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Rayleigh Wave Propagation on the Partially Saturated Poro-Thermo-Viscoelastic Half-Space Based on Fractional Order Viscoelasticity

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Abstract

This paper probes into the propagation characteristics of Rayleigh waves in a partially saturated, porous, thermo-viscoelastic half-space, with full consideration of the fractional viscoelastic effect and thermal coupling effect. A fractional Zener model is introduced to depict the thermo-viscoelastic mechanical behavior of the solid skeleton by constructing a complete set of governing equations that include mass balance, generalized Darcy's law, momentum balance, and generalized heat conduction. Field equations are derived by means of Helmholtz vector decomposition, and the dispersion equation, and the phase velocity expression of Rayleigh waves are obtained by combining the traction-free and adiabatic boundary conditions of the medium. The impacts of key material properties, such as medium saturation, intrinsic permeability, medium viscoelasticity, and thermal expansion coefficient, on the dispersion feature of Rayleigh waves are discussed in detail. Numerical analysis results show that an increase in the thermal expansion coefficient will lead to a rise in Rayleigh wave phase velocity, in which the increase in P1 compressional wave velocity plays a dominant role among the velocities of various types of waves. Meanwhile, the attenuation coefficient of Rayleigh waves presents a decreasing trend and gradually tends to be stable with the growth of the thermal expansion coefficient. Similarly, the phase velocity of Rayleigh waves also increases with the rise in fractional order index, which is jointly dominated by the velocity enhancement of P1 waves and S waves. In addition, the attenuation coefficient of Rayleigh waves increases first and then decreases with the increase in fractional order index and reaches the peak value when the fractional order index is about 0.4. The research results reveal the influence of laws of thermal expansion characteristics and viscoelasticity on Rayleigh wave propagation and provide theoretical support for the analysis of wave propagation characteristics in porous media in relevant engineering applications.



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1. Introduction

Viscoelastic constitutive models have become core tools for seismic wave propagation and dynamic analysis of oil–gas reservoirs [1–3]. They can effectively describe the frequency-dependent stress–strain behavior and historical memory effects in geological media. Traditional integer-order viscoelastic models (e.g., Maxwell, Kelvin–Voigt) are

capable of characterizing basic attenuation properties but fail to capture the multiscale coupling effects between the solid skeleton and pore fluids in porous media [4].

By introducing the Caputo derivative [5], fractional-order models extend constitutive relations to the nonlocal domain, significantly improving the accuracy of dispersion and attenuation descriptions in complex media. Recent advances in nonlocal analysis have provided rigorous mathematical foundations for such frameworks. Specifically, Sharma [6] established a comprehensive existence and nonexistence theory for mixed local–nonlocal elliptic problems involving both classical and fractional Laplacian operators under mixed Dirichlet–Neumann boundary conditions using variational methods. This rigorous mathematical treatment of nonlocal operators and mixed boundary value problems directly supports the validity of the Caputo derivative formalism used in our fractional Zener model. The fractional Zener model [7–9] can continuously simulate the elastic-to-viscoelastic transition behavior by adjusting relaxation time and fractional order, making it highly suitable for gas–liquid–solid three-phase partially saturated media [10,11]. This paper investigates the dispersion feature of Rayleigh waves in porous half-spaces by integrating fractional thermo-viscoelastic theory with three-phase coupling mechanisms, and its results can directly serve engineering scenarios such as medium parameter inversion in seismic exploration and thermo-mechanical coupling analysis of geothermal reservoirs [12,13].

The theoretical framework for wave propagation in porous media originated from Biot's theory of elastic wave propagation in saturated media [1], which laid the foundation for subsequent research by proposing the propagation mechanisms of fast/slow P-waves and shear waves. White revealed the controlling role of microscale fluid distribution on wave attenuation through the patchy saturation model [14]. Dvorkin and Nur further combined Biot's macroflow with the microscale mechanism of squirt flow to propose the BISQ model [15], successfully explaining the strong dispersion observed in partially saturated rocks. However, these models are all based on integer-order viscoelastic assumptions and fail to accurately describe the influence of historical strains on the solid skeleton [16]. With the development of fractional calculus, Mainardi systematically established fractional-order viscoelastic constitutive relations [5], providing a mathematical framework for complex-medium modeling. Wei et al. introduced the fractional-order Zener model into liquid-saturated porous layers [7,17], finding that the fractional-order parameter and permeability jointly regulate the attenuation of reflected wave energy, and the looseness of the interface significantly affects high-frequency waves. In the field of thermo-mechanical coupling, Liu et al. established Rayleigh wave dispersion equations in thermoelastic media [12,18]. Recent developments in generalized thermoelasticity have extended classical frameworks to incorporate non-classical heat conduction laws that are critical for describing transient thermal responses in geological media. Notably, Chen [19] systematically analyzed the global existence, asymptotic behavior, and finite-time blow-up of solutions for a class of generalized thermoelastic systems with p-Laplacian heat conduction. This work demonstrates the growing interest in extending thermoelastic theory beyond the classical Fourier law and provides important theoretical context for our adoption of the Tzou dual-phase-lag generalized heat conduction law. In multiphase flow coupling, Santos [20] and Garg [21] and Nayfeh [22] developed dynamic models for dual-fluid saturated porous media but did not address the interaction between gas compressibility and heat conduction.

Notably, few current studies have simultaneously considered thermal effects, fractional viscoelasticity, and three-phase coupling when investigating Rayleigh waves in partially saturated porous media. Most existing works either ignore thermal–mechanical–flow coupling, rely on integer-order viscoelastic models, or focus only on fully saturated systems—creating a critical gap in addressing the complexity of real geological media.

The core novelty of this work lies in bridging this gap: this study establishes a porous half-space model that integrates a fractional-order viscoelastic skeleton, pore liquid–gas two-phase fluid, and generalized heat conduction. Its primary objective is to derive the thermo-mechanical-flow-coupled Rayleigh wave dispersion equation and systematically clarify how thermal expansion and viscoelastic parameters regulate Rayleigh wave phase velocity and attenuation.

To achieve this goal, first establish the governing equations of the model, including mass balance, generalized Darcy’s law, momentum balance, and generalized heat conduction. Then introduce the fractional Zener model to characterize the viscoelastic behavior of the solid skeleton, and derive the wave equations through Helmholtz decomposition. Finally, combined with adiabatic and traction-free boundary conditions, we obtain the dispersion equation and analyze the influences of key parameters on Rayleigh wave propagation. This research is expected to provide more accurate theoretical support for engineering problems involving wave propagation in complex porous media.

2. Basic Equations of Unsaturated Porous Thermo-Viscoelastic Media

Partially saturated porous media are composed of three phases, namely, solid grains (s), pore liquid (l), and pore gas (g). The liquid saturation degree S^l represents the fraction of void volume occupied by the liquid phase, while the gas saturation degree notes the fraction of void volume occupied by the gas phase, with the constraint $S^l + S^g = 1$.

2.1. Governing Equations

When there are no nonconvective terms between components, the partial mass conservation equation for the multiphase medium is as follows [22–24].

$$\frac{\partial[(1 - n^s)\rho^s]}{\partial t} + \nabla \cdot [(1 - n^s)\rho^s \dot{\mathbf{u}}^s] = 0 \quad (1a)$$

$$\frac{\partial(n^s S^l \rho^l)}{\partial t} + \nabla \cdot (n^s S^l \rho^l \dot{\mathbf{u}}^l) = 0 \quad (1b)$$

$$\frac{\partial(n^s S^g \rho^g)}{\partial t} + \nabla \cdot (n^s S^g \rho^g \dot{\mathbf{u}}^g) = 0 \quad (1c)$$

where n^s refers to porosity, and the notations ρ^b and $\mathbf{u}^b$ represent the density and displacement vector of phase b ($b = s, l, g$) respectively.

In general, the products of spatial derivatives and time derivatives of dependent variables are terms of higher order relative to their time derivatives alone. Thus, by expanding Equation (1), the mass balance equations can be re-expressed as

$$-\frac{\partial n^s}{\partial t} + (1 - n^s) \frac{\partial \rho^s}{\rho^s \partial t} + (1 - n^s) \nabla \cdot \dot{\mathbf{u}}^s = 0 \quad (2a)$$

$$n^s \frac{\partial S^l}{\partial t} + S^l \frac{\partial n^s}{\partial t} + n^s S^l \frac{\partial \rho^l}{\rho^l \partial t} + n^s S^l \nabla \cdot \dot{\mathbf{u}}^l = 0 \quad (2b)$$

$$-n^s \frac{\partial S^l}{\partial t} + (1 - S^l) \frac{\partial n^s}{\partial t} + n^s (1 - S^l) \frac{\partial \rho^g}{\rho^g \partial t} + n^s (1 - S^l) \nabla \cdot \dot{\mathbf{u}}^g = 0 \quad (2c)$$

In partially saturated porous media, the densities of the solid phase ρ^s and liquid phase ρ^l may be written as [25]

$$\rho^s = \rho^{s0} \exp \left[\beta_{sp} (p_s - p_{ref}^0) - \beta_{sT} (T - T_0) \right] \quad (3a)$$

$$\rho^l = \rho^{l0} \exp[\beta_{wp}(p_w - p_{w0}) - \beta_{wT}(T - T_0)] \quad (3b)$$

where ρ^{s0} denotes the reference density of solid particles under reference pressure p_{ref}^0 and reference temperature T_0 , ρ^{l0} is the reference density of the liquid phase under the reference pore water pressure p_{w0} , the two parameter sets β_{sp} , β_{wp} and β_{sT} , β_{wT} are the compressibility and thermal expansion coefficients of the solid particles and the liquid phase, respectively; and p_s and p_w refer to the mean net pressure, pore–liquid pressure, and pore–gas pressure, respectively.

In partially saturated porous media, the density of the gas phase ρ^g can be formulated as [25]

$$\rho^g = \frac{M_a p_g}{RT} \quad (3c)$$

where p_g refers to pore–gas pressure, $M_a = 0.0288 \text{ kg/mol}$ denotes the molar mass of dry air, $R = 8.3144 \text{ J/(mol}\cdot\text{K)}$ stands for the universal gas constant, and T stands for the medium temperature of the partially saturated poro thermo-elastic medium.

The time-dependent variations in densities can be derived via the chain rule, leading to the following expressions:

$$\frac{\partial \rho^s}{\partial t} = \beta_{sp} \rho^s \frac{\partial p_s}{\partial t} - \beta_{sT} \rho^s \frac{\partial T}{\partial t} \quad (4a)$$

$$\frac{\partial \rho^l}{\partial t} = \beta_{wp} \rho^l \frac{\partial p_w}{\partial t} - \beta_{wT} \rho^l \frac{\partial T}{\partial t} \quad (4b)$$

$$\frac{\partial \rho^g}{\partial t} = \frac{M_a}{RT} \frac{\partial p_g}{\partial t} - \frac{M_a p_g}{RT^2} \frac{\partial T}{\partial t} \quad (4c)$$

Given that the saturation S^l is dependent on matric suction ψ ($\psi = p_g - p_w$) and temperature T , the time evolution equation of saturation can be expressed as

$$\frac{\partial S^l}{\partial t} = \frac{\partial S^l}{\partial \psi} \left(\frac{\partial p_g}{\partial t} - \frac{\partial p_w}{\partial t} \right) + \frac{\partial S^l}{\partial T} \frac{\partial T}{\partial t} \quad (5)$$

using the soil–water-retention curve model proposed by Van Genuchten [26], the derivative of saturation S^l with respect to matric suction ψ can be derived as

$$\frac{\partial S^l}{\partial \psi} = -\chi m d (S_{sat}^l - S_{res}^l) (S_e)^{\frac{m+1}{m}} \left[(S_e)^{-\frac{1}{m}} - 1 \right]^{\frac{d-1}{d}} \quad (6)$$

where χ , m and d serve as material parameters of V-G model; the effective water saturation S_e is given by $S_e = (S^l - S_{res}^l) / (S_{sat}^l - S_{res}^l)$. S_{res}^l and S_{sat}^l are the residual saturation and full saturation of the medium, respectively.

Meanwhile, the matrix suction ψ can be regarded as a function of the actual temperature T and the reference temperature T_0 [27]:

$$\psi = \psi(T_0) \exp[-\beta_\psi (T - T_0)] \quad (7)$$

where $\psi(T_0)$ denoted the matrix suction at the reference temperature T_0 , while β_ψ represents the surface tension temperature-dependent coefficient.

Based on Equations (5)–(7), the time derivative of saturation S^l is formulated as

$$\frac{\partial S^l}{\partial t} = A_s \left(\frac{\partial p_g}{\partial t} - \frac{\partial p_w}{\partial t} \right) - \beta_\psi A_s \chi^{-1} (S_e)^{-\frac{1}{m}} \left[(S_e)^{-\frac{1}{m}} - 1 \right]^{\frac{1}{d}} \frac{\partial T}{\partial t} \quad (8)$$

where $A_s = -\chi m d (S_{sat}^l - S_{res}^l) (S_e)^{\frac{m+1}{m}} \left[(S_e)^{-\frac{1}{m}} - 1 \right]^{\frac{d-1}{d}}$.

