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Article

Availability Analysis of an Unreliable Single Unit System Operating in a Doubly Stochastic Shock Environment

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Abstract

This paper considers an unreliable system together with a repairman operating in a doubly stochastic environment. The system operates in a stochastic environment which alternates between two levels, namely less load and heavy load. The intrinsic life-time of the system is exponentially distributed with varying means depending on whether the system is working in less load period or heavy load period. Environment dependent shocks arrive to the system according to a Poisson process with different rates depending on whether the environment is in less load or in heavy load period. At the occurrence of a shock, the operating system fails and it is immediately taken to the repair facility and the repair commences instantaneously. The repair time is exponentially distributed whose mean is dependent on the type of failure and the level of the environment. After each repair, the system begins to function in heavy load environment or less load environment according to the level of the environment at the time of completion of the repair. The repair facility is unaffected by shocks. Kolmogorov equations governing the behaviour of the system are derived and the probability distribution of the states is obtained. The availability function is also obtained and the model is highlighted with a numerical illustration.

Keywords: availability; doubly stochastic environment; poisson shock model; load-dependent failure; environment-dependent repair

MSC: 90B25

1. Introduction

Shock models have been used to study several cases such as earthquake occurrences, reliability of mechanical systems, production systems and inventory systems. The monograph of Nakagawa [1] cites a huge number of research articles that have been published in the past on shock models. Gaver [2] introduced the notion of stochastic environment in studying a class of stochastic models for system reliability. Esary et al. [3] studied some shock models for obtaining life time distribution of a component which is subjected to Poisson shocks. A-Hameed and Proschan [4] obtained the life distribution of a device subject to a sequence of shocks occurring randomly in time according to a non-stationary pure birth process. Råde [5] studied a parallel reliability system which is subject to shocks generated by a renewal point process. Shanthikumar and Sumita [6] introduced shock models which can be applied to study several random phenomena such as occurrences of earthquakes,

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failure of mechanical systems, production systems and inventory systems. Gut [7] described cumulative shock models by developing a theory of stopped two-dimensional random walks and obtained certain limit theorems for lifetime/failure time of a system subjected to random shocks. Petakos and Tsapelas [8] developed reliability analysis for systems operating in a random environment where shocks occur and cause component failure in a specific way. Skoulakis [9] proposed and obtained some performance measures for a general shock model for a reliability system. Wang and Zhang [10] obtained an optimal replacement policy for a shock model with two-type failures. In his monograph, Nakagawa [11] summarized his research work on maintenance policies and reliability properties for system reliability models by using stochastic processes. Zhao and Nakagawa [12] surveyed some of the research work done on advanced maintenance techniques with shock and damage models in computer systems and mechanical systems. Munoli and Suhas [13] studied a problem of modelling and assessment of survival probability of a component experiencing two types of shocks, one type causing mild damage to the system not affecting the functioning of the system, and the other type causing complete failure of the system. Wu and Cui [14] developed two Markov renewal shock models with multiple failure mechanisms and obtained reliability functions and other reliability indices. Zhao et al. [15] considered the problem of obtaining maintenance policy for mission-oriented systems subject to degradation and external shocks. Wu et al. [16] developed two novel critical shock models based on Markov renewal processes and obtained the reliability function and the mean time to failure. Hu and Zhu [17] presented a review of system reliability models with random shocks and uncertainty citing numerous articles on reliability problems. Hussien and El-Sherbeny [18] have studied the reliability and availability of an unreliable one-unit system with a single technician to perform repair and maintenance in the presence of shocks occurring at random times. These shocks damage the unit and degrade its performance cumulatively over time.

