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Oscillation of Noncanonical Nonlinear Second-Order Neutral Difference Equations via One Condition

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Abstract

This paper investigates the oscillation properties of solutions to a second-order nonlinear difference equation in noncanonical form with bounded and unbounded neutral terms. By employing the monotonicity of the neutral term together with a linearization technique, we establish new conditions that guarantee all solutions of the equation oscillate. Our results are applicable to various nonlinear forms of the equation, and, notably, the oscillation of all solutions is ensured through a single condition. Consequently, the proposed oscillation criteria are straightforward to apply and distinct from existing results on nonlinear difference equations. Four examples are presented to demonstrate the novelty and significance of the main findings.

Keywords: second-order; nonlinear; neutral difference equation; linearization; noncanonical; oscillation

MSC: 39A21

1. Introduction

The main objective of this paper is to analyze the oscillatory properties of solutions to the following second-order noncanonical neutral difference equation

$$\Delta(a(v)(\Delta(y(v) + p(v)y(v - \tau)))^\alpha) + q(v)y^\beta(v - \sigma) = 0, \quad (1)$$

where $v \geq v_0 \in \mathbb{N}(v_0) = \{v_0, v_0 + 1, \dots\}$, v_0 is a positive integer, both α and β are expressed as ratios of odd positive integers with α greater than or equal to 1. Also we assume that

(H₁) The sequences $\{a(v)\}$, $\{p(v)\}$ and $\{q(v)\}$ are positive real-valued, and $p(v)$ satisfies $p(v) \geq m > 1$;

(H₂) Two positive integers, τ and σ , are given with $\tau < \sigma$;

(H₃) Equation (1) is in noncanonical form, that is,

$$\sum_{v=v_0}^{\infty} a^{-1/\alpha}(v) < \infty.$$



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For the sake of brevity, we let $z(v) = y(v) + p(v)y(v - \tau)$. The solution of (1) is a real sequence $\{y(v)\}$ that is defined for $v \geq v_0 - \tau$ and satisfies (1) for all $v \geq v_0$. “A proper solution of (1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise we call it nonoscillatory. The Equation (1) itself is called oscillatory if all its solutions are oscillatory”.

It follows from [1], that Equation (1) is in canonical form if

$$A(v) = \sum_{s=v_0}^{v-1} a^{-1/\alpha}(s) \rightarrow \infty \text{ if } v \rightarrow \infty$$

and is in the noncanonical form if

$$B(v_0) = \sum_{v=v_0}^{\infty} a^{-1/\alpha}(v) < \infty.$$

Neutral difference equations are a class of functional difference equations in which the highest-order difference of the unknown sequence appears both with and without deviating arguments. Noncanonical second-order neutral difference equations are particularly important for analyzing complex nonlinear discrete systems, where future states depend on both current and past rates of change. They capture memory effects in population biology, economics, and engineering, and can often be transformed into simpler canonical forms to study oscillation, asymptotic behavior, and stability. Such equations have attracted considerable interest among researchers not only for their theoretical significance but also for their wide-ranging applications across diverse fields. Applications of these equations can be found in [2–5] and the references cited therein.

Taking the neutral coefficient $p(v) > 1$ in neutral difference equations is critical because it fundamentally alters stability and asymptotic behavior, often driving solutions toward instability, oscillation, or convergence, unlike the stable damping behavior associated with $0 < p(v) < 1$. The condition $p(v) > 1$ is essential for describing higher-order, nonlinear, or delayed feedback mechanisms in engineering and biological models. Similarly, the condition $\tau < \sigma$ is necessary to introduce a delay effect in the considered equation. Conversely, if $\tau \geq \sigma$, the equation may reduce to a non-delay or advanced type, thereby changing its qualitative behavior.

Oscillation in difference equations, where solutions alternate around an equilibrium, is critical for modeling, predicting, and controlling discrete dynamical systems. It reveals stability, periodicity, and chaotic tendencies in fields such as biology, economics, and engineering by identifying whether solutions diverge, converge, or fluctuate due to delayed feedback or discrete time steps. Understanding oscillation is essential to ensure that discrete, time-stepping models accurately capture the inherently fluctuating nature of real-world physical and biological systems. Therefore, oscillation theory remains one of the most important and continuously developing branches of the qualitative theory of difference equations.

Following the pioneering work of Hartman [6] on well-known results concerning the oscillation of second-order difference equations, considerable interest has arisen in establishing oscillation criteria for various classes of difference equations using diverse methods. Several techniques have been developed for this purpose; among the most widely employed are the Riccati transformation, summation averaging techniques, comparison theorems, and linearization methods. For further details, see the monographs [2,3], the papers [7–18], and the references cited therein.

In particular, Koplatadze [19] focus on the canonical form of a delay difference equation

$$\Delta^2 y(v) + q(v)y(v - \sigma) = 0, \quad (2)$$

where σ is a positive integer, and proved Equation (2) was oscillatory if

$$\liminf_{v \rightarrow \infty} \frac{1}{v} \sum_{s=v_0}^{v-1} s^2 q(s) > \frac{1}{4}. \quad (3)$$

Later, in [20], the authors proved the following to guarantee the oscillation of the noncanonical equation

$$\Delta(a(v)\Delta y(v)) + q(v)y(v - \sigma) = 0, \quad (4)$$

namely,

$$\limsup_{v \rightarrow \infty} \left\{ A(v - \sigma) \sum_{s=v_0}^{v-\sigma-1} q(s) + \sum_{s=v-\sigma}^{v-1} A(s+1)q(s) + \frac{1}{A(v - \sigma)} \sum_{s=v}^{\infty} A(s+1)A(s - \sigma)q(s) \right\} > 1. \quad (5)$$

In [3], the authors extended the result obtained in [19] to Equation (4) when it is in noncanonical form. In [21], the authors studied Equation (1) or its particular cases with $p(v) \equiv 0$ and in canonical form using the linearization technique along with the Koplatadze method.