By substituting Equation (4a) into Equation (2a), the time derivative of porosity is given by

$$\frac{\partial n^s}{\partial t} = (1 - n^s) \left(\nabla \cdot \dot{\mathbf{u}}^s + \beta_{sp} \dot{p}_s - \beta_{sT} \frac{\partial T}{\partial t} \right) \quad (9)$$

Equation (9) indicates that porosity is influenced by temperature, pressure, and deformation imposed on the partially saturated porous media. However, considering the compressibility of the solid skeleton is negligibly smaller than other terms in Equation (9), the simplification $\beta_{sp} = 0$ is subsequently employed [25]. This assumption is reasonable for engineering scenarios such as medium and shallow strata under low pressure levels.

By substituting Equations (4b,c), (8) and (9) into Equation (2b,c), the resulting equations are obtained as

$$A_{11} \dot{p}_w + A_{12} \dot{p}_g + A_{13} \nabla \cdot \dot{\mathbf{u}}^s + A_{14} \nabla \cdot \dot{\mathbf{u}}^l + A_{15} \nabla \cdot \dot{\mathbf{u}}^g + A_{16} \dot{T} = 0 \quad (10a)$$

$$A_{21} \dot{p}_w + A_{22} \dot{p}_g + A_{23} \nabla \cdot \dot{\mathbf{u}}^s + A_{24} \nabla \cdot \dot{\mathbf{u}}^l + A_{25} \nabla \cdot \dot{\mathbf{u}}^g + A_{26} \dot{T} = 0 \quad (10b)$$

where coefficients $A_{11} - A_{26}$ are defined as

$$\begin{aligned} A_{11} &= n^s S^l \beta_{wp}, A_{12} = n^s (1 - S^l) \frac{M_a}{\rho^g RT}, A_{13} = 1 - n^s, A_{14} = n^s S^l \\ A_{15} &= n^s (1 - S^l), A_{16} = -(1 - n^s) \beta_{sT} - n^s S^l \beta_{wT} - n^s (1 - S^l) \frac{M_a p_g}{\rho^g RT^2} \\ A_{21} &= n^s S^l (1 - S^l) \beta_{wp} - n^s A_s, A_{22} = n^s A_s - n^s S^l (1 - S^l) \frac{M_a}{\rho^g RT}, A_{23} = 0 \\ A_{24} &= -A_{25} = n^s S^l (1 - S^l) \\ A_{26} &= -n^s \beta_\psi A_s \chi^{-1} \left(S_e^{-\frac{1}{m}} - 1 \right)^{\frac{1}{d}} + n^s S^l (1 - S^l) \left(\frac{M_a p_g}{\rho^g RT^2} - \beta_{wT} \right) \end{aligned}$$

By solving Equation (10a,b) yields simplified expressions for pore-liquid pressure and pore-air pressure as

$$-\dot{p}_w = a_{11} \nabla \cdot \dot{\mathbf{u}}^s + a_{12} \nabla \cdot \dot{\mathbf{u}}^l + a_{13} \nabla \cdot \dot{\mathbf{u}}^g + a_{14} \dot{T} \quad (11a)$$

$$-\dot{p}_g = a_{21} \nabla \cdot \dot{\mathbf{u}}^s + a_{22} \nabla \cdot \dot{\mathbf{u}}^l + a_{23} \nabla \cdot \dot{\mathbf{u}}^g + a_{24} \dot{T} \quad (11b)$$

where coefficients $a_{11} - a_{24}$ are expressed as

$$\begin{aligned} a_{11} &= \frac{A_{22} A_{13}}{A_{22} A_{11} - A_{12} A_{21}}, a_{12} = \frac{A_{22} A_{14} - A_{12} A_{24}}{A_{22} A_{11} - A_{12} A_{21}}, a_{13} = \frac{A_{22} A_{15} - A_{12} A_{25}}{A_{22} A_{11} - A_{12} A_{21}} \\ a_{14} &= \frac{A_{22} A_{16} - A_{12} A_{26}}{A_{22} A_{11} - A_{12} A_{21}}, a_{21} = \frac{A_{21} A_{13}}{A_{21} A_{12} - A_{11} A_{22}}, a_{22} = \frac{A_{21} A_{14} - A_{11} A_{24}}{A_{21} A_{12} - A_{11} A_{22}} \\ a_{23} &= \frac{A_{21} A_{15} - A_{11} A_{25}}{A_{21} A_{12} - A_{11} A_{22}}, a_{24} = \frac{A_{21} A_{16} - A_{11} A_{26}}{A_{21} A_{12} - A_{11} A_{22}} \end{aligned}$$

The generalized Darcy's law is usually used to characterize the seepage movement of liquid and gas in the pores of partially saturated porous media. For isotropic media, neglecting the action of gravity, the seepage equations can be derived as [25,28]

$$n^s S^l (\dot{\mathbf{u}}^l - \dot{\mathbf{u}}^s) = -\frac{k_r^l k}{\mu_l} (\nabla p_w + \rho^l \ddot{\mathbf{u}}^l) \quad (12a)$$

$$n^s S^g (\dot{\mathbf{u}}^g - \dot{\mathbf{u}}^s) = -\frac{k_r^g k}{\mu_g} (\nabla p_g + \rho^g \ddot{\mathbf{u}}^g) \quad (12b)$$

where k represents intrinsic permeability. The parameters μ_l , μ_g and k_r^l , k_r^g denote the dynamic viscosities and relative permeabilities of the liquid and gas phase, respectively. Their specific definitions are as follows [25,29]:

$$\mu_l = (243.18 \times 10^{-7}) 10^{\frac{247.8}{T-140}}, \mu_g = 1.48 \times 10^{-6} \frac{\sqrt{T}}{1 + 119/T} \quad (13)$$

$$k_r^l = \sqrt{S_e} \left[1 - \left(1 - S_e^{\frac{1}{m}} \right)^m \right]^2, k_r^g = \sqrt{1 - S_e} \left(1 - S_e^{\frac{1}{m}} \right)^{2m} \quad (14)$$

From Equation (12a,b), the following expressions can be derived:

$$-\nabla p_w = \frac{\mu_l}{k_r^l k} n^s S^l (\dot{\mathbf{u}}^l - \dot{\mathbf{u}}^s) + \rho^l \ddot{\mathbf{u}}^l \quad (15a)$$

$$-\nabla p_g = \frac{\mu_g}{k_r^g k} n^s (1 - S^l) (\dot{\mathbf{u}}^g - \dot{\mathbf{u}}^s) + \rho^g \ddot{\mathbf{u}}^g \quad (15b)$$

2.2. Constitutive Equations

In the absence of body forces, the momentum balance equation for partially saturated porous media is expressed as

$$\sigma_{ij,j} = \bar{\rho}^s \ddot{u}_i^s + \bar{\rho}^l \ddot{u}_i^l + \bar{\rho}^g \ddot{u}_i^g \quad (16)$$

in which σ_{ij} is the total stress, and the three apparent densities $\bar{\rho}^s$, $\bar{\rho}^l$ and $\bar{\rho}^g$ can be written as $\bar{\rho}^s = (1 - n^s) \rho^s$, $\bar{\rho}^l = n^s S^l \rho^l$ and $\bar{\rho}^g = n^s S^g \rho^g$, respectively.

The effective stress for partially saturated soils can be described using the single stress-state variable proposed by Bishop, expressed in the form of the Cauchy effective stress tensor or Bishop stress as

$$\sigma_{ij} = \sigma'_{ij} - p \delta_{ij} \quad (17)$$

where δ_{ij} is the Kronecker delta, and the pore pressure p is expressed as

$$p = \alpha_{eff} p_w + (1 - \alpha_{eff}) p_g \quad (18)$$

where α_{eff} is an effective stress parameter denoting the fraction of matric suction that contributes to the effective stress. Schrefler proposed approximating α_{eff} as identical to the saturation S^l [30]. Although strictly speaking, the relationship between them should be more complex and nonlinear, this assumption is employed herein to simplify the model.

Considering the strain effects of pore–fluid pressure and temperature change on solid grains, the constitutive relation for a homogeneous porous medium is given by

$$\sigma'_{ij} = c_{ijkl} (\varepsilon_{kl} - \varepsilon_{kl}^p - \varepsilon_{kl}^T) \quad (19)$$

where ε_{kl}^p represents the strain induced by pore–fluid pressure, and ε_{kl}^T stands for the strain resulting from temperature variation; both can be further elaborated as

$$\varepsilon_{11}^p = \varepsilon_{22}^p = \varepsilon_{33}^p = -\frac{1}{3K_s} p \quad (20)$$

$$\varepsilon_{kl}^T = \delta_{kl} \beta_T \theta \quad (21)$$

where K_s denotes the compressibility modulus of solid grains, β_T is the thermal expansion coefficient for partially saturated porous media, and $\theta = T - T_0$ represents the temperature change.

After a series of mathematical manipulations, we derive the equation as follows:

$$\sigma'_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_v\delta_{ij} + \frac{K_b}{K_s}p\delta_{ij} - 3K_b\beta_T\theta\delta_{ij} \quad (22)$$

where λ and μ refer to the Lamb's constants, $\varepsilon_v = \nabla \cdot \mathbf{u}^s$ represents the volumetric strain, and $K_b = \lambda + \frac{2\mu}{3}$ denotes the bulk modulus of the soil skeleton.

Inserting Equation (22) into Equation (17) yields the following equation:

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_v\delta_{ij} - \alpha p\delta_{ij} - \lambda'\theta\delta_{ij} \quad (23)$$

where $\alpha = 1 - \frac{K_b}{K_s}$ and $\lambda' = 3K_b\beta_T$.

Inserting Equation (23) into Equation (16) and following a series of mathematical manipulations, the momentum balance equation for partially saturated poro-thermo-elastic media is derived as

$$\begin{aligned} \mu\nabla^2\mathbf{u}^s + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}^s) - \alpha\alpha_{eff}\nabla p_w - \alpha(1 - \alpha_{eff})\nabla p_g - \lambda'\nabla T \\ = \bar{\rho}^s\ddot{\mathbf{u}}^s + \bar{\rho}^l\ddot{\mathbf{u}}^l + \bar{\rho}^g\ddot{\mathbf{u}}^g \end{aligned} \quad (24)$$

2.3. Fractional Zener Model of Viscoelastic Solid Frame

Traditional poroelastic models are based on the premise that the solid skeleton is purely elastic. In fact, the solid skeleton of porous media such as clay-containing sandstones has obvious viscoelastic characteristics. Considering the time-harmonic nature of relevant mechanical quantities, the elastic coefficients λ , μ and K_s ought to be substituted with complex moduli $\bar{\lambda}(\omega)$, $\bar{\mu}(\omega)$ and $\bar{K}_s(\omega)$. The bulk moduli of the liquid and gas phase, μ_l and μ_g , are not strictly constants; however, their contribution to attenuation is minimal, allowing them to be considered frequency-independent [2,31]. In his early research, Stoll treated the forms of $\bar{K}_b(\omega)$ and $\bar{\mu}(\omega)$ as complex constants as a special case. Carcione characterized complex moduli using the generalized Zener model, which consists of several Zener models (or Standard Linear Solid models) connected in parallel [32]. Nevertheless, increasing the number of Zener models typically leads to a greater number of model parameters that need to be determined. To minimize the number of model parameters, the fractional Zener model emerges as the most promising option [33,34].

In the classical viscoelastic model, the spring represents the elastic body following Hooke's law, and the dashpot represents the viscous body obeying Newton's law. A spring connected in series with a dashpot forms the Maxwell model, and a spring connected in parallel with a dashpot constitutes the Kelvin model. The Zener model (Standard Linear Solid) makes up for the defects of the Maxwell model (no stress relaxation) and the Kelvin model (no strain creep), which is constructed by connecting a spring in series with a Kelvin model or in parallel with a Maxwell model. The constitutive equation of the Zener model is given by [35]

$$\sigma + \tau_\sigma \frac{d\sigma}{dt} = G_0\varepsilon + G_0\tau_\varepsilon \frac{d\varepsilon}{dt} \quad (25)$$

in which σ and ε represent stress and strain, respectively. τ_ε and τ_σ stand for strain relaxation time and stress relaxation time, respectively, while G_0 denotes the relaxation modulus as $t \rightarrow \infty$.

The fractional derivative acts as an extension of the integer-order derivative and can capture the historical dependency of physical phenomena. Replacing the integer-order time

derivative with a fractional derivative usually introduces a time-nonlocal characteristic. In this study, the constitutive equation of the Zener model (i.e., Equation (25)) is modified by incorporating the fractional derivative, which is expressed as

$$\sigma + (\tau_\sigma)^\gamma {}^C_0 D_t^\gamma \sigma = G_0 \varepsilon + G_0 (\tau_\varepsilon)^\gamma {}^C_0 D_t^\gamma \varepsilon, 0 < \gamma \leq 1 \quad (26)$$

in which ${}^C_0 D_t^\gamma$ represents the time-domain Caputo fractional derivative operator, formulated as [35]

$${}^C_0 D_t^\gamma f(t) = {}_0 I_t^{1-\gamma} f^{(1)}(t) = \begin{cases} \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-\tau)^{-\gamma} f^{(1)}(\tau) d\tau, & 0 < \gamma < 1 \\ f^{(1)}(t), & \gamma = 1 \end{cases} \quad (27)$$

where $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ ($\text{Re}(z) > 0$) denotes the Gamma function; γ indicates the fractional derivative order; $f^{(1)}(t)$ stands for the first-order integer derivative; and ${}_0 I_t^\gamma$ represents the γ -order Riemann–Liouville fractional integral, expressed as [35]

$${}_0 I_t^\gamma f(t) = \begin{cases} \frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} f(\tau) d\tau, & 0 < \gamma < 1 \\ f(t), & \gamma = 1 \end{cases} \quad (28)$$

Under thermodynamic constraints, Bagley and Torvik proved that the fractional orders associated with stress and strain in Equation (26) must be identical. Notably, this model degenerates into the classical Zener model when $\gamma = 1$.

