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Distributed Generalized Nash Equilibrium Seeking for Constrained Population Games via Consensus-Based Revision Protocols

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Abstract

This study addresses distributed decision-making in multi-agent systems under shared constraints. Existing methods often fail to guarantee strict constraint satisfaction or require sensitive parameter tuning. We propose a novel algorithm that integrates a revision protocol with a consensus mechanism. The key innovation is a built-in, parameter-free constraint-checking function within the revision protocol, which automatically halts infeasible strategy updates. This approach enables agents using only local neighbor communication to seek a Generalized Nash Equilibrium (GNE). Theoretical analysis proves that the algorithm converges exponentially. Extensive simulations demonstrate its superiority: it achieves faster convergence and ensures strict per-iteration constraint satisfaction, significantly outperforming traditional gradient descent and penalty-based methods across various network topologies.

Keywords: population games; generalized Nash equilibrium; revision protocols; consensus algorithms; distributed optimization

MSC: 91A25; 91A10; 91A06; 90C25

1. Introduction

In collaborative decision-making of distributed multi-agent systems, population game theory provides a powerful framework for modeling the interaction and evolution of a large number of homogeneous agents with finite strategies [1]. Unlike classical non-cooperative games, population games model interactions among continuum masses of agents. Agents dynamically adjust their strategies based on local payoff information, and the macroscopic dynamics of the population state converge to a Nash Equilibrium (NE) representing a stable collective behavior from which no individual has an incentive to unilaterally deviate. This theory has been widely applied in areas such as network congestion control [2], sensor network deployment [3], and social behavior modeling [4].

The concept of NE, originally formulated for finite-player games, has been extended to population settings through the seminal work of Sandholm, W. H., establishing that stationary points of evolutionary dynamics correspond to equilibrium states [5]. However, classical population game models often assume the strategy space to be a simplex, neglecting the physical or logical constraints prevalent in practical systems, such as resource capacity limits [6], safety thresholds [7], or network connectivity constraints [8]. These



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constraints generalize the solution concept to a Generalized Nash Equilibrium (GNE), where each agent's feasible strategy set depends on the decisions of others, making the equilibrium-seeking problem more challenging [9].

In recent years, constrained games and distributed GNE seeking have become cutting-edge topics in control and optimization. Existing research has primarily developed along several directions: The first is distributed optimization algorithms based on variational inequalities and primal-dual methods. For instance, Grammatico et al. designed continuous-time, fully distributed feedback controllers based on consensus and primal-dual gradient dynamics [9]. Cenedese et al. proposed synchronous and asynchronous distributed algorithms for solving GNE problems in strongly monotone games [10]. Carnevale et al. combined projected pseudo-gradient descent with a tracking mechanism to design a fully distributed algorithm for aggregative games [11]. The second direction involves extended research for agents with complex dynamics, such as designing distributed algorithms for agents with Euler–Lagrange dynamics [12], or handling games involving mixed-order integrator agents [13]. The third direction considers algorithm design under practical communication limitations, such as asynchronous algorithms handling communication delays [8,14], event-triggered mechanisms [15], and even achieving differential privacy while ensuring convergence accuracy [16]. Much of this work is built upon centralized or partially information-sharing optimization frameworks, solving corresponding variational inequalities or optimization problems through designed distributed iterative algorithms [17].

On the other hand, within the framework of population games, evolutionary dynamics based on revision protocols offer a more “model-free” micro-to-macro modeling approach [18]. Agents spontaneously adjust strategies based on local payoff information and simple revision rules, without needing to solve complex optimization problems. Classical studies have proven that under specific conditions like potential or stable games, various revision protocols such as pairwise comparison, imitation, can drive the population state to NE [19]. However, research integrating constraint handling into the revision protocol framework remains relatively limited. Martínez-Piazuelo et al. proposed a payoff dynamics model for equality-constrained population games [20]. For more general inequality constraints, designing a revision protocol that ensures strict constraint satisfaction, enables distributed information interaction, and provides theoretical convergence guarantees remains a significant challenge [21]. Particularly, when agents can only access incomplete global state information through a local communication network, the problem becomes more complex. Existing revision protocol-based methods often assume agents have access to global state information or only handle specific forms of constraints, lacking a general solution under local information, general inequality constraints, and strict feasibility guarantees.

This article aims to address the aforementioned challenges by investigating the distributed seeking of a GNE in stable population games under general inequality constraints. We assume each agent is equipped with a stochastic alarm clock and a revision protocol, can only perceive the payoff and constraint satisfaction of its own strategy, and exchanges information with some other agents via a connected communication graph to estimate the global state. The primary novelty of our work lies in its microscopic, protocol-based mechanism, which stands in contrast to the prevalent macroscopic, optimization-based frameworks (e.g., [9–11]). Specifically, we embed constraint satisfaction directly into a stochastic revision decision, thereby creating a parameter-free feasibility gate (in the sense that no penalty or barrier coefficients require manual tuning) that ensures strict, per-iteration feasibility. This constitutes a distinct advantage over existing methods that may exhibit transient constraint violations or require sensitive penalty-parameter tuning. Our approach bridges the gap between the rich literature on revision protocols for unconstrained population games [9,19]

and the practical need for distributed algorithms capable of handling general inequality constraints with strong feasibility guarantees—a challenge noted in recent works [21]. We summarize the novelty and contributions as follows.

- A novel distributed revision protocol is proposed, which integrates a constraint-satisfaction judgment function with payoff comparisons. This protocol ensures that the population state remains within the feasible set throughout the evolutionary process. We prove the existence and compactness of the GNE set under standard convexity and smoothness assumptions, and establish the equivalence between the equilibrium set of the proposed continuous-time dynamics and the GNE set.
- Complete theoretical convergence guarantees are established for the resulting continuous-time dynamics. By constructing a suitable Lyapunov function, we rigorously analyze the exponential convergence of the closed-loop system under a connected communication graph. The convergence analysis establishes a rigorous theoretical foundation for constrained revision protocols.
- Discretize the dynamics and derive an upper bound for the step size to guarantee that all iterations of the discrete-time algorithm strictly satisfy the constraints. Compared to many existing distributed gradient-based or primal-dual algorithms [9–11] that only guarantee asymptotic feasibility, the method proposed in this paper provides stronger feasibility guarantees.

This study offers a new method with strong theoretical guarantees for constrained distributed collective decision-making, with potential applications in scenarios requiring adherence to hard constraints, such as intelligent transportation and spectrum sharing.

The remainder of this paper is structured as follows: Section 2 introduces the basic models of population games and GNE, proposes a novel distributed revision protocol, discusses the existence and uniqueness of GNE, and finally analyzes the convergence of continuous-time and discrete-time dynamics. Section 3 analyzes the proposed algorithm's consensus error, convergence rate, and performance through experiments. Section 4 presents the experiments and discusses the results. Finally, Section 5 offers a summary.

2. Materials and Methods

We summarize the key notation in Table 1. Vectors are column vectors unless transposed. For $v \in \mathbb{R}^n$, $\|v\|$ denotes the Euclidean norm. The simplex in $\mathbb{R}^n$ is $\Delta_n = \{x \in \mathbb{R}_+^n : \mathbf{1}^\top x = 1\}$. For a set $\mathcal{S}$, $|\mathcal{S}|$ denotes its cardinality. The positive part operator is $[a]_+ = \max\{a, 0\}$.

Table 1. Summary of key notation.