The main feature of shock models studied in the aforementioned research papers is the arrival pattern of the shocks experienced by reliability systems over the time axis. The arrival pattern is modelled as a stochastic point process on the time axis $[0, \infty)$ (see Snyder [19], Srinivasan [20], Jacobsen [21], Sigman [22]). Each shock renders a random amount of damage to the system. The amount of each damage may be small or large depending on the behaviour of the system. If the system continues to function with a decreased effectiveness after a shock, then the damage is considered to be mild. On the other hand, if the system breaks down at the occurrence of a shock, then the damage is considered to be fatal. Instead of allowing the system to function with a decreased effectiveness after the occurrence of a mild shock, it is reasonable to halt the system as a preventive measure and perform maintenance repair to bring back the system to a new one. Furthermore, the system may operate under the influence of a random environment in the sense that there may be heavy load and less load alternately on the system. The life time of the system may not be same in heavy load and less load intervals. Frequencies of occurrence of shocks may be different in heavy load and less load intervals.

Petakos and Tsapelas [8], in their model, do not incorporate environment-dependent shock arrivals with environment-dependent intrinsic failure and repair simultaneously. Råde [5] and Skoulakis [9] analysed reliability systems only under a single environment. Shanthikumar and Sumita [6] developed general shock models with correlated renewal sequences not determined by external Markov environment. Sigman [22] gives a general theory of stationary marked point processes but does not address doubly stochastic shock-modulated repairable systems. In this paper, we study a stochastic model of a reliability system subjected to shocks in a random environment wherein the intrinsic failure rate, the shock arrival rate and the repair rate are jointly modulated by a two-state Markov environment, so that all three time-scales respond to the load level. Further, failures are classified dichotomously into intrinsic and shock-induced types, and the repair rate depends both on the type of failure and on the environment level at the time of failure. To the best of our knowledge, this combination has not been analysed in closed form in the prior literature.

The plan of the paper is as follows: In Section 2, the stochastic model of a reliability system is proposed. Equations governing the system are derived in Section 3. Section 4 provides a solution for the probability distribution of the states. The availability function is obtained in Section 5. A reliability analysis is presented in Section 6. Section 7 derives the mean time to failure. Section 8 introduces the efficiency measure. Section 9 highlights the model with a numerical illustration. A conclusion is included in Section 10.

2. Description of the Stochastic Model

We consider a single component (one unit) reliability system together with a repairman operating in a doubly stochastic environment. The system operates in a stochastic environment which alternates between two levels, namely less load (level 0) and heavy load (level 1). Shocks arrive to the system according to a Poisson process with rate η_0 or η_1 depending on whether the environment is in level 0 or in level 1 respectively. At the occurrence of a shock, the operating unit fails and it is immediately taken to the repair facility and the repair commences instantaneously. The repair time is random and immediately after repair, the unit becomes a new unit. The repair time of a failed unit is exponentially distributed with mean

$\frac{1}{\gamma_{00}}$, if the unit failure has occurred in less load environment due to life-failure
$\frac{1}{\gamma_{01}}$, if the unit failure has occurred in heavy load environment due to life-failure
$\frac{1}{\gamma_{10}}$, if the unit failure has occurred in less load environment due to shock
$\frac{1}{\gamma_{11}}$, if the unit failure has occurred in heavy load environment due to shock

After each repair, the unit begins to function in heavy load environment or less load environment according to the level of the environment at the time of completion of the repair. The repair facility is unaffected by shocks. The life-time of the unit is exponentially distributed with mean $\frac{1}{\mu_0}$ or $\frac{1}{\mu_1}$ depending on whether the unit is operating in less load environment or heavy load environment.

The exponential distributions are assumed for the intrinsic life-time and the repair time fundamentally because the memoryless property makes the joint process $\{(X(t), Y(t)) : t \geq 0\}$ a continuous-time Markov chain on a finite state space, and hence enables the closed-form Laplace-domain solution and the explicit availability and reliability expressions obtained in the paper.

If the exponential assumption is relaxed—for example, by adopting a Weibull life time to capture wear-out, or a general repair-time distribution—the closed-form availability would in general be lost. This generalization may be taken up as a direction for future research.