The Fourier transform of a continuous function $f(t)$ integrable over the entire real line is defined as [36]

$$\mathcal{F}\{f(t)\} = \bar{f}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega\tau} f(\tau) d\tau \quad (29)$$

For integer-order derivatives and convolution integrals, the subsequent relations are valid

$$\mathcal{F}\{f^{(n)}(t)\} = (i\omega)^n \mathcal{F}\{f(t)\} \quad (30)$$

$$\mathcal{F}\{f(t) * g(t)\} = \mathcal{F}\{f(t)\} \mathcal{F}\{g(t)\} \quad (31)$$

where $f^{(n)}(t)$ represents the n -th order integer derivative, and the symbol $*$ denotes the convolution operation. The Caputo fractional derivative, i.e., Equation (27), can be reformulated as

$${}^C_0 D_t^\gamma f(t) = h_+(t) * f^{(1)}(t) \quad (32)$$

where

$$h_+(t) = \begin{cases} \frac{t^{-\gamma}}{\Gamma(1-\gamma)}, & t > 0 \\ 0, & t \leq 0 \end{cases} \quad (33)$$

By setting $x = i\omega t$ and utilizing the definition of the Gamma function, the Fourier transform of $h_+(t)$ is given by

$$\begin{aligned} \mathcal{F}\{h_+(t)\} &= \int_{-\infty}^{+\infty} h_+(t) e^{-i\omega t} dt = \int_0^{+\infty} \frac{t^{-\gamma}}{\Gamma(1-\gamma)} e^{-i\omega t} dt \\ &= \frac{1}{\Gamma(1-\gamma)} \int_0^{+\infty} \left(\frac{x}{i\omega}\right)^{-\gamma} e^{-x} d\left(\frac{x}{i\omega}\right) = \frac{(i\omega)^{\gamma-1}}{\Gamma(1-\gamma)} \int_0^{+\infty} x^{-\gamma} e^{-x} dx = (i\omega)^{\gamma-1} \end{aligned} \quad (34)$$

If $f(t)$ satisfies the condition of the initial value being zero, $f(t)$ tends to equal zero as $t \rightarrow \infty$, and both $f(t)$ and its relevant derivatives possess sufficient smoothness over the

defined interval and meet the integrability requirements. Employing expressions (30), (31), (32) and (34), the Fourier transform of the γ -order Caputo fractional derivative is

$$\mathcal{F}\left\{{}_0^C D_t^\gamma f(t)\right\} = \mathcal{F}\{h_+(t)\}\mathcal{F}\{f^{(1)}(t)\} = (i\omega)^{\gamma-1}(i\omega)\mathcal{F}\{f(t)\} = (i\omega)^\gamma \mathcal{F}\{f(t)\} \quad (35)$$

By applying the Fourier transform to Equation (26), we obtain

$$\bar{\sigma}(\omega) + (i\omega\tau_\sigma)^\gamma \bar{\sigma}(\omega) = G_0 \bar{\varepsilon}(\omega) + G_0 (i\omega\tau_\varepsilon)^\gamma \bar{\varepsilon}(\omega) \quad (36)$$

where $\bar{\sigma}(\omega) = \mathcal{F}\{\sigma(t)\}$ and $\bar{\varepsilon}(\omega) = \mathcal{F}\{\varepsilon(t)\}$.

The complex modulus $\bar{G}(\omega)$ of the fractional Zener model is expressible as

$$\bar{G}(\omega) = \frac{\bar{\sigma}(\omega)}{\bar{\varepsilon}(\omega)} = G_0 \frac{1 + (i\omega\tau_\varepsilon)^\gamma}{1 + (i\omega\tau_\sigma)^\gamma} \quad (37)$$

in which i denotes the imaginary unit and ω represents the angular frequency.

Analogously, the complex moduli $\bar{\lambda}(\omega)$, $\bar{K}_s(\omega)$ and $\bar{\mu}(\omega)$ can be presented as

$$\bar{\lambda}(\omega) = \lambda_0 \frac{1 + (i\omega\tau_{\varepsilon 1})^{\gamma_1}}{1 + (i\omega\tau_{\sigma 1})^{\gamma_1}} = \lambda_0 \Omega_1(\omega), \quad 0 \leq \gamma_1 \leq 1 \quad (38)$$

$$\bar{K}_s(\omega) = K_{s0} \frac{1 + (i\omega\tau_{\varepsilon 2})^{\gamma_2}}{1 + (i\omega\tau_{\sigma 2})^{\gamma_2}} = K_{s0} \Omega_2(\omega), \quad 0 \leq \gamma_2 \leq 1 \quad (39)$$

$$\bar{\mu}(\omega) = \mu_0 \frac{1 + (i\omega\tau_{\varepsilon 3})^{\gamma_3}}{1 + (i\omega\tau_{\sigma 3})^{\gamma_3}} = \mu_0 \Omega_3(\omega), \quad 0 \leq \gamma_3 \leq 1 \quad (40)$$

where λ_0 (bulk strain) and μ_0 (shear strain) are the relaxation Lamé constants and K_{s0} is the relaxation compressibility modulus for $t \rightarrow \infty$. Here, γ_1 , γ_2 and γ_3 denote the fractional derivative orders for the complex moduli of λ , K_s and μ , respectively. $\tau_{\varepsilon 1}/\tau_{\sigma 1}$, $\tau_{\varepsilon 2}/\tau_{\sigma 2}$ and $\tau_{\varepsilon 3}/\tau_{\sigma 3}$ represent the relaxation times of bulk, compressibility and shear strain/stress, respectively.

2.4. Thermodynamic Governing Equations

Using the generalized non-Fourier heat conduction law put forward by Tzou [37,38].

$$\mathbf{q}(\mathbf{x}, t + \tau_q) = -K \nabla T(\mathbf{x}, t + \tau_\theta) \quad (41)$$

where $\mathbf{q}$ denotes the heat flux vector, K represents the thermal conductivity, τ_q stands for the phase-lag of the heat flux, and τ_θ refers to the phase-lag of the temperature gradient, with the constraint that $0 \leq \tau_\theta < \tau_q$.

The heat conduction equation for partially saturated poro thermo-elastic media takes the following form:

$$K \left(1 + \tau_\theta \frac{\partial}{\partial t}\right) \nabla^2 T = \left(1 + \tau_q \frac{\partial}{\partial t}\right) \left(\hat{m} \frac{\partial T}{\partial t} + \lambda' T_0 \frac{\partial \varepsilon_v}{\partial t} - \beta_T T_0 \frac{\partial p}{\partial t}\right) \quad (42)$$

where the weighted specific heat $\hat{m}$ for partially saturated soil is given by

$$\hat{m} = (1 - n^s) \rho^s c_s + n^s S^l \rho^l c_l + n^s S^g \rho^g c_g \quad (43)$$

in which c_b ($b = s, l, g$) denotes the specific heat capacity of phase b .

2.5. Wave Equation of Poro-Thermo-Viscoelastic Model

Let $\mathbf{u}^f = n^s S^l (\mathbf{u}^l - \mathbf{u}^s)$ and $\mathbf{u}^a = n^s S^g (\mathbf{u}^g - \mathbf{u}^s)$. By combining Equations (11), (15), (24) and (42), and incorporating the viscoelastic properties of the solid frame, the governing equations are modified by introducing frequency-dependent complex moduli. The wave equations for partially saturated porothermoviscoelastic media are finally derived as follows:

$$\mu(\omega) \nabla^2 \mathbf{u}^s + \left(\bar{\lambda}^*(\omega) + \bar{\mu}(\omega) \right) \nabla (\nabla \cdot \mathbf{u}^s) + \bar{D}_1(\omega) \nabla (\nabla \cdot \mathbf{u}^f) + \bar{D}_2(\omega) \nabla (\nabla \cdot \mathbf{u}^a) + \bar{D}_3(\omega) \nabla T = \rho \ddot{\mathbf{u}}^s + \rho^l \ddot{\mathbf{u}}^f + \rho^g \ddot{\mathbf{u}}^a \quad (44a)$$

$$B_1 \nabla (\nabla \cdot \mathbf{u}^s) + B_2 \nabla (\nabla \cdot \mathbf{u}^f) + B_3 \nabla (\nabla \cdot \mathbf{u}^a) + B_4 \nabla T = \rho^l \ddot{\mathbf{u}}^s + \vartheta^l \ddot{\mathbf{u}}^f + v^l \dot{\mathbf{u}}^f \quad (44b)$$

$$B_5 \nabla (\nabla \cdot \mathbf{u}^s) + B_6 \nabla (\nabla \cdot \mathbf{u}^f) + B_7 \nabla (\nabla \cdot \mathbf{u}^a) + B_8 \nabla T = \rho^g \ddot{\mathbf{u}}^s + \vartheta^g \ddot{\mathbf{u}}^a + v^g \dot{\mathbf{u}}^a \quad (44c)$$

$$\bar{C}_1(\omega) \nabla \cdot (\dot{\mathbf{u}}^s + \tau_q \ddot{\mathbf{u}}^s) + C_2 \nabla \cdot (\dot{\mathbf{u}}^f + \tau_q \ddot{\mathbf{u}}^f) + C_3 \nabla \cdot (\dot{\mathbf{u}}^a + \tau_q \ddot{\mathbf{u}}^a) + C_4 (\dot{T} + \tau_q \ddot{T}) = K (\nabla^2 T + \tau_\theta \nabla^2 \dot{T}) \quad (44d)$$

where

$$\begin{aligned} \bar{\lambda}^*(\omega) &= \bar{\lambda}(\omega) + \bar{\alpha}(\omega) \alpha_{eff} B_1 + \bar{\alpha}(\omega) (1 - \alpha_{eff}) B_5, \rho = \bar{\rho}^s + \bar{\rho}^l + \bar{\rho}^g \\ \bar{D}_1(\omega) &= \bar{\alpha}(\omega) \alpha_{eff} B_2 + \bar{\alpha}(\omega) (1 - \alpha_{eff}) B_6, \bar{D}_2(\omega) = \bar{\alpha}(\omega) \alpha_{eff} B_3 + \bar{\alpha}(\omega) (1 - \alpha_{eff}) B_7 \\ \bar{D}_3(\omega) &= \bar{\alpha}(\omega) \alpha_{eff} B_4 + \bar{\alpha}(\omega) (1 - \alpha_{eff}) B_8 - \bar{\lambda}'(\omega), \\ \bar{C}_1(\omega) &= \bar{\lambda}'(\omega) T_0 + \beta_T T_0 [B_1 \alpha_{eff} + B_5 (1 - \alpha_{eff})], C_2 = \beta_T T_0 [B_2 \alpha_{eff} + B_6 (1 - \alpha_{eff})] \\ C_3 &= \beta_T T_0 [B_3 \alpha_{eff} + B_7 (1 - \alpha_{eff})], C_4 = \hat{m} + \beta_T T_0 [B_4 \alpha_{eff} + B_8 (1 - \alpha_{eff})] \\ B_1 &= \frac{A_{22}}{A_{22} A_{11} - A_{12} A_{21}}, B_2 = \frac{A_{22} A_{14} - A_{12} A_{24}}{A_{14} (A_{22} A_{11} - A_{12} A_{21})}, B_3 = \frac{A_{22} A_{15} - A_{12} A_{25}}{A_{15} (A_{22} A_{11} - A_{12} A_{21})} \\ B_4 &= \frac{A_{22} A_{16} - A_{12} A_{26}}{A_{22} A_{11} - A_{12} A_{21}}, B_5 = \frac{A_{21}}{A_{21} A_{12} - A_{11} A_{22}}, B_6 = \frac{A_{21} A_{14} - A_{11} A_{24}}{A_{14} (A_{21} A_{12} - A_{11} A_{22})} \\ B_7 &= \frac{A_{21} A_{15} - A_{11} A_{25}}{A_{15} (A_{21} A_{12} - A_{11} A_{22})}, B_8 = \frac{A_{21} A_{16} - A_{11} A_{26}}{A_{21} A_{12} - A_{11} A_{22}}, \vartheta^l = \frac{\rho^l}{n^s S^l}, \vartheta^g = \frac{\rho^g}{n^s (1 - S^l)} \\ v^l &= \frac{\mu_l}{k_r^l k}, v^g = \frac{\mu_g}{k_r^g k} \end{aligned}$$

in which $\bar{\alpha}(\omega)$ and $\bar{\lambda}'(\omega)$ denote the value of α and λ' when complex moduli $\bar{\lambda}(\omega)$, $\bar{K}_s(\omega)$ and $\bar{\mu}(\omega)$ are considered.

