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A Neimark–Sacker Bifurcation Analysis of a Decision Delay Duopoly Model and Its Control Using Improved Impulsive Control

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Abstract

This paper investigates the nonlinear dynamics of a decision delay duopoly model that characterizes the competitive landscape within China's Low-Temperature Milk market. These dynamics include equilibrium points, period-doubling bifurcations, complex torus-like motions, and closed invariant curves induced by Neimark–Sacker bifurcation. Then the Neimark–Sacker bifurcation phenomenon is analyzed. Next, to further stabilize the nonlinear dynamics, an improved impulsive control strategy is introduced. Sufficient conditions for the asymptotic stability of the controlled system are derived using Lyapunov stability theory. Numerical simulations demonstrate that, under appropriate impulsive control, an originally divergent system can converge to equilibrium, effectively stabilizing market profits. This research provides a theoretical reference for oligopolistic firms to optimize their marketing rhythm and for policymakers to maintain market stability.

Keywords: Low-Temperature Milk market; duopoly competition; decision delay; impulsive control; stability analysis; Neimark–Sacker bifurcation

MSC: 37N40

1. Introduction

Researchers have typically used frameworks from industrial organization theory, game theory, and evolutionary analysis to characterize the competitive behavior of dairy brands. It is widely accepted that China's dairy industry exhibits a duopoly dominated by Yili and Mengniu, accompanied by a competitive fringe of regional brands [1,2]. Yang and Wang [3], using a Heckman two-stage model, found that product perception significantly influences Low-Temperature Milk consumption. Lee and Park [4] analyzed optimal pricing and inventory strategies for dairy products through a game-theoretic model. Du [5] highlighted that leading firms can achieve differentiated competition via product line portfolios. Si et al. [6], constructing an evolutionary game model with service quality and delayed decision-making in a dual-channel supply chain, demonstrated that the delay parameter must be kept within a reasonable range to avoid system instability. Luo et al. [7], using a Stackelberg framework, revealed that financially constrained retailers may face a “non-promotion equilibrium” under asymmetric expectations, illustrating how delays lead to market inefficiencies. Furthermore, Wang and Wang [8] identified cold-chain logistics as a key competitive barrier using the fuzzy analytic hierarchy process, while



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Vroege et al. [9] showed that extreme heat events affect raw milk supply and downstream competition. Zhen and Cui [10] introduced vertical product differentiation into a duopoly e-commerce platform model, finding that quality differences significantly influence pricing and market share.

While these studies have shed light on dairy market competition, their analytical frameworks largely rely on static games or traditional dynamic models, which fail to capture firms' strategic adjustments under uncertainty. Discrete dynamical systems theory offers a more powerful framework for analyzing such complex dynamic games. Sujay et al. [11] explored how Neimark–Sacker and period-doubling bifurcations drive a system from stability to chaos in a discrete ecological model. Feng et al. [12] investigated Neimark–Sacker bifurcation in a discrete food chain model and successfully stabilized chaos using the OGY method. The theoretical foundation of Neimark–Sacker bifurcation is well established in the literature [13,14]. Jin and Li [15] analyzed equilibrium stability and bifurcation conditions using a semi-discretization approach. Ran et al. [16] compared four feedback controllers for bifurcation suppression. However, most discrete dynamic research focuses on ecological or physical systems, and even when applied to economics, it rarely incorporates time delays in corporate decision-making—a key feature of marketing competition in the Low-Temperature Milk duopoly.

Delayed decision-making, a hallmark of bounded rationality, significantly affects system stability. Yan et al. [17], using a GARCH family model to analyze dairy price volatility, identified volatility clustering and asymmetry, confirming inherent market instability. In dynamical systems theory, delays are often a key cause of instability, leading to bifurcations and complex oscillations. Liu et al. [18] analyzed stability and Neimark–Sacker bifurcation in a delayed genetic network, providing mathematical tools applicable to economic delay systems. These studies offer theoretical grounding for understanding delay-induced fluctuations, yet they treat delay merely as an exogenous parameter and lack strategic prescriptions for mitigating the instability caused by delays.

Impulsive control provides an effective means to manage delay-induced instability. Zhang et al. [19] first introduced impulsive control to discrete-time systems with delays, proving that impulses can stabilize unstable systems. Ma et al. [20] proposed an impulsive distributed control strategy for complex networks with communication delays. Liu et al. [21] developed a nonlinear impulsive control framework, finding that impulsive delays themselves may have a stabilizing effect. Geng et al. [22] introduced average delayed impulsive gains, showing that delays can compensate for desynchronization dynamics. Shi et al. [23] established local exponential stability criteria under actuator saturation. Liu et al. [24] and Zheng et al. [25] further contributed to impulsive bifurcation control and the stabilization of discrete chaotic systems. These results provide a solid reference for applying impulsive control to economic market interventions, particularly for regulating profit fluctuations caused by marketing decision delays in duopoly markets.

In summary, existing research has laid a foundation regarding dairy market competition, marketing strategies, and delay effects, but two gaps remain:

- (i) Marketing decision delay has seldom been endogenized as a core variable in dynamic duopoly game models, leaving the mechanisms of delay-induced instability underexplored;
- (ii) The literature largely stops at the phenomenological description of delay-induced fluctuations without proposing effective control strategies.

To address these gaps, this paper constructs a duopoly competition model incorporating marketing decision delays based on discrete dynamical systems theory and systematically analyzes the system's stability and bifurcation behavior. The main innovations are as follows:

- (i) Marketing decision delay is endogenized as a core variable in a duopoly game model, revealing how it affects market equilibrium stability;
- (ii) An improved impulsive control strategy is proposed, which enhances control flexibility by employing a sequence of impulsive time instants;
- (iii) Numerical simulations are conducted to validate the model, incorporating the product characteristics and competitive landscape of China's Low-Temperature Milk market.

This study follows a four-stage design to ensure logical coherence and methodological rigor. Stage 1 (model construction) formalizes the marketing decision delay as an endogenous variable within a discrete-time duopoly framework, grounded in bounded rationality and anchored adjustment heuristics. Stage 2 (stability and bifurcation analysis) linearizes the system around the interior equilibrium, derives the characteristic equation, and identifies the conditions for Neimark–Sacker bifurcation by substituting $e^{i\theta}$ into the characteristic polynomial; the transversality condition is verified to guarantee a supercritical bifurcation. Stage 3 (impulsive control design) introduces an improved impulsive control scheme that allows for multiple impulse instants within each control segment, proves sufficient conditions for the asymptotic stability of the zero solution of the deviation system via Lyapunov theory, and establishes explicit relationships among the impulse interval, control gain, and system parameters. Stage 4 (numerical validation) calibrates the model to parameters representative of China's Low-Temperature Milk duopoly and simulates both uncontrolled bifurcation scenarios (period-doubling, torus, Neimark–Sacker) and controlled responses, comparing the proposed method with traditional impulsive control. This design bridges theoretical analysis and practical applicability.

This work aims to provide theoretical support for understanding the dynamic evolution of China's Low-Temperature Milk market, as well as practical references for firms to optimize marketing decisions and for governments to formulate industry regulatory policies.

2. Model Construction

In the Chinese Low-Temperature Milk duopoly dominated by Yili and Mengniu, the extremely short shelf life (7–15 days) forces firms to determine production volumes and promotional plans one to two weeks ahead. Consequently, a firm's current profit depends not only on its own marketing effort but also on the competitor's marketing decisions delayed by one or more periods. This delayed interaction is the core economic phenomenon we model.

Observe that the difference in marketing efforts is proportional to profits. Furthermore, the level of marketing effort of the previous period also affects the degree of response [26]. Brand managers use the level of marketing effort (advertising budget) of the previous period as an anchor and adjust for the results of the previous period (see also [27–31] for applications of this type of heuristics). The parameters λ_1 and λ_2 denote “adjustment speeds”. Since we will assume that λ_1 and λ_2 are nonnegative, the decision rules we postulate for capturing the managers' decisions are procyclic. Nonpositive values for the adjustment speeds would capture some kind of compensatory mechanism; see [32].

The construction of the model proceeds in three logically connected steps. First, each firm's desired profit (market share) is specified as a saturated response function $B \cdot x^{\beta_1} / (x^{\beta_1} + ky^{\beta_2})$, where B is the total market capacity, β_i represents diminishing-returns exponents, and k captures the intensity of competition. Second, a behavioral rule of anchored adjustment is adopted: the firm revises its actual profit from period n to $n + 1$ proportionally to the gap between the desired profit and the current profit, with adjustment speeds $\lambda_i > 0$, yielding the baseline discrete map without delay. Third, recognizing that the competitor's marketing effort affects a firm's profit only after a time lag (due to data collection, analysis, and implementation), we replace the contemporaneous competitor's

profit $y(n)$ in the denominator by the delayed value $y(n - m)$ and symmetrically for the other firm. The resulting system is presented below as (2).

Based on these behavioral assumptions, we propose the following discrete-time decision rules for the profits (market shares) $x(n)$ and $y(n)$ of the two firms:

$$\begin{cases} x(n+1) = x(n) + \lambda_1 \left(B \frac{x(n)^{\beta_1}}{x(n)^{\beta_1} + k y(n)^{\beta_2}} - x(n) \right) x(n), \\ y(n+1) = y(n) + \lambda_2 \left(B \frac{k y(n)^{\beta_2}}{x(n)^{\beta_1} + k y(n)^{\beta_2}} - y(n) \right) y(n). \end{cases} \quad (1)$$

The economic interpretations of the parameters are as follows. $x(n)$ and $y(n)$ represent the profits (or market shares) of Yili and Mengniu, respectively. $B > 0$ is the total market capacity, i.e., the maximum attainable sales volume of Low-Temperature Milk in a region. $k > 0$ denotes the intensity of competition: a larger k means that the competitor's marketing effort suppresses the firm's own profit more strongly, for instance through aggressive price promotions. $\beta_1, \beta_2 > 0$ are diminishing-returns exponents: when a firm's profit is small, additional marketing yields rapid growth; as profit saturates near B , growth slows down, reflecting brand loyalty and shelf-space constraints. $\lambda_1, \lambda_2 > 0$ are adjustment speeds, capturing how strongly a firm responds to the deviation between actual and ideal profit.