Assumptions and Notation

Let $X(t)$ be the state of the unit at time t. We define

$$X(t) = \begin{cases} 0 & \text{if the unit is in the failed state at time } t \text{ due to its life-failure.} \\ 1 & \text{if the unit is in the failed state at time } t \text{ due to shock occurrence .} \\ 2 & \text{if the unit is in the working state at time } t \text{ .} \end{cases}$$

Let $Y(t)$ be the state of the environment at time t. We define

$$Y(t) = \begin{cases} 0 & \text{if the environment is loaded mildly at time } t\,. \\ 1 & \text{if the environment is loaded heavily at time } t\,. \end{cases}$$

Let $\{Y(t)|t \geq 0\}$ be a two-state Markov process with transition probabilities defined by

$$Pr[Y(t+\Delta) = 1|Y(t) = 0] = \alpha\Delta + o(\Delta), Pr[Y(t+\Delta) = 0|Y(t) = 1] = \beta\Delta + o(\Delta).$$

Let $V(t) = (X(t), Y(t))$. Then, $\{V(t)|t \geq 0\}$ is a two-dimensional Markov process with state space $\Omega = \{(i,j)|i = 0,1,2; j = 0,1\}$. Assume that at time $t = 0$, the load of the environment is heavy and repair of unit is just completed. Then $V(0) = (2,1)$. Define

$$P(i,j,t) \quad = \quad Pr[V(t) = (i,j)|V(0) = (2,1)], (i,j) \in \Omega.$$

3. Governing Equations

The governing equations are derived by applying the law of total probability over an infinitesimal interval $(t, t+dt]$.

Using probability law for additive events, and the transition diagram for each state of the model, we have derived the governing equations as shown below (see Figure 1):

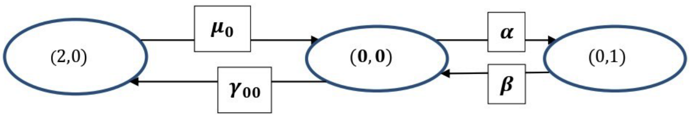

Figure 1. In-Flow & Out-Flow Centered at (0,0).

Consider

$$P(0,0,t+dt) = P(0,0,t)(1 - \alpha dt - \gamma_{00}dt) + P(0,1,t)\beta dt + P(2,0,t)\mu_0 dt + o(dt),$$
$$\frac{P(0,0,t+dt) - P(0,0,t)}{dt} = -(\alpha + \gamma_{00})P(0,0,t) + \beta P(0,1,t) + \mu_0 P(2,0,t) + \frac{o(dt)}{dt},$$

Taking the limit as $dt \to 0$, we have

$$P'(0,0,t) = -(\alpha + \gamma_{00})P(0,0,t) + \beta P(0,1,t) + \mu_0 P(2,0,t). \tag{1}$$

(see Figure 2)

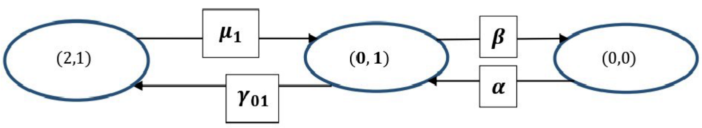

Figure 2. In-Flow & Out-Flow Centered at (0,1).

Consider

$$P(0,1,t+dt) = P(0,1,t)(1 - \beta dt - \gamma_{01}dt) + P(0,0,t)\alpha dt + P(2,1,t)\mu_1 dt + o(dt)$$
$$\frac{P(0,1,t+dt) - P(0,1,t)}{dt} = -(\beta + \gamma_{01})P(0,1,t) + \alpha P(0,0,t) + \mu_1 P(2,1,t) + \frac{o(dt)}{dt},$$

Taking the limit as $dt \to 0$, we have

$$P'(0,1,t) = -(\beta + \gamma_{01})P(0,1,t) + \alpha P(0,0,t) + \mu_1 P(2,1,t). \tag{2}$$

(see Figure 3)

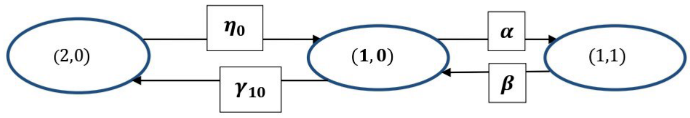

Figure 3. In-Flow & Out-Flow Centered at (1,0).