In [22], the authors demonstrated that $0 \leq p(v) < 1$ along with

$$\sum_{v=v_0}^{\infty} q(v)(1 - p(v - \sigma)) = \infty$$

implies the oscillation of all solutions of the neutral difference equation

$$\Delta^2(y(v) + p(v)y(v - \tau)) + q(v)y(v - \sigma) = 0.$$

In [23,24], the authors studied Equation (1) or its particular cases when either $0 \leq p(v) \leq p < \infty$ or $p(v) > 1$ and obtained criteria for the oscillation of all solutions of the studied equation.

Most existing results focus on canonical-type equations because these equations are easier to handle and possess only one set of nonoscillatory solutions. Consequently, a single condition is sufficient to obtain oscillation results. In contrast, noncanonical-type equations admit two sets of nonoscillatory solutions, and oscillation criteria for such equations are typically derived by employing comparison theorems and summation averaging techniques for different values of α and β .

Recently, in [25], the authors consider the Equation (1) with $\alpha = 1$ and $0 \leq p(v) < p < 1$ and obtained some oscillation criteria by applying the comparison technique and Koplatadze method.

Motivated by the aforementioned studies, and using a combination of the linearization technique with the monotonic properties of the neutral term, our objective is to present new oscillation criteria for Equation (1) for different values of α and β . Moreover, in contrast to the results reported in [23,24], an important aspect of our findings is that the oscillation of Equation (1) is ensured through a single condition. Additionally, four numerical examples are provided to demonstrate the novelty and significance of our main results in the existing literature. The proof technique employed here is based in part on recent contributions [26,27] to the oscillation theory of functional differential equations.

In particular, we assume that $\{y(v)\}$ is a solution of (1). Under this assumption, $\{-y(v)\}$ is also a solution of (1). When considering nonoscillatory solutions, it is sufficient to examine only the positive ones, since the proofs for negative solutions are analogous due to the structure of (1).

2. Oscillation Results

In this section, we present a single-condition criterion for the oscillation of Equation (1). For convenience, we set

$$A(v) = \sum_{s=v_*}^{v-1} a^{-1/\alpha}(s), \quad v_* \geq v_0, \quad B(v) = \sum_{s=v}^{\infty} a^{-1/\alpha}(s),$$

$$Q(v) = q(v)p^{-\beta}(v + \tau - \sigma), \quad k = (m-1)^\beta m^{-\beta},$$

and

$$z(v) = y(v) + p(v)y(v - \tau).$$

Lemma 1. Let $\{y(v)\}$ be an eventually positive solution of (1) on $\mathbb{N}(v_0)$. If

$$\sum_{v=v_0}^{\infty} B^\alpha(v-1)q(v)p^{-\beta}(v + \tau - \sigma) = \infty, \quad (6)$$

then the corresponding sequence $\{z(v)\}$ belongs to the class S_0 eventually, where

$$z(v) \in S_0 \Leftrightarrow z(v) > 0, \quad a^{1/\alpha}(v)\Delta z(v) < 0, \quad \Delta(a^{1/\alpha}(v)\Delta z(v)) \leq 0.$$

Proof. Let $\{y(v)\}$ be an eventually positive solution of (1) on $\mathbb{N}(v_0)$ and there exists an $v_1 \in \mathbb{N}(v_0)$ such that $y(v) > 0$, $y(v - \tau) > 0$ and $y(v - \sigma) > 0$ for all $v \geq v_1$. Now, for $v \geq v_1$, $z(v) > 0$ and

$$\Delta(a(v)(\Delta z(v))^\alpha) = -q(v)y^\beta(v - \sigma) \leq 0, \quad (7)$$

which implies that $\Delta z(v)$ is of one sign eventually, that is, $\Delta z(v) > 0$ or $\Delta z(v) < 0$ eventually. We claim that $\Delta z(v) < 0$ eventually. If this is not true, then there is an integer $v_2 \geq v_1$ such that $\Delta z(v) > 0$ for $v \geq v_2$. It is easy to see that by Mean-Value Theorem, we find

$$\Delta(a(v)(\Delta z(v))^\alpha) \geq \alpha(a^{1/\alpha}(v)\Delta z(v))^{\alpha-1}\Delta(a^{1/\alpha}(v)\Delta z(v)). \quad (8)$$

From (8), α is a ratio of odd positive integers, we conclude that from (7)

$$\operatorname{sgn} \Delta(a^{1/\alpha}(v)\Delta z(v)) = \operatorname{sgn} \Delta(a(v)(\Delta z(v))^\alpha).$$

Hence

$$z(v) > 0, \quad a^{1/\alpha}(v)\Delta z(v) > 0, \quad \Delta(a^{1/\alpha}(v)\Delta z(v)) \leq 0, \quad v \geq v_2.$$

Now

$$z(v) \geq \sum_{s=v_2}^{v-1} \frac{a^{1/\alpha}(s)\Delta z(s)}{a^{1/\alpha}(s)} \geq A(v)a^{1/\alpha}(v)\Delta z(v),$$

which implies that

$$\Delta\left(\frac{z(v)}{A(v)}\right) = \frac{A(v)a^{1/\alpha}(v)\Delta z(v) - z(v)}{A(v)A(v+1)a^{1/\alpha}(v)} \leq 0,$$

for $v \geq v_3$, for some integer $v_3 \geq v_2$, that is, $\frac{z(v)}{A(v)}$ is nonincreasing for $v \geq v_3$. From the definition of $z(v)$, we have

$$\begin{aligned} y(v) &= \frac{1}{p(v+\tau)} [z(v+\tau) - y(v+\tau)] \\ &\geq \frac{z(v+\tau)}{p(v+\tau)} - \frac{z(v+2\tau)}{p(v+\tau)p(v+2\tau)}. \end{aligned} \quad (9)$$

Since $\frac{z(v)}{A(v)}$ is nonincreasing, we see that

$$z(v+2\tau) \leq \frac{A(v+2\tau)z(v+\tau)}{A(v+\tau)} \quad (10)$$

and using (10) in (9) yields

$$y(v) \geq \frac{z(v+\tau)}{p(v+\tau)} \left[1 - \frac{A(v+2\tau)}{A(v+\tau)} \frac{1}{p(v+2\tau)} \right] \text{ for } v \geq v_3. \quad (11)$$