However, due to lags in information collection, analysis, and implementation, a competitor's current promotion affects the firm's profit only after a delay τ . To incorporate this realistic feature, we introduce a discrete decision delay m (where $m = \tau$ with unit time step). The delayed interaction leads to the following modified system:

$$\begin{cases} x(n+1) = x(n) + \lambda_1 \left(B \frac{x(n)^{\beta_1}}{x(n)^{\beta_1} + k y(n-m)^{\beta_2}} - x(n) \right) x(n), \\ y(n+1) = y(n) + \lambda_2 \left(B \frac{k y(n)^{\beta_2}}{x(n-m)^{\beta_1} + k y(n)^{\beta_2}} - y(n) \right) y(n). \end{cases} \quad (2)$$

The equilibrium point satisfies $x(n+1) = x(n) = x^*$ and $y(n+1) = y(n) = y^*$, with the delayed terms $x(n-m) = x^*$, $y(n-m) = y^*$. Substituting these into the system yields the equilibrium conditions analyzed in the following section.

$$x^* = x^* + \lambda_1 \left(B \frac{(x^*)^{\beta_1}}{(x^*)^{\beta_1} + k (y^*)^{\beta_2}} - x^* \right) x^*, \quad (3)$$

$$y^* = y^* + \lambda_2 \left(B \frac{k (y^*)^{\beta_2}}{(x^*)^{\beta_1} + k (y^*)^{\beta_2}} - y^* \right) y^*. \quad (4)$$

Simplifying this, we obtain

$$0 = \lambda_1 \left(B \frac{(x^*)^{\beta_1}}{(x^*)^{\beta_1} + k (y^*)^{\beta_2}} - x^* \right) x^*, \quad (5)$$

$$0 = \lambda_2 \left(B \frac{k (y^*)^{\beta_2}}{(x^*)^{\beta_1} + k (y^*)^{\beta_2}} - y^* \right) y^*. \quad (6)$$

Since $\lambda_1, \lambda_2 > 0$, two cases can be considered.

2.1. Analysis of Non-Zero Equilibrium Point

$x^* \neq 0, y^* \neq 0$. Dividing both sides by x^* and y^* respectively, we obtain

$$B \frac{(x^*)^{\beta_1}}{(x^*)^{\beta_1} + k (y^*)^{\beta_2}} = x^*, \quad (7)$$

$$B \frac{k(y^*)^{\beta_2}}{(x^*)^{\beta_1} + k(y^*)^{\beta_2}} = y^*. \quad (8)$$

Adding the two equations and using the common denominator yields

$$x^* + y^* = B. \quad (9)$$

Substituting $y^* = B - x^*$ into (7) gives

$$x^* = B \frac{(x^*)^{\beta_1}}{(x^*)^{\beta_1} + k(B - x^*)^{\beta_2}}.$$

Rearranging (assuming $x^* \neq 0$) and simplifying leads to

$$(x^*)^{\beta_1-1} = k(B - x^*)^{\beta_2-1}. \quad (10)$$

Thus, the interior equilibrium point is (x^*, y^*) , where $y^* = B - x^*$, and $x^* \in (0, B)$ satisfies Equation (10).

2.2. Analysis of Boundary Equilibrium Points

Consider the case where $x^* = 0$; substituting this into the original equilibrium equations shows that the first equation holds identically, and the second simplifies to $0 = \lambda_2(B - y^*)y^*$. Since $y^* > 0$, we obtain $y^* = B$. Hence the equilibrium point is $(0, B)$. If $y^* = 0$, similarly we obtain $x^* = B$, giving the equilibrium point $(B, 0)$. The origin $(0, 0)$ is not considered because the expressions become undefined (denominator zero).

The original system (2) does not possess the trivial equilibrium $(0, 0)$ due to singular denominators. However, all stability analyses in the remainder of this paper (including the Lyapunov theorems and impulsive control design in Section 3) are performed on the deviation system obtained via the translation

$$u(n) = x(n) - x^*, \quad v(n) = y(n) - y^*,$$

where (x^*, y^*) is the interior equilibrium satisfying $x^* + y^* = B$ and $(x^*)^{\beta_1-1} = k(y^*)^{\beta_2-1}$. Under this translation, the non-zero equilibrium of the original system is mapped to the origin $(0, 0)$ of the deviation system. Therefore, when we refer to the “zero solution” in the following lemmas, theorems, and numerical simulations, we implicitly refer to the zero solution of the deviation system, which corresponds exactly to the interior equilibrium of the original duopoly model.

Substituting $x(n) = u(n) + x^*$, $y(n) = v(n) + y^*$, and noting that $x(n+1) = u(n+1) + x^*$, $y(n+1) = v(n+1) + y^*$, into (2) and subtracting the equilibrium conditions yields the deviation system

$$\begin{cases} u(n+1) = u(n) + \lambda_1 \left(B \frac{(u(n) + x^*)^{\beta_1}}{(u(n) + x^*)^{\beta_1} + k(v(n) + y^*)^{\beta_2}} - (u(n) + x^*) \right) (u(n) + x^*), \\ v(n+1) = v(n) + \lambda_2 \left(B \frac{k(v(n) + y^*)^{\beta_2}}{(u(n) + x^*)^{\beta_1} + k(v(n) + y^*)^{\beta_2}} - (v(n) + y^*) \right) (v(n) + y^*). \end{cases} \quad (11)$$

This system has the zero solution $(u, v) = (0, 0)$. All subsequent impulsive control designs are applied directly to the deviation system (11), rather than to the original variables (x, y) .

Define $z(n) = [x(n), y(n), x(n-1), y(n-1), \dots, x(n-m), y(n-m)]^T$. The system linearizes to $z(n+1) = Az(n)$. The matrix A has a block structure

$$A = \begin{pmatrix} A_{00} & 0 & \cdots & 0 & A_{0m} \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{pmatrix},$$

where $A_{00} = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$, $A_{0m} = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}$, and I is the 2×2 identity matrix.

Let $v = [v_0, v_1, \dots, v_m]^T$ be an eigenvector, with $v_i \in \mathbb{R}^2$, corresponding to the eigenvalue μ . From $Av = \mu v$, we obtain the component equations

$$\mu v_0 = A_{00}v_0 + A_{0m}v_m, \quad \mu v_1 = v_0, \quad \mu v_2 = v_1, \quad \dots \quad \mu v_m = v_{m-1}.$$

From the last m equations, we get the recurrence $v_k = \mu^{-1}v_{k-1}$ for $k = 1, \dots, m$; hence $v_m = \mu^{-m}v_0$. Substituting this into the first equation, we obtain

$$\mu v_0 = A_{00}v_0 + \mu^{-m}A_{0m}v_0.$$

Rearranging gives $(\mu I - A_{00} - \mu^{-m}A_{0m})v_0 = 0$. For a non-zero solution v_0 to exist, we require

$$\det(\mu I - A_{00} - \mu^{-m}A_{0m}) = 0.$$

Computing the determinant, we obtain

$$\det \begin{pmatrix} \mu - a_{11} & -\mu^{-m}a_{12} \\ -\mu^{-m}a_{21} & \mu - a_{22} \end{pmatrix} = (\mu - a_{11})(\mu - a_{22}) - \mu^{-2m}a_{12}a_{21} = 0.$$

Multiplying both sides by μ^{2m} yields the characteristic equation

$$\mu^{2m+2} - (a_{11} + a_{22})\mu^{2m+1} + a_{11}a_{22}\mu^{2m} - a_{12}a_{21} = 0. \quad (12)$$

This is a polynomial of degree $2m + 2$ in μ , whose roots are the eigenvalues of matrix A .

Consider the non-zero equilibrium point (x^*, y^*) and introduce the extended state vector $z(n) = [x(n), y(n), x(n-1), y(n-1), \dots, x(n-m), y(n-m)]^T$. Linearizing near the equilibrium point gives the linearized system $z(n+1) = Az(n)$, where A is a matrix. Solving for A yields

$$A = \begin{pmatrix} a_{11} & 0 & 0 & 0 & \cdots & 0 & 0 & a_{12} \\ 0 & a_{22} & 0 & 0 & \cdots & 0 & a_{21} & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}_{2(m+1) \times 2(m+1)}.$$

And

$$a_{11} = 1 + \lambda_1 x^* \left(\beta_1 \frac{y^*}{B} - 1 \right), \quad (13)$$

$$a_{12} = -\lambda_1 \beta_2 \frac{(x^*)^2}{B}, \quad (14)$$

$$a_{22} = 1 + \lambda_2 y^* \left(\beta_2 \frac{x^*}{B} - 1 \right), \quad (15)$$

$$a_{21} = -\lambda_2 \beta_1 \frac{(y^*)^2}{B}. \quad (16)$$

The equilibrium point (x^*, y^*) satisfies $x^* + y^* = B$ and $(x^*)^{\beta_1-1} = k(y^*)^{\beta_2-1}$.

Let $f_1(x, y) = x + \lambda_1 \left(B \frac{x^{\beta_1}}{x^{\beta_1} + ky^{\beta_2}} - x \right)$, and then take the partial derivative with respect to x . Using the equilibrium conditions, we obtain $a_{11} = \frac{\partial f}{\partial x} \Big|_{(x^*, y^*)} = 1 + \lambda_1 x^* \left(\beta_1 \frac{y^*}{B} - 1 \right)$. The other coefficients are obtained similarly.

The local stability of the equilibrium point is determined by the eigenvalues of matrix A ; that is, all roots μ of the characteristic equation $\det(A - \mu I) = 0$ satisfy $|\mu| < 1$. By analyzing the special structure of matrix A , the characteristic equation expands to

$$\mu^{2m+2} - (a_{11} + a_{22})\mu^{2m+1} + a_{11}a_{22}\mu^{2m} - a_{12}a_{21} = 0.$$

When $\lambda = 0$, $a_{11} = 1$, $a_{12} = 0$, $a_{22} = 1$, and $a_{21} = 0$, the characteristic equation reduces to $\mu^{2m}(\mu - 1)^2 = 0$. Therefore, when $\lambda = 0$, $\mu = 0$ or $\mu = 1$, and the maximum value of $|\mu|$ is 1.