3. Rayleigh Wave

Using the Cartesian coordinate system in Figure 1, Rayleigh waves travel in the $x - z$ plane along the boundary of a partially saturated poro-thermo-viscoelastic half-space. The positive z -axis points into the interior of poro-thermo-viscoelastic medium.

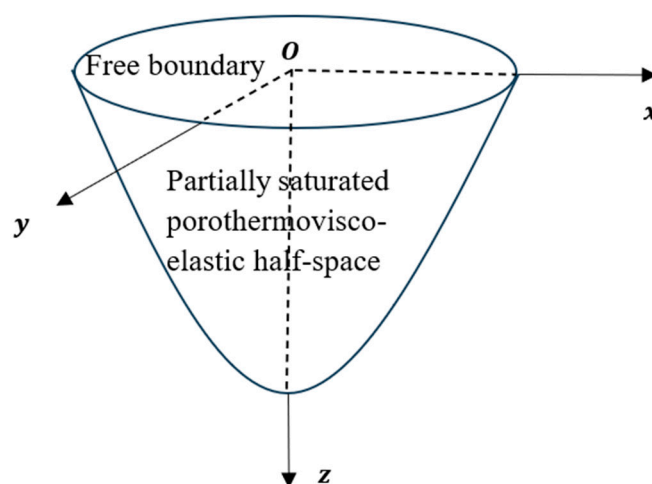


Figure 1. The semi-infinite system of partially saturated poro-thermo-viscoelastic media.

3.1. Field Equations

Based on the Helmholtz vector decomposition, the vector field decomposes into the gradient of scalar fields (ψ_s, ψ_f and ψ_a) and the curl of vector fields ($\mathbf{H}_s, \mathbf{H}_f$ and $\mathbf{H}_a$):

$$\mathbf{u}^r = \nabla \psi_r + \nabla \times \mathbf{H}_r, \quad r = s, f, a \quad (45)$$

where $\mathbf{H}_r = H_r \mathbf{i}_2$.

Substituting Equation (45) into Equation (44) and applying divergence and curl operators to both sides of the resultant equations, we derive the subsequent equations:

$$\begin{aligned} & \left(\bar{\lambda}^*(\omega) + 2\bar{\mu}(\omega) \right) \nabla^2 \psi_s + \bar{D}_1(\omega) \nabla^2 \psi_f + \bar{D}_2(\omega) \nabla^2 \psi_a + \bar{D}_3(\omega) T \\ & = \rho \ddot{\psi}_s + \rho^l \ddot{\psi}_f + \rho^g \ddot{\psi}_a \end{aligned} \quad (46a)$$

$$B_1 \nabla^2 \psi_s + B_2 \nabla^2 \psi_f + B_3 \nabla^2 \psi_a + B_4 T = \rho^l \ddot{\psi}_s + \rho^l \ddot{\psi}_f + v^l \dot{\psi}_f \quad (46b)$$

$$B_5 \nabla^2 \psi_s + B_6 \nabla^2 \psi_f + B_7 \nabla^2 \psi_a + B_8 T = \rho^g \ddot{\psi}_s + \rho^g \ddot{\psi}_a + v^g \dot{\psi}_a \quad (46c)$$

$$\begin{aligned} & \bar{C}_1(\omega) \nabla^2 (\dot{\psi}_s + \tau_q \ddot{\psi}_s) + C_2 \nabla^2 (\dot{\psi}_f + \tau_q \ddot{\psi}_f) + C_3 \nabla^2 (\dot{\psi}_a + \tau_q \ddot{\psi}_a) + C_4 (\dot{T} + \tau_q \ddot{T}) \\ & = K \nabla^2 (T + \tau_\theta \dot{T}) \end{aligned} \quad (46d)$$

$$\rho \ddot{H}_s + \rho^l \ddot{H}_f + \rho^g \ddot{H}_a = \bar{\mu}(\omega) \nabla^2 H_s \quad (46e)$$

$$\rho^l \ddot{H}_s + \rho^l \ddot{H}_f + v^l \dot{H}_f = 0 \quad (46f)$$

$$\rho^g \ddot{H}_s + \rho^g \ddot{H}_a + v^g \dot{H}_a = 0 \quad (46g)$$

Suppose the potential solutions for plane waves take the form:

$$\psi_r = A_r \exp[-\eta_1 z + i(k_R x - \omega t)] \quad (47a)$$

$$H_r = B_r \exp[-\eta_2 z + i(k_R x - \omega t)] \quad (47b)$$

$$T = A_T \exp[-\eta_1 z + i(k_R x - \omega t)] \quad (47c)$$

where k_R denotes the complex wave number of a Rayleigh wave, and $\omega = 2\pi f$ is the angular frequency (f stands for frequency), A_r ($r = s, f, a$), A_T and B_r are amplitudes of corresponding potentials. Decay exponents are defined as $\eta_1 = \sqrt{k_R^2 - k_p^2}$ and $\eta_2 = \sqrt{k_R^2 - k_s^2}$,

where $\eta > 0$ to ensure decay into the half-space, and with k_p (compressional wave, including the thermal wave) and k_s (shear wave) as their complex wave numbers.

Substituting Equation (47) into Equation (46) and carrying out algebraic manipulations to yield the dispersion equations for body waves in partially saturated poro-thermo-viscoelastic media gives

$$\begin{pmatrix} \bar{b}_{11}(\omega) & \bar{b}_{12}(\omega) & \bar{b}_{13}(\omega) & \bar{b}_{14}(\omega) \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ \bar{b}_{41}(\omega) & b_{42} & b_{43} & b_{44} \end{pmatrix} \begin{pmatrix} A_s \\ A_f \\ A_a \\ A_T \end{pmatrix} = 0 \quad (48a)$$

$$\begin{pmatrix} \bar{c}_{11}(\omega) & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} B_s \\ B_f \\ B_a \end{pmatrix} = 0 \quad (48b)$$

where

$$\begin{aligned} \bar{b}_{11}(\omega) &= \rho\omega^2 - \left(\bar{\lambda}^*(\omega) + 2\bar{\mu}(\omega)\right)k_p^2, \bar{b}_{12}(\omega) = \rho^l\omega^2 - \bar{D}_1(\omega)k_p^2 \\ \bar{b}_{13}(\omega) &= \rho^g\omega^2 - \bar{D}_2(\omega)k_p^2, \bar{b}_{14}(\omega) = \bar{D}_3(\omega), b_{21} = \rho^l\omega^2 - B_1k_p^2 \\ b_{22} &= \vartheta^l\omega^2 + v^l i\omega - B_2k_p^2, b_{23} = -B_3k_p^2, b_{24} = B_4, b_{31} = \rho^g\omega^2 - B_5k_p^2, b_{32} = -B_6k_p^2 \\ b_{33} &= \vartheta^g\omega^2 + v^g i\omega - B_7k_p^2, b_{34} = B_8, \bar{b}_{41}(\omega) = \bar{C}_1(\omega)(i\omega + \tau_q\omega^2)k_p^2 \\ b_{42} &= C_2(i\omega + \tau_q\omega^2)k_p^2, b_{43} = C_3(i\omega + \tau_q\omega^2)k_p^2, b_{44} = K(1 - \tau_\theta i\omega)k_p^2 - C_4(i\omega + \tau_q\omega^2) \\ \bar{c}_{11}(\omega) &= \rho\omega^2 - \bar{\mu}(\omega)k_s^2, c_{12} = \rho^l\omega^2, c_{13} = \rho^g\omega^2, c_{21} = \rho^l\omega^2, c_{22} = \vartheta^l\omega^2 + v^l i\omega \\ c_{31} &= \rho^g\omega^2, c_{33} = \vartheta^g\omega^2 + v^g i\omega, c_{23} = c_{32} = 0 \end{aligned}$$

For Equation (48a,b) to have non-trivial solutions, the determinant of their coefficient matrices must vanish, leading to the dispersion equations. At a given frequency ω , 8 and 2 complex wavenumber roots (in the form $k_p = \text{Re}(k_p) + i\text{Im}(k_p)$) can be solved for Equations (48a) and (48b) respectively. Here, the real part corresponds to the propagation wavenumber of body waves, and the imaginary part reflects attenuation characteristics—since wave amplitude decreases along the propagation direction, the imaginary part must be greater than zero. Thus, Equation (48a) yields only four physically meaningful roots: the first three are complex wavenumbers of compressional waves (denoted as P1, P2, and P3 in descending order of wave speed) and the fourth is that of the thermal wave (denoted as k_{p4}). Equation (48b) has only one physically meaningful root, i.e., the complex wavenumber of the shear S-wave. Correspondingly, the frequency-dependent phase velocities of compressional waves (including the thermal wave) and the shear wave are defined as $v_{pn} = \frac{\omega}{\text{Re}(k_{pn})}$ ($n = 1, 2, 3, 4$) and $v_s = \omega / \text{Re}(k_s)$ respectively, while their attenuation coefficients are $\alpha_{pn} = \text{Im}(k_{pn})$ ($n = 1, 2, 3, 4$) and $\alpha_s = \text{Im}(k_s)$ respectively.

Rayleigh waves arise from the superposition of compressional and shear waves at the free surface of a partially saturated poro-thermo-viscoelastic half-space [39,40]. However, thermal waves also participate in forming the Rayleigh wave in such media. Thus, Equation (47a–c) can be rewritten as

$$\psi_s = \sum_{n=1}^4 \left[A_s^{(n)} \exp\left(-\eta_1^{(n)} z\right) \right] \exp[i(k_R x - \omega t)] \quad (49a)$$

$$\psi_f = \sum_{n=1}^4 \left[A_f^{(n)} \exp\left(-\eta_1^{(n)} z\right) \right] \exp[i(k_R x - \omega t)] \quad (49b)$$

$$\psi_a = \sum_{n=1}^4 \left[A_a^{(n)} \exp(-\eta_1^{(n)} z) \right] \exp[i(k_R x - \omega t)] \quad (49c)$$

$$T = \sum_{n=1}^4 \left[A_T^{(n)} \exp(-\eta_1^{(n)} z) \right] \exp[i(k_R x - \omega t)] \quad (49d)$$

$$H_s = B_s \exp(-\eta_2 z) \exp[i(k_R x - \omega t)] \quad (49e)$$

$$H_f = B_f \exp(-\eta_2 z) \exp[i(k_R x - \omega t)] \quad (49f)$$

$$H_a = B_a \exp(-\eta_2 z) \exp[i(k_R x - \omega t)] \quad (49g)$$

where $\eta_1^{(n)} = \sqrt{k_R^2 - k_{pn}^2}$ and $n = 1, 2, 3, 4$ correspond to the p_1 , p_2 , p_3 and p_4 waves, respectively.

From Equation (48a,b), the amplitude ratios of potential functions for each wave are derived as

$$\bar{\delta}_{fp}^{(n)}(\omega) = \frac{A_f^{(n)}}{A_s^{(n)}} = \frac{\bar{d}_{11}^{(n)}(\omega)\bar{d}_{15}^{(n)}(\omega) - \bar{d}_{12}^{(n)}(\omega)\bar{d}_{16}^{(n)}(\omega)}{\bar{d}_{13}^{(n)}(\omega)\bar{d}_{16}^{(n)}(\omega) - \bar{d}_{14}^{(n)}(\omega)\bar{d}_{15}^{(n)}(\omega)} \quad (50a)$$

$$\bar{\delta}_{ap}^{(n)}(\omega) = \frac{A_a^{(n)}}{A_s^{(n)}} = \frac{\bar{d}_{21}^{(n)}(\omega)\bar{d}_{25}^{(n)}(\omega) - \bar{d}_{22}^{(n)}(\omega)\bar{d}_{26}^{(n)}(\omega)}{\bar{d}_{23}^{(n)}(\omega)\bar{d}_{26}^{(n)}(\omega) - \bar{d}_{24}^{(n)}(\omega)\bar{d}_{25}^{(n)}(\omega)} \quad (50b)$$

$$\bar{\delta}_{Tp}^{(n)}(\omega) = \frac{A_T^{(n)}}{A_s^{(n)}} = \frac{\bar{d}_{31}^{(n)}(\omega)\bar{d}_{35}^{(n)}(\omega) - \bar{d}_{32}^{(n)}(\omega)\bar{d}_{36}^{(n)}(\omega)}{\bar{d}_{33}^{(n)}(\omega)\bar{d}_{36}^{(n)}(\omega) - \bar{d}_{34}^{(n)}(\omega)\bar{d}_{35}^{(n)}(\omega)} \quad (50c)$$

$$\delta_{fs} = \frac{B_f}{B_s} = -\frac{c_{12}}{c_{22}} \quad (50d)$$

$$\bar{\delta}_{as}(\omega) = \frac{B_a}{B_s} = \frac{c_{21}c_{12} - \bar{c}_{11}(\omega)c_{22}}{c_{13}c_{22}} \quad (50e)$$