Consider

$$P(1,0,t+dt) = P(1,0,t)(1-\alpha dt - \gamma_{10}dt) + P(1,1,t)\beta dt + P(2,0,t)\eta_0 dt + o(dt),$$
$$\frac{P(1,0,t+dt)-P(1,0,t)}{dt} = -(\alpha+\gamma_{10})P(1,0,t) + \beta P(1,1,t) + \eta_0 P(2,0,t) + \frac{o(dt)}{dt},$$

Taking the limit as $dt \to 0$, we have

$$P^{'}(1,0,t) = -(\alpha+\gamma_{10})P(1,0,t) + \beta P(1,1,t) + \eta_0 P(2,0,t). \tag{3}$$

(see Figure 4)

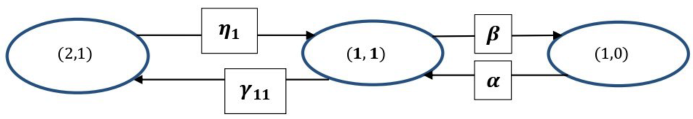

Figure 4. In-Flow & Out-Flow Centered at (1,1).

Consider

$$P(1,1,t+dt) = P(1,1,t)(1-\beta dt - \gamma_{11}dt) + P(1,0,t)\alpha dt + P(2,1,t)\eta_1 dt + o(dt),$$
$$\frac{P(1,1,t+dt)-P(1,1,t)}{dt} = -(\beta+\gamma_{11})P(1,1,t) + \alpha P(1,0,t) + \eta_1 P(2,1,t) + \frac{o(dt)}{dt},$$

Taking the limit as $dt \to 0$, we have

$$P^{'}(1,1,t) = -(\beta+\gamma_{11})P(1,1,t) + \alpha P(1,0,t) + \eta_1 P(2,1,t). \tag{4}$$

(see Figure 5)

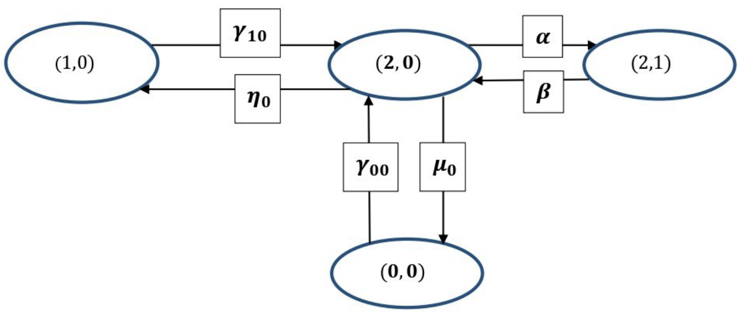

Figure 5. In-Flow & Out-Flow Centered at (2,0).

Consider

$$\begin{aligned}P(2,0,t+dt) &= P(2,0,t)(1-\mu_0 dt-\alpha dt-\eta_0 dt)+P(2,1,t)\beta dt\\ &\quad +P(0,0,t)\gamma_{00}dt+P(1,0,t)\gamma_{10}dt+o(dt),\\ \frac{P(2,0,t+dt)-P(2,0,t)}{dt} &= -(\mu_0+\alpha+\eta_0)P(2,0,t)+\beta P(2,1,t)\\ &\quad +\gamma_{00}P(0,0,t)+\gamma_{10}P(1,0,t)+\frac{o(dt)}{dt},\end{aligned}$$

Taking the limit as $dt \to 0$, we have

$$P^{'}(2,0,t) = -(\mu_0+\alpha+\eta_0)P(2,0,t)+\beta P(2,1,t)+\gamma_{00}P(0,0,t)+\gamma_{10}P(1,0,t). \tag{5}$$

(see Figure 6)

