}$ of (27), we obtain
$$A - \varepsilon \leq \frac{\alpha_2}{l - \varepsilon}\Gamma - \alpha_2(\Gamma - \varepsilon)^{1+1/\alpha_2}.$$
Since ε is arbitrary, we arrive at
$$A \leq \frac{\alpha_2}{l}\Gamma - \alpha_2 \Gamma^{1+1/\alpha_2}.$$
Let
$$X := \frac{\alpha_2}{l},\ x := \alpha_2,\ U := \Gamma,\ \text{and}\ \gamma := \alpha_2.$$
Using the inequality
$$XU - YU^{1+1/\gamma} \leq \frac{X^{\gamma+1}}{Y^{\gamma}} \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}, \quad X, Y > 0, \tag{28}$$
we conclude that
$$A \leq \frac{1}{l^{\alpha_2+1}} \left(\frac{\alpha_2}{\alpha_2 + 1} \right)^{\alpha_2+1},$$
which gives us the contradiction with (16).

Now if case (II) of Lemma 1 holds, then by Lemma 2, it follows that $x(t)$ is convergent as $t \to \infty$, $t \in \mathbb{T}$, which completes the proof. □

3. Examples

The following examples illustrate the importance of the findings presented in this paper. Please note that the underlying time scale $\mathbb{T}$ is not specified explicitly, since the example is formulated for general time scales. The deviating function $g : \mathbb{T} \to \mathbb{T}$ is also not specified explicitly and is assumed to satisfy the general hypotheses stated in the Introduction; in particular,
$$\lim_{\substack{t\to\infty \\ t\in\mathbb{T}}} g(t) = \infty.$$

Example 1. *Consider the nonlinear third-order functional dynamic equation*
$$\left\{ \sqrt[7]{t} \sqrt[7]{\left(\left[\frac{x^{\Delta}(t)}{\sqrt[5]{t |x^{\Delta}(t)|}} \right]^{\Delta} \right)^5} \right\}^{\Delta} + \frac{1}{t} \frac{x(g(t))}{\sqrt[7]{|x(g(t))|^3}} = 0, \tag{29}$$
where
$$\alpha_1 = \frac{4}{5},\ \alpha_2 = \frac{5}{7},\ r_1(t) = \frac{1}{\sqrt[5]{t}},\ r_2(t) = \sqrt[7]{t},\ q(t) = \frac{1}{t}.$$
It is not difficult to see that conditions (2) hold since
$$\int_{t_0}^{\infty} \left(\frac{1}{r_1(\tau)} \right)^{1/\alpha_1} \Delta\tau = \int_{t_0}^{\infty} \sqrt[4]{\tau} \Delta\tau = \infty,$$

and

$$\int_{t_0}^{\infty}\left(\frac{1}{r_2(\tau)}\right)^{1/\alpha_2}\Delta\tau=\int_{t_0}^{\infty}\frac{\Delta\tau}{\sqrt[5]{\tau}}=\infty,$$

due to [Example 5.60] [7]. Using Pötzsche chain rule, we obtain

$$\begin{aligned}H_1(t,t_0)&=\int_{t_0}^{t}\left(\frac{1}{r_2(\tau)}\right)^{1/\alpha_2}\Delta\tau=\int_{t_0}^{t}\frac{\Delta\tau}{\sqrt[5]{\tau}}\\&\geq\frac{5}{4}\int_{t_0}^{t}\left(\sqrt[5]{\tau^4}\right)^{\Delta}\Delta\tau=\frac{5}{4}\left(\sqrt[5]{t^4}-\sqrt[5]{t_0^4}\right),\end{aligned}$$

which implies that

$$\begin{aligned}&\liminf_{t\to\infty}r_2^{1/\alpha_2}(t)H_1(t,t_0)H_2^{\alpha_2}(\varphi(t),t_0)q(t)\\&\geq\frac{5}{4}\liminf_{t\to\infty}\frac{\sqrt[5]{t}}{t}\left(\sqrt[5]{t^4}-\sqrt[5]{t_0^4}\right)\sqrt[7]{H_2^5(\varphi(t),t_0)}=\infty.\end{aligned}$$

Then, by Theorem 1, every solution of (29) is oscillatory or convergent as $t\to\infty$.

Remark 1. *This example verifies that all the conditions of Theorem 1 are satisfied, and hence every solution of Equation (29) is oscillatory or convergent. It also illustrates the applicability of the obtained criteria in a general time scales.*

Example 2. *Consider a third-order half-linear functional dynamic equation for $t\in[t_0,\infty)_{\mathbb{T}}$,*

$$\left\{t^4\left(\left[\sqrt[5]{t}\left(\sqrt[5]{x^{\Delta}(t)}\right)^2\right]^{\Delta}\right)^5\right\}^{\Delta}+\frac{\beta}{l^6}\left(\frac{5}{6}\right)^6\frac{1}{tH_2^3(\varphi(t),t_0)}x^2(g(t))=0,\qquad(30)$$

where all quantities appearing in this example, such as H_i, $i=1,2$, and the deviating function g, are defined as in the preceding sections. The constant $\beta>0$ is chosen to ensure that the assumption of Theorem 2 is satisfied. It is evident that conditions (2) hold due to

$$\int_{t_0}^{\infty}\left(\frac{1}{r_1(\tau)}\right)^{1/\alpha_1}\Delta\tau=\int_{t_0}^{\infty}\frac{\Delta\tau}{\sqrt{\tau}}=\infty,$$

and

$$\int_{t_0}^{\infty}\left(\frac{1}{r_2(\tau)}\right)^{1/\alpha_2}\Delta\tau=\int_{t_0}^{\infty}\frac{\Delta\tau}{\sqrt[5]{\tau^4}}=\infty,$$