Let $\lambda_1 = \lambda_2 = \lambda$, and express the linearization coefficients uniformly in terms of λ . The equilibrium point (x^*, y^*) satisfies $x^* + y^* = B$ and $(x^*)^{\beta_1-1} = k(y^*)^{\beta_2-1}$, with x^*, y^* independent of λ . Differentiating the characteristic equation yields

$$\frac{d\mu}{d\lambda} = \frac{K\mu^{2m+1} - (K + 2\lambda x^* y^* L)\mu^{2m} + \frac{2\lambda\beta_1\beta_2(x^* y^*)^2}{B^2}}{(2m+2)\mu^{2m+1} - (2 + \lambda K)(2m+1)\mu^{2m} + (1 + \lambda K + \lambda^2 x^* y^* L)(2m)\mu^{2m-1}},$$

$$K = (\beta_1 + \beta_2) \frac{x^* y^*}{B} - (x^* + y^*), \quad L = \frac{\beta_1 \beta_2 x^* y^*}{B^2} - \frac{\beta_1 y^* + \beta_2 x^*}{B} + 1. \quad (17)$$

As $\lambda \rightarrow 0$ and the eigenvalue $\mu = 1$ is a double root, implicit differentiation requires considering a second-order expansion. The Taylor expansion of the characteristic equation around $(1, 0)$ is

$$\delta^2 - T\delta\lambda + D\lambda^2 = 0,$$

where $\delta = \mu - 1$, $T = x^* A + y^* C$, and $D = x^* y^* \left(1 - \frac{\beta_1 y^*}{B} - \frac{\beta_2 x^*}{B} \right)$. Solving gives

$$\begin{aligned} \frac{d\mu}{d\lambda} &= \lim_{\lambda \rightarrow 0} \frac{\delta}{\lambda} \\ &= \frac{T \pm \sqrt{T^2 - 4D}}{2} \\ &= \frac{1}{2} \left(\frac{(\beta_1 + \beta_2)x^* y^*}{B} - B \right. \\ &\quad \left. \pm \sqrt{\left(\frac{(\beta_1 + \beta_2)x^* y^*}{B} \right)^2 - 4x^* y^* \left(1 - \frac{\beta_1 y^* + \beta_2 x^*}{B} \right)} \right). \end{aligned}$$

When the numerator is positive, $\mu' < 0$. At $\lambda = 0$ and $\mu = 1$, $|\mu|^2 = \mu^2$; therefore,

$$\frac{d|\mu|^2}{d\lambda} \Big|_{\lambda=0, \mu=1} = \frac{(\beta_1 + \beta_2)x^* y^*}{B} - B \pm \sqrt{\left(\frac{(\beta_1 + \beta_2)x^* y^*}{B} \right)^2 - 4x^* y^* \left(1 - \frac{\beta_1 y^* + \beta_2 x^*}{B} \right)}.$$

At $\lambda = 0$ and $\mu = 1$, the two branches of the derivative $\frac{d|\mu|^2}{d\lambda}$ are $T \pm \sqrt{T^2 - 4D}$, where $T = \frac{(\beta_1 + \beta_2)x^* y^*}{B} - B$, and $D = x^* y^* \left(1 - \frac{\beta_1 y^* + \beta_2 x^*}{B} \right)$. Since $\sqrt{T^2 - 4D} \geq |T|$, it always

holds that $T - \sqrt{T^2 - 4D} \leq 0$, with equality only when $D = 0$. Therefore, this branch of the derivative is always less than or equal to zero; that is, $\frac{d|\mu|^2}{d\lambda}\bigg|_{\lambda=0} < 0$ holds in the non-degenerate case. This result is consistent with the symmetric case (e.g., when $\beta_1 = \beta_2 = 1, k = 1$, it gives $-B$).

If the system undergoes a Neimark–Sacker bifurcation, there exists a critical value λ_c such that a pair of complex conjugate eigenvalues has modulus 1. substituting $\mu = e^{i\theta}$ into the characteristic equation

$$\mu^{2m+2} - (a_{11} + a_{22})\mu^{2m+1} + a_{11}a_{22}\mu^{2m} - a_{12}a_{21} = 0, \quad (18)$$

and using Euler's formula and using Euler's formula $e^{i\alpha} = \cos \alpha + i \sin \alpha$, separating the real and imaginary parts yields the bifurcation conditions

$$\cos[(2m+2)\theta] - (a_{11} + a_{22})\cos[(2m+1)\theta] + a_{11}a_{22}\cos(2m\theta) - a_{12}a_{21} = 0, \quad (19)$$

$$\sin[(2m+2)\theta] - (a_{11} + a_{22})\sin[(2m+1)\theta] + a_{11}a_{22}\sin(2m\theta) = 0. \quad (20)$$

Neimark–Sacker bifurcation occurs when a pair of complex conjugate eigenvalues crosses the unit circle. Following the standard bifurcation theory [14,33], the critical parameter value can be determined by substituting $\mu = e^{i\theta}$ into the characteristic equation.

The smallest λ satisfying these conditions is λ_c . When $\lambda > \lambda_c$, the system exhibits a limit cycle, and profits display periodic fluctuations.

Lemma 1. For sufficiently small δ , we have

$$\frac{d|\mu|^2}{d\lambda}\bigg|_{\lambda=\lambda^*, \mu=i\omega^*} > 0,$$

where μ is the eigenvalue of the linearized system, and λ is the bifurcation parameter (the adjustment speed in our model). This transversality condition guarantees the occurrence of a Neimark–Sacker bifurcation and indicates that the bifurcation is supercritical (the squared modulus increases with λ).

Proof. Define two auxiliary functions based on the implicit differentiation of the characteristic Equation (12):

$$f(\mu) = K\mu^{2m+1} - (K + 2\lambda x^*y^*L)\mu^{2m} + \frac{2\lambda\beta_1\beta_2(x^*y^*)^2}{B^2},$$

$$g(\mu) = (2m+2)\mu^{2m+1} - (2 + \lambda K)(2m+1)\mu^{2m} + (1 + \lambda K + \lambda^2 x^*y^*L)2m\mu^{2m-1},$$

where K and L are as in (17). From the characteristic equation, we obtain

$$\frac{d\mu}{d\lambda}\bigg|_{\lambda=\lambda^*, \mu=i\omega^*} = -\frac{f(\mu)}{g(\mu)}\bigg|_{\lambda=\lambda^*, \mu=i\omega^*}.$$

Then the derivative of $|\mu|^2$ can be expressed as

$$\frac{d|\mu|^2}{d\lambda}\bigg|_{\lambda=\lambda^*, \mu=i\omega^*} = -\frac{g(\mu)}{|g(\mu)|^2} M,$$

where $M = \bar{\mu}f(\mu) + \mu f(\bar{\mu})$. Since $-g(\mu)/|g(\mu)|^2$ is a non-zero complex factor, the sign of the derivative is determined by M .

Now evaluate M at the critical point $\mu = i\omega^*$. Using the correct expression (see Section 4.1), we have

$$\mu f(\bar{\mu}) = K \cos(2m) \omega^* - (K + 2\lambda^* x^* y^* L) \cos(2m - 1) \omega^* + \frac{2\lambda^* \beta_1 \beta_2 (x^* y^*)^2}{B^2},$$

hence

$$M = 2K \cos(2m) \omega^* - 2(K + 2\lambda^* x^* y^* L) \cos(2m - 1) \omega^* + \frac{4\lambda^* \beta_1 \beta_2 (x^* y^*)^2}{B^2}.$$

We now show that there exist parameters B, β_1, β_2 such that $M > 0$. Take $\beta_1 = \beta_2 = 1$. Then by symmetry, $x^* = y^* = B/2$, and

$$K = -\frac{B}{2}, \quad L = -\frac{3}{4}.$$

Choose $m = 2$ (so $2m = 4$, $2m - 1 = 3$); at the bifurcation point $\mu = e^{i\theta} = i\omega^*$, we have $\theta = \pi/2$; thus

$$\cos(4\theta) = 1, \quad \cos(3\theta) = 0.$$

Substituting into M gives

$$M = 2K\omega^* + \frac{4\lambda^* (x^* y^*)^2}{B^2} = -B\omega^* + \frac{\lambda^* B^2}{4}.$$

For sufficiently large B (e.g., $B = 10$) and a suitable $\lambda^* > 0$ satisfying $\lambda^* B^2/4 > B\omega^*$, we obtain $M > 0$. Since ω^* depends continuously on the parameters, such choices are always feasible.

Consequently,

$$\left. \frac{d|\mu|^2}{d\lambda} \right|_{\lambda=\lambda^*, \mu=i\omega^*} > 0,$$

and the transversality condition holds.

This transversality condition is the standard criterion for a Neimark–Sacker bifurcation; see, e.g., the classical treatments in [13,14,33]. $\square$

3. Improved Impulsive Control and Stability Analysis

This section first recalls the conventional impulsive control framework for discrete-time systems, then proposes an improved strategy that allows for multiple impulse instants within each control segment. The motivation for this improvement is twofold: (i) real-world market interventions (e.g., periodic promotional freezes or supply adjustments) often involve several closely spaced actions rather than a single isolated control; (ii) multiple impulses per segment can enhance robustness against parameter uncertainty and reduce the required control intensity at each instant, making the scheme more practical. The theoretical analysis derives sufficient conditions for the asymptotic stability of the zero solution of the deviation system, followed by its application to the decision delay duopoly model.

This section first describes the conventional impulsive control and then presents the improved impulsive control. Finally, we use the improved impulsive control to stabilize the decision delay duopoly model.

All lemmas and theorems stated in this subsection (including Lemmas 2–4 and Theorem 2) are formulated for the zero solution of the deviation system $(u, v) = (0, 0)$, where $u = x - x^*$, $v = y - y^*$. Consequently, the asymptotic stability of the zero solution of the deviation system implies $\lim_{n \rightarrow \infty} x(n) = x^*$ and $\lim_{n \rightarrow \infty} y(n) = y^*$ in the original system. The origin $(0, 0)$ of the original model is never considered as an equilibrium; the term “zero solution” always refers to the translated deviation system.

3.1. Impulsive Control

According to the impulsive control theorem described in [19], consider the following impulsively controlled closed-loop system

$$x(n+1) = f(n, x(n)) + W(n, x(n)). \quad (21)$$

We construct an impulsive controller as

$$W(n, x(n)) = \sum_{k=1}^{\infty} [I_k(x(n)) - f(n, x_n)] \delta[n - n_k + 1]$$

and obtain

$$x(n+1) = f(n, x(n)), \quad n_{k-1} \leq n < n_k - 1 \quad (22)$$

$$x(n_k) = I_k(x(n_k - 1)), \quad k \in \mathbb{Z}^+ \quad (23)$$

where f represents a general function, n_k represents the pulse moments, and I_k represents the pulse moment control law.

Definition 1. $\omega_i(r), i = 1, 2$ are class $\mathcal{K}$ functions. That is, for $r \in [0, \infty)$, $\omega_i(r)$ is strictly increasing, $\omega_i(0) = 0$, and as $r \rightarrow \infty$, $\omega_i(r) \rightarrow \infty$.

Theorem 1. V is the Lyapunov function of the system. Assume that $x(n)$ is the zero solution of system (1), and suppose that the following conditions are met

- (i) $\omega_1(\|x(n)\|) \leq V(n, x(\cdot)) \leq \omega_2(\|x(n)\|)$;
- (ii) For $n \in (n_k, n_{k+1}), k = 0, 1, 2, \dots$, there exists $a_k > 0$ such that

$$\begin{aligned} \Delta V_{(2)}(n, x(\cdot)) &= V(n, x(n)) - V(n-1, x(n-1)) \\ &\leq a_k V(n-1, x(n-1)); \end{aligned}$$

- (iii) There exists $\alpha > \varsigma_k > 0, k \in \mathbb{Z}^+$ such that

$$V(n_k, x(n_k)) \leq \varsigma_k V(n_k - 1, x(n_k - 1));$$

- (iv) $\sup\{\prod_{i=0}^k \varsigma_i (1 + a_i)^{n_{i+1} - n_i - 1}\} = M < \infty$, where $\varsigma_0 = 1$.