where $\bar{d}_{11}^{(n)} \sim \bar{d}_{36}^{(n)}$ ($n = 1, 2, 3, 4$) are expressed as

$$\bar{d}_{11}^{(n)}(\omega) = b_{31}^{(n)}b_{43}^{(n)} - \bar{b}_{41}^{(n)}(\omega)b_{33}^{(n)}, \bar{d}_{12}^{(n)}(\omega) = \bar{b}_{11}^{(n)}(\omega)b_{23}^{(n)} - b_{21}^{(n)}\bar{b}_{13}^{(n)}(\omega)$$

$$\bar{d}_{13}^{(n)}(\omega) = \bar{b}_{12}^{(n)}(\omega)b_{23}^{(n)} - b_{22}^{(n)}\bar{b}_{13}^{(n)}(\omega), \bar{d}_{14}^{(n)} = b_{32}^{(n)}b_{43}^{(n)} - b_{42}^{(n)}b_{33}^{(n)}$$

$$\bar{d}_{15}^{(n)}(\omega) = \bar{b}_{14}^{(n)}(\omega)b_{23}^{(n)} - b_{24}^{(n)}\bar{b}_{13}^{(n)}(\omega), \bar{d}_{16}^{(n)} = b_{34}^{(n)}b_{43}^{(n)} - b_{44}^{(n)}b_{33}^{(n)}$$

$$\bar{d}_{21}^{(n)}(\omega) = b_{31}^{(n)}b_{42}^{(n)} - \bar{b}_{41}^{(n)}(\omega)b_{32}^{(n)}, \bar{d}_{22}^{(n)}(\omega) = \bar{b}_{11}^{(n)}(\omega)b_{22}^{(n)} - b_{21}^{(n)}\bar{b}_{12}^{(n)}(\omega)$$

$$\bar{d}_{23}^{(n)}(\omega) = \bar{b}_{13}^{(n)}(\omega)b_{22}^{(n)} - b_{23}^{(n)}\bar{b}_{12}^{(n)}(\omega), \bar{d}_{24}^{(n)} = b_{33}^{(n)}b_{42}^{(n)} - b_{43}^{(n)}b_{32}^{(n)}$$

$$\bar{d}_{25}^{(n)}(\omega) = \bar{b}_{14}^{(n)}(\omega)b_{22}^{(n)} - b_{24}^{(n)}\bar{b}_{12}^{(n)}(\omega), \bar{d}_{26}^{(n)} = b_{34}^{(n)}b_{42}^{(n)} - b_{44}^{(n)}b_{32}^{(n)}, \bar{d}_{31}^{(n)}(\omega) = \bar{d}_{11}^{(n)}(\omega)$$

$$\bar{d}_{32}^{(n)}(\omega) = \bar{d}_{12}^{(n)}(\omega), \bar{d}_{33}^{(n)}(\omega) = \bar{b}_{14}^{(n)}(\omega)b_{23}^{(n)} - b_{24}^{(n)}\bar{b}_{13}^{(n)}(\omega), \bar{d}_{34}^{(n)} = b_{34}^{(n)}b_{43}^{(n)} - b_{44}^{(n)}b_{33}^{(n)}$$

$$\bar{d}_{35}^{(n)}(\omega) = \bar{b}_{12}^{(n)}(\omega)b_{23}^{(n)} - b_{22}^{(n)}\bar{b}_{13}^{(n)}(\omega), \bar{d}_{36}^{(n)} = b_{32}^{(n)}b_{43}^{(n)} - b_{42}^{(n)}b_{33}^{(n)}$$

3.2. Boundary Conditions

For the boundary of the poro-thermo-viscoelastic medium at $z = 0$ it is assumed to be traction-free, liquid-permeable, gas-permeable, and thermally insulated, resulting in a zero normal derivative of temperature at this boundary. Accordingly, the boundary conditions at $z = 0$ are expressed as

$$\sigma_{zz} = 0, \sigma_{xz} = 0, p_w = 0, p_g = 0, \frac{\partial T}{\partial z} = 0 \quad (51)$$

where

$$\sigma_{zz} = \bar{\lambda}(\omega) \left(\frac{\partial^2 \psi_s}{\partial x^2} + \frac{\partial^2 \psi_s}{\partial z^2} \right) + 2\bar{\mu}(\omega) \left(\frac{\partial^2 \psi_s}{\partial z^2} + \frac{\partial^2 H_s}{\partial x \partial z} \right) - \bar{\lambda}'(\omega) T \quad (52a)$$

$$\sigma_{xz} = \bar{\mu}(\omega) \left(2 \frac{\partial^2 \psi_s}{\partial x \partial z} + \frac{\partial^2 H_s}{\partial x^2} - \frac{\partial^2 H_s}{\partial z^2} \right) \quad (52b)$$

$$p_w = -B_1 \left(\frac{\partial^2 \psi_s}{\partial x^2} + \frac{\partial^2 \psi_s}{\partial z^2} \right) - B_2 \left(\frac{\partial^2 \psi_f}{\partial x^2} + \frac{\partial^2 \psi_f}{\partial z^2} \right) - B_3 \left(\frac{\partial^2 \psi_a}{\partial x^2} + \frac{\partial^2 \psi_a}{\partial z^2} \right) - B_4 T \quad (52c)$$

$$p_g = -B_5 \left(\frac{\partial^2 \psi_s}{\partial x^2} + \frac{\partial^2 \psi_s}{\partial z^2} \right) - B_6 \left(\frac{\partial^2 \psi_f}{\partial x^2} + \frac{\partial^2 \psi_f}{\partial z^2} \right) - B_7 \left(\frac{\partial^2 \psi_a}{\partial x^2} + \frac{\partial^2 \psi_a}{\partial z^2} \right) - B_8 T \quad (52d)$$

By utilizing Equations (49), (51) and (52), the subsequent linear relations concerning the amplitudes of potentials can be derived.

$$\begin{pmatrix} \bar{l}_{11}(\omega) & \bar{l}_{12}(\omega) & \bar{l}_{13}(\omega) & \bar{l}_{14}(\omega) & \bar{l}_{15}(\omega) \\ \bar{l}_{21}(\omega) & \bar{l}_{22}(\omega) & \bar{l}_{23}(\omega) & \bar{l}_{24}(\omega) & \bar{l}_{25}(\omega) \\ \bar{l}_{31}(\omega) & \bar{l}_{32}(\omega) & \bar{l}_{33}(\omega) & \bar{l}_{34}(\omega) & l_{35} \\ \bar{l}_{41}(\omega) & \bar{l}_{42}(\omega) & \bar{l}_{43}(\omega) & \bar{l}_{44}(\omega) & l_{45} \\ \bar{l}_{51}(\omega) & \bar{l}_{52}(\omega) & \bar{l}_{53}(\omega) & \bar{l}_{54}(\omega) & l_{55} \end{pmatrix} \begin{pmatrix} A_s^{(1)} \\ A_s^{(2)} \\ A_s^{(3)} \\ A_s^{(4)} \\ B_s \end{pmatrix} = 0 \quad (53)$$

where the coefficient matrix of Equation (53) is designated as L and L is a 5×5 matrix. The elements of matrix L are given as follows:

$$\begin{aligned} \bar{l}_{11}(\omega) &= \bar{\lambda}(\omega) k_{p1}^2 + 2\bar{\mu}(\omega) (k_{p1}^2 - k_R^2) + \bar{\lambda}'(\omega) \bar{\delta}_{Tp}^{(1)}(\omega) \\ \bar{l}_{12}(\omega) &= \bar{\lambda}(\omega) k_{p2}^2 + 2\bar{\mu}(\omega) (k_{p2}^2 - k_R^2) + \bar{\lambda}'(\omega) \bar{\delta}_{Tp}^{(2)}(\omega) \\ \bar{l}_{13}(\omega) &= \bar{\lambda}(\omega) k_{p3}^2 + 2\bar{\mu}(\omega) (k_{p3}^2 - k_R^2) + \bar{\lambda}'(\omega) \bar{\delta}_{Tp}^{(3)}(\omega) \\ \bar{l}_{14}(\omega) &= \bar{\lambda}(\omega) k_{p4}^2 + 2\bar{\mu}(\omega) (k_{p4}^2 - k_R^2) + \bar{\lambda}'(\omega) \bar{\delta}_{Tp}^{(4)}(\omega) \\ \bar{l}_{15}(\omega) &= 2\bar{\mu}(\omega) i k_R \sqrt{k_R^2 - k_s^2}, \bar{l}_{21}(\omega) = 2\bar{\mu}(\omega) i k_R \sqrt{k_R^2 - k_{p1}^2}, \bar{l}_{22}(\omega) = 2\bar{\mu}(\omega) i k_R \sqrt{k_R^2 - k_{p2}^2} \\ \bar{l}_{23}(\omega) &= 2\bar{\mu}(\omega) i k_R \sqrt{k_R^2 - k_{p3}^2}, \bar{l}_{24}(\omega) = 2\bar{\mu}(\omega) i k_R \sqrt{k_R^2 - k_{p4}^2}, \bar{l}_{25}(\omega) = \bar{\mu}(\omega) (2k_R^2 - k_s^2) \\ \bar{l}_{31}(\omega) &= k_{p1}^2 (B_1 + B_2 \bar{\delta}_{fp}^{(1)}(\omega) + B_3 \bar{\delta}_{ap}^{(1)}(\omega)) - B_4 \bar{\delta}_{Tp}^{(1)}(\omega) \\ \bar{l}_{32}(\omega) &= k_{p2}^2 (B_1 + B_2 \bar{\delta}_{fp}^{(2)}(\omega) + B_3 \bar{\delta}_{ap}^{(2)}(\omega)) - B_4 \bar{\delta}_{Tp}^{(2)}(\omega) \\ \bar{l}_{33}(\omega) &= k_{p3}^2 (B_1 + B_2 \bar{\delta}_{fp}^{(3)}(\omega) + B_3 \bar{\delta}_{ap}^{(3)}(\omega)) - B_4 \bar{\delta}_{Tp}^{(3)}(\omega) \\ \bar{l}_{34}(\omega) &= k_{p4}^2 (B_1 + B_2 \bar{\delta}_{fp}^{(4)}(\omega) + B_3 \bar{\delta}_{ap}^{(4)}(\omega)) - B_4 \bar{\delta}_{Tp}^{(4)}(\omega) \\ \bar{l}_{41}(\omega) &= k_{p1}^2 (B_5 + B_6 \bar{\delta}_{fp}^{(1)}(\omega) + B_7 \bar{\delta}_{ap}^{(1)}(\omega)) - B_8 \bar{\delta}_{Tp}^{(1)}(\omega) \\ \bar{l}_{42}(\omega) &= k_{p2}^2 (B_5 + B_6 \bar{\delta}_{fp}^{(2)}(\omega) + B_7 \bar{\delta}_{ap}^{(2)}(\omega)) - B_8 \bar{\delta}_{Tp}^{(2)}(\omega) \\ \bar{l}_{43}(\omega) &= k_{p3}^2 (B_5 + B_6 \bar{\delta}_{fp}^{(3)}(\omega) + B_7 \bar{\delta}_{ap}^{(3)}(\omega)) - B_8 \bar{\delta}_{Tp}^{(3)}(\omega) \\ \bar{l}_{44}(\omega) &= k_{p4}^2 (B_5 + B_6 \bar{\delta}_{fp}^{(4)}(\omega) + B_7 \bar{\delta}_{ap}^{(4)}(\omega)) - B_8 \bar{\delta}_{Tp}^{(4)}(\omega) \\ \bar{l}_{51}(\omega) &= \bar{\delta}_{Tp}^{(1)}(\omega) \sqrt{k_R^2 - k_{p1}^2}, \bar{l}_{52}(\omega) = \bar{\delta}_{Tp}^{(2)}(\omega) \sqrt{k_R^2 - k_{p2}^2}, \bar{l}_{53}(\omega) = \bar{\delta}_{Tp}^{(3)}(\omega) \sqrt{k_R^2 - k_{p3}^2} \end{aligned}$$

$$\bar{l}_{54}(\omega) = \bar{\delta}_{Tp}^{(4)}(\omega) \sqrt{k_R^2 - k_{p4}^2}, l_{35} = l_{45} = l_{55} = 0$$

The condition for the nontriviality of solutions for Equation (53) is that the determinant of matrix L must be equal to zero. Therefore, the dispersion equation for complex wave number k_R of the Rayleigh wave in the partially saturated poro-thermo-viscoelastic media can be obtained. Furthermore, the velocity and attenuation coefficient of the Rayleigh wave can be represented as $v_R = \omega / \text{Re}(k_R)$ and $\alpha_R = \text{Im}(k_R)$, respectively.

4. Numerical Results and Discussion

To investigate the propagation characteristics of the Rayleigh wave in partially saturated poro-thermo-viscoelastic media, velocity-related parameters of Rayleigh wave are examined via numerical examples. Physical parameter values for the subsequent numerical examples are presented in Tables 1 and 2.

Table 1. Thermophysical properties of the partially saturated porous media [18,41].