due to [Example 5.60] [7]. Using Pötzsche chain rule, we obtain

$$\begin{aligned}H_1(t,t_0)&=\int_{t_0}^{t}\left(\frac{1}{r_2(\tau)}\right)^{1/\alpha_2}\Delta\tau=\int_{t_0}^{t}\frac{\Delta\tau}{\sqrt[5]{\tau^4}}\\&\geq5\int_{t_0}^{t}\left(\sqrt[5]{\tau}\right)^{\Delta}\Delta\tau=5\left(\sqrt[5]{t}-\sqrt[5]{t_0}\right),\end{aligned}$$

and so,

$$\begin{aligned}&\liminf_{t\to\infty}r_2^{1/\alpha_2}(t)H_1(t,t_0)H_2^{\alpha_2}(\varphi(t),t_0)q(t)\\&\geq\frac{5\beta}{l^6}\left(\frac{5}{6}\right)^6\liminf_{t\to\infty}\frac{\sqrt[5]{t^4}\left(\sqrt[5]{t}-\sqrt[5]{t_0}\right)}{t}=\frac{5\beta}{l^6}\left(\frac{5}{6}\right)^6.\end{aligned}$$

Therefore, by applying Theorem 2, every solution of (30) is oscillatory or convergent as $t \to \infty$ if $\beta > \dfrac{1}{5}$.

Remark 2. *This example shows that the assumptions of Theorem 2 hold. Hence, if $\beta > \dfrac{1}{5}$, every solution of (30) is either oscillatory or convergent. The proposed criteria build on earlier results and remain effective for half-linear third-order dynamic equations on time scales, while also providing new conditions that extend existing oscillation results.*

Example 3. *Consider the nonlinear third-order advanced dynamic equation*

$$\left\{ t^{\alpha_2 - 1}\phi_{\alpha_2}\left(\left[t^{\alpha_1 - 1}\phi_{\alpha_1}\left(x^{\Delta}(t) \right) \right]^{\Delta} \right) \right\}^{\Delta} + \frac{\beta\alpha_2^{\alpha_2}}{tH_2^{\alpha_2}(t,t_0)}\phi_{\alpha}(x(g(t))) = 0,\ g(t) \geq t, \tag{31}$$

where $\alpha_1 > 0$, $\alpha_2 \geq 1$, and where all quantities appearing in this example, such as H_i, $i = 1,2$, and the deviating function g are defined as in the preceding sections. The constant $\beta > 0$ is chosen to ensure that the assumption of Theorem 2 is satisfied. It is not difficult to see that conditions (2) holds since

$$\int_{t_0}^{\infty} \left(\frac{1}{r_i(\tau)} \right)^{1/\alpha_i} \Delta\tau = \int_{t_0}^{\infty} \frac{\Delta\tau}{\tau^{1-1/\alpha_i}} = \infty,\ i = 1,2,$$

due to [Example 5.60] [7]. By Pötzsche chain rule, we see

$$\begin{aligned} H_1(t,t_0) &= \int_{t_0}^{t} \left(\frac{1}{r_2(\tau)} \right)^{1/\alpha_2} \Delta\tau = \int_{t_0}^{t} \frac{\Delta\tau}{\tau^{1-1/\alpha_2}} \\ &\geq \alpha_2 \int_{t_0}^{t} \left(\tau^{1/\alpha_2} \right)^{\Delta} \Delta\tau = \alpha_2 \left(t^{1/\alpha_2} - t_0^{1/\alpha_2} \right). \end{aligned}$$

Hence,

$$\begin{aligned} &\liminf_{t\to\infty} r_2^{1/\alpha_2}(t) H_1(t,t_0) H_2^{\alpha_2}(\varphi(t),t_0) q(t) \\ &\geq \beta\alpha_2^{\alpha_2+1} \liminf_{t\to\infty} \frac{t^{1/\alpha_2} - t_0^{1/\alpha_2}}{t^{1/\alpha_2}} = \beta\alpha_2^{\alpha_2+1}. \end{aligned}$$

Then, using Theorem 2, every solution of (31) is oscillatory or convergent as $t \to \infty$ if $\beta > \frac{1}{l^{\alpha_2+1}} \frac{1}{(\alpha_2+1)^{\alpha_2+1}}$.

Remark 3. *This example shows that all the assumptions of Theorem 2 are satisfied; hence every solution of equation (31) is oscillatory or convergent under the stated condition on β. In particular, it shows that the proposed criteria are effective for advanced third-order dynamic equations with $g(t) \geq t$ on time scales. This illustrates their generality and provides an extension of existing oscillation results for second-order equations (see, e.g., [4–6,37,38]), which have been widely studied in the classical continuous setting.*

4. Conclusions and Discussion

The obtained results establish new Kneser-type oscillation criteria for third-order half-linear dynamic equations on general time scales. In particular, in the continuous case $\mathbb{T} = \mathbb{R}$, the results yield new oscillation criteria for the corresponding third-order differential equations, complementing the well-developed theory for second-order equations. The approach based on Riccati-type transformations allows the treatment of equations with deviating arguments, which are not fully covered in the existing literature. Consequently, the results provide extensions beyond classical oscillation criteria while remaining applicable to both continuous and discrete cases. The effectiveness of the proposed criteria is illustrated through several examples, including delay and advanced cases.

An interesting direction for future research is the development of Kneser-type criteria for third-order equations in the noncanonical case, i.e.,

$$\int_{t_0}^{\infty}\left(\frac{1}{r_i(\tau)}\right)^{1/\alpha_i}\Delta\tau<\infty,\ i=1,2.$$

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