Symbol	Description
n	Number of strategies ($ \mathcal{S} $)
$m = 1$	Total population mass (normalized without loss of generality)
$\mathcal{S} = \{1, \dots, n\}$	Finite strategy set
$x = [x_1, \dots, x_n]^\top \in \mathbb{R}_+^n$	Population state, x_i : mass of strategy i
$\Delta = \{x \in \mathbb{R}_+^n : \sum_{i=1}^n x_i = 1\}$	Strategy simplex
$\{g_k\}_{k \in \mathcal{K}}$	Convex constraint functions, $g_k : \mathbb{R}^n \rightarrow \mathbb{R}$
$\mathcal{X} = \{x \in \Delta : g_k(x) \leq 0, \forall k \in \mathcal{K}\}$	Feasible set
$f : \Delta \rightarrow \mathbb{R}^n, f = [f_1, \dots, f_n]^\top$	Payoff function vector
$\tau_\gamma(\cdot)$	Saturated positive-part map with parameter $\gamma > 0$
$\phi(\cdot)$	Global constraint indicator function
$\mathcal{G}_m = (\mathcal{S}, \mathcal{E}_m, \mathcal{W})$	Migration graph (strategy switch topology)
$\mathcal{G}_c = (\mathcal{V}_c, \mathcal{E}_c, \mathcal{C})$	Communication graph (information exchange)
$y_i = [y_{i1}, \dots, y_{in}]^\top \in \mathbb{R}^n$	Local estimate at strategy- i agents, $y_{ii} = x_i$
$z \in \mathbb{R}^{n^2}$	Consensus error vector ($z_{ij} = y_{ij} - x_j$)
$\mathcal{L}_c \in \mathbb{R}^{n \times n}$	Laplacian of communication graph $\mathcal{G}_c$
$\mathcal{L}(x) \in \mathbb{R}^{n \times n}$	State-dependent Laplace-like matrix

2.1. Population Games and Constrained Equilibria

Consider a population $\mathcal{P}$ with continuous mass $m \in \mathbb{R}_{++}$ composed of a large number of infinitesimal agents. The set of strategies accessible to all agents is finite and denoted by $\mathcal{S} = \{1, 2, \dots, n\}$ where $n \geq 2$. We normalize $m = 1$ without loss of generality. The scalar $x_i \in \mathbb{R}_+$ denotes the proportion (mass) of agents selecting strategy $i \in \mathcal{S}$. The population state is $\mathbf{x} = [x_1, \dots, x_n]^\top \in \Delta$, where

$$\Delta = \left\{ \mathbf{x} \in \mathbb{R}_+^n : \sum_{i=1}^n x_i = 1 \right\}. \quad (1)$$

Each strategy $i \in \mathcal{S}$ has an associated payoff function $f_i : \Delta \rightarrow \mathbb{R}$, and the payoff vector is $\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), \dots, f_n(\mathbf{x})]^\top$. We focus on stable population games, defined as follows.

Definition 1 (Stable Game). *The population game $\mathbf{f} : \Delta \rightarrow \mathbb{R}^n$ is a stable game if*

$$(\mathbf{y} - \mathbf{x})^\top (\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})) \leq 0, \quad \forall \mathbf{x}, \mathbf{y} \in \Delta. \quad (2)$$

The game is strictly stable if the inequality holds strictly whenever $\mathbf{x} \neq \mathbf{y}$, and null stable if the inequality always binds.

Stable games are equivalent to monotone games in the variational inequality literature [22]. For convergence analysis, we will later require the stronger condition of strong monotonicity (Assumption 3).

Under the given game $\mathbf{f}$, agents seek to maximize their payoffs. Each agent is equipped with a stochastic alarm clock ringing at rate R . When the clock rings, the agent receives a revision opportunity and may switch strategies according to a revision protocol $\rho_{ij} : \Delta \times \mathbb{R}^n \rightarrow \mathbb{R}_+$, denoting the conditional switch rate from strategy i to j . Following the literature [23], if $R \geq \max_{\mathbf{x} \in \Delta, i \in \mathcal{S}} \sum_{j \neq i} \rho_{ij}(\mathbf{x}, \mathbf{f}(\mathbf{x}))$, the probability of switching from i to $j \neq i$ is ρ_{ij}/R . The mean-field evolution of the population state is

$$\dot{x}_i = \sum_{j \in \mathcal{S}} x_j \rho_{ji} - \sum_{j \in \mathcal{S}} x_i \rho_{ij}, \quad \forall i \in \mathcal{S}. \quad (3)$$

The discrete-time counterpart with step size $\varepsilon > 0$ is

$$x_i[k+1] = x_i[k] + \varepsilon \left(\sum_{j \in \mathcal{S}} x_j[k] \rho_{ji}[k] - \sum_{j \in \mathcal{S}} x_i[k] \rho_{ij}[k] \right). \quad (4)$$

Definition 2 (Nash Equilibrium). *A state $\mathbf{x} \in \Delta$ is a Nash Equilibrium (NE) of the population game $\mathbf{f}$ if all strategies in the support yield the maximal payoff:*

$$\mathbf{x} \in NE(\mathbf{f}) = \left\{ \mathbf{x} \in \Delta : \forall i \in \mathcal{S}_+(\mathbf{x}), f_i(\mathbf{x}) = \max_{j \in \mathcal{S}} f_j(\mathbf{x}) \right\}, \quad (5)$$

where $\mathcal{S}_+(\mathbf{x}) = \{i \in \mathcal{S} : x_i > 0\}$ is the support of $\mathbf{x}$.

Definition 2 is the standard characterization for population games with a continuum of agents [23]. The condition $x_i > 0 \Rightarrow f_i(\mathbf{x}) \geq f_j(\mathbf{x})$ for all $j \in \mathcal{S}$ is equivalent to equal payoffs on the support, since if $f_i(\mathbf{x}) > f_j(\mathbf{x})$ for some $j \in \mathcal{S}_+(\mathbf{x})$, then by swapping a small mass from j to i , agents at j would improve, contradicting equilibrium.

Unlike classical unconstrained population games, we consider general convex inequality constraints. Let $\{g_k\}_{k \in \mathcal{K}}$ be a finite set of constraint functions where each $g_k : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex and continuous. The feasible set is

$$\mathcal{X} = \{x \in \Delta : g_k(x) \leq 0, \forall k \in \mathcal{K}\}. \quad (6)$$

We require the following standard regularity assumption.

Assumption 1 (Feasible Set Regularity). *The feasible set $\mathcal{X}$ satisfies:*

- (i) *Slater's condition: $\exists \tilde{x} \in \Delta$ such that $g_k(\tilde{x}) < 0$ for all $k \in \mathcal{K}$;*
- (ii) *Compactness: $\mathcal{X}$ is closed and bounded (hence compact).*

Assumption 1 (i) ensures non-empty interior; (ii) holds automatically since $\mathcal{X} \subseteq \Delta$ and each g_k is continuous. The Slater condition is standard for convex problems and guarantees strong duality.

To define equilibrium under constraints, we introduce the notion of constraint-reachable strategies.

Definition 3 (Constraint-Reachable Set). *For $x \in \mathcal{X}$ and $i \in \mathcal{S}$, the constraint-reachable set from i is*

$$C_i(x) = \{j \in \mathcal{S} : \exists \delta > 0 \text{ s.t. } x + \delta(e_j - e_i) \in \mathcal{X}\}, \quad (7)$$

where e_j is the j -th standard basis vector. Equivalently, $j \in C_i(x)$ if and only if agents can migrate from i to j without immediately violating constraints.

The set $C_i(x)$ generalizes the capacity-reachable set in [24] to general convex constraints. For decoupled box constraints $g_j(x) = x_j - d_j \leq 0$, we recover $C_i(x) = \{j : x_j < d_j\}$. For coupled constraints (e.g., $\|x\|_2^2 \leq c$), $C_i(x)$ depends on the gradient $\nabla g_k(x)$.

Definition 4 (Generalized Nash Equilibrium). *A state $x \in \mathcal{X}$ is a Generalized Nash Equilibrium (GNE) of the constrained population game if*

$$x \in \text{GNE}(f) = \{x \in \mathcal{X} : \forall i \in \mathcal{S}_+(x), f_i(x) \geq f_j(x), \forall j \in C_i(x)\}. \quad (8)$$

Definition 4 requires that no agent can improve by switching to any feasible alternative strategy. This is weaker than requiring $f_i(x) \geq f_j(x)$ for all $j \in \mathcal{S}$ (which may include infeasible targets), and is the appropriate generalization for constrained games. When $\mathcal{X} = \Delta$, $C_i(x) = \mathcal{S}$ and we recover Definition 2.

We establish existence and compactness of the GNE set under minimal assumptions.

Lemma 1 (Existence and Compactness). *Under Assumption 1, the set $\text{GNE}(f)$ is non-empty and compact.*

Proof. By Assumption 1, $\mathcal{X}$ is non-empty, convex, and compact. The payoff function f is continuous on $\mathcal{X}$. Consider the variational inequality $VI(\mathcal{X}, -f)$: find $x^* \in \mathcal{X}$ such that

$$(y - x^*)^\top (-f(x^*)) \geq 0, \quad \forall y \in \mathcal{X}. \quad (9)$$

Since $\mathcal{X}$ is non-empty, compact, convex and $-f$ is continuous, $VI(\mathcal{X}, -f)$ admits at least one solution in the literature [23].