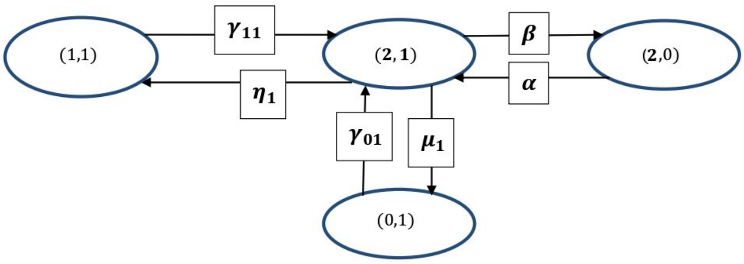

Figure 6. In-Flow & Out-Flow Centered at (2,1).

Consider

$$\begin{aligned}P(2,1,t+dt) &= P(2,1,t)(1-\mu_1 dt-\beta dt-\eta_1 dt)+P(2,0,t)\alpha dt\\ &\quad +P(0,1,t)\gamma_{01}dt+P(1,1,t)\gamma_{11}dt+o(dt),\\ \frac{P(2,1,t+dt)-P(2,1,t)}{dt} &= -(\mu_1+\beta+\eta_1)P(2,1,t)+\alpha P(2,0,t)\\ &\quad +\gamma_{01}P(0,1,t)+\gamma_{11}P(1,1,t)+\frac{o(dt)}{dt},\end{aligned}$$

Taking the limit as $dt \to 0$, we have

$$P^{'}(2,1,t) = -(\mu_1+\beta+\eta_1)P(2,1,t)+\alpha P(2,0,t)+\gamma_{01}P(0,1,t)+\gamma_{11}P(1,1,t). \tag{6}$$

In the above Equations (1)–(6), we have used the notation

$$\frac{d}{dt}P(i,j,t) = P^{'}(i,j,t).$$

4. Transient State Probabilities

Using Laplace transform technique, we obtain the transient solution for the state probabilities. Denoting the Laplace transform of $P(i,j,t)$ by $P^*(i,j,s)$, the governing Equations (1)–(6) yield,

$$\begin{aligned}(s+\alpha+\gamma_{00})P^*(0,0,s) &= \beta P^*(0,1,s)+\mu_0 P^*(2,0,s), &(7)\\ (s+\beta+\gamma_{01})P^*(0,1,s) &= \alpha P^*(0,0,s)+\mu_1 P^*(2,1,s), &(8)\\ (s+\alpha+\gamma_{10})P^*(1,0,s) &= \beta P^*(1,1,s)+\eta_0 P^*(2,0,s), &(9)\\ (s+\beta+\gamma_{11})P^*(1,1,s) &= \alpha P^*(1,0,s)+\eta_1 P^*(2,1,s), &(10)\\ (s+\mu_0+\alpha+\eta_0)P^*(2,0,s) &= \beta P^*(2,1,s)+\gamma_{00}P^*(0,0,s)+\gamma_{10}P^*(1,0,s), &(11)\\ (s+\mu_1+\beta+\eta_1)P^*(2,1,s) &= 1+\alpha P^*(2,0,s)+\gamma_{01}P^*(0,1,s)+\gamma_{11}P^*(1,1,s). &(12)\end{aligned}$$

Using (7)–(10), we obtain

$$
\begin{aligned}
P^*(0,0,s) &= \frac{\mu_0(s+\beta+\gamma_{01})}{D_0(s)}P^*(2,0,s) + \frac{\mu_1\beta}{D_0(s)}P^*(2,1,s), && (13)\\
P^*(0,1,s) &= \frac{\mu_0\alpha}{D_0(s)}P^*(2,0,s) + \frac{\mu_1(s+\alpha+\gamma_{00})}{D_0(s)}P^*(2,1,s), && (14)\\
P^*(1,0,s) &= \frac{\beta\eta_1}{D_1(s)}P^*(2,1,s) + \frac{\eta_0(s+\beta+\gamma_{11})}{D_1(s)}P^*(2,0,s), && (15)\\
P^*(1,1,s) &= \frac{\alpha\eta_0}{D_1(s)}P^*(2,0,s) + \frac{\eta_1(s+\alpha+\gamma_{10})}{D_1(s)}P^*(2,1,s), && (16)
\end{aligned}
$$

where

$$
\begin{aligned}
D_0(s) &= (s+\alpha+\gamma_{00})(s+\beta+\gamma_{01}) - \alpha\beta,\\
D_1(s) &= (s+\alpha+\gamma_{10})(s+\beta+\gamma_{11}) - \alpha\beta.
\end{aligned}
$$