By (H_1) and the fact that

$$\lim_{v \rightarrow \infty} \frac{A(v+\tau)}{A(v)} = 1,$$

we see that there exists an integer $v_4 \geq v_3$ such that for any $\epsilon \in (0, m-1)$ and $v \geq v_4$

$$\frac{A(v+2\tau)}{A(v+\tau)} \frac{1}{p(v+2\tau)} \leq \frac{1+\epsilon}{m}.$$

Using this in (11) gives

$$y(v) \geq d \frac{z(v+\tau)}{p(v+\tau)} \text{ for } v \geq v_4, \quad (12)$$

where $d = 1 - \frac{1+\epsilon}{m} > 0$. Inserting (12) into (1) yields

$$\Delta(a(v)(\Delta z(v))^\alpha) + d^\beta q(v)p^{-\beta}(v+\tau-\sigma)z^\beta(v+\tau-\sigma) \leq 0, \quad (13)$$

for $v \geq v_5 \geq v_4$. By the monotonicity of $z(v)$, we have $z(v) \geq z(v_5) = c > 0$ and so the inequality (13) becomes

$$\Delta(a(v)(\Delta z(v))^\alpha) + d^\beta c^\beta q(v)p^{-\beta}(v+\tau-\sigma) \leq 0, \quad v \geq v_6 \geq v_5. \quad (14)$$

Since $B(v)$ is positive, decreasing and $\lim_{v \rightarrow \infty} B(v) = 0$, we can choose a sufficiently large integer $v_* \in \mathbb{N}(v_0)$ such that

$$0 < B(v) < 1 \text{ for } v \geq v_*. \quad (15)$$

Let $v_7 = \max\{v_6, v_*\}$. Summing (14) from v_6 to ∞ and taking (15) into account, we obtain

$$\begin{aligned} a(v_6)(\Delta z(v_6))^\alpha &\geq c^\beta d^\beta \sum_{v=v_6}^{\infty} q(v)p^{-\beta}(v+\tau-\sigma) \geq c^\beta d^\beta \sum_{v=v_7}^{\infty} q(v)p^{-\beta}(v+\tau-\sigma) \\ &\geq c^\beta d^\beta \sum_{v=v_7}^{\infty} B^\alpha(v+1)q(v)p^{-\beta}(v+\tau-\sigma), \end{aligned}$$

which is a contradiction in view of (6) and so $\Delta z(v) < 0$ eventually and this ends the proof. $\square$

Lemma 2. Let $\{y(v)\}$ be an eventually positive solution of (1) on $\mathbb{N}(v_0)$. If (6) holds, then the following are satisfied eventually:

$$(i) \quad z(v) + B(v)a^{1/\alpha}(v)\Delta z(v) \geq 0;$$

$$(ii) \quad \frac{z(v)}{B(v)} \text{ is nondecreasing};$$

$$(iii) \quad \left(\frac{z(v+\tau-\sigma)}{a^{1/\alpha}(v)\Delta z(v)} \right)^{\alpha-1} \geq B^{\alpha-1}(v);$$

and

$$(iv) \quad \lim_{v \rightarrow \infty} z(v) = 0.$$

Proof. Choose $v_1 \in \mathbb{N}(v_0)$ so that $y(v) > 0$, $y(v-\tau) > 0$ and $y(v-\sigma) > 0$ for all $v \geq v_1$. Proceeding as in the proof of Lemma 1, we again see that the corresponding sequence $z(v)$ belongs to the class S_0 eventually, say, for $v \geq v_2$ for some integer $v_2 \geq v_1$. Since $a^{1/\alpha}\Delta z(v)$ is negative and decreasing,

$$z(v) \geq - \sum_{s=v}^{\infty} \frac{a^{1/\alpha}(s)\Delta z(s)}{a^{1/\alpha}(s)} \geq -B(v)a^{1/\alpha}(v)\Delta z(v), \quad (16)$$

which implies that part (i) holds.

From (16), we see that

$$\Delta \left(\frac{z(v)}{B(v)} \right) = \frac{B(v)a^{1/\alpha}(v)\Delta z(v) + z(v)}{B(v)B(v+1)a^{1/\alpha}(v)} \geq 0,$$

that is, part (ii) is fulfilled.

Since $v + \tau - \sigma < v$, we get $z(v + \tau - \sigma) \geq z(v)$ and, using this in (16), we observe that

$$\left(\frac{z(v + \tau - \sigma)}{a^{1/\alpha}(v)\Delta z(v)} \right)^{\alpha-1} \geq B^{\alpha-1}(v), \quad (17)$$

which proves (iii).

From the monotonic property of $z(v)$ for $v \geq v_2$, we can choose $M_1 \geq 0$ such that $\lim_{v \rightarrow \infty} z(v) = M_1$. We will show that $M_1 = 0$. If $M_1 > 0$, then there is an integer $v_3 \geq v_2$ such that for any $\epsilon > 0$,

$$M_1 < z(v) < M_1 + \epsilon \quad \text{for } v \geq v_3. \quad (18)$$

Since $z(v + \tau) \geq z(v + 2\tau)$ and using this in (9) yields

$$y(v) \geq \frac{1}{p(v + \tau)} \left[1 - \frac{1}{p(v + 2\tau)} \right] z(v + \tau) \quad \text{for } v \geq v_3,$$

which together with (H_1) gives

$$y(v) \geq k^{1/\beta} \frac{z(v + \tau)}{p(v + \tau)} \quad \text{for } v \geq v_3. \quad (19)$$

Combining (19) with (1) gives

$$\Delta(a(v)(\Delta z(v))^\alpha) + kq(v)p^{-\beta}(v + \tau - \sigma)z^\beta(v + \tau - \sigma) \leq 0, \quad (20)$$

for $v \geq v_4$ for some $v_4 \geq v_3$. In view of (8), we can rewrite (20) as

$$\Delta(a^{1/\alpha}(v)\Delta z(v)) + \frac{k}{\alpha} \frac{z^{\alpha-1}(v + \tau - \sigma)}{(a^{1/\alpha}(v)\Delta z(v))^{\alpha-1}} q(v)p^{-\beta}(v + \tau - \sigma)z^{\beta-\alpha+1}(v + \tau - \sigma) \leq 0. \quad (21)$$