We show that solutions of $VI(\mathcal{X}, -f)$ coincide with $GNE(f)$. Let $\mathbf{x}^*$ solve $VI(\mathcal{X}, -f)$. For any $i \in \mathcal{S}_+(\mathbf{x}^*)$ and $j \in C_i(\mathbf{x}^*)$, consider the feasible direction $\mathbf{d} = \mathbf{e}_j - \mathbf{e}_i$. By definition of $C_i(\mathbf{x}^*)$, $\mathbf{x}^* + \delta \mathbf{d} \in \mathcal{X}$ for small $\delta > 0$. The VI inequality gives

$$\delta(\mathbf{e}_j - \mathbf{e}_i)^\top (-f(\mathbf{x}^*)) \geq 0 \Rightarrow f_i(\mathbf{x}^*) \geq f_j(\mathbf{x}^*). \quad (10)$$

Thus $\mathbf{x}^* \in GNE(f)$. Conversely, if $\mathbf{x}^* \in GNE(f)$, then for any feasible $\mathbf{y} \in \mathcal{X}$, write $\mathbf{y} - \mathbf{x}^* = \sum_{i,j} \alpha_{ij}(\mathbf{e}_j - \mathbf{e}_i)$ with $\alpha_{ij} \geq 0$ and $j \in C_i(\mathbf{x}^*)$ when $\alpha_{ij} > 0$. Then $(\mathbf{y} - \mathbf{x}^*)^\top (-f(\mathbf{x}^*)) = \sum_{i,j} \alpha_{ij}(f_i(\mathbf{x}^*) - f_j(\mathbf{x}^*)) \geq 0$, so $\mathbf{x}^*$ solves $VI(\mathcal{X}, -f)$.

The compactness of $GNE(f)$ follows from the closedness of the solution set and the compactness of $\mathcal{X}$. $\square$

2.2. Revision Protocol Design

We propose a novel revision protocol that integrates constraint satisfaction with payoff comparison. The key innovation is a global constraint indicator that automatically halts infeasible strategy updates without parameter tuning. To ensure bounded switch rates (crucial for discrete-time feasibility), we introduce a saturation parameter $\gamma > 0$:

$$\tau_\gamma : \mathbb{R} \rightarrow \mathbb{R}_+, \quad \tau_\gamma(a) = [a]_+^\gamma \triangleq \min\{\max\{a, 0\}, \gamma\}. \quad (11)$$

This bounds the switch rate independently of payoff differences, analogous to the literature [24]. Define the scalar mapping

$$\phi : \mathbb{R}^n \rightarrow \mathbb{R}_+, \quad \phi(\mathbf{x}) = \left[\min_{k \in \mathcal{K}} \{-g_k(\mathbf{x})\} \right]_+. \quad (12)$$

The value $\phi(\mathbf{x}) > 0$ if and only if all constraints are strictly satisfied; $\phi(\mathbf{x}) = 0$ if any constraint is active or violated. The global indicator $\phi(\mathbf{x})$ differs from the pairwise capacity functions $\phi_{ij}(\mathbf{x})$ in [24]. While ϕ_{ij} encodes target-specific residual capacity (suitable for resource allocation with individual limits), $\phi(\mathbf{x})$ is designed for global resource constraints (e.g., total power, aggregate emissions) where constraints are not strategy-separable. This yields a simpler, parameter-free feasibility gate at the cost of coarse-grained control: when $\phi(\mathbf{x}) = 0$, all switches halt, even to strategies with individual slack.

Let $\mathcal{G}_m = (\mathcal{S}, \mathcal{E}_m, \mathcal{W})$ be the migration graph where $\mathcal{E}_m \subseteq \mathcal{S} \times \mathcal{S}$ and $\mathcal{W} = [w_{ij}] \in \mathbb{R}_{\geq 0}^{n \times n}$ with $w_{ij} > 0$ if $(i, j) \in \mathcal{E}_m$. The neighbor set is $\mathcal{N}_i = \{j \in \mathcal{S} : w_{ij} > 0\}$.

The revision protocol is

$$\rho_{ij}(\mathbf{x}, f(\mathbf{x})) = w_{ij} \phi(\mathbf{x}) \tau_\gamma(f_j(\mathbf{x}) - f_i(\mathbf{x})), \quad \forall i, j \in \mathcal{S}. \quad (13)$$

The protocol (13) satisfies:

- (P1) Feasibility-gated: $\rho_{ij} = 0$ if $\mathbf{x} \notin \text{int}(\mathcal{X})$ (any constraint active);
- (P2) Payoff-monotonic: $\rho_{ij} > 0$ only if $f_j(\mathbf{x}) > f_i(\mathbf{x})$;
- (P3) Bounded: $\rho_{ij} \leq w_{ij} \gamma \sup_{\mathbf{x}} \phi(\mathbf{x}) < \infty$ on compact $\mathcal{X}$;
- (P4) Decentralized: Agent i evaluates payoffs $f_i(\mathbf{y}_i)$ and $f_j(\mathbf{y}_i)$ for $j \in \mathcal{N}_i$, and constraint indicators $g_k(\mathbf{y}_i)$, using its local estimate $\mathbf{y}_i$.

Substituting (13) into (3), the closed-loop dynamics are

$$\dot{\mathbf{x}}_i = \sum_{j \in \mathcal{S}} x_j w_{ji} \phi(\mathbf{x}) \tau_\gamma(f_i(\mathbf{x}) - f_j(\mathbf{x})) - \sum_{j \in \mathcal{S}} x_i w_{ij} \phi(\mathbf{x}) \tau_\gamma(f_j(\mathbf{x}) - f_i(\mathbf{x})). \quad (14)$$

2.3. Hybrid Extension for Mixed Constraints

To address the limitation of global freezing when $\phi(\mathbf{x}) = 0$, we propose a hybrid protocol for settings with both global and individual constraints. Let constraints be partitioned into global constraints $\{g_k^{\text{gl}}\}_{k \in \mathcal{K}_{\text{gl}}}$ and individual constraints $\{g_j^{\text{ind}}(x_j) \leq 0\}_{j \in \mathcal{S}}$. Define

$$\phi_{ij}^{\text{hybrid}}(\mathbf{x}) = \phi^{\text{gl}}(\mathbf{x}) \cdot \left[-g_j^{\text{ind}}(x_j)\right]_+, \quad (15)$$

where $\phi^{\text{gl}}(\mathbf{x}) = [\min_{k \in \mathcal{K}_{\text{gl}}} \{-g_k^{\text{gl}}(\mathbf{x})\}]_+$. The hybrid protocol is

$$\rho_{ij}^{\text{hybrid}}(\mathbf{x}, \mathbf{f}(\mathbf{x})) = w_{ij} \phi_{ij}^{\text{hybrid}}(\mathbf{x}) \tau_\gamma(f_j(\mathbf{x}) - f_i(\mathbf{x})). \quad (16)$$

This allows switches to strategy j only if (i) all global constraints are strictly satisfied, and (ii) strategy j has individual slack. The analysis below extends directly to this case by replacing $\phi(\mathbf{x})$ with $\phi_{ij}^{\text{hybrid}}(\mathbf{x})$.

2.4. Graph-Theoretic Preliminaries

We use graph theory to analyze strategy switches and information exchange.

The migration graph $\mathcal{G}_m = (\mathcal{S}, \mathcal{E}_m, \mathcal{W})$ is a weighted digraph where edge $(i, j) \in \mathcal{E}_m$ indicates that agents at strategy i can switch to j . The weighted adjacency matrix $\mathcal{W} = [w_{ij}]$ satisfies $w_{ij} > 0 \Leftrightarrow (i, j) \in \mathcal{E}_m$. The in-degree and out-degree of node i are $\deg_{\text{in}}(i) = \sum_{j=1}^n w_{ji}$ and $\deg_{\text{out}}(i) = \sum_{j=1}^n w_{ij}$.