S^l	S_{res}^l	S_{sat}^l	m	d	n^s	F (Hz)	k (m ²)
0.6	0.05	1.0	0.5	2	0.4	1000	1×10^{-10}
ρ^s (kg/m ³)		ρ^l (kg/m ³)		ρ^g (kg/m ³)		K (J/s/m/K)	
2650		1000		1.3		0.4	
β_{wp} (Pa ⁻¹)		β_{sT} (K ⁻¹)		β_ψ (K ⁻¹)		β_T (K ⁻¹)	
4.58×10^{-10}		7.80×10^{-6}		2.09×10^{-3}		1.0×10^{-4}	
χ (Pa ⁻¹)		τ_θ (s)		τ_q (s)		T (K)	
0.0001		1.5×10^{-7}		2.0×10^{-7}		293.2	
C_s (J/kg/K)		C_l (J/kg/K)		C_g (J/kg/K)		T_0 (K)	
1000		4180		1900		300	
						β_{wT} (K ⁻¹)	
						2.10×10^{-4}	

Table 2. Viscoelastic parameters [7,33].

λ_o (Pa)	K_{so} (Pa)	μ_o (Pa)	γ_1	γ_2	γ_3
4.4×10^9	3.6×10^{10}	2.8×10^9	0.5	0.5	0.5
τ_{e1} (s)	τ_{e2} (s)	τ_{e3} (s)	$\tau_{\sigma 1}$ (s)	$\tau_{\sigma 2}$ (s)	$\tau_{\sigma 3}$ (s)
5.0×10^{-2}	5.0×10^{-2}	5.0×10^{-2}	1.0×10^{-2}	1.0×10^{-2}	1.0×10^{-2}

Figure 2 illustrates the variation law of the Rayleigh wave phase velocity v_R with respect to the fractional order index γ under different thermal expansion coefficients β_T (specifically, $\beta_T = 0.0001 \text{ K}^{-1}$, 0.0003 K^{-1} and 0.0004 K^{-1}). It can be seen that, regardless of the value of β_T , v_R exhibits a monotonically increasing trend as γ increases. This is because when the fractional order index γ increases, the viscoelastic response of the solid skeleton transitions from “viscosity-dominated” to “elasticity-dominated”; in general, the higher the proportion of elasticity in the material’s viscoelastic response, the greater the wave velocity. Consequently, the equivalent stiffness of the skeleton increases, which in turn promotes the rise of v_R . In addition, for the same γ , the higher the β_T , the greater the v_R , which implies that the thermal expansion coefficient also exerts a positive regulatory effect on the phase velocity of Rayleigh waves.

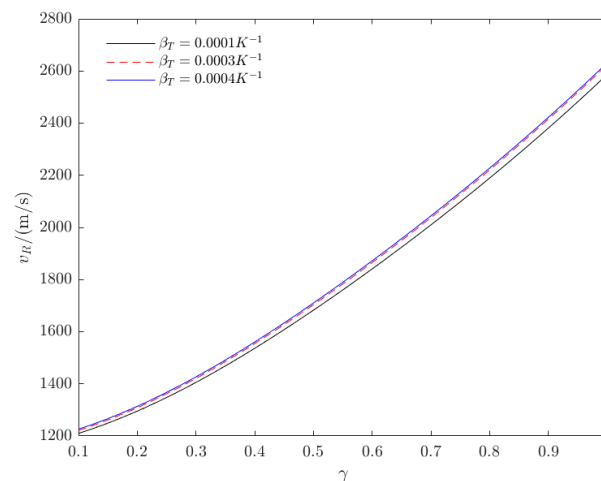


Figure 2. Variation in the phase velocity of the Rayleigh wave against the fractional order index for different thermal expansion coefficients.

Since the Rayleigh wave can be regarded as the superposition of compressional waves and shear waves, in order to study how the increase in the fractional order index leads to the monotonic increase in the Rayleigh wave velocity, let us respectively plot and analyze the variation trends with frequency of four compressional waves (P1, P2, P3 waves and T wave) and shear wave (S wave) under different fractional order indices.

Figure 3 shows the influence of the fractional order index on four compressive waves. It can be seen that the fractional order index has almost no effect on the wave velocities of P2, P3, and thermal waves. While for the P1 wave, as can be seen from Figure 3a, at low frequency (1 Hz), the wave velocities under the four fractional order indices are very close. As the frequency increases, the curves gradually separate. When $\gamma = 0.8$, the wave velocity rises more steeply in the low frequency region. At higher frequencies, each curve approaches a constant value, and the larger the fractional order index, the larger the final constant value. This indicates that the fractional order index γ has a significant influence on the wave velocity of the P1 wave. Larger γ values lead to higher wave velocities at high frequencies and affect the growth rate of wave velocity at low frequencies.

As can be seen from Figure 4, the influence of the fractional order index on the S wave is basically the same as that on the P1 wave. Thus, when $\gamma \rightarrow 1$, the wave velocities of the P1 wave and the S wave increase accordingly, while the wave velocities of the P2, P3, and T waves remain almost unchanged. Therefore, the wave velocity of the Rayleigh wave, which is composed of the superposition of these waves, also increases monotonically.

Figure 5 shows the influence of the thermal expansion coefficients on the phase velocity of the Rayleigh wave. It can be seen that as the thermal expansion coefficients increase, the wave velocity of the Rayleigh wave shows an increasing trend. Meanwhile, Figure 5a also indicates that the phase velocity of the Rayleigh wave increases with the increase in porosity.

Figures 6 and 7 investigate the influence of thermal expansion coefficients on the phase velocities of each component wave of the Rayleigh wave (four compressional waves and one shear wave), respectively. It is found that as the thermal expansion coefficient increases, the phase velocity of the P1 wave shows an increasing trend, while that of the thermal wave (T wave) shows a decreasing trend, and the phase velocities of the other three waves remain almost unchanged. Moreover, the phase velocity of the P1 wave is much higher than that of the T wave. From this, it can be inferred that the P1 wave dominates the superposition contribution to the Rayleigh wave's phase velocity. This mechanism thus provides an

explanation for the phenomenon observed in Figure 5, where the phase velocity of the Rayleigh wave increases with the increase in the thermal expansion coefficient.

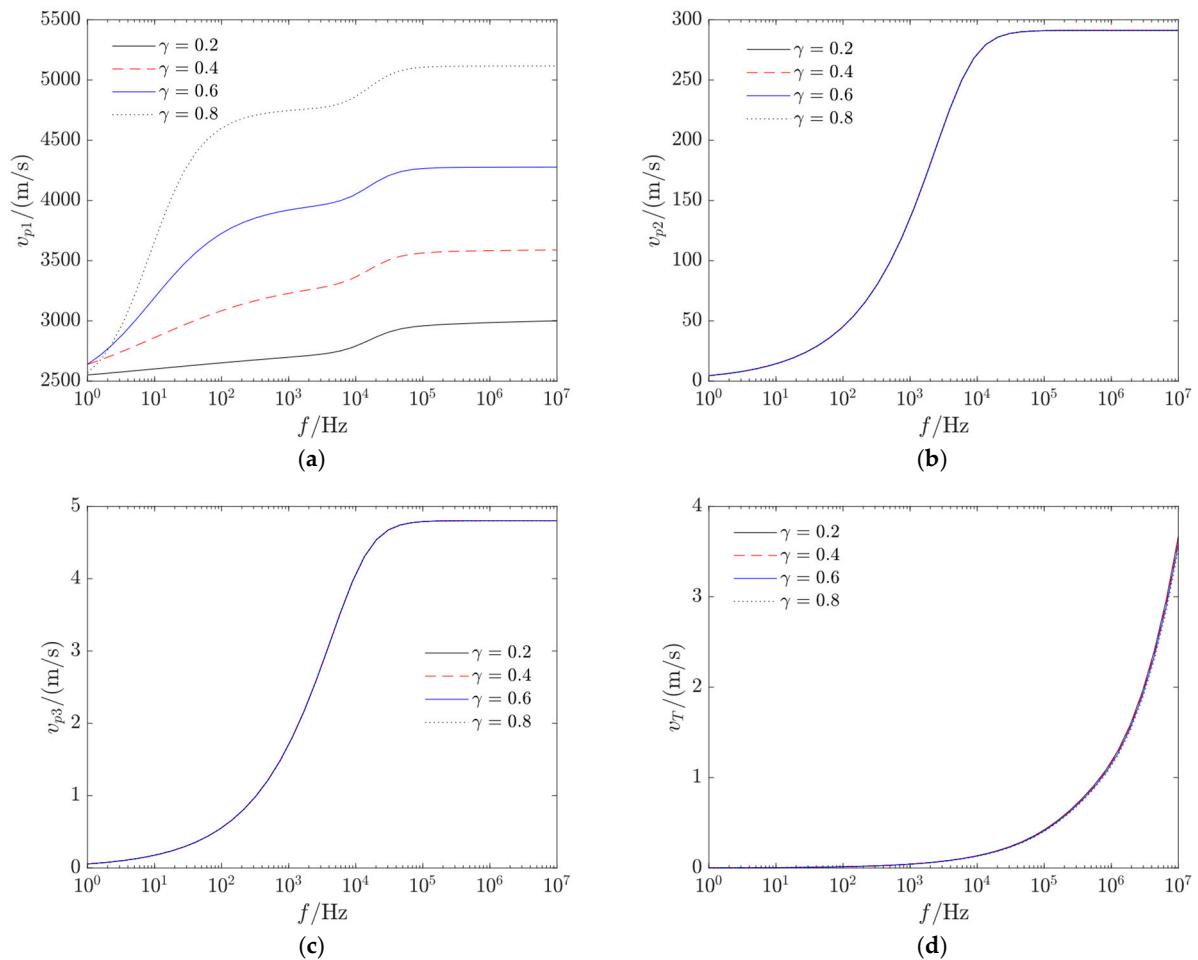


Figure 3. Variations of compressional wave phase velocity with frequency under different fractional order indices: (a) P1 wave; (b) P2 wave; (c) P3 wave; (d) thermal wave.

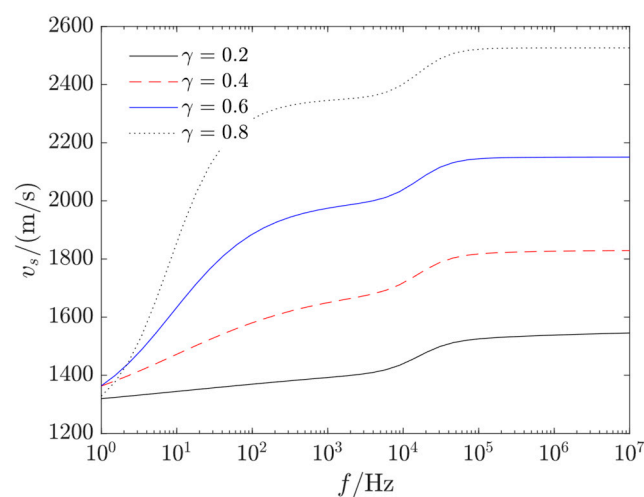


Figure 4. Variation in the phase velocity of the S wave against frequency for different fractional order indices.

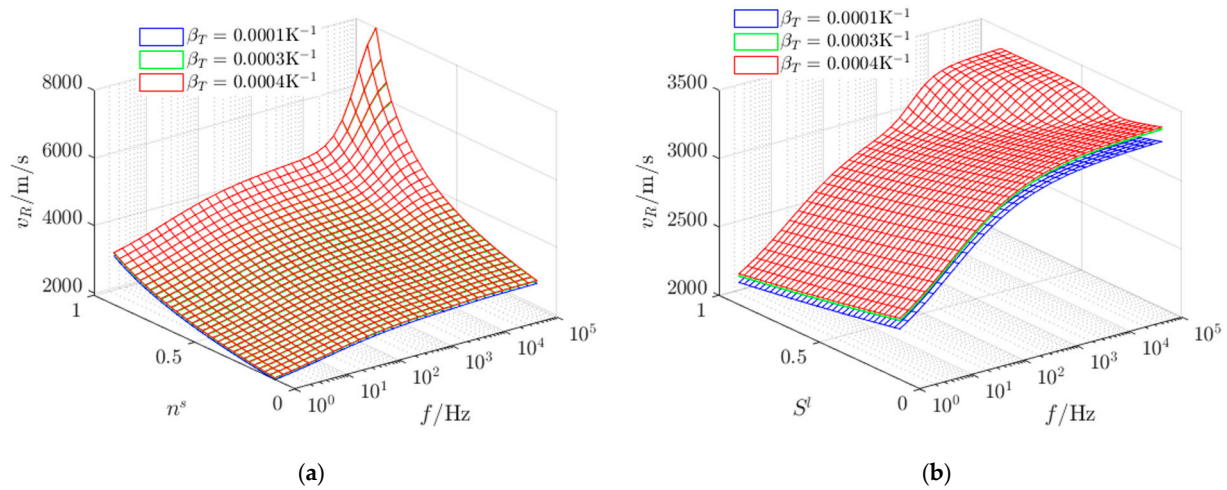


Figure 5. The influence of the thermal expansion coefficients on the phase velocity of the Rayleigh wave: (a) Under the joint regulation of porosity and frequency; (b) Under the joint regulation of saturation and frequency.