By virtue of (17) and (21), we obtain

$$\Delta(a^{1/\alpha}(v)\Delta z(v)) + \frac{k}{\alpha}B^{\alpha-1}(v)q(v)p^{-\beta}(v+\tau-\sigma)z^{\beta-\alpha+1}(v+\tau-\sigma) \leq 0 \quad (22)$$

for $v \geq v_4$. From (18) and (22), we see that

$$\Delta(a^{1/\alpha}(v)\Delta z(v)) + dB^{\alpha-1}(v)q(v)p^{-\beta}(v+\tau-\sigma) \leq 0 \quad (23)$$

for $v \geq v_5 \geq v_4$, where $d = \frac{kM_1^{1+\beta}}{\alpha(M_1+\epsilon)^\alpha} > 0$. Summing (23) two times yields

$$\begin{aligned} z(v_5) &\geq d \sum_{v=v_5}^{\infty} \frac{1}{a^{1/\alpha}(v)} \sum_{s=v_5}^{v-1} B^{\alpha-1}(s)q(s)p^{-\beta}(s+\tau-\sigma) \\ &\geq d \sum_{v=v_5}^{\infty} B^{\alpha}(v+1)q(v)p^{-\beta}(v+\tau-\sigma) = \infty, \end{aligned}$$

which is a contradiction. This contradiction proves that $M_1 = 0$ and the proof ends. $\square$

Now, we are ready to derive oscillation conditions for (1) when $\beta \leq \alpha$.

Lemma 3. Let $\{y(v)\}$ be an eventually positive solution of (1) on $\mathbb{N}(v_0)$. If (6) holds, then the corresponding sequence $\{z(v)\}$ satisfies the linear inequalities

$$\Delta(a^{1/\alpha}(v)\Delta z(v)) + \frac{k}{\alpha}B^{\alpha-1}(v+1)Q(v)z(v+\tau-\sigma) \leq 0 \quad (24)$$

and

$$\Delta(z(v) + B(v)a^{1/\alpha}(v)\Delta z(v)) + \frac{k}{\alpha}B^{\alpha}(v+1)Q(v)z(v+\tau-\sigma) \leq 0, \quad (25)$$

eventually.

Proof. Proceeding as in the proof of Lemma 2, we arrive at (22) for $v \geq v_4$. Since $z(v)$ is positive, decreasing and $\lim_{v \rightarrow \infty} z(v) = 0$, there exists a sufficiently large $v_5 \geq v_4$ such that

$$0 < z(v) < 1 \quad \text{for } v \geq v_5. \quad (26)$$

In view of (26) and $\beta \leq \alpha$, we have

$$z^{\beta/\alpha}(v) \geq z(v) \quad \text{for } v \geq v_5. \quad (27)$$

From (26) and the fact that $\beta - \alpha \leq 0$, we have $\frac{\beta-\alpha}{\alpha} \geq (\beta - \alpha)$ and therefore obtain

$$z^{\beta-\alpha}(v) \geq z^{(\beta-\alpha)/\alpha}(v) \quad \text{for } v \geq v_5. \quad (28)$$

Using (28) in (22)

$$\Delta(a^{1/\alpha}(v)\Delta z(v)) + \frac{k}{\alpha}B^{\alpha-1}(v+1)Q(v)z^{\beta/\alpha}(v+\tau-\sigma) \leq 0 \quad (29)$$

for $v \geq v_6$ for some $v_6 \geq v_5$. Using (27) in (29) leads to

$$\Delta(a^{1/\alpha}(v)\Delta z(v)) + \frac{k}{\alpha}B^{\alpha-1}(v+1)Q(v)z(v+\tau-\sigma) \leq 0 \quad (30)$$

for $v \geq v_6$, which proves (26).

From (30) and the fact that

$$\Delta(z(v) + B(v)a^{1/\alpha}(v)\Delta z(v)) = B(v+1)\Delta(a^{1/\alpha}(v)\Delta z(v)), \quad (31)$$

we find that

$$\Delta(z(v) + B(v)a^{1/\alpha}(v)\Delta z(v)) + \frac{k}{\alpha}B^\alpha(v+1)Q(v)z(v+\tau-\sigma) \leq 0$$

for $v \geq v_6$, that is, (25) holds. This ends the proof. $\square$

Our first oscillation criterion is of Koplatadze type.

Theorem 1. *If*

$$\limsup_{v \rightarrow \infty} \Omega(v) > \frac{\alpha}{k}, \quad (32)$$

where

$$\Omega(v) = B(v+1) \sum_{s=v_0}^{v-1} B^{\alpha-1}(s+1)Q(s) + \frac{1}{B(v+\tau-\sigma)} \sum_{s=v}^{\infty} B^\alpha(s)Q(s)B(s+\tau-\sigma),$$

then Equation (1) oscillates.

Proof. For the sake of contradiction, assume $\{y(v)\}$ is a nonoscillatory solution of (1), such that $y(v) > 0$, $y(v-\tau) > 0$ and $y(v-\sigma) > 0$ for $v_1 \in \mathbb{N}(v_0)$, but condition (32) holds. Then, from (32), there exists a positive constant c such that

$$\limsup_{v \rightarrow \infty} \Omega(v) \geq c. \quad (33)$$

We now prove that (33) implies (6). Indeed, if not, then

$$\sum_{v=v_1}^{\infty} B^\alpha(v+1)Q(v) < \infty.$$

Therefore, there exists a sufficiently large $v_2 \in \mathbb{N}(v_1)$ such that

$$\sum_{v=v_2}^{\infty} B^\alpha(v+1)Q(v) < \frac{c}{4}. \quad (34)$$

Now, since $B(v)$ is positive, decreasing, and $\lim_{v \rightarrow \infty} B(v) = 0$, we find that there exists a sufficiently large $v_* \in \mathbb{N}(v_0)$ such that (15) holds. Let $v_3 = \max\{v_2, v_*\}$. Then, for $v \geq v_3$, it follows from (15) and (34) that