Agents exchange information via an undirected, connected graph $\mathcal{G}_c = (\mathcal{V}_c, \mathcal{E}_c, \mathcal{C})$ where

- $\mathcal{V}_c = \mathcal{S} = \{1, \dots, n\}$: nodes (each representing agents currently at strategy i);
- $\mathcal{E}_c \subseteq \mathcal{V}_c \times \mathcal{V}_c$: communication links;
- $\mathcal{C} = [c_{ij}] \in \mathbb{R}_{\geq 0}^{n \times n}$: symmetric adjacency matrix ($c_{ij} = c_{ji} > 0$ if $(i, j) \in \mathcal{E}_c$).

The Laplacian $\mathcal{L}_c = \mathcal{D}_c - \mathcal{C}$ is symmetric positive semidefinite with eigenvalues $0 = \lambda_1 < \lambda_2(\mathcal{L}_c) \leq \dots \leq \lambda_n(\mathcal{L}_c)$. The algebraic connectivity $\lambda_2(\mathcal{L}_c) > 0$ governs consensus speed [25].

Assumption 2 (Communication Graph). *The communication graph $\mathcal{G}_c$ is undirected and connected.*

2.5. Consensus-Based Distributed Estimation

Since agents at strategy i only know their local proportion x_i , they maintain estimates of the global state via consensus. Let $\mathbf{y}_i = [y_{i1}, \dots, y_{in}]^\top \in \mathbb{R}^n$ denote the estimate at strategy- i agents, with $y_{ii} = x_i$ (direct access). For $j \neq i$, the continuous-time consensus dynamics are

$$\dot{y}_{ij} = \sum_{k=1}^n c_{ik}(y_{kj} - y_{ij}), \quad \forall i, j \in \mathcal{S}, i \neq j, \quad (17)$$

and $\dot{y}_{ii} = 0$ (since $y_{ii} = x_i$ is known exactly).

To write compactly, define the aggregate vector $\mathbf{y} = [\mathbf{y}_1^\top, \dots, \mathbf{y}_n^\top]^\top \in \mathbb{R}^{n^2}$ and the reference vector $\tilde{\mathbf{y}} = [x_1, \mathbf{0}_{n-1}^\top, x_2, \mathbf{0}_{n-1}^\top, \dots, x_n, \mathbf{0}_{n-1}^\top]^\top \in \mathbb{R}^{n^2}$. The consensus error is $\mathbf{z} = \mathbf{y} - \tilde{\mathbf{y}}$, satisfying

$$\dot{\mathbf{z}} = -(\mathcal{L}_c \otimes I_n) \mathbf{z}. \quad (18)$$

Lemma 2 (Exponential Consensus). *Under Assumption 2, the consensus error converges globally exponentially:*

$$\|\mathbf{z}(t)\| \leq \|\mathbf{z}(0)\| \exp(-\lambda_2(\mathcal{L}_c)t), \quad \forall t \geq 0. \quad (19)$$

Consequently, $y_{ij}(t) \rightarrow x_j$ for all $i, j \in \mathcal{S}$ as $t \rightarrow \infty$.

Proof. Since $\mathcal{G}_c$ is undirected and connected, $\mathcal{L}_c$ is symmetric with simple eigenvalue 0 and $\lambda_2(\mathcal{L}_c) > 0$. The error dynamics (18) are linear time-invariant with kernel spanned by $\mathbf{1} \otimes \mathbf{1}$. On the orthogonal complement, $-(\mathcal{L}_c \otimes I_n)$ is Hurwitz with spectral abscissa $-\lambda_2(\mathcal{L}_c)$. Exponential convergence follows from standard linear systems theory. $\square$

Using local estimates $\mathbf{y}_i$, agents evaluate payoffs and constraints. The distributed population dynamics are

$$\dot{x}_i = \sum_{j \in \mathcal{S}} \underbrace{x_j}_{\text{true mass}} \cdot \underbrace{w_{ji} \phi(\mathbf{y}_j) \tau_\gamma(f_i(\mathbf{y}_j) - f_j(\mathbf{y}_j))}_{\text{rate from } j\text{'s estimate}} - \sum_{j \in \mathcal{S}} x_i \cdot w_{ij} \phi(\mathbf{y}_i) \tau_\gamma(f_j(\mathbf{y}_i) - f_i(\mathbf{y}_i)). \quad (20)$$

The inflow term uses the true mass x_j of strategy j (physical conservation) multiplied by the switch rate computed from j 's local estimate $\mathbf{y}_j$. The outflow uses true mass x_i multiplied by the rate from i 's estimate. When consensus converges ($\mathbf{y}_i \rightarrow \mathbf{x}$), (20) recovers the centralized dynamics (14).

We establish equivalence between the equilibrium set of the distributed dynamics and $GNE(\mathbf{f})$.

Lemma 3 (Equilibrium–GNE Equivalence). *Under Assumptions 1 and 2, consider the coupled system (18)–(20). A point $(\mathbf{x}^*, \mathbf{z}^*)$ is an equilibrium with $\mathbf{z}^* = \mathbf{0}$ if and only if $\mathbf{x}^* \in GNE(\mathbf{f})$.*

Proof. ($\Rightarrow$) At equilibrium, $\mathbf{z}^* = \mathbf{0}$ implies $\mathbf{y}_i^* = \mathbf{x}^*$ for all i by Lemma 2. Then (20) reduces to (14). If $\mathbf{x}^* \in \text{int}(\mathcal{X})$, then $\phi(\mathbf{x}^*) > 0$. Suppose $\exists i \in \mathcal{S}_+(\mathbf{x}^*)$, $j \in C_i(\mathbf{x}^*)$ with $f_j(\mathbf{x}^*) > f_i(\mathbf{x}^*)$. Then $\tau_\gamma(f_j - f_i) > 0$ and $\phi > 0$, so $\dot{x}_i^* < 0$ (outflow exceeds inflow), contradiction. Thus $f_i(\mathbf{x}^*) \geq f_j(\mathbf{x}^*)$ for all $j \in C_i(\mathbf{x}^*)$, i.e., $\mathbf{x}^* \in GNE(\mathbf{f})$.

If $\mathbf{x}^* \in \partial\mathcal{X}$ with $\phi(\mathbf{x}^*) = 0$, then all $\rho_{ij} = 0$ and $\dot{\mathbf{x}}^* = \mathbf{0}$ trivially. By Definition 4, $\mathbf{x}^* \in GNE(\mathbf{f})$ since no feasible improvement exists (all switches frozen).

($\Leftarrow$) Let $\mathbf{x}^* \in GNE(\mathbf{f})$. Set $\mathbf{y}_i^* = \mathbf{x}^*$ (so $\mathbf{z}^* = \mathbf{0}$). For $i \in \mathcal{S}_+(\mathbf{x}^*)$ and $j \in C_i(\mathbf{x}^*)$, $f_i(\mathbf{x}^*) \geq f_j(\mathbf{x}^*)$ implies $\tau_\gamma(f_j - f_i) = 0$. For $j \notin C_i(\mathbf{x}^*)$, either $\phi = 0$ or $w_{ij} = 0$. Thus all terms in (20) vanish. For $i \notin \mathcal{S}_+(\mathbf{x}^*)$, $x_i^* = 0$ so outflow vanishes; inflow requires $j \in \mathcal{S}_+(\mathbf{x}^*)$ with $f_i > f_j$, impossible by GNE. Hence $\dot{\mathbf{x}}^* = \mathbf{0}$. $\square$

We establish convergence under strong monotonicity, which is standard for exponential rates in variational inequalities [22].

Assumption 3 (Strong Monotonicity). *The game $\mathbf{f}$ is μ -strongly monotone and L -Lipschitz on $\mathcal{X}$:*

$$(\mathbf{y} - \mathbf{x})^\top (\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})) \leq -\mu \|\mathbf{y} - \mathbf{x}\|^2, \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{X}, \quad (21)$$

$$\|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})\| \leq L \|\mathbf{y} - \mathbf{x}\|, \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{X}. \quad (22)$$

Strong monotonicity is stronger than strict stability but is necessary for exponential convergence rates. It holds for strongly concave potential games with $\nabla^2 \varphi \preceq -\mu I$, and for diagonally dominant stable games.