Consequently, (11) and (12) yield

$$
\begin{aligned}
P^*(2,0,s) &= \frac{A(s)}{D_2(s)}P^*(2,1,s) && (17)\\
P^*(2,1,s) &= \frac{D_0(s)D_1(s)}{D_3(s)} + \frac{B(s)}{D_3(s)}P^*(2,0,s), && (18)
\end{aligned}
$$

where

$$
\begin{aligned}
A(s) &= \beta D_0(s)D_1(s) + \mu_1\beta\gamma_{00}D_1(s) + \beta\eta_1\gamma_{10}D_0(s)\\
B(s) &= \alpha D_0(s)D_1(s) + \mu_0\alpha\gamma_{01}D_1(s) + \alpha\eta_0\gamma_{11}D_0(s)\\
D_2(s) &= (s+\mu_0+\alpha+\eta_0)D_0(s)D_1(s) - \mu_0\gamma_{00}(s+\beta+\gamma_{01})D_1(s)\\
&- \eta_0\gamma_{10}(s+\beta+\gamma_{11})D_0(s)\\
D_3(s) &= (s+\mu_1+\beta+\eta_1)D_0(s)D_1(s) - \mu_1\gamma_{01}(s+\alpha+\gamma_{00})D_1(s)\\
&- \eta_1\gamma_{11}(s+\alpha+\gamma_{10})D_0(s).
\end{aligned}
$$

Substituting (17) into (18), we get

$$
P^*(2,1,s) = \frac{D_0(s)D_1(s)D_2(s)}{D_2(s)D_3(s) - A(s)B(s)}. \tag{19}
$$

Substituting (19) into (17), we get

$$
P^*(2,0,s) = \frac{A(s)D_0(s)D_1(s)}{D_2(s)D_3(s) - A(s)B(s)}. \tag{20}
$$

Substituting (19) and (20) into (13)–(16), we get

$$
\begin{aligned}
P^*(0,0,s) &= \frac{\mu_0(s+\beta+\gamma_{01})A(s)D_1(s) + \mu_1\beta D_1(s)D_2(s)}{D_2(s)D_3(s) - A(s)B(s)}, && (21)\\
P^*(0,1,s) &= \frac{\mu_0\alpha A(s)D_1(s) + \mu_1(s+\alpha+\gamma_{00})D_1(s)D_2(s)}{D_2(s)D_3(s) - A(s)B(s)}, && (22)\\
P^*(1,0,s) &= \frac{\beta\eta_1 D_0(s)D_2(s) + \eta_0(s+\beta+\gamma_{11})A(s)D_0(s)}{D_2(s)D_3(s) - A(s)B(s)}, && (23)\\
P^*(1,1,s) &= \frac{\alpha\eta_0 A(s)D_0(s) + \eta_1(s+\alpha+\gamma_{10})D_0(s)D_2(s)}{D_2(s)D_3(s) - A(s)B(s)}. && (24)
\end{aligned}
$$