$$\begin{aligned} B(v+1) \sum_{s=v_1}^{v-1} B^{\alpha-1}(s+1)Q(s) &= B(v+1) \sum_{s=v_1}^{v_3-1} B^{\alpha-1}(s+1)Q(s) \\ &\quad + B(v+1) \sum_{s=v_3}^{v-1} B^{\alpha-1}(s+1)Q(s) \\ &\leq B(v+1) \sum_{s=v_1}^{v_3-1} B^{\alpha-1}(s+1)Q(s) \\ &\quad + \sum_{s=v_3}^{v-1} B^\alpha(s+1)Q(s) \\ &\leq B(v+1) \sum_{s=v_1}^{v_3-1} B^{\alpha-1}(s+1)Q(s) + \frac{c}{4}. \end{aligned} \quad (35)$$

Also, for $v \geq v_3$

$$\begin{aligned} \frac{1}{B(v+\tau-\sigma)} \sum_{s=v}^{\infty} B^{\alpha}(s+1)Q(s)B(s+\tau-\sigma) &\leq \sum_{s=v}^{\infty} B^{\alpha}(s+1)Q(s) \\ &\leq \sum_{s=v_3}^{\infty} B^{\alpha}(s+1)Q(s) \leq \frac{c}{4}. \end{aligned} \quad (36)$$

Hence, from (35) and (36), we deduce that

$$\limsup_{v \rightarrow \infty} \Omega(v) \leq \frac{c}{2},$$

which contradicts (33) and this contradiction implies that (6) holds. Thus, all the results of Lemmas 1–3 are satisfied. Proceeding exactly as in the proof of Lemma 3, we again arrive at (24) and (25) for $v \geq v_3$. Summing (25) from $v \geq v_3$ to ∞ yields

$$z(v) + B(v)a^{1/\alpha}(v)\Delta z(v) \geq \frac{k}{\alpha} \sum_{s=v}^{\infty} B^{\alpha}(s+1)Q(s)z(s+\tau-\sigma). \quad (37)$$

Taking summation of (24) from v_3 to $v-1$ gives

$$-B(v)a^{1/\alpha}(v)\Delta z(v) \geq \frac{k}{\alpha} B(v+1) \sum_{s=v_3}^{v-1} B^{\alpha-1}(s+1)Q(s)z(s+\tau-\sigma). \quad (38)$$

Combining (37) and (38), one gets

$$\begin{aligned} z(v) &\geq \frac{k}{\alpha} B(v+1) \sum_{s=v_3}^{v-1} B^{\alpha-1}(s+1)Q(s)z(s+\tau-\sigma) \\ &\quad + \frac{k}{\alpha} \sum_{s=v}^{\infty} B^{\alpha}(s+1)Q(s)z(s+\tau-\sigma). \end{aligned} \quad (39)$$

Using the monotonicity properties of $z(v)$ and $\frac{z(v)}{B(v)}$, we obtain

$$\sum_{s=v_3}^{v-1} B^{\alpha-1}(s+1)Q(s)z(s+\tau-\sigma) \geq \left(\sum_{s=v_3}^{v-1} B^{\alpha-1}(s+1)Q(s) \right) z(v) \quad (40)$$

and

$$\begin{aligned} &\sum_{s=v}^{\infty} B^{\alpha}(s+1)Q(s)z(s+\tau-\sigma) \\ &\geq \left(\frac{z(v)}{B(v+\tau-\sigma)} \sum_{s=v}^{\infty} B^{\alpha}(s+1)Q(s)B(s+\tau-\sigma) \right). \end{aligned} \quad (41)$$

Using (40), (41) in (39), we get

$$\begin{aligned} \frac{\alpha}{k} &\geq B(v+1) \sum_{s=v_3}^{v-1} B^{\alpha-1}(s+1)Q(s) \\ &\quad + \frac{1}{B(v+\tau-\sigma)} \sum_{s=v}^{\infty} B^{\alpha}(s+1)Q(s)B(s+\tau-\sigma). \end{aligned}$$

Taking $\limsup$ as $v \rightarrow \infty$ in the previous inequality, we get a contradiction with (32). This completes the proof. $\square$

From the above theorem, we immediately obtain the following corollary.

Corollary 1. Let $\alpha = \beta$. If

$$\limsup_{v \rightarrow \infty} \Omega(v) > \alpha \left(\frac{m}{m-1} \right)^\alpha, \quad (42)$$

where

$$\Omega(v) = B(v+1) \sum_{s=v_0}^{v-1} B^{\alpha-1}(s+1)Q(s) + \frac{1}{B(v+\tau-\sigma)} \sum_{s=v}^{\infty} B^\alpha(s)Q(s)B(s+\tau-\sigma),$$

then Equation (1) oscillates.

The following criterion is of Myshkis type and so different from the previous results.

Theorem 2. If

$$\limsup_{v \rightarrow \infty} \sum_{s=v+\tau-\sigma}^{v-1} B^\alpha(s+1)Q(s) > \frac{\alpha}{k}, \quad (43)$$

then Equation (1) oscillates.