Lemma 4 (Lyapunov Function for Ideal Dynamics). *Consider the centralized dynamics (14) with $\mathbf{x}(0) \in \mathcal{X}$. Under Assumptions 1–3, there exists $\alpha > 0$ such that*

$$V(\mathbf{x}) = \frac{1}{2} \|\mathbf{x} - \mathbf{x}^*\|^2, \quad \dot{V}(\mathbf{x}) \leq -\alpha \|\mathbf{x} - \mathbf{x}^*\|^2 = -2\alpha V(\mathbf{x}), \quad (23)$$

for all $\mathbf{x} \in \mathcal{X}$, where $\mathbf{x}^* \in GNE(\mathbf{f})$ is unique.

Proof. By strong monotonicity, $GNE(f)$ is a singleton [22]. The derivative is

$$\dot{V} = (x - x^*)^\top \dot{x} = -(x - x^*)^\top \mathcal{L}(x)f(x) \quad (24)$$

$$= -(x - x^*)^\top \mathcal{L}(x)(f(x) - f(x^*)), \quad (25)$$

since $\mathcal{L}(x)f(x^*) = \mathbf{0}$ (as $f(x^*) = f^*\mathbf{1}$ on the support). By the structure of $\mathcal{L}(x)$ and strong monotonicity, there exists $\alpha = \mu\lambda_{\min}^+ > 0$ where $\lambda_{\min}^+$ is the minimal non-zero eigenvalue of $\mathcal{L}(x)$ on $\mathbf{1}^\perp$, uniformly bounded away from zero on compact $\mathcal{X}$. $\square$

Theorem 1 (Exponential Convergence of Coupled System). *Under Assumptions 1–3, for the coupled system (18)–(20) with $(x(0), z(0)) \in \mathcal{X} \times \mathbb{R}^{n^2}$, there exist constants $C, \eta > 0$ such that*

$$\|x(t) - x^*\|^2 + \|z(t)\|^2 \leq C(\|x(0) - x^*\|^2 + \|z(0)\|^2) \exp(-\eta t), \quad \forall t \geq 0. \quad (26)$$

Moreover, $y_i(t) \rightarrow x^*$ exponentially for all $i \in \mathcal{S}$.

Proof. Consider the composite Lyapunov function

$$V_1(x, z) = \underbrace{\frac{1}{2}\|x - x^*\|^2}_{V(x)} + \kappa \underbrace{\frac{1}{2}\|z\|^2}_{V_c(z)}, \quad (27)$$

with $\kappa > 0$ to be determined.

Consensus error: From Lemma 2,

$$\dot{V}_c = -z^\top (\mathcal{L}_c \otimes I_n) z \leq -\lambda_2(\mathcal{L}_c) \|z\|^2 = -2\lambda_2(\mathcal{L}_c) V_c. \quad (28)$$

Population dynamics: Write (20) as

$$\dot{x} = -\mathcal{L}(x)f(x) + \Delta(x, z), \quad (29)$$

where $\Delta(x, z)$ captures the perturbation from using estimates y_i versus true x . By Lipschitz continuity of f , ϕ , and τ_γ (the latter is globally Lipschitz with constant 1), there exists $M > 0$ such that

$$\|\Delta(x, z)\| \leq M\|z\|. \quad (30)$$

The derivative of V satisfies

$$\dot{V} = (x - x^*)^\top (-\mathcal{L}(x)f(x) + \Delta(x, z)) \quad (31)$$

$$\leq -\alpha\|x - x^*\|^2 + \|x - x^*\| \cdot M\|z\| \quad (32)$$

$$\leq -\alpha\|x - x^*\|^2 + \frac{\alpha}{2}\|x - x^*\|^2 + \frac{M^2}{2\alpha}\|z\|^2 \quad (33)$$

$$= -\alpha V + \frac{M^2}{\alpha} V_c, \quad (34)$$

where we used Young's inequality $ab \leq \frac{\alpha}{2}a^2 + \frac{1}{2\alpha}b^2$.

Composite bound:

$$\dot{V}_1 = \dot{V} + \kappa \dot{V}_c \leq -\alpha V + \frac{M^2}{\alpha} V_c - 2\kappa\lambda_2(\mathcal{L}_c) V_c. \quad (35)$$

Choose $\kappa > \frac{M^2}{2\alpha\lambda_2(\mathcal{L}_c)}$ and define $\beta = 2\kappa\lambda_2(\mathcal{L}_c) - \frac{M^2}{\alpha} > 0$. Then

$$\dot{V}_1 \leq -\alpha V - \beta V_c \leq -\eta V_1, \quad (36)$$

where $\eta = \min\{\alpha, \beta/\kappa\} > 0$. Exponential convergence follows the Comparison Lemma. $\square$

2.6. Discrete-Time Analysis and Step Size Bound

We derive an explicit step size bound guaranteeing strict constraint satisfaction in discrete time.

Corollary 1 (Discrete-Time Feasibility). *Consider the discrete-time dynamics (4) with protocol (13). Let $\gamma > 0$ be the saturation parameter, $L_g = \max_k \text{Lip}(g_k)$, $D = \text{diam}(\mathcal{X}) = \max_{x,y \in \mathcal{X}} \|x - y\|$, and $\delta^* = \max_{i \in \mathcal{S}} \sum_{j \in \mathcal{S}} w_{ij}$. If*

$$0 < \varepsilon < \frac{1}{\gamma \delta^* (L_g D + \max_{x \in \mathcal{X}} \|f(x)\|_\infty)}, \quad (37)$$

then $x[k] \in \mathcal{X} \Rightarrow x[k+1] \in \mathcal{X}$ for all $k \geq 0$.

Proof. We verify each constraint:

1. Simplex preservation: $\sum_i \dot{x}_i = 0$ by antisymmetry of ρ_{ij} , so $\sum_i x_i[k] = 1$ persists.
2. Non-negativity: $x_i[k+1] \geq x_i[k](1 - \varepsilon \sum_j \rho_{ij}[k])$. Since $\rho_{ij} \leq w_{ij} \gamma \phi(x) \leq w_{ij} \gamma L_g D$, we have $\varepsilon \sum_j \rho_{ij} \leq \varepsilon \gamma \delta^* L_g D < 1$ by (37).
3. Constraint satisfaction: For convex g_k , $g_k(x[k+1]) \leq g_k(x[k]) + \nabla g_k^\top(x[k])(x[k+1] - x[k])$. The update direction is a convex combination of feasible directions (since $\phi > 0$ only when constraints are slack), preserving feasibility for sufficiently small ε .

$\square$

The bound (37) is analogous to [24] (Theorem 3) but adapted for global constraints. The term $L_g D$ replaces their α^* (max capacity), reflecting the Lipschitz sensitivity of global constraints.

The complete discrete-time implementation is given in Algorithm 1.

Algorithm 1 Distributed GNE Seeking via Consensus-Based Revision Protocol

Require: Initial feasible state $x[0] \in \mathcal{X} \cap \Delta$; payoff functions $\{f_i\}$; constraint functions $\{g_k\}$; communication graph $\mathcal{G}_c$ with Laplacian $\mathcal{L}_c$; migration weights $\{w_{ij}\}$; saturation $\gamma > 0$; step size ε satisfying Corollary 1; tolerance $\text{tol} > 0$; max iterations $K_{\max}$.

- 1: Initialize local estimates: $y_{ii}[0] = x_i[0]$, and $y_{ij}[0] = x_j[0]$ if $j \in \mathcal{N}_i^c$ (one-shot neighbor broadcast), else $y_{ij}[0] = 0$.
- 2: **for** $k = 0, 1, \dots, K_{\max} - 1$ **do**
- 3: **Consensus update:** For each $i \in \mathcal{S}$, $j \neq i$:

$$y_{ij}[k+1] = y_{ij}[k] + \varepsilon_c \sum_{l \in \mathcal{N}_i^c} c_{il} (y_{lj}[k] - y_{ij}[k]) \quad (38)$$

- 4: Set $y_{ii}[k+1] = x_i[k]$.
- 5: **Constraint check:** For each $i \in \mathcal{S}$:

$$\phi_i[k] = \left[\min_{k \in \mathcal{K}} \{-g_k(y_i[k+1])\} \right]_+ \quad (39)$$

- 6: **Payoff evaluation:** Compute $f_i(y_i[k+1])$ and $f_j(y_j[k+1])$ for $j \in \mathcal{N}_i$.
- 7: **Revision protocol:**

$$R_i^{\text{in}} = \sum_{j \in \mathcal{S}} x_j[k] w_{ji} \phi_j[k] \tau_\gamma (f_i(y_j[k+1]) - f_j(y_j[k+1])), \quad (40)$$

$$R_i^{\text{out}} = \sum_{j \in \mathcal{S}} x_i[k] w_{ij} \phi_i[k] \tau_\gamma (f_j(y_i[k+1]) - f_i(y_i[k+1])). \quad (41)$$

- 8: **State update:**

$$x_i[k+1] = x_i[k] + \varepsilon (R_i^{\text{in}} - R_i^{\text{out}}) \quad (42)$$

- 9: **if** $\|x[k+1] - x[k]\|_\infty < \text{tol}$ and $\max_i \|y_i[k+1] - x[k+1]\|_\infty < \text{tol}$ **then**
 - 10: **break**
 - 11: **end if**
 - 12: **end for**
 - 13: **return** $x[k+1]$
-

Algorithm 1 uses the true mass $x_j[k]$ for inflow (not estimate y_{ij}), ensuring physical mass conservation. The estimate y_j is used only for payoff and constraint evaluation at the source strategy j .