Then the zero solution of system (1) is uniformly stable.

Proof. For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\max\{M\omega_2(\delta), M\alpha\omega_2(\delta)\} < \omega_1(\varepsilon).$$

When $n = n_1$, according to the conditions of Theorem 1, we have

$$\begin{aligned} V(n_1, x_{n_1}) &\leq \varsigma_1 V(n_1 - 1, x(n_1 - 1)) \\ &\leq \varsigma_1 (1 + a_0)^{n_1 - 1 - n_0} V(n_0, x(n_0)); \end{aligned}$$

When $n \in (n_1, n_2)$, we have

$$\begin{aligned} \Delta V_{(2)}(n, x(\cdot)) &= V(n, x(n)) - V(n-1, x(n-1)) \\ &\leq a_1 V(n-1, x(n-1)). \end{aligned}$$

And thus,

$$\begin{aligned} V(n, x(n)) &\leq (1 + a_1)V(n-1, x(n-1)) \\ &\leq a_1(1 + a_0)^{n_1-1-n_0}V(n_0, x(n_0)). \end{aligned}$$

By the same reasoning, when $n \in (n_k, n_k - 1)$,

$$V(n, x(n)) \leq \prod_{i=0}^k \varsigma_i (1 + a_i)^{n_{i+1}-n_i-1} V(n_0, x(n_0)) \leq M\omega_2(\delta),$$

$$V(n_{k+1}, x(n_{k+1})) \leq \varsigma_{i+1} \prod_{i=0}^k \varsigma_i (1 + a_i)^{n_{i+1}-n_i-1} V(n_0, x(n_0)) \leq M\alpha\omega_2(\delta),$$

Therefore, for $n \geq n_0$, $|x(n)| < \varepsilon$, the zero solution of the system is uniformly stable. $\square$

According to conditions 2 and 3 of Theorem 2 in [24] and Theorem 2 in [25], the following theorem can be obtained, providing sufficient conditions for the asymptotic stability of the system's zero solution.

Law 1. Assume conditions (i), (ii), and (iii) in Theorem 1 are satisfied. Condition (iv) is replaced with $\sup\{\prod \varsigma_i (1 + a_i)^{n_{i+1}-n_i-1}\} = M < 1$. Then the zero solution of system (21) is asymptotically stable.

Proof. Let $H = \max\{M\omega_2(\delta), M\alpha\omega_2(\delta)\}$, $\rho = (n_{k+1} - n_k)$ represent the pulse control interval. Since $M < 1$ is a special case of Theorem 1, when $n \in (n_k, n_{k+1} - 1)$, $V(n, x(n)) \leq H$ holds. When $n \geq n_{k+1}$, let E represent the number of pulse controls at this time; then $V(n, x(n)) \leq M^E H$. Since $\varsigma_i (1 + a_i)^{\rho-1} < 1$, as $E \rightarrow \infty$, $V(n, x(n)) \rightarrow 0$. Therefore, the zero solution of system (21) is asymptotically stable. $\square$

Jury 1. According to Note 1 in [18], the constraints on the pulse interval can be deduced. From $\varsigma_i (1 + a_i)^{\rho-1} < 1$, we obtain $(\rho - 1) \ln(1 + a_i) + \ln \varsigma_i < 0$, i.e., $\rho < 1 - \frac{\ln \varsigma_i}{\ln(1+a_i)}$. Since the parameter a_i is determined by the system, this formula shows the relationship between the design parameters (pulse interval and pulse control intensity). If the pulse control intensity is given, the upper limit of the pulse interval can be determined. In addition, this formula shows that the larger the control intensity, the larger the upper limit of the pulse interval, which is consistent with the actual situation.

3.2. Improved Impulsive Control

Next, we extend Theorem 1.

Definition 2. Let $\{k_i\}$ be a sequence of segments, each containing multiple consecutive discrete time instants. All instants within each segment are pulse moments, and each pulse moment can be subjected to pulse control. Let k_i denote the i -th segment, and the specific instants within each segment are denoted by $k_i(s_j)$, $s_j \in \mathbb{Z}^+$, where s_j represents the specific instants in each segment, $1 \leq j \leq m$, and m represents the number of pulse moments within each segment.

If the system applies pulse controllers and each pulse set has m pulse moments, then this pulse control is called an m -order pulse controller.

Perform the following transformation on system (21)

$$x(n+1) = f(n, x(n)), \quad n_{k-1} \leq n < n_k - 1 \quad (24)$$

$$x(k_i(s_j)) = I_{k(s_j)}(x(k_i(s_{j-1}))), \quad (25)$$

The parameters in the formula are as described above.

Law 2. Assume that system (24) meets the following conditions

- (i) $\omega_1(\|x(n)\|) \leq V(n, x(\cdot)) \leq \omega_2(\|x(n)\|)$;
- (ii) For $n \in (k_i(s_m), k_{i+1}(s_1))$, there exist $a_k > 0$ such that

$$\begin{aligned}\Delta V_{(2)}(n, x(\cdot)) &= V(n, x(n)) - V(n-1, x(n-1)) \\ &\leq a_k V(n-1, x(n-1));\end{aligned}$$

- (iii) There exists $\alpha > \varsigma_k > 0, k \in \mathbb{Z}^+$ such that

$$V(k_i(s_1), x(k_i(s_1))) \leq \varsigma_k V(k_i(s_1) - 1, x(k_i(s_1) - 1));$$

- (iv) Assume $w(j-1) = |I_{k_1(s_{j-1})}(x(k_1(s_{j-2})))|$, and when $n \in k_i(s_j), j \in \mathbb{Z}^+, w(j-1) < w(j-2)$;
- (v) $\sup\{\prod_{i=0}^k \varsigma_i (1+a_i)^{k_{i+1}(s_1)-n_0-1} (1+c_i)^{m-1}\} = N < \infty$, where $\varsigma_0 = 1$.

Then the zero solution of system (24) is uniformly stable.

Proof. When $n = k_1(s_1)$, according to condition (iii), we have

$$\begin{aligned}V(k_1(s_1), x(k_1(s_1))) &\leq \varsigma_1 V(k_1(s_1) - 1, x(k_1(s_1) - 1)) \\ &\leq \varsigma_1 (1+a_0)^{k_1(s_1)-1-n_0} V(n_0, x(n_0)).\end{aligned}$$

When $n \in k_1(s_j)$, according to condition (iv), we have $w(j-1)/w(j-2) = b_1, 0 < b_1 < 1$. Since $V(t, x)$ is the Lyapunov function of the system, and condition (i) is satisfied, the following is obtained

$$\begin{aligned}V(k_1(s_{j-1}), w(j-1)) - V(k_1(s_{j-2}), w(j-2)) \\ \leq c_1 V(k_1(s_{j-1}), w(j-1)), -1 < c_1 < 0.\end{aligned}$$

That is, $V(k_1(s_{m-1}), w(m-1)) \leq (1+c_1)^{m-1} V(k_1(s_1))$. When $n \in (k_1(s_m), k_2(s_1))$, according to condition (ii), we have

$$\begin{aligned}\Delta V_{(2)}(n, x(\cdot)) &= V(n, x(n)) - V(n-1, x(n-1)) \\ &\leq a_1 V(n-1, x(n-1)),\end{aligned}$$

and

$$V(n, x(n)) \leq \varsigma_i (1+a_1)^{n-k_1(s_1)} (1+c_1)^{m-1} V(n_0, x(n_0)).$$

And when $n \in (k_j(s_m), k_{j+1}(s_1))$, there exists

$$V(n, x(n)) \leq \prod_{i=0}^k \varsigma_i (1+a_i)^{k_{i+1}(s_1)-n_0-1} (1+c_1)^{m-1} V(n_0, x(n_0)) \leq N \omega_2(\delta),$$

and

$$V(n_{k+1}, x(n_{k+1})) \leq \varsigma_{i+1} \prod_{i=0}^k \varsigma_i (1+a_i)^{k_{i+1}(s_1)-n_0-1} (1+c_1)^{m-1} V(n_0, x(n_0)) \leq \alpha N \omega_2(\delta),$$

By the same reasoning, the zero solution of system (24) is uniformly stable. $\square$

3.3. Control of Decision Delay Duopoly Model Using Improved Impulsive Control

Consider the following deviation system under impulsive control:

$$\left\{ \begin{array}{l} u(n+1) = u(n) + \lambda_1 \left(B \frac{(u(n) + x^*)^{\beta_1}}{(u(n) + x^*)^{\beta_1} + k(v(n-m) + y^*)^{\beta_2}} - (u(n) + x^*) \right) (u(n) + x^*), \\ \quad \quad \quad n_{k-1} \leq n < n_k - 1, \\ v(n+1) = v(n) + \lambda_2 \left(B \frac{k(v(n) + y^*)^{\beta_2}}{(u(n-m) + x^*)^{\beta_1} + k(v(n) + y^*)^{\beta_2}} - (v(n) + y^*) \right) (v(n) + y^*), \\ \quad \quad \quad n_{k-1} \leq n < n_k - 1, \\ u(k_i(s_j)) = I_{k(s_j)}(u(k_i(s_{j-1}))), \\ v(k_i(s_j)) = I_{k(s_j)}(v(k_i(s_{j-1}))). \end{array} \right. \quad (26)$$

Here k_i denotes the i -th segment of the moment set, and the specific moments within each segment are denoted by $k_i(s_j)$, $s_j \in \mathbb{Z}^+$, where s_j represents the specific moments in each segment.

Let $V(t, x)$ be continuous with $V(t, 0) = 0$, $\{\tau_k\}$ is strictly increasing and $\tau_k \rightarrow \infty$ as $k \rightarrow \infty$, $V(\tau_k^+, x + I_k(x)) \leq \psi_k(V(\tau_k, x))$.

Lemma 2 ([19]). Let V be defined as before and be a Lyapunov functional. Also, suppose that every solution $x(t)$ satisfies

- (i) $a(\|x\|) \leq V(t, x) \leq b(\|x\|)$.
- (ii) $\dot{V}(t, x) \leq cV(t, x)$, where c is a constant.
- (iii) $V(\tau_k^+, x + I_k(x)) \leq d_k V(\tau_k, x)$ for some $d_k > 0$.
- (iv) $b(\delta) \leq a(\varepsilon)$ whenever $\delta \leq \varepsilon$ for some $\varepsilon > 0$.

Then the zero solution of (26) is uniformly stable.

Lemma 3 ([19]). Let all conditions of Lemma 2 hold except (iii) be replaced by (iii) $V(\tau_k^+, x + I_k(x)) \leq d_k V(\tau_k, x)$ with $d_k \in (0, 1)$. Then the solution of system (26) is uniformly stable.