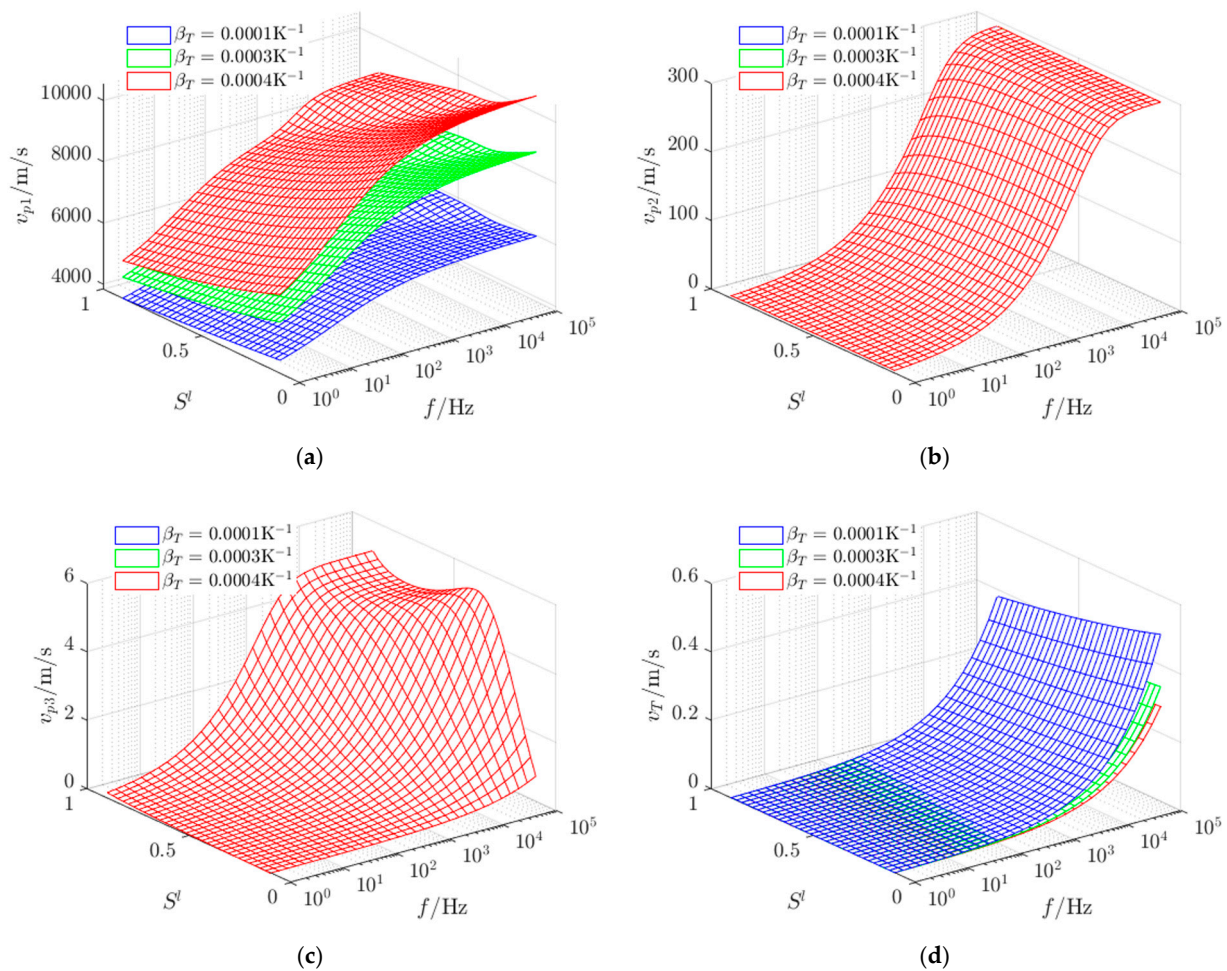


Figure 6. The influence of the thermal expansion coefficients on the phase velocities of four compression waves: (a) P1 wave; (b) P2 wave; (c) P3 wave; (d) thermal wave.

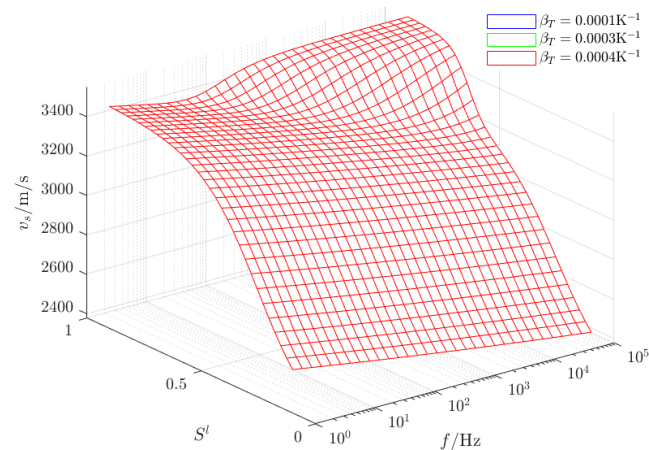


Figure 7. The influence of the thermal expansion coefficients on the phase velocities of the shear wave.

Figure 8 illustrates the variation in the Rayleigh wave's phase velocity with saturation for different frequencies. It shows that the phase velocity decreases linearly as saturation increases; however, at high frequencies ($f = 500$ Hz, $f = 1000$ Hz), the phase velocity begins to rise after reaching a specific saturation threshold. Notably, the higher the frequency, the lower the saturation required for the occurrence of this inflection point.

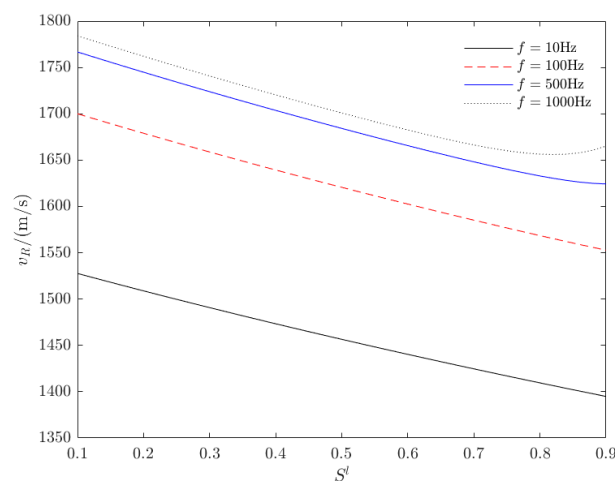


Figure 8. Effect of liquid saturation on Rayleigh wave velocity at different frequencies.

Figure 9 depicts the variational trends of the Rayleigh wave's phase velocity with inherent permeability under varying saturations in partially saturated poro-thermo-viscoelastic media. Saturations are set as 0.01, 0.6, 0.7, and 0.8, with other material properties kept constant as listed in Table 1. As illustrated in Figure 9, the phase velocity stays constant in both high and low permeability regions, yet it rises sharply with permeability growth in the intermediate permeability range. Eventually, the phase velocities converge to a uniform value across different saturation levels. Moreover, higher saturation shifts the phase velocity variation curve toward the low-permeability zone. Calculation results further indicate that at extremely low saturation ($S^l = 0.01$), the phase velocity remains unaffected by inherent permeability. This is because the viscous and inertial coupling effects between gas–liquid phases and solid particles are negligible when the soil approaches dryness.

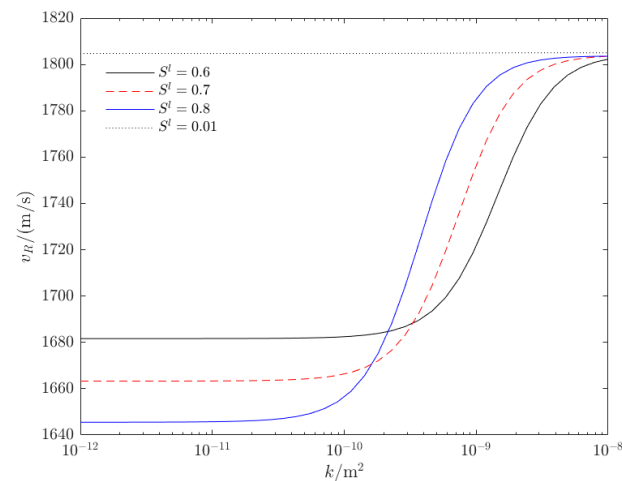


Figure 9. Effect of intrinsic permeability on Rayleigh wave velocity at different liquid saturations.

Figure 10 shows the attenuation characteristics of Rayleigh waves. It can be observed that as the fractional order index increases, the attenuation coefficient of Rayleigh waves exhibits a trend of first increasing and then decreasing, with the attenuation coefficient reaching a peak around the fractional order index $\gamma = 0.4$. As the thermal expansion coefficients increase, the attenuation coefficient of Rayleigh waves shows a decreasing trend and gradually levels off. In the low-frequency range, saturation and porosity have almost no effect on the attenuation coefficient of Rayleigh waves. In the high-frequency range, when saturation reaches a certain threshold, the attenuation coefficient of Rayleigh waves increases with the increase in saturation. In contrast, the attenuation coefficient of Rayleigh waves decreases as porosity increases.

Figures 11 and 12 show the effects of the fractional order index on the attenuation coefficients of the five body waves. It can be observed that the fractional order index has no effect on the attenuation of P2 and P3 waves. As the fractional order index increases, the attenuation coefficient of P4 waves increases; meanwhile, the lower the fractional order index, the larger the peak value of the attenuation coefficient of P1 waves in the mid-to-high frequency range. Moreover, at high frequencies with a relatively small fractional order index, as saturation increases, the attenuation coefficient of P1 waves shows a trend of first decreasing, then increasing, then decreasing again, and then increasing again. The attenuation coefficient of S waves decreases as the fractional order index increases. Additionally, it can be seen that the attenuation coefficients of P2, P3, and P4 waves are much higher than those of P1 and S waves.

Figure 13 compares the wave velocity results obtained from the elastic model, the integer-order Zener model ($\gamma = 1$), and the fractional-order Zener model proposed in this paper. Unlike integer-order derivatives, fractional-order derivatives exhibit a nonlocal property, which makes them more suitable for characterizing the history-dependent viscoelastic behavior of materials. By merely adjusting the fractional exponent of the fractional-order Zener model, the proposed model can describe any transitional state between classical elasticity and Newtonian viscosity. Therefore, the present fractional Zener model represents a more flexible viscoelastic constitutive model.

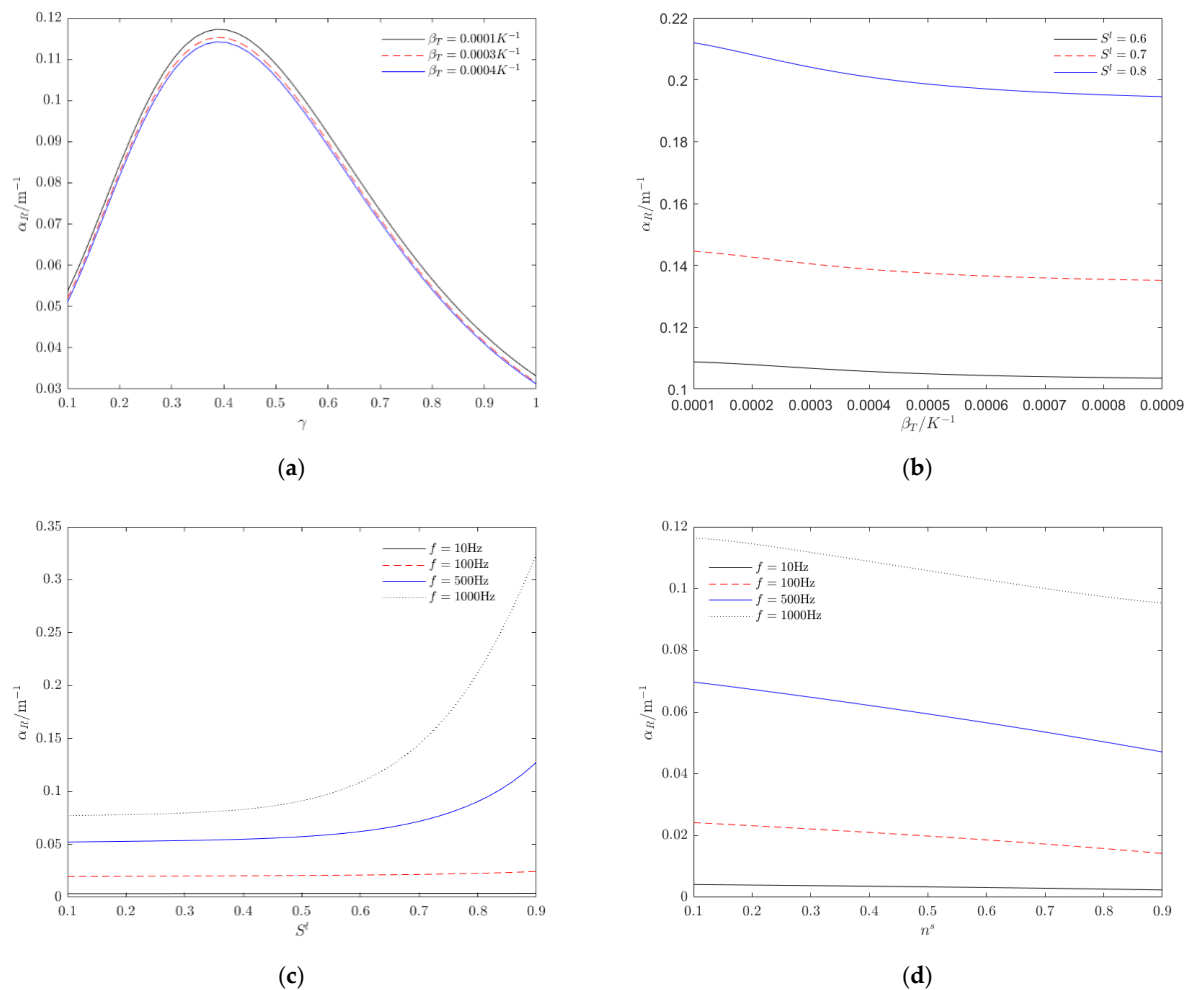


Figure 10. Variations of Rayleigh wave attenuation coefficient under different material parameters: (a) fractional order index; (b) thermal expansion coefficient; (c) saturation; (d) porosity.