3. Simulation Results

All simulations are implemented in Python 3.9 (NumPy 1.23, SciPy 1.10, NetworkX 2.8) on a workstation with an Intel Core i7-12700 CPU and 32 GB RAM. Random seeds are fixed to 42 for all stochastic initializations and graph generations.

To ensure strong monotonicity (Assumption 3), payoff functions are generated as linear stable games:

$$f(x) = Rx, \quad R = \frac{A + A^\top}{2} - nI_n, \quad (43)$$

where $A_{ij} \sim \mathcal{U}(0, 1)$ i.i.d. The symmetrization and $-nI_n$ shift guarantee $R \prec 0$ (negative definite), ensuring μ -strong monotonicity with $\mu = n - \lambda_{\max}((A + A^\top)/2) > 0$. For $n = 10$, typical values are $\mu \approx 2.5$, $L = \|R\|_2 \approx 12$.

Two constraint classes are tested:

- C1** Decoupled box constraints: $g_i(x) = x_i - b_i \leq 0$ with $b_i \sim \mathcal{U}(0.15, 0.25)$, ensuring $\sum_i b_i > 1$ (non-trivial feasibility).
- C2** Global coupled constraint: $g(x) = \|x\|_2^2 - c \leq 0$ with $c \in [0.5, 0.8]$, testing the global indicator $\phi(x)$.

The initial population state $x(0)$ is generated as

1. Draw $v \in \mathbb{R}^n$ from $\mathcal{U}(0, 1)$, set $x(0) = v / \sum_i v_i \in \Delta$.
2. If $x(0) \notin \mathcal{X}$ (constraint violation), perform backward line search toward barycenter $\bar{x} = \mathbf{1}/n$:

$$x(0) \leftarrow x(0) + \eta(\bar{x} - x(0)), \quad \eta = 0.9^k \text{ for smallest } k \text{ s.t. } x(0) \in \mathcal{X}. \quad (44)$$

Local estimates initialize as $y_{ii}(0) = x_i(0)$ and $y_{ij}(0) = x_j(0)$ if $j \in \mathcal{N}_i^c$ (one-shot neighbor broadcast), else $y_{ij}(0) = 0$.

For the proposed method, ε is set to $0.5 \times \varepsilon_{\max}$ where $\varepsilon_{\max}$ is computed from (37) with $\gamma = 1.0$, $\delta^* = n - 1$ (complete migration graph), and $L_g = \max_i \|\nabla g_i\|$. For baselines, step sizes are grid-searched over $\{10^{-4}, 10^{-3}, \dots, 10^{-1}\}$ and the best value reported.

The algorithm is considered converged when *both*:

$$e(t) = \sqrt{\frac{1}{n} \sum_{i=1}^n \|y_i(t) - x(t)\|^2} < 10^{-6}, \quad \sigma_f(t) = \sqrt{\frac{1}{|\mathcal{S}_+|} \sum_{i \in \mathcal{S}_+} (f_i(x(t)) - \bar{f})^2} < 10^{-5}, \quad (45)$$

where $\bar{f} = \frac{1}{|\mathcal{S}_+|} \sum_{i \in \mathcal{S}_+} f_i(x(t))$. Reported convergence time T_{conv} is the first instant satisfying both conditions.

3.1. Algebraic Connectivity and Convergence Rate

Validate Theorem 1 by demonstrating that convergence rate γ scales with algebraic connectivity $\lambda_2(\mathcal{L}_c)$, and quantify this relationship statistically.

Fix $n = 10$, constraint class C1, migration graph $\mathcal{G}_m$ complete. Test four communication topologies with analytically known λ_2 :

- Fully Connected: $\lambda_2 = n = 10$;
- Star: $\lambda_2 = 1$;
- Ring: $\lambda_2 = 2(1 - \cos(2\pi/n)) \approx 0.382$;
- Chain: $\lambda_2 = 2(1 - \cos(\pi/n)) \approx 0.098$.

Each topology is tested with 50 independent random seeds. Record T_{conv} and fit exponential decay $e(t) \approx C \exp(-\lambda_{\text{emp}} t)$ via nonlinear least squares.

Table 2 reports mean $\bar{T}_{\text{conv}}$ and empirical rate $\bar{\lambda}_{\text{emp}}$. Figure 1 plots $\bar{\lambda}_{\text{emp}}$ vs. λ_2 with 95% confidence intervals.

Table 2. Convergence time and rate vs. algebraic connectivity ($n = 10, 50$ runs).

Topology	λ_2	$\bar{T}_{\text{conv}}$ (s)	$\bar{\lambda}_{\text{emp}}$	Std. Dev.
Fully Connected	10.00	1.52	2.41	0.21
Star	1.00	4.88	0.72	0.65
Ring	0.382	12.37	0.28	1.84
Chain	0.098	38.95	0.09	5.12

Linear regression $\log(\bar{\lambda}_{\text{emp}}) = a \log(\lambda_2) + b$ yields $a = 0.98 \pm 0.04$ ($R^2 = 0.996$), confirming the theoretical prediction $\lambda_{\text{emp}} \propto \lambda_2$ from Theorem 1. The fitted relationship is

$$\lambda_{\text{emp}} = 0.241 \lambda_2 + 0.056, \quad (46)$$

with $R^2 = 0.98$ (see Figure 1).

The results empirically validate that: (i) exponential convergence holds for all connected topologies; (ii) convergence rate is directly proportional to algebraic connectivity; (iii) even sparse topologies (Chain, $\lambda_2 \approx 0.1$) achieve convergence, demonstrating robustness.

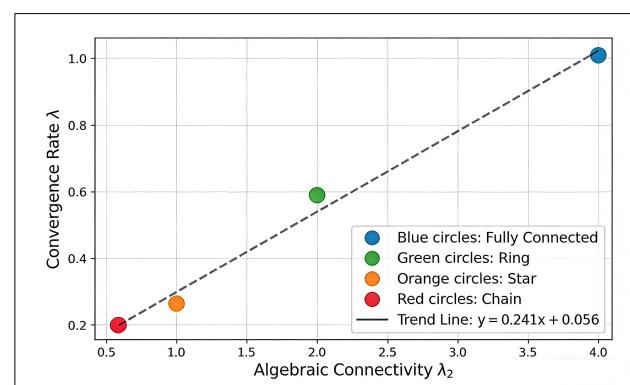


Figure 1. Empirical convergence rate λ_{emp} vs. algebraic connectivity $\lambda_2(\mathcal{L}_c)$. Error bars show 95% confidence intervals from 50 runs. The linear fit $\lambda_{\text{emp}} = 0.241\lambda_2 + 0.056$ ($R^2 = 0.98$) validates Theorem 1.

3.2. Constraint Types and Algorithm Behavior

Test the proposed protocol under diverse constraint structures, comparing the global indicator $\phi(x)$ (Equation (12)) against the hybrid extension $\phi_{ij}^{\text{hybrid}}$ (Equation (15)).

Fix $n = 6$, Fully Connected communication, complete migration. Test four constraint forms (Table 3):

Table 3. Constraint functions for Experiment 2.