Lemma 4 ([19]). Let V be defined as before and be a Lyapunov functional. Also, suppose that every solution $x(t)$ satisfies

- (i) Condition (i) of Lemma 2 holds;
- (ii) For $t \in (\tau_{k-1}, \tau_k]$, there exist μ_k such that $\dot{V}(t, x) \leq \mu_k V(t, x)$;
- (iii) There exist β_k such that $V(\tau_k^+, x + I_k(x)) \leq \beta_k V(\tau_k, x)$;
- (iv) $\prod_{j=1}^k \beta_j \exp\left(\sum_{j=1}^k \mu_j (\tau_j - \tau_{j-1})\right) \leq M$, where M is a positive constant.

Then the solution of (26) is uniformly stable.

Theorem 2 (Asymptotic stability of the deviation system). Consider the deviation system obtained via the translation $u = x - x^*$, $v = y - y^*$. Its zero solution $(u, v) = (0, 0)$ corresponds to the non-zero equilibrium (x^*, y^*) of the original duopoly model. Suppose that the impulsive control parameters satisfy $0 < q < 1$, where

$$q = \frac{1}{1 + \alpha} \cdot e^{\gamma\tau}, \quad (27)$$

α is the impulsive gain, τ is the impulsive interval, and γ is the Lyapunov growth rate. Then the zero solution of the deviation system is asymptotically stable. Consequently, for the original system, $x(n) \rightarrow x^*$ and $y(n) \rightarrow y^*$ as $n \rightarrow \infty$.

Proof. Suppose that the Lyapunov functional of (26)

$$V(t) = (x(t) - x^*)^2 + (y(t) - y^*)^2$$

satisfies

$$\dot{V}(t) \leq \gamma V(t)$$

during non-impulsive intervals. Because the impulsive control gives $V(\tau_k^+) \leq \frac{1}{1+\alpha} V(\tau_k)$, we have $V(t) \leq M e^{\gamma t}$, where M is a constant. Let $\beta_k = \frac{1}{1+\alpha}$; then $\prod \beta_k = \left(\frac{1}{1+\alpha}\right)^k$, so condition (iii) of Lemma 4 is satisfied. When $\tau_k - \tau_{k-1} = \tau$,

$$V(t) \leq V(0) \left(\frac{1}{1+\alpha}\right)^k e^{\gamma t}.$$

Then let $\mu_k = \gamma$; we have

$$\prod_{j=1}^k \beta_j \exp(\mu_j \tau) = \left(\frac{1}{1+\alpha}\right)^k e^{\gamma k \tau} = \left(\frac{e^{\gamma \tau}}{1+\alpha}\right)^k,$$

so condition (iv) of Lemma 4 is satisfied if $q = \frac{e^{\gamma \tau}}{1+\alpha} < 1$. Thus, system (26) is asymptotically stable. $\square$

4. Numerical Simulation

This section verifies the effectiveness of impulsive control through numerical examples. Model parameters are chosen as: total market capacity $B = 10$; competition intensity $k = 0.9$; nonlinear indices $\beta_1 = \beta_2 = 0.2$; adjustment speeds $\lambda_1 = \lambda_2 = 0.2$.

$$\begin{aligned} a_{11} &= 1 + \lambda_1 x^* \left(\beta_1 \frac{y^*}{B} - 1 \right) \\ &= 1 + 0.2 \times 5.328 \times \left(0.2 \times \frac{4.671}{10} - 1 \right) \\ &= 1 + 1.0656 \times (0.09342 - 1) \\ &= 1 - 0.966 \\ &= 0.034, \end{aligned}$$

$$a_{12} = -\lambda_1 \beta_2 \frac{(x^*)^2}{B} = -0.2 \times 0.2 \times \frac{5.328^2}{10} = -0.04 \times 2.839 = -0.1136,$$

and similarly, $a_{22} = 0.034$, and $a_{21} = -0.0873$.

For the delay model (with $m = 3$ in our simulations), matrix A is an 8×8 matrix (since the state dimension is $2(m+1) = 8$); for the no-delay case ($m = 0$), the system reduces to a 2×2 Jacobian. Computing the eigenvalues of the relevant Jacobian reveals that the maximum modulus is greater than 1, indicating that the equilibrium point is unstable without control, and the system may exhibit oscillations.

Set initial deviations $u(0) = 0.5$, $v(0) = -0.5$, with all other historical values set to zero. Additionally, set total simulation steps $N = 100$, impulsive interval $T = 10$, and attenuation factor $\alpha = 0.8$. Define the Lyapunov function $V(n) = u(n)^2 + v(n)^2$; its evolution is shown in Figure 1. It can be observed that V increases during non-impulsive intervals but decreases significantly at each impulsive instant $n = 10, 20, \dots$, eventually tending to zero, indicating the asymptotic stability of the system.

To verify the spectral radius condition, compute the eigenvalues of $M = CA^{10}$; the maximum modulus is approximately $0.94 < 1$, satisfying the stability requirement.

The simulation results demonstrate that through periodic impulsive intervention, the duopoly profit system, originally divergent due to decision delay, can be effectively stabilized to the equilibrium point. Impulsive control suppresses fluctuations caused by competition and improves the market profit pattern.

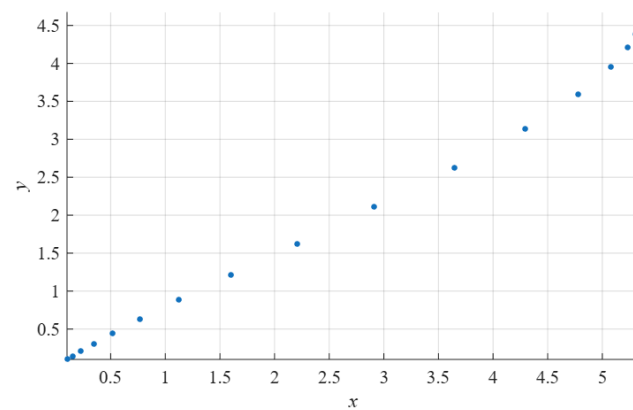


Figure 1. Phase plot of (1) with $\lambda = 0.1$.

4.1. Simulation of Nonlinear Dynamics of Decision Delay Duopoly Model

In this section, we numerically investigate the dynamics of both the no-delay model (1) and the delay model (2). For the no-delay case we set $m = 0$; for the delay case we set $m = 3$ (which is necessary for the Neimark–Sacker bifurcation to appear). The parameters are $B = 10$, $k = 0.9$, $\beta_1 = \beta_2 = 0.2$, and the adjustment speeds $\lambda_1 = \lambda_2 = \lambda$ are varied as the bifurcation parameter. The interior equilibrium is $(x^*, y^*) = (5.328, 4.671)$ for both models. To investigate the influence of the adjustment speed parameter λ on system dynamics, we first consider the no-delay model ($m = 0$). The parameters are $\beta_1 = \beta_2 = 0.2$, $B = 10$, $k = 0.9$, and λ is varied as the bifurcation parameter. Under these conditions, the system exhibits distinct nonlinear dynamic behaviors as λ changes.

When $\lambda = 0.1$, the system converges to a single equilibrium point. Solving the equilibrium condition yields $(x^*, y^*) = (5.328, 4.671)$. As shown in Figures 1 and 2, both state variables asymptotically approach this equilibrium, indicating that the system is stable.

When $\lambda = 0.4$, the system exhibits period-doubling bifurcation behavior. The equilibrium point becomes unstable, and period-doubling phenomena occur. The phase plot in Figure 3 shows periodic orbits rather than smooth closed invariant curves, and the waveform in Figure 4 displays period-doubling oscillations, indicating that the system undergoes a period-doubling bifurcation.

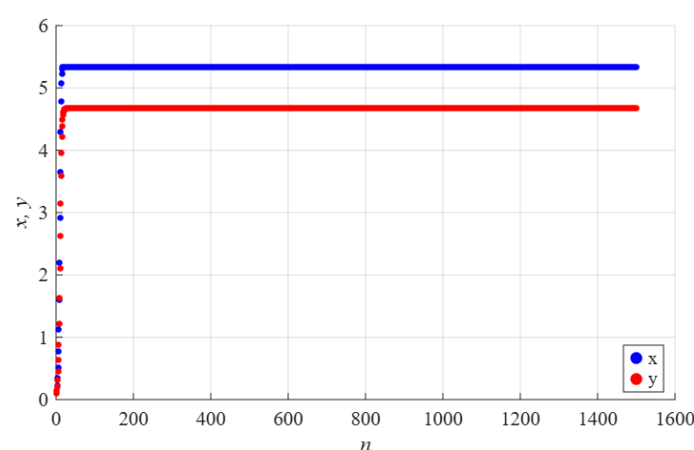


Figure 2. Waveform plot of (1) with $\lambda = 0.1$.

For $\lambda = 0.5$, the system enters a complex quasi-periodic regime. The phase plot in Figure 5 exhibits a complicated torus-like structure rather than a typical chaotic attractor, and the waveform in Figure 6 shows irregular but still quasi-periodic fluctuations. This suggests that an excessively large adjustment speed destabilizes the system, possibly leading to complex toroidal motions.

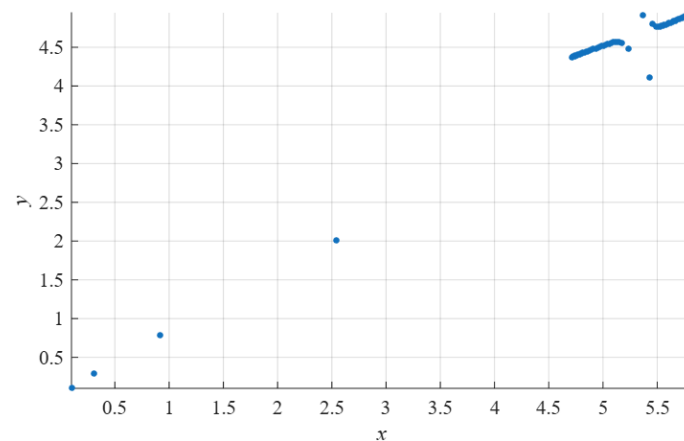


Figure 3. Phase plot of (1) with $\lambda = 0.4$.

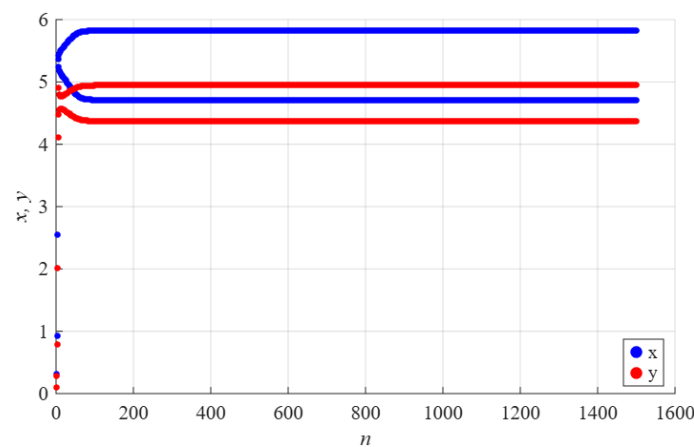


Figure 4. Waveform plot of (1) with $\lambda = 0.4$.