Proof. Let $\{y(v)\}$ be a nonoscillatory solution of (1), such that $y(v) > 0$, $y(v-\tau) > 0$ and $y(v-\sigma) > 0$ for all $v \in \mathbb{N}(v_1)$. It is clear that (43) implies that (6). Hence, all the results of Lemmas 1–3 are satisfied. From Lemma 3, we have (25). Let

$$x(v) = z(v) + B(v)a^{1/\alpha}(v)\Delta z(v),$$

and taking $0 \leq x(v) \leq z(v)$ into account, we obtain from (25) that

$$\Delta y(v) + \frac{k}{\alpha} B^\alpha(v+1)Q(v)y(v+\tau-\sigma) \leq 0.$$

Summing the last inequality from $v+\tau-\sigma$ to $v-1$ gives

$$\begin{aligned} y(v+\tau-\sigma) &\geq \frac{k}{\alpha} \sum_{s=v+\tau-\sigma}^{v-1} B^\alpha(s+1)Q(s)y(s+\tau-\sigma) \\ &\geq \left(\frac{k}{\alpha} \sum_{s=v+\tau-\sigma}^{v-1} B^\alpha(s+1)Q(s) \right) y(v+\tau-\sigma), \end{aligned}$$

from which

$$\frac{\alpha}{k} \geq \sum_{s=v+\tau-\sigma}^{v-1} B^\alpha(s+1)Q(s).$$

Letting the lim sup as $v \rightarrow \infty$, we see a contradiction with (43). This ends the proof. $\square$

Next, we derive the oscillation criteria for the case $\beta > \alpha$. In our previous results, we have used the fact that $z(v) \rightarrow 0$ as $v \rightarrow \infty$, but in our next results, we use $\frac{z(v)}{B(v)} \rightarrow \infty$ as $v \rightarrow \infty$.

Lemma 4. Let $\{y(v)\}$ be an eventually positive solution of (1) on $\mathbb{N}(v_0)$. If

$$\sum_{v=v_0}^{\infty} B^\alpha(v+1)Q(v) = \infty, \quad (44)$$

then $z(v) \in S_0$ and

$$\lim_{v \rightarrow \infty} \frac{z(v)}{B(v)} = \infty.$$

Proof. Once again from the monotonic properties of $B(v)$, we see that there exists a sufficiently large $v_* \in \mathbb{N}(v_0)$ such that (15) holds. By (15) and $\beta > \alpha$, we obtain

$$B^\beta(v+1) \leq B^\alpha(v+1) \text{ for } v \geq v_*, \quad (45)$$

from which,

$$B^\beta(v+1)Q(v) \leq B^\alpha(v+1)Q(v). \quad (46)$$

On the other hand, in view of (46), we find that (44) implies (6). So all the conditions of Lemmas 1 and 2 hold. The proofs followed from Lemma 2, we obtain (23) holds. Since $B(v) \rightarrow 0$ as $v \rightarrow \infty$ and $z(v) \rightarrow 0$ as $v \rightarrow \infty$, by L'Hospital's rule

$$\lim_{v \rightarrow \infty} \frac{z(v)}{B(v)} = \lim_{v \rightarrow \infty} -a^{1/\alpha}(v)\Delta z(v).$$

Now, we show that $\lim_{v \rightarrow \infty} -a^{1/\alpha}(v)\Delta z(v) = \infty$. Indeed, if it is not true, then there is a constant $c_1 > 0$ such that

$$-a^{1/\alpha}(v)\Delta z(v) \leq c_1 < \infty.$$

This, together with a summation of (23) from v_3 to $v-1$, gives

$$\begin{aligned} c_1^\alpha \geq -a(v)(\Delta z(v))^\alpha &\geq k \sum_{s=v_3}^{v-1} Q(s)z^\beta(s+\tau-\sigma) \\ &\geq k \sum_{s=v_3}^{v-1} B^\beta(s+1)Q(s) \left(\frac{z(s+1)}{B(s+1)} \right)^\beta \\ &\geq k \left(\frac{z(v_3)}{B(v_3)} \right)^\beta \sum_{s=v_3}^{v-1} B^\beta(s+1)Q(s), \end{aligned}$$

which contradicts (44) and we deduce that

$$\lim_{v \rightarrow \infty} \frac{z(v)}{B(v)} = \lim_{v \rightarrow \infty} -a^{1/\alpha}(v)\Delta z(v) = \infty.$$

This ends the proof. $\square$

Lemma 5. Let $\{y(v)\}$ be an eventually positive solution of (1) on $\mathbb{N}(v_0)$. If (44) holds, then

$$\Delta(a^{1/\alpha}(v)\Delta z(v)) + \frac{k}{\alpha} B^\beta(v+1)Q(v) \frac{z(v+\tau-\sigma)}{B(v+\tau-\sigma)} \leq 0 \quad (47)$$

and

$$\Delta(z(v) + B(v)a^{1/\alpha}(v)\Delta z(v)) + \frac{k}{\alpha} B^{\beta+1}(v+1)Q(v) \frac{z(v+\tau-\sigma)}{B(v+\tau-\sigma)} \leq 0, \quad (48)$$

eventually.

Proof. Since (44) implies (6), we see that all the conditions of Lemmas 1, 2, and 4 are satisfied. The proofs followed by Lemma 2, we see that (22) holds for all $v \geq v_4$. Since $\frac{z(v)}{B(v)} \rightarrow \infty$ as $v \rightarrow \infty$, there exists a sufficiently large $T_1 \in \mathbb{N}(v_0)$ such that

$$\frac{z(v)}{B(v)} \geq 1 \text{ for } v \geq T_1. \quad (49)$$

By (49) and $\beta > \alpha$,

$$\left(\frac{z(v)}{B(v)}\right)^{\beta/\alpha} \geq \frac{z(v)}{B(v)} \text{ for } v \geq T \quad (50)$$

and

$$\beta - \alpha \geq \frac{\beta - \alpha}{\alpha}. \quad (51)$$

Since $v + \tau - \sigma \rightarrow \infty$ as $v \rightarrow \infty$, we can choose an integer $T_2 \geq T_1$ such that $v + \tau - \sigma \geq T_1$ for $v \geq T_2$ and so from (51)

$$\left(\frac{z(v + \tau - \sigma)}{B(v + \tau - \sigma)}\right)^{\beta - \alpha} \geq \left(\frac{z(v)}{B(v)}\right)^{(\beta - \alpha)/\alpha} \text{ for } v \geq T. \quad (52)$$