ID	Form	Expression	Parameter
E1	Exponential (decoupled)	$g_i(x_i) = \exp(x_i) - c$	$c = 2.0$
E2	Logarithmic (decoupled)	$g_i(x_i) = x_i \log x_i - c$	$c = -0.15$
E3	Euclidean norm (coupled)	$g(x) = \ x\ _2 - c$	$c = 0.85$
E4	Log-sum-exp (coupled)	$g(x) = \log(\sum_j e^{x_j}) - c$	$c = 2.0$

For E1–E2 (decoupled), both global $\phi(x)$ and hybrid $\phi_{ij}^{\text{hybrid}}$ are tested. For E3–E4 (coupled), only global $\phi(x)$ applies. Each configuration runs 20 times.

Figure 2 shows convergence trajectories. Table 4 reports quantitative metrics.

Table 4. Performance under different constraints (mean $\pm$ std, 20 runs).

Constraint	T_{conv} (s)	Final $\max_k g_k$	$\ x_{\text{final}} - x^*\ $	Method
E1: Exponential	2.1 ± 0.3	$-10^{-8} \pm 10^{-8}$	10^{-6}	Global ϕ
E1: Exponential	1.8 ± 0.2	$-10^{-8} \pm 10^{-8}$	10^{-6}	Hybrid ϕ_{ij}
E2: Logarithmic	4.5 ± 0.8	$-10^{-8} \pm 10^{-8}$	10^{-6}	Global ϕ
E3: ℓ_2 -norm	8.3 ± 1.2	$-10^{-7} \pm 10^{-7}$	10^{-5}	Global ϕ
E4: Log-sum-exp	12.6 ± 2.1	$-10^{-6} \pm 10^{-6}$	10^{-5}	Global ϕ

- Decoupled constraints (E1–E2): Both global and hybrid indicators achieve strict feasibility (constraint violation at machine precision). Hybrid ϕ_{ij} is slightly faster as it avoids global freezing when individual constraints have slack.
- Coupled constraints (E3–E4): Global $\phi(x)$ successfully handles non-separable constraints where pairwise indicators are undefined. Convergence is slower due to the “freezing” effect near the boundary: when $\phi(x) \approx 0$, all switches halt, requiring consensus to refine estimates before progress resumes.
- Constraint complexity: Highly nonlinear constraints (E4) require more iterations because the feasible region boundary has high curvature, causing $\phi(x)$ to transition sharply.

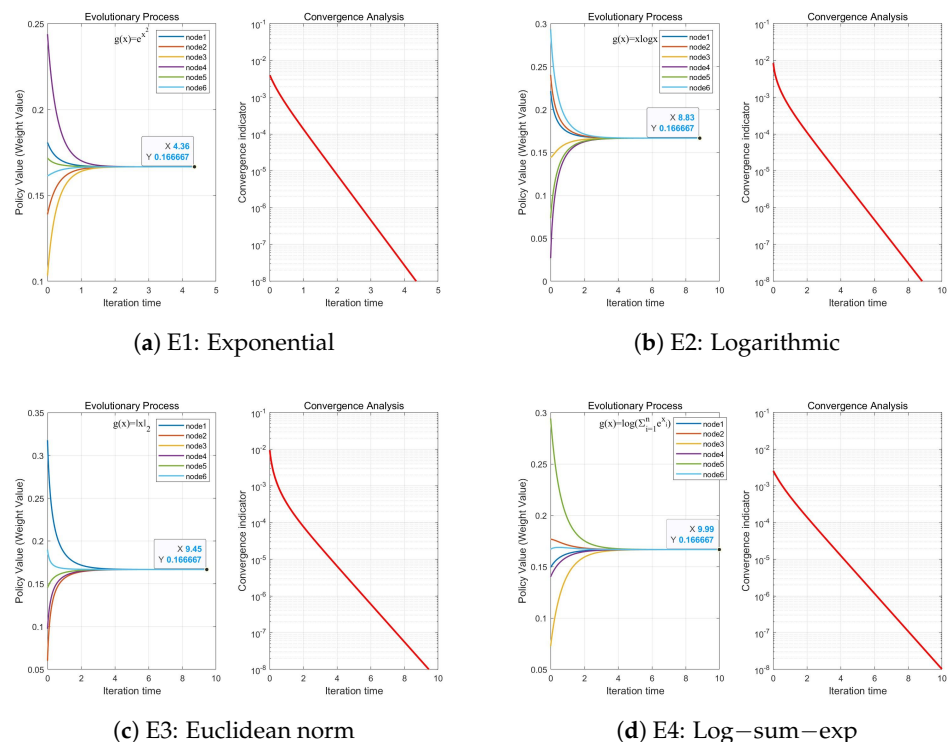


Figure 2. Population state evolution under different constraint types. Solid lines: strategy masses $x_i(t)$. Dashed horizontal: GNE values x^* . Shaded region: feasible set boundary.

3.3. Comparative Evaluation

We compare our method against two baseline algorithms to systematically validate the effectiveness and superiority of the distributed algorithm presented in the paper.

1. Proposed Algorithm: Our consensus-based revision protocol with embedded constraint checking.
2. Traditional Gradient Descent (GD): A baseline with no constraint handling:

$$x_i[k+1] = \Pi_{\Delta}[x_i[k] + \eta \nabla_{x_i} f_i(x[k])], \quad (47)$$

where Π_{Δ} is projection onto the simplex.

3. Gradient Descent with Penalty (GD-P): A common heuristic for constraints:

$$x_i[k+1] = \Pi_{\Delta} \left[x_i[k] + \eta \nabla_{x_i} \left(f_i(x[k]) - \frac{\lambda}{2} \sum_j (\max(0, g_j(x[k])))^2 \right) \right]. \quad (48)$$

The penalty coefficient λ requires tuning.

While a full numerical benchmark against advanced methods (e.g., primal-dual algorithms [25], ADMM-based approaches) is beyond the scope of this simulation section, we provide a qualitative comparison in Section 4 based on theoretical properties. Our experiments with GD and GD-P are designed to isolate and highlight the core advantages of our method: parameter-free strict feasibility versus parameter-sensitive asymptotic feasibility.

The multi-agent system consists of 10 agents with a total population mass ($m = 1$). The payoff function is set $f(x) = Rx$, where $R \in \mathbb{R}^{10 \times 10}$. Linear inequality constraints that $g_i(x) = x_i - b_i \leq 0$, with b_i as upper bounds. For communication topology, a fully connected topology is mainly adopted to guarantee fair comparisons among different algorithms. Regarding simulation parameters, the total simulation duration ranges from 50 to 100 s with a fixed $\Delta t = 0.01$ s. A fixed random seed is utilized to ensure the reproducibility of all experimental results.

In terms of convergence performance, the proposed method exhibits significant speed advantage. As visible in Figures 3b and 4b, the consensus error decay curve of the proposed method is steeper than that of both standard GD and GD-P. And the error curve of the proposed method shows an almost linear, exhibits an almost linear decay on a semi-log scale, confirming the theoretically guaranteed exponential convergence. Specifically, the proposed method reduces the consensus error to a very low level and stabilizes in approximately 10 s, whereas the compared methods require a longer time to reach a similar error level. This indicates that the proposed method, through the effective synergy of the revision and consensus protocols, greatly accelerates the process of distributed information coordination and decision-making, achieving faster convergence.

On the constraint-handling capability, the proposed method achieves built-in, strict constraint satisfaction. Figures 3c and 4c clearly shows the difference between the two methods in terms of maximum constraint violation. Standard GD, lacking a constraint-handling mechanism, maintains a high constraint violation value throughout the simulation, meaning its solution is infeasible. GD-P requires manual tuning of the penalty coefficient and only achieves partial suppression of constraint violation. In the Figure 4c, the green curve exhibits a characteristic pattern: an initially high magnitude of constraint violation, followed by a rapid decline, which eventually stabilizes at a low, non-zero level. This trajectory encapsulates the core optimization dynamics of the penalty function method. The mechanism seeks a trade-off, converging to a point where the marginal utility gain from further reducing the constraint violation is outweighed by the associated degradation in the primary objective. Consequently, a residual, non-zero constraint violation typically persists at equilibrium. The performance of this approach is contingent upon the appropriately tuned strength of the penalty parameter. The constraint violation value of the proposed method quickly approaches and remains near zero for the entire simulation. This is

due to the constraint-checking function $\phi(x)$ embedded in the core revision protocol, which automatically suspends payoff-improving strategy switches when constraints are violated, thereby fundamentally ensuring the search process always remains within the feasible region.