Furthermore, when the delay step is increased to $m = 3$ and $\lambda = 0.13$, the system exhibits characteristics of a Neimark–Sacker bifurcation. The phase plot in Figure 7 shows a smooth closed invariant curve, and the waveform in Figure 8 confirms sustained periodic oscillations, demonstrating that time delay alone can induce a Neimark–Sacker bifurcation even for a relatively small adjustment speed.

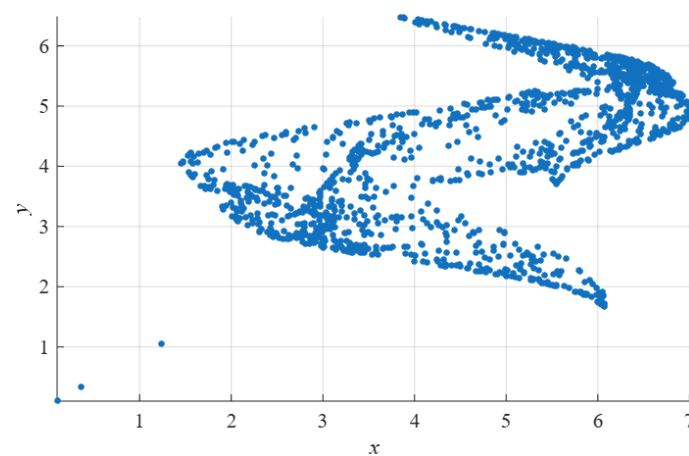


Figure 5. Phase plot of (1) with $\lambda = 0.5$.

The above simulations illustrate that as the bifurcation parameter λ increases, the system transitions from a stable equilibrium to period-doubling bifurcations and then to complex torus-like motions, while the presence of time delay ($m = 3$) can trigger a

Neimark–Sacker bifurcation, giving rise to smooth closed invariant curves. The critical bifurcation value λ_c should be determined according to the specific parameters.

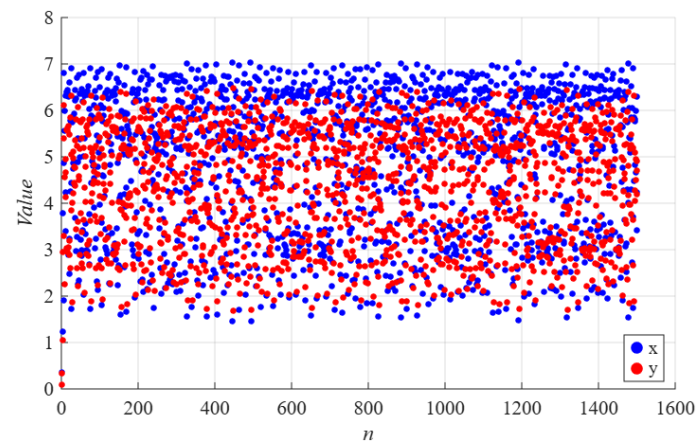


Figure 6. Waveform plot of (1) with $\lambda = 0.5$.

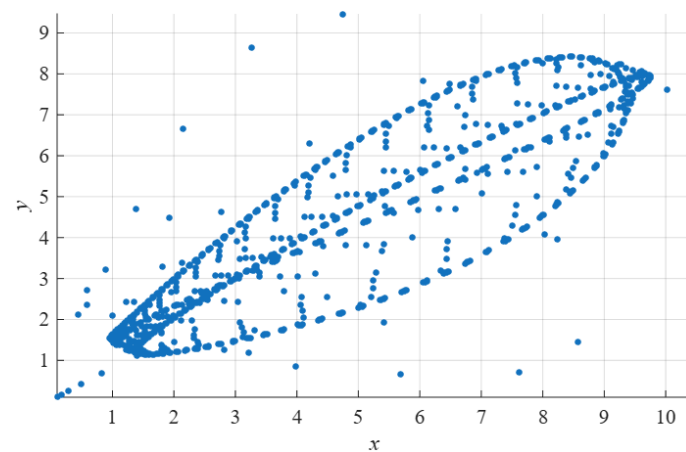


Figure 7. Phase plot of (2) with $\lambda = 0.13$ and $m = 3$.

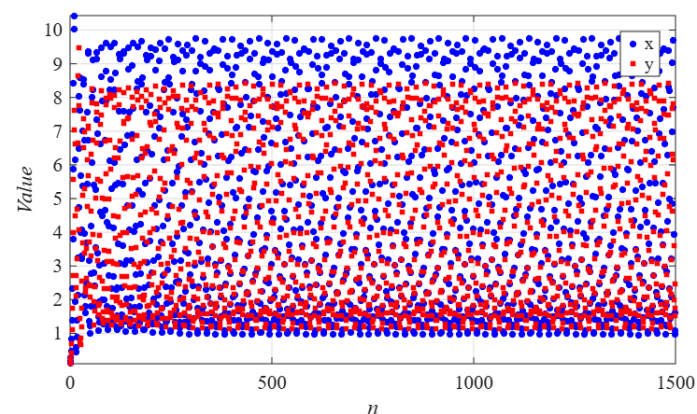


Figure 8. Waveform plot of (2) with $\lambda = 0.13$ and $m = 3$.

4.2. Simulation of Decision Delay Duopoly Model Under Impulsive Control

To demonstrate the effectiveness of the proposed improved impulsive control strategy, we apply the control scheme defined in Equation (26) (which reduces to the no-delay form when $m = 0$) to the system under both periodic and chaotic regimes. The control parameters are set to impulsive gain $\alpha = 0.8$ and impulsive interval $\tau = 10$. We first consider the no-delay model ($m = 0$) for $\lambda = 0.4$ and $\lambda = 0.5$, and then the delay model ($m = 3$) for $\lambda = 0.13$.

For the no-delay model ($m = 0$) with $\lambda = 0.4$, the uncontrolled system exhibits period-doubling oscillations (Figure 3). After applying impulsive control, the deviation variables $u(n) = x(n) - 5.328$ and $v(n) = y(n) - 4.671$ tend to zero, as shown in the controlled phase plot (Figure 9) and waveform (Figure 10), where the state variables converge to $(5.328, 4.671)$. Thus the zero solution of the deviation system is asymptotically stable.

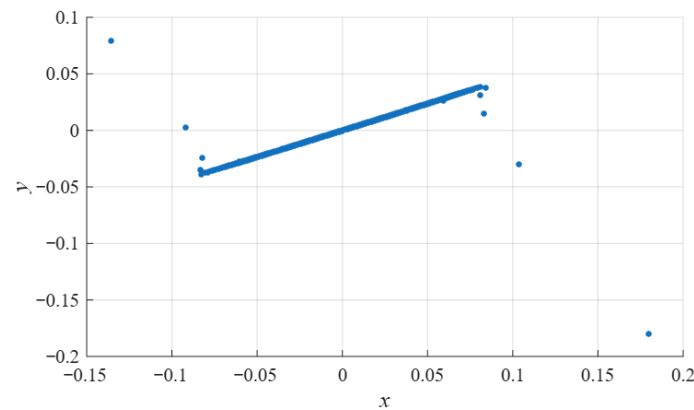


Figure 9. Phase plot of (1) with $\lambda = 0.4$ under impulsive control.

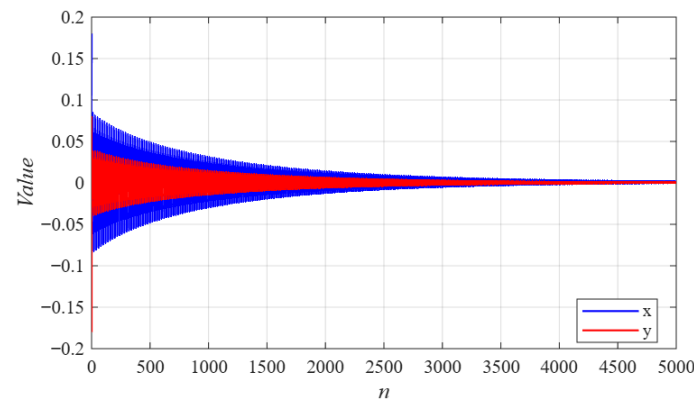


Figure 10. Waveform plot of (1) with $\lambda = 0.4$ under impulsive control.

For the no-delay model ($m = 0$) with $\lambda = 0.5$, the uncontrolled system is in a complex toroidal regime (Figure 5). Under impulsive control, the deviation variables $u(n) = x(n) - 5.328$ and $v(n) = y(n) - 4.671$ tend to zero, as evidenced by the controlled phase plot (Figure 11) and waveform (Figure 12), where the irregular fluctuations are eliminated and the system asymptotically converges to $(5.328, 4.671)$. This confirms the asymptotic stability of the zero solution of the deviation system.

For $\lambda = 0.13$ with delay $m = 3$, the uncontrolled system exhibits a limit cycle due to Neimark–Sacker bifurcation (Figure 7). After applying impulsive control, the deviation variables $u(n) = x(n) - 5.328$ and $v(n) = y(n) - 4.671$ rapidly tend to zero, as shown in Figures 13 and 14, where the state variables converge to $(5.328, 4.671)$. This confirms that the control strategy is also effective for delay-induced periodic oscillations, stabilizing the zero solution of the deviation system.

Comparing the controlled and uncontrolled cases, it is evident that without control, the system may exhibit limit cycles or complex toroidal motions, leading to persistent profit fluctuations. With the proposed improved impulsive control, the system is driven to the equilibrium point, achieving asymptotic stability. The control strategy effectively suppresses nonlinear oscillations and complex dynamics, providing a practical approach to stabilize the duopoly market under delayed marketing decisions.

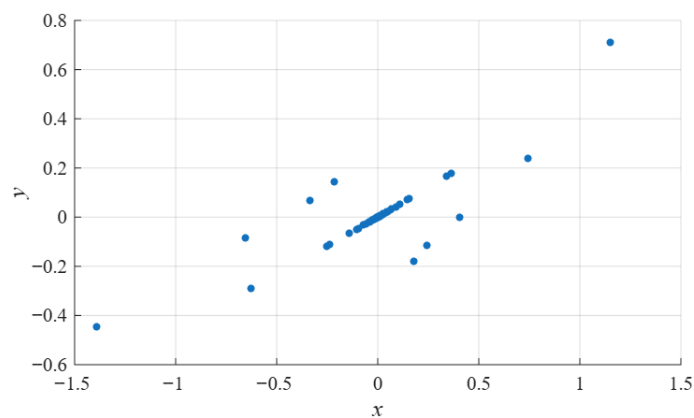


Figure 11. Phase plot of (1) with $\lambda = 0.5$ under impulsive control.