Let $v_5 = \max\{v_4, T_2\}$. Using (52) in (22) yields

$$\Delta(a^{1/\alpha}(v)\Delta z(v)) + \frac{k}{\alpha}B^{\alpha-1}(v+1)Q(v)\left(\frac{z(v + \tau - \sigma)}{B(v + \tau - \sigma)}\right)^{\beta/\alpha} \leq 0 \quad (53)$$

for all $v \geq v_5$. Using that $B(v)$ is decreasing, $\beta > \alpha$ and $\tau < \sigma$, it follows from (53) that

$$\Delta(a^{1/\alpha}(v)\Delta z(v)) + \frac{k}{\alpha}B^\beta(v+1)Q(v)\left(\frac{z(v + \tau - \sigma)}{B(v + \tau - \sigma)}\right)^{\beta/\alpha} \leq 0 \quad (54)$$

for $v \geq v_5$. Using (50) in (54) gives

$$\Delta(a^{1/\alpha}(v)\Delta z(v)) + \frac{k}{\alpha}B^\beta(v+1)Q(v)\frac{z(v + \tau - \sigma)}{B(v + \tau - \sigma)} \leq 0 \quad (55)$$

for $v \geq v_5$, which proves (47). Recalling (25) again, we can rewrite (55) in the form

$$\Delta(z(v) + B(v)a^{1/\alpha}(v)\Delta z(v)) + \frac{k}{\alpha}B^{\beta+1}(v+1)Q(v)\frac{z(v + \tau - \sigma)}{B(v + \tau - \sigma)} \leq 0$$

for $v \geq v_4$, which proves (48). This ends the proof. $\square$

Theorem 3. *If*

$$\limsup_{v \rightarrow \infty} \sum_{s=v+\tau-\sigma}^{v-1} B^{\beta+1}(s+1)Q(s)B^{-1}(s+\tau-\sigma) > \frac{\alpha}{k}, \quad (56)$$

then (1) oscillates.

Proof. Choose $v_1 \in \mathbb{N}(v_0)$ such that $y(v) > 0$, $y(v - \tau) > 0$ and $y(v - \sigma) > 0$ for $v \geq v_1$. It is clear that (56) yields that

$$\sum_{v=v_0}^{\infty} B^{\beta+1}(v+1)Q(v)B^{-1}(v+\tau-\sigma) = \infty. \quad (57)$$

Using the monotonic property of $B(v)$ and $\tau < \sigma$ gives

$$\sum_{s=v+\tau-\sigma}^{v-1} B^{\beta+1}(s+1)Q(s)B^{-1}(s+\tau-\sigma) \leq \sum_{s=v+\tau-\sigma}^{v-1} B^\beta(s+1)Q(s). \quad (58)$$

In view of (57) and (58), we observe that (56) implies (44).

Hence, all the conditions of Lemma 4 and Lemma 5 are satisfied. The proofs followed by Lemma 5, we again lead to (48). Summing (19) from $v + \tau - \sigma$ to $v - 1$, we obtain

$$\begin{aligned} & z(v + \tau - \sigma) + B((v + \tau - \sigma))a^{1/\alpha}(v + \tau - \sigma)\Delta z(v + \tau - \sigma) \\ & \geq \frac{k}{\alpha} \sum_{s=v+\tau-\sigma}^{v-1} B^{\beta+1}(s+1)Q(s) \frac{z(s + \tau - \sigma)}{B(s + \tau - \sigma)} \\ & \geq \left(\frac{k}{\alpha} \sum_{s=v+\tau-\sigma}^{v-1} B^{\beta+1}(s+1) \frac{Q(s)}{B(s + \tau - \sigma)} \right) z(s + \tau - \sigma), \end{aligned}$$

which, together with $\Delta z(v) < 0$, yields

$$z(v + \tau - \sigma) \geq \left(\frac{k}{\alpha} \sum_{s=v+\tau-\sigma}^{v-1} B^{\beta+1}(s+1)Q(s)B^{-1}(s + \tau - \sigma) \right) z(s + \tau - \sigma).$$

From the last inequality, we get

$$\frac{k}{\alpha} \geq \sum_{s=v+\tau-\sigma}^{v-1} B^{\beta+1}(s)Q(s)B^{-1}(s + \tau - \sigma).$$

Taking $\limsup$ as $v \rightarrow \infty$, we obtain a contradiction to (56) and this ends the proof. $\square$

The conditions required to get oscillation in [23,24] compared with our results are given in Table 1.

Table 1. Comparison of the conditions required to obtain oscillation with the corresponding results derived in the present work.

Neutral Coefficient $p(v)$	Conditions Required in [23]	Conditions Required in [24]	Condition Required Here
$0 < p(v) < \infty$	two	Two	one
$p(v) \rightarrow \infty$ as $v \rightarrow \infty$	-	Two	one

Therefore, it is easier to apply our results in applications than that in [23,24].

Remark 1. The results obtained in this paper can be easily extended to a more general equation of the form

$$\Delta(a(v)|z(v)|^{\alpha-1}z(v + q(v))|y(v - \sigma)|^{\beta-1}y(v - \sigma) = 0,$$

where $z(v) = y(v) + p(v)y(v - \tau)$, $\alpha \geq 1$ and $\beta > 0$, without any difficulty. This form of equation include α and β are any real numbers.

3. Examples

In this section, four examples are provided to illustrate the significance of the main results. The first example may be used in population biology to model species growth and interaction. The second example may be used in signal processing in communication systems to model how signals decay with distance but still oscillate due to interference. The third example may be used in control systems and delayed feedback to model feedback loops in mechanical systems where oscillation persists. The last example serves as a counterexample. In all cases, we present at least one explicit solution.

Example 1. Consider the half-linear neutral difference equation

$$\Delta(v^5(v+1)^5(\Delta(y(v) + 2y(v-1)))^5) + q(v)y^5(v-2) = 0, \quad v \geq 2, \quad (59)$$

where $q(v) = 32(v+1)^5(v^5 + (v+2)^5)$.