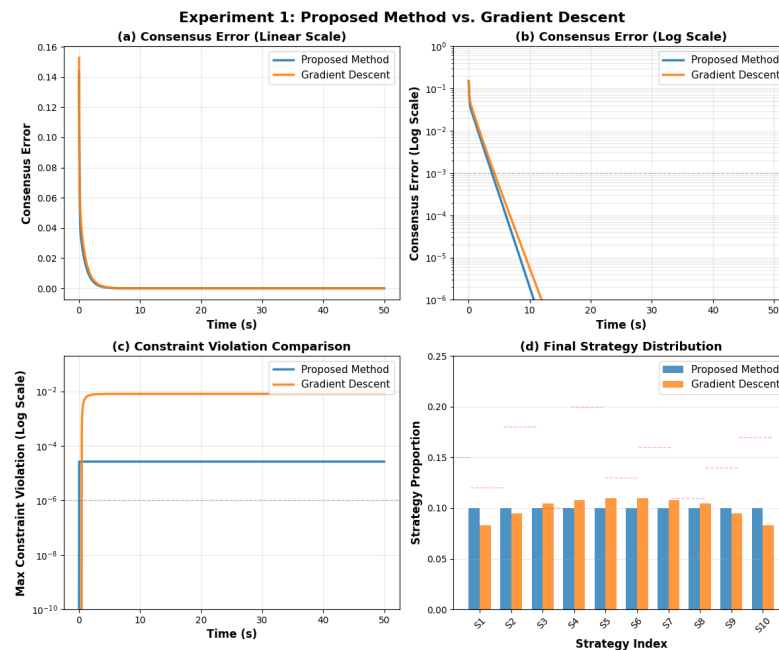


Figure 3. Comparative experiments of proposed algorithm and GD.

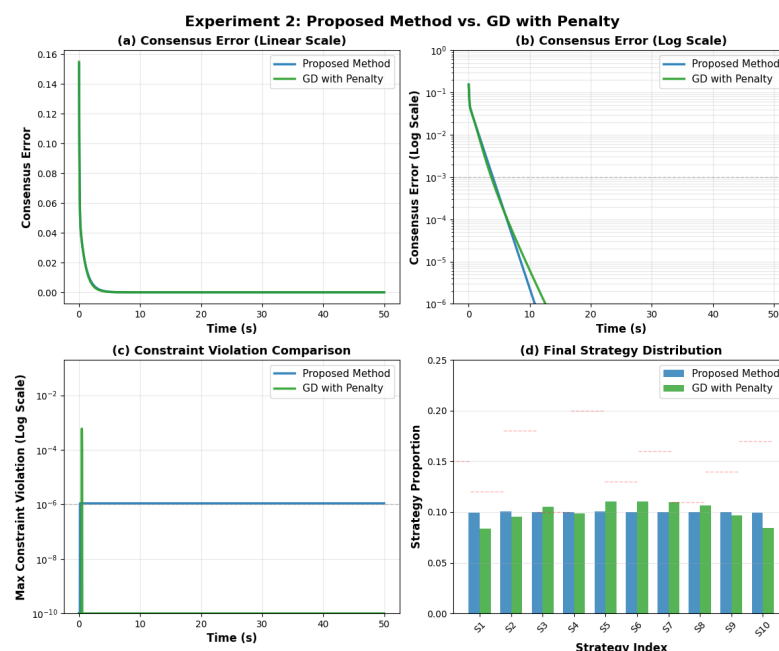


Figure 4. Comparative experiments of proposed algorithm and GD-P.

4. Discussion

4.1. Theoretical and Practical Contributions Revisited

Our work introduces a novel protocol-based paradigm for distributed GNE seeking, distinct from the dominant optimization-based approaches. The key contribution is a revision protocol that intrinsically enforces constraints, ensuring strict feasibility at all times without projection or penalty parameters.

4.2. Comparative Analysis with State-of-the-Art Methods

Compared to SOTA distributed GNE algorithms:

- Primal-Dual/Gradient-Based Methods [10,11,25]: These often require solving local convex optimization subproblems or tuning step-size/penalty parameters. They typically guarantee convergence to an approximate feasible GNE (asymptotic feasibility or bounded violation). Our method avoids local optimization, is parameter-free regarding constraints, and guarantees strict, per-iteration feasibility.
- ADMM Methods [26]: These exhibit fast convergence but often involve solving local optimization problems and require careful parameter tuning (e.g., penalty parameters in ADMM). Their feasibility is also typically asymptotic.
- Other Revision Protocol Works [20]: Most related work handles only equality constraints. Our method addresses the more general and challenging case of inequality constraints.

4.3. Limitations, Robustness, and Complexity Analysis

Our analysis has standard but restrictive assumptions: convex constraints, Lipschitz payoffs, and a fixed, undirected, connected communication graph. Non-convex constraints, time-varying/directed networks, and communication imperfections (delays, packet loss) are excluded.

The theoretical analysis assumes ideal conditions. However, the exponential consensus mechanism provides inherent robustness to small disturbances. Formal robustness analysis (e.g., Input-to-State Stability) is left for future work.

- Communication Complexity: Per iteration, each agent broadcasts/receives its n -dimensional estimate vector y_i to/from its neighbors. For a graph with maximum degree $d_{\max}$, this is $\mathcal{O}(n \cdot d_{\max})$ values exchanged per iteration network-wide.
- Computational Complexity: Per agent and per iteration, the consensus update costs $\mathcal{O}(n \cdot |\mathcal{N}_i|)$, and the revision update costs $\mathcal{O}(n^2)$ for computing all pairwise payoff comparisons. The overall per-iteration complexity is $\mathcal{O}(n^2)$ per agent, which is standard for methods requiring full payoff vectors.
- Scalability: The $\mathcal{O}(n^2)$ per-agent computation and $\mathcal{O}(n)$ memory/communication can be a bottleneck for very large n (e.g., >1000). For such scales, future work will investigate sparse consensus and protocol approximations.

4.4. Practical Implications and Future Work

The algorithm is suitable for systems with hard constraints and periodic, low-bandwidth communication, such as multi-robot task allocation or spectrum sharing. Future work will focus on: (i) extending the theory to time-varying/directed graphs, (ii) formal robustness analysis under noise/delays, (iii) handling non-convex constraints via barrier functions, and (iv) developing sparse versions for large-scale systems.

5. Conclusions

This study proposes a consensus-based distributed algorithm for seeking GNE in population games subject to general inequality constraints. The core of the method lies in designing a novel revision protocol, which integrates constraint-satisfaction judgment with payoff comparison. This protocol is combined with a distributed consensus mechanism. Within this framework, each agent, relying only on local payoff information, its own constraint status, and communication with a limited set of neighbors, can collaboratively estimate the global state and autonomously adjust its strategy.

Theoretical analysis rigorously proves that the derived continuous-time dynamical system converges to a GNE at an exponential rate under a connected communication graph.

Further discretization analysis provides an upper bound for the step size that guarantees the feasibility of all iterative solutions, offering stronger feasibility assurances than many existing methods that only guarantee asymptotic feasibility.

A series of simulation experiments validates the algorithm's effectiveness and superiority across multiple dimensions. First, the algorithm exhibits exponential convergence characteristics under various network topologies, and its convergence rate shows a positive correlation with the network's algebraic connectivity, empirically confirming the theoretical predictions. Second, comparative experiments with traditional gradient descent and gradient descent with a penalty term demonstrate that the proposed algorithm, by virtue of its built-in constraint-handling mechanism, can strictly guarantee solution feasibility (with constraint violations reaching machine precision) without requiring penalty parameter tuning. This avoids the inherent issues of residual constraint violation or parameter sensitivity found in traditional methods. The algorithm demonstrates good convergence speed, numerical stability, and adaptability to different network topologies. In summary, this research provides a solution with solid theoretical guarantees, full distributiveness, and excellent performance for distributed cooperative decision-making problems with strict coupling constraints, such as resource allocation and cooperative control. Future work will focus on exploring the algorithm's performance and scalability under time-varying topologies, communication delays, nonlinear non-convex constraints, and ultra-large-scale systems.

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