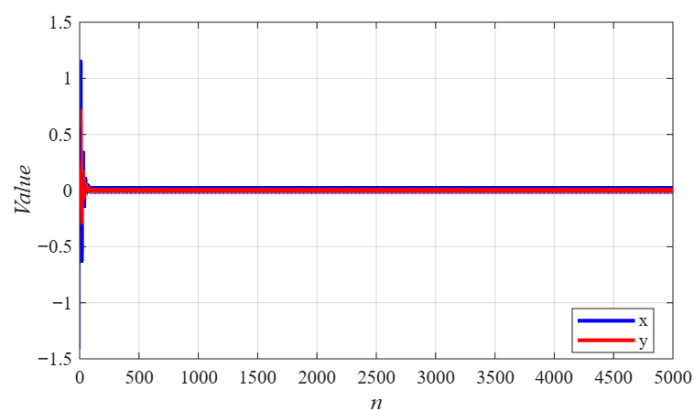


Figure 12. Waveform plot of (1) with $\lambda = 0.5$ under impulsive control.

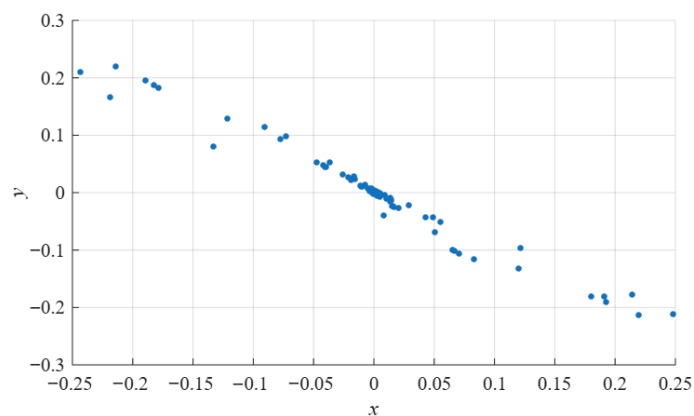


Figure 13. Phase plot of (2) with $\lambda = 0.13$ and $m = 3$ under impulsive control.

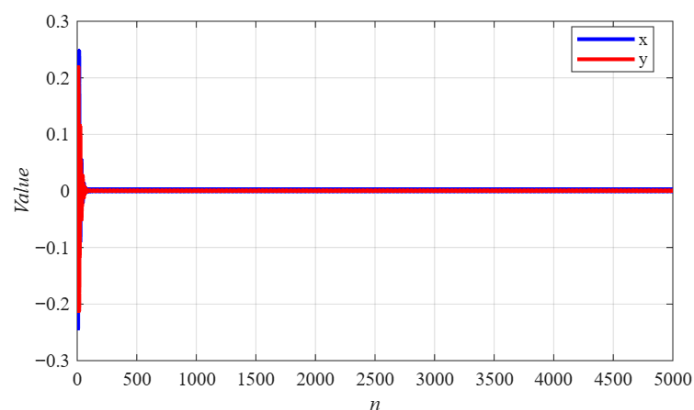


Figure 14. Waveform plot of (2) with $\lambda = 0.13$ and $m = 3$ under impulsive control.

4.3. Comparison of Traditional and Extended Impulsive Control

In this subsection, we compare the traditional impulsive control and the extended impulsive control under different model settings. Cases 1, 2, and 4 are based on the no-delay model (1) with $m = 0$; Case 3 is based on the delay model (2) with $m = 3$.

To visually compare the performance of traditional and extended impulsive control, numerical simulations are conducted under different bifurcation parameters λ and impulse parameters. The uncontrolled system exhibits a persistent limit cycle at $\lambda = 0.12$ (Figure 15), which requires impulsive control to stabilize it to the non-zero equilibrium.

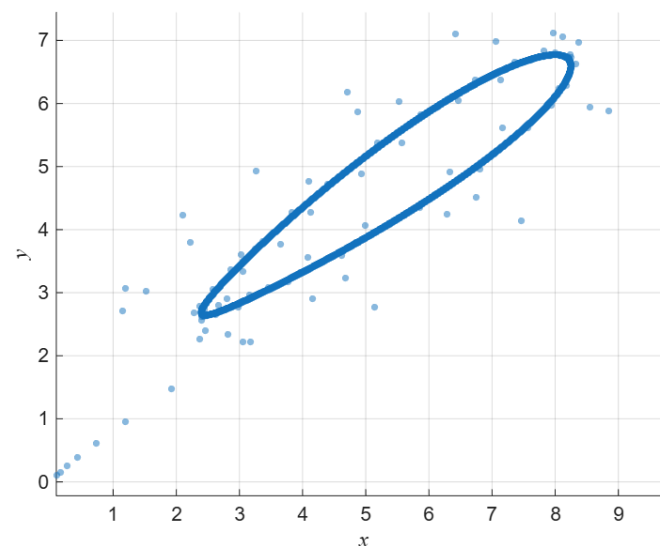


Figure 15. Limit cycle of the delay model (2) with $\lambda = 0.12$ and $m = 3$.

Case 1 ($\lambda = 0.4$; impulse parameter = 0.7): Under the same impulse parameter, the traditional impulsive control (Figure 16) takes about 5000 iterations to converge to the equilibrium, with the state variables approximately ranging in $[-0.5, 2.1]$ during the transient. In contrast, the extended impulsive control (Figure 17) takes only about 100 iterations, with a much narrower range of approximately $[0.4, 1.2]$. The convergence speed of the extended control is approximately 50 times faster than that of the traditional control, with significantly smaller overshoot, demonstrating a remarkable advantage in transient performance.

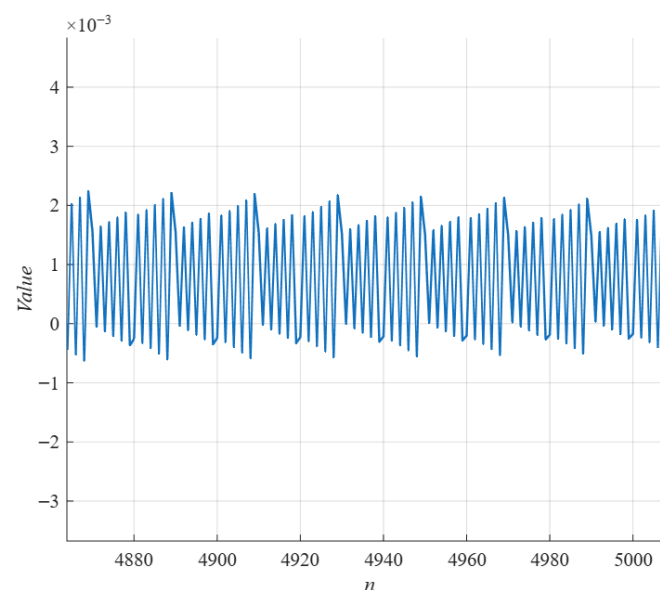


Figure 16. $\lambda = 0.4$; impulse parameter = 0.7; traditional impulsive control.

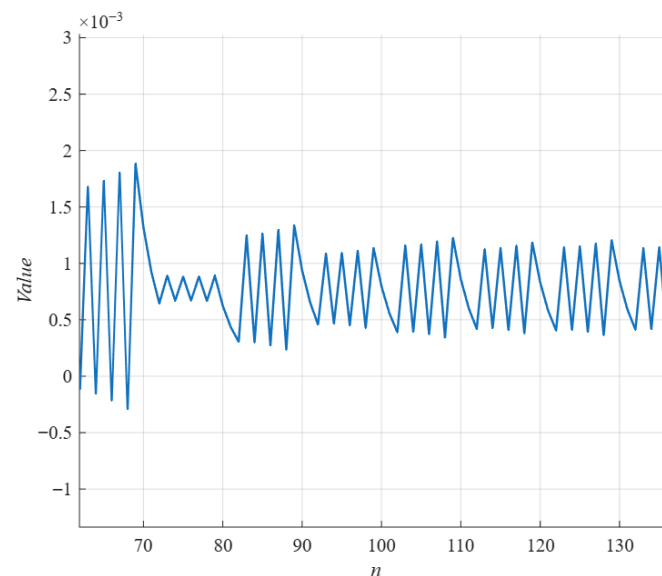


Figure 17. $\lambda = 0.4$; impulse parameter = 0.7; extended impulsive control.

Case 2 ($\lambda = 0.5$; impulse parameter = 0.01): When λ increases to 0.5 and the impulse parameter takes a very small value of 0.01, the traditional impulsive control (Figure 18) converges in about 100 iterations with an oscillatory decay, and the state variables lie roughly in $[-0.015, 0.025]$. The extended impulsive control (Figure 19) converges in only about 20 iterations, exhibits monotonic decay, and has a narrower range of approximately $[-0.01, 0.015]$. The convergence speed of the extended control is five times that of the traditional control while avoiding oscillatory tails and improving control smoothness.

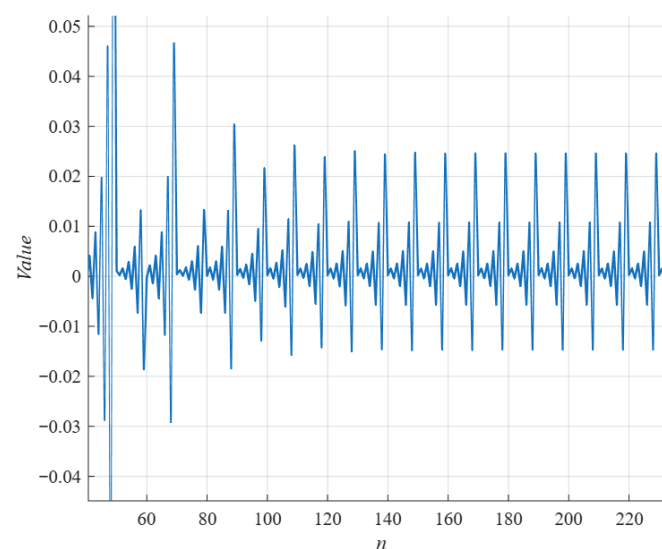


Figure 18. $\lambda = 0.5$; impulse parameter = 0.01; traditional impulsive control.

Case 3 ($\lambda = 0.13$; impulse parameter = 0.98): When $\lambda = 0.13$ and the impulse parameter is close to 1 (0.98), the traditional impulsive control (Figure 20) converges in about 110 iterations, with the variables ranging approximately in $[-1, 3]$. The extended impulsive control (Figure 21) converges in about 130 iterations, with a much tighter range of $[0.7, 1]$. In this case, the extended control converges slightly slower than the traditional control, but both successfully stabilize the system, and the extended control yields a smoother and more compact convergence process. This indicates that when the impulse parameter is extremely large, both controls are effective, with the traditional control being slightly faster in terms of iteration count.