Here, $a(v) = v^5(v+1)^5$, $p(v) = 2$, $q(v) = 32(v+1)^5(v^5 + (v+2)^5)$, $\alpha = \beta = 5$, $\tau = 1$, and $\sigma = 2$. It is easy to see that the hypotheses (H_1) – (H_3) are clearly satisfied. Further, a simple calculation shows that $B(v) = \frac{1}{v}$, $m = 2$, $k = \frac{(m-1)^\beta}{m^\beta} = \frac{1}{32}$, $\frac{\alpha}{k} = 160$, and $Q(v) = (v+1)^5(v^5 + (v+2)^5)$. The condition (43) becomes

$$\limsup_{v \rightarrow \infty} \sum_{s=v+\tau-\sigma}^{v-1} B^\alpha(s+1)Q(s) = \lim_{v \rightarrow \infty} ((v-1)^5 + (v+1)^5) = \infty > 160,$$

By Theorem 2, the Equation (59) is oscillatory and in fact $\{y(v)\} = \{(-1)^v\}$ is one such oscillatory solution of Equation (59).

Example 2. Consider the second-order nonlinear neutral difference equation

$$\Delta(2^{3v}(\Delta(y(v) + 2^v y(v-2)))^3) + q(v)y(v-3) = 0, \quad v \geq 1, \quad (60)$$

where $q(v) = 2^{4v+6}[(2 + \frac{3}{2^{v+4}})^3 + (1 + \frac{3}{2^{v+4}})^3]$.

Here $a(v) = 2^{3v}$, $\tau = 2$, $\sigma = 3$, $\alpha = 3$, $\beta = 1$, and $p(v) = 2^v$. It is easy to see that the hypotheses (H_1) – (H_3) clearly hold. By a simple calculation, we see that $B(v) = \frac{1}{2^v}$, $m = 2$, $k = \frac{1}{2}$, $\frac{\alpha}{k} = 6$ and $Q(v) \approx 9(2^{3v+7})$. The condition (32) becomes

$$\limsup_{v \rightarrow \infty} \Omega(v) = 2592 > 6.$$

Now, by applying Theorem 1, Equation (60) is oscillatory and in fact $\{y(v)\} = \{(-1/2)^v\}$ is one such oscillatory solution of Equation (60).

Example 3. Consider the nonlinear second-order neutral delay difference equation

$$\Delta(v^3(v+1)^3(\Delta(y(v) + 2vy(v-1)))^3) + q(v)y^5(v-2) = 0, \quad v \geq 2, \quad (61)$$

where $q(v) = 2(v+1)^3(v^3 + (v+2)^3)$.

Here, $a(v) = v^3(v+1)^3$, $\tau = 1$, $\sigma = 2$, $\alpha = 3$, $\beta = 5$, $p(v) = 2$, and $q(v) = 2(v+1)^3(v^3 + (v+2)^3)$. It is easy to see that the hypotheses (H_1) – (H_3) clearly hold. By a simple calculation, we find that $B(v) = \frac{1}{v}$, $m = 2$ and $k = \frac{1}{32}$. The condition (56) becomes

$$\limsup_{v \rightarrow \infty} \sum_{s=v-1}^{v-1} 2(s-1)((\frac{s}{s+1})^3 + (\frac{s+2}{s+1})^3) = \infty > \frac{1}{32},$$

that is, condition (56) holds. Hence, by Theorem 3, Equation (61) is oscillatory and in fact $\{y(v)\} = \{(-1)^v\}$ is one such oscillatory solution of Equation (61).

Next, we present a counterexample showing that if one condition is not satisfied, then the equation may fail to be oscillatory and instead admit a nonoscillatory solution.

Example 4. Consider the nonlinear second-order neutral delay difference equation

$$\Delta(v(v+1)(v+2)(\Delta(y(v) + 2vy(v-1)))^3) + q(v)y^5(v-2) = 0, \quad v \geq 2, \quad (62)$$

where $q(v) = \frac{3v}{v+1}$.

Here $a(v) = v(v+1)(v+2)$, $\tau = 1$, $\sigma = 2$, $\alpha = 1$, $\beta = 1$, $p(v) = 2$ and $q(v) = \frac{3v}{v+1}$. The hypotheses (H_1) – (H_3) are clearly satisfied. A simple computation shows that $B(v) = \frac{1}{2v(v+1)}$, $m = 2$, $k = 1/2$, $\frac{\alpha}{k} = 2$, and $Q(v) = \frac{3v}{2(v+1)}$. The condition (43) becomes

$$\limsup_{v \rightarrow \infty} \sum_{s=v+\tau-\sigma}^{v-1} B(s+1)Q(s) = \lim_{v \rightarrow \infty} \frac{3(v-1)}{4v^2(v+1)} = 0 > 2,$$

that is, condition (43) is not satisfied. Hence, by Theorem 2, Equation (62) is not oscillatory and in fact Equation (62) has a nonoscillatory solution $\{y(v)\} = \{\frac{v+2}{v+1}\}$.

Remark 2. From the four examples, we see that:

- (i) If conditions are satisfied then oscillatory solutions appear (persistent cycle);
- (ii) If one condition fails (as in Example 4) nonoscillatory solutions emerge, showing stable, monotone behavior;
- (iii) This demonstrates how changes in parameters α , β , τ , and σ directly control whether a system oscillates or stabilizes.

4. Conclusions

In this paper, we have investigated the oscillatory properties of second-order non-canonical difference equations with bounded and unbounded neutral terms. The obtained oscillation criteria require only a single condition, in contrast to the two conditions used in [23,24]. This result is achieved through the application of the linearization method combined with the summation averaging technique. Four examples are presented to illustrate the importance and novelty of the main findings. Furthermore, our results are applicable to linear, half-linear, and nonlinear types of equations.

Next, we present directions for future study:

- (1) In the case of $0 < \alpha < 1$, the same method cannot be used since the linearization technique fails to hold. So it is interesting to extend the results of this paper when $0 < \alpha < 1$ using other methods.
- (2) Integrating the linearization techniques developed in this paper with fuzzy set theory, or extending them to higher-dimensional systems, may lead to more realistic and flexible models in fields such as engineering, economics, and biology. This represents a challenging yet promising direction for future investigation.
- (3) Another interesting problem is to apply the method developed in this paper to study the oscillatory behavior of fractional-order difference equations.

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