Figure 14 illustrates the influences of thermal expansion and viscoelasticity on the attenuation index of Rayleigh waves in different frequency bands. Under small perturbations of the same magnitude applied to both the thermal expansion coefficient and the fractional-order index γ , the vertical axis represents the difference in the attenuation index before and after perturbation. It can be seen that the attenuation of Rayleigh waves is more sensitive to variations in viscoelasticity than to variations in thermal expansion. In the low-frequency band, for instance at $f = 180$ Hz, the variation in the attenuation index induced by the viscoelastic perturbation is 0.0042 m^{-1} , while that caused by the thermal expansion coefficient perturbation is $2.69 \times 10^{-5} \text{ m}^{-1}$. At this frequency, both factors exert a relatively limited influence on Rayleigh wave attenuation.

To verify the correctness of the solution in this paper, fractional order indices $\gamma_1 = \gamma_2 = \gamma_3 = 0$ (defined in Equations (38)–(40)), and the relaxation times are set to zero to degrade the model in this paper to a general thermoelastic model. Figure 15 shows the comparison of Rayleigh wave velocities at different frequencies calculated by the degraded model in this paper with the simulation results of Liu et al. [12]. The results indicate that the variation trends of the Rayleigh wave obtained by the model in this paper are consistent with those in the literature by Liu et al. [12], but the absolute value has some deviation. For example, when $f = 18,000$ Hz, the wave velocity obtained by the degenerated model in this paper is 1200.514 m/s , while the wave velocity reported in Reference [12] is 1200.216 m/s , with a relative error of approximately 0.025%.

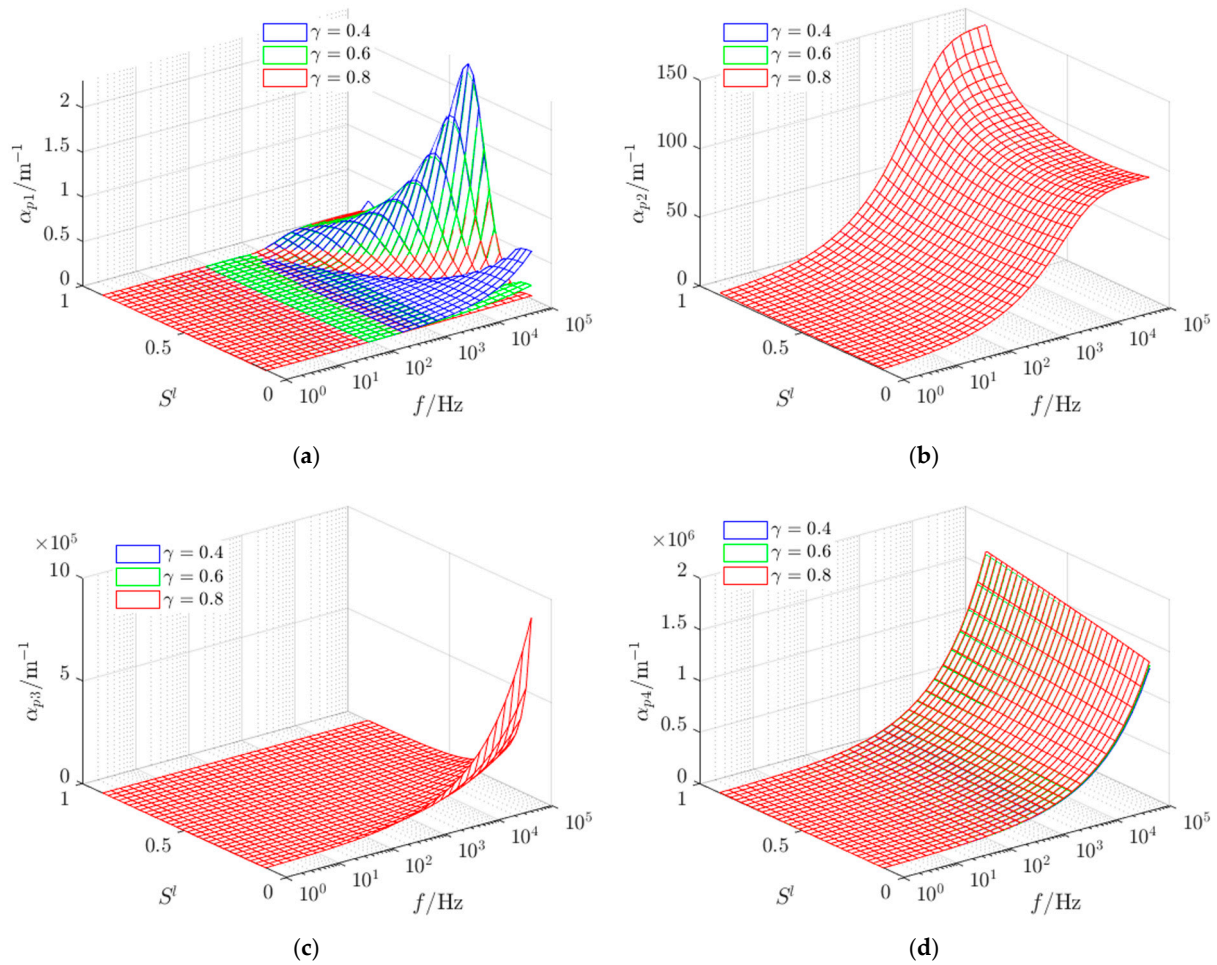


Figure 11. The effects of the fractional order index on the attenuation coefficient of four compression waves: (a) P1 wave; (b) P2 wave; (c) P3 wave; (d) thermal wave.

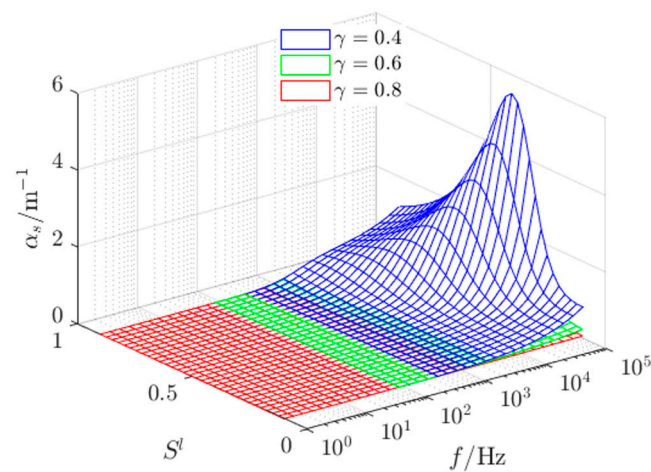


Figure 12. The effects of the fractional order index on the attenuation coefficient of the shear wave.

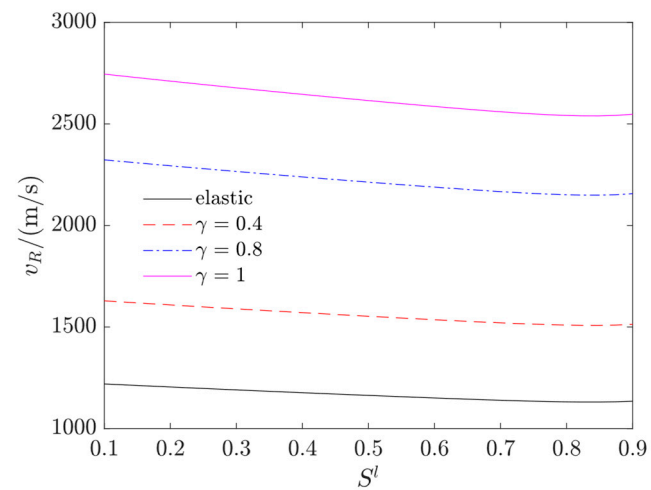


Figure 13. Comparison of the elastic model, integer-order Zener model, and the fractional-order Zener model.

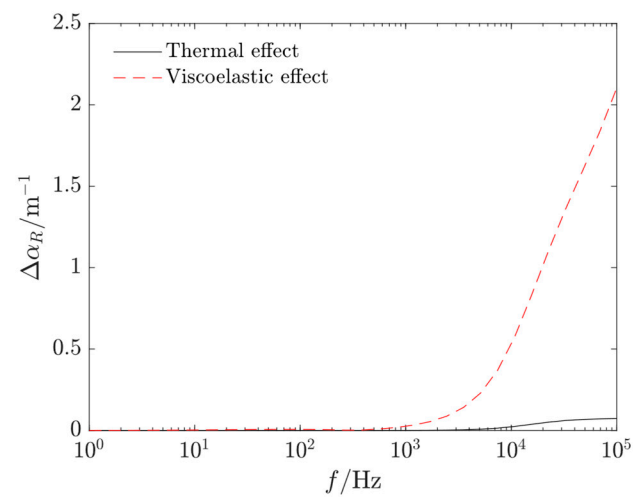


Figure 14. Sensitivity analysis of thermal and viscoelastic effects on the attenuation index of Rayleigh waves in different frequency bands.

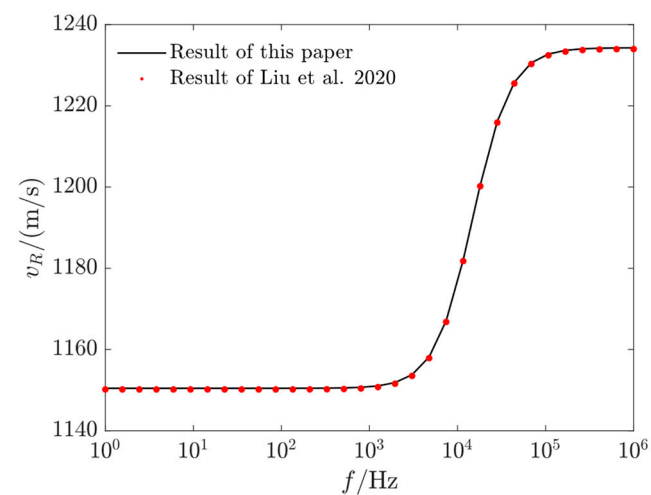


Figure 15. Comparison curves of the phase velocity of the Rayleigh wave [12].

5. Conclusions

This study focuses on the influence of thermo-viscoelastic coupling on the dispersion feature of the Rayleigh wave in partially saturated porous media with the solid–fluid–gas three-phase system. A thermo-viscoelastic porous half-space model is established based on the fractional Zener model, and the main conclusions are as follows:

- (a) The phase velocities of Rayleigh waves, P1 waves, and S waves all increase as the fractional order index rises, while the phase velocities of P2 waves, P3 waves, and T waves are almost unaffected by the fractional order index. Additionally, as the thermal expansion coefficient increases, the phase velocity of Rayleigh waves also increases accordingly.
- (b) The attenuation coefficient of Rayleigh waves first increases and then decreases in γ increase in fractional order index γ (peaking at $\gamma \approx 0.4$), and decreases with the increase in thermal expansion coefficient. Among body waves, the increase of γ has no effect on the attenuation of P2 and P3 waves, but it makes the attenuation of P4 waves (thermal waves) increase and the attenuation of S waves decrease. The attenuation peak of P1 waves in the medium-high frequency range increases with the decrease of γ . Additionally, the attenuations of P2, P3, and P4 waves are significantly greater than those of P1 and S waves.
- (c) The phase velocities of Rayleigh waves increase with frequency and permeability but decrease with saturation. At very low saturation, permeability scarcely affects wave velocity. For attenuation, in the low-frequency range, saturation and porosity barely affect the Rayleigh wave attenuation coefficient. In the high-frequency range, the attenuation coefficient decreases with porosity and first stabilizes, then increases with saturation.

The thermal expansion coefficient and viscoelasticity are key factors controlling the propagation and attenuation of the Rayleigh wave. Both regulate wave propagation and attenuation characteristics through solid–liquid coupling and skeleton stiffness. This study provides theoretical support for thermo-mechanical coupling wave analysis under complex geological conditions.

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