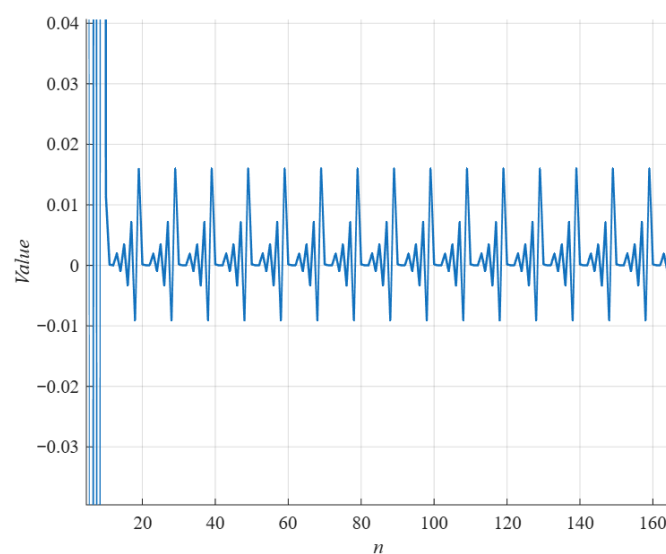


Figure 19. $\lambda = 0.5$; impulse parameter = 0.01; extended impulsive control.

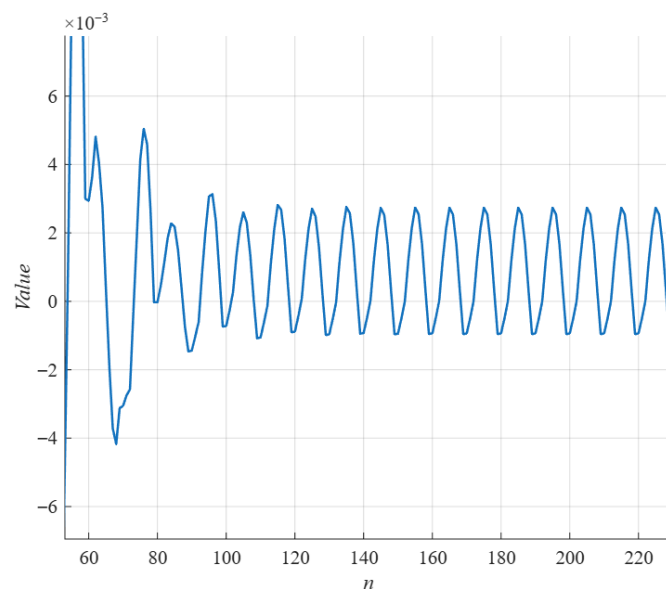


Figure 20. $\lambda = 0.13$; impulse parameter = 0.98; traditional impulsive control.

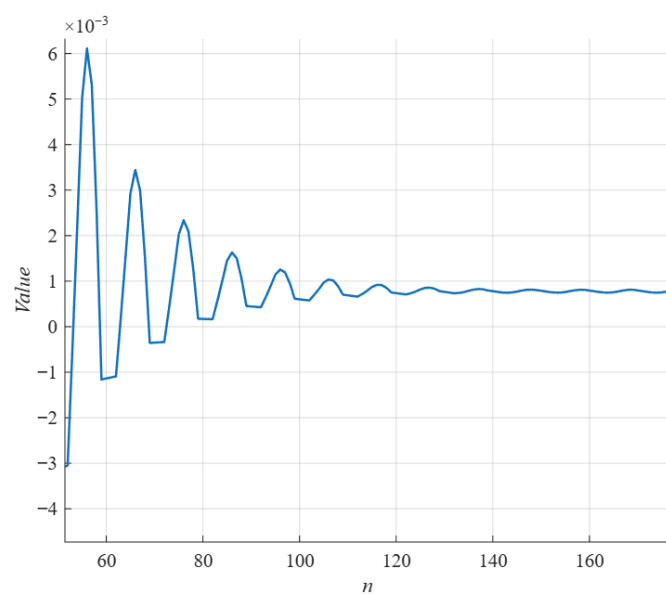


Figure 21. $\lambda = 0.13$; impulse parameter = 0.98; extended impulsive control.

Case 4 ($\lambda = 0.5$; impulse parameter = 0.05): Most notably, for $\lambda = 0.5$ and impulse parameter = 0.05, the traditional impulsive control (Figure 22) completely fails to stabilize the system: the state variables remain divergent or exhibit large sustained oscillations, fluctuating approximately in $[-0.27, 1.2]$. In contrast, the extended impulsive control (Figure 23) successfully stabilizes the system in only about 30 iterations. This comparison directly demonstrates the decisive robustness advantage of the extended impulsive control in parameter regimes where the traditional control fails.

Summarizing the above four parameter combinations: when the traditional control is effective, the extended impulsive control exhibits faster convergence in most cases (Cases 1 and 2, with speed-up factors of 50 and 5, respectively); only when the impulse parameter is extremely close to 1 (Case 3) does the extended control converge slightly slower, yet it remains effective and produces a narrower convergence range. Moreover, when the traditional control fails (Case 4), the extended control still quickly stabilizes the system. Therefore, the proposed extended impulsive control strategy is more robust and superior over a wider range of impulse parameters and system conditions.

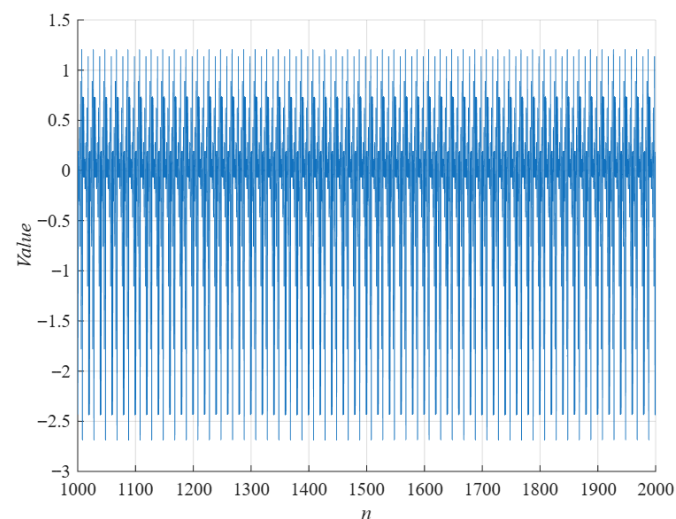


Figure 22. $\lambda = 0.5$; impulse parameter = 0.05; traditional impulsive control (fails to stabilize).

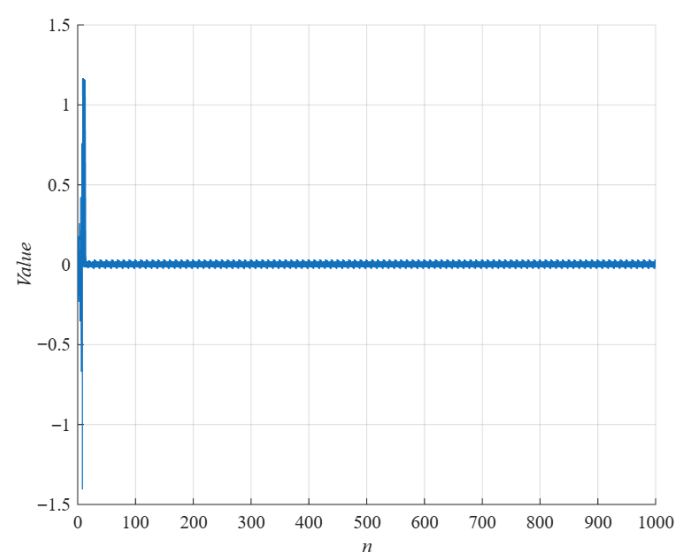


Figure 23. $\lambda = 0.5$; impulse parameter = 0.05; extended impulsive control (successfully stabilizes).

In all controlled cases, the deviation variables (u, v) asymptotically approach zero, implying that the original output variables (x, y) of the duopoly model converge to the desired

non-zero equilibrium (5.328, 4.671). The discrete delayed competition model and the extended impulsive control method constructed in this paper provide a theoretical basis for solving the dynamic instability problem caused by delayed marketing decisions in China's Low-Temperature Milk duopoly market. Future work may further consider impulsive control under stochastic disturbances and the complex dynamics of multi-oligopoly games.

This paper addresses the dynamic instability problem arising from the delayed marketing decisions of firms in China's Low-Temperature Milk duopoly market. A discrete competition model with time delay was constructed, the stability and bifurcation characteristics of the uncontrolled system were analyzed, and an impulsive control strategy was introduced for stabilization. Theoretical analysis shows that by designing appropriate impulsive attenuation factors and intervals, the controlled system can be rendered asymptotically stable. Numerical simulations verify the effectiveness of impulsive control. This research provides a theoretical basis for oligopolistic firms to optimize their marketing rhythm and for governments to formulate market intervention policies. Future work could consider impulsive control under stochastic disturbances and the complex dynamics of multi-oligopoly games.

5. Conclusions

Based on the theoretical analysis and numerical simulations conducted in this study, we constructed a discrete duopoly competition model incorporating marketing decision delays and systematically investigated the stability of the system and the mechanism by which delays induce profit fluctuations. The results show that when the adjustment speed parameter exceeds a critical threshold, the uncontrolled system loses stability through a Neimark–Sacker bifurcation, leading to periodic oscillations or more complex dynamical behaviors. Specifically, as λ increases from 0.1 to 0.4 and then to 0.5, the system transitions from a stable equilibrium to period-doubling oscillations and finally to complex torus-like motions. Time delay further contributes to periodic fluctuations.

To address this issue, an improved impulsive control strategy is introduced. Using Lyapunov stability theory, sufficient conditions for the asymptotic stability of the controlled system are derived, and an explicit relationship between the impulsive interval, control gain, and system parameters is established. Numerical simulations further demonstrate that, with appropriately designed impulsive intervention, the originally divergent system can be effectively stabilized at the equilibrium point, significantly suppressing profit volatility. A comparative analysis of controlled and uncontrolled cases confirms the effectiveness of the proposed method in eliminating limit cycles and toroidal motions.

This research not only reveals the intrinsic mechanism through which delays in marketing decisions affect market stability in the context of China's Low-Temperature Milk duopoly market but also provides a quantifiable proactive control approach for oligopolistic firms to optimize their marketing rhythm and for regulators to maintain orderly market competition. Future work may extend the model to consider stochastic disturbances, heterogeneous decision-making rules, or multi-oligopoly structures.

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Data Availability Statement: The data presented in this study are available on request from the corresponding author. The data are not publicly available due to the nature of theoretical simulation and numerical analysis, which do not generate shareable raw experimental datasets.

Conflicts of Interest: The authors declare no conflicts of interest.

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