We now proceed to invert (19)–(24). First, we consider the fact that, by (19)–(24), each of $P^*(i,j,s), (i,j) \in \Omega$ is a rational function of the form $\frac{F_1(s)}{F_2(s)}$, where the degree of $F_1(s)$ is less than that of $F_2(s)$. In fact, $F_2(s) = D_2(s)D_3(s) - A(s)B(s)$ is a 10th degree polynomial in s. Since $F_2(0) = 0$, one of the zeros is 0. Let the other zeros be $\xi_i, i = 1, 2, \ldots, 9$. Then, we have

$$D_2(s)D_3(s) - A(s)B(s) = s\Pi_{j=1}^{9}(s - \xi_j).$$

After simplification, we get

$$\begin{aligned}
D_0(s) &= s^2 + d_{01}s + d_{02}, \\
D_1(s) &= s^2 + d_{11}s + d_{12}, \\
D_0(s)D_1(s) &= s^4 + d_{011}s^3 + d_{012}s^2 + d_{013}s + d_{014}, \\
A(s) &= \beta s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4, \\
B(s) &= \alpha s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4, \\
A(s)B(s) &= \alpha\beta s^8 + c_1 s^7 + c_2 s^6 + c_3 s^5 + c_4 s^4 + c_5 s^3 + c_6 s^2 + c_7 s + c_8, \\
D_2(s) &= s^5 + l_1 s^4 + l_2 s^3 + l_3 s^2 + l_4 s + l_5, \\
D_3(s) &= s^5 + m_1 s^4 + m_2 s^3 + m_3 s^2 + m_4 s + m_5, \\
D_2(s)D_3(s) &= s^{10} + n_1 s^9 + n_2 s^8 + n_3 s^7 + n_4 s^6 + n_5 s^5 \\
&\quad + n_6 s^4 + n_7 s^3 + n_8 s^2 + n_9 s + n_{10}, \\
D_2(s)D_3(s) - A(s)B(s) &= s^{10} + n_1 s^9 + r_2 s^8 + r_3 s^7 + r_4 s^6 + r_5 s^5 \\
&\quad + r_6 s^4 + r_7 s^3 + r_8 s^2 + r_9 s + r_{10},
\end{aligned}$$

where

$$\begin{aligned}
d_{01} &= \alpha + \beta + \gamma_{00} + \gamma_{01}, \\
d_{02} &= (\alpha + \gamma_{00})(\beta + \gamma_{01}) - \alpha\beta, \\
d_{11} &= \alpha + \beta + \gamma_{10} + \gamma_{11}, \\
d_{12} &= (\alpha + \gamma_{10})(\beta + \gamma_{11}) - \alpha\beta, \\
d_{011} &= d_{11} + d_{01}, \\
d_{012} &= d_{12} + d_{01}d_{11} + d_{02}, \\
d_{013} &= d_{01}d_{12} + d_{02}d_{11}, \\
d_{014} &= d_{02}d_{12}, \\
a_1 &= \beta d_{011}, \\
a_2 &= \beta d_{012} + \mu_1\beta\gamma_{00} + \beta\eta_1\gamma_{10}, \\
a_3 &= \beta d_{013} + \mu_1\beta\gamma_{00}d_{11} + \beta\eta_1\gamma_{10}d_{01}, \\
a_4 &= \beta d_{014} + \mu_1\beta\gamma_{00}d_{12} + \beta\eta_1\gamma_{10}d_{02}, \\
b_1 &= \alpha d_{011}, \\
b_2 &= \alpha d_{012} + \mu_0\alpha\gamma_{01} + \alpha\eta_0\gamma_{11}, \\
b_3 &= \alpha d_{013} + \mu_0\alpha\gamma_{01}d_{11} + \alpha\eta_0\gamma_{11}d_{01}, \\
b_4 &= \alpha d_{014} + \mu_0\alpha\gamma_{01}d_{12} + \alpha\eta_0\gamma_{11}d_{02}, \\
c_1 &= (b_1\beta + \alpha a_1), \\
c_2 &= (b_2\beta + a_1 b_1 + \alpha a_2), \\
c_3 &= (b_3\beta + a_1 b_2 + a_2 b_1 + \alpha a_3), \\
c_4 &= (b_4\beta + a_1 b_3 + a_2 b_2 + a_3 b_1 + \alpha a_4), \\
c_5 &= (a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1), \\
c_6 &= (a_2 b_4 + a_3 b_3 + a_4 b_2), \\
c_7 &= (a_3 b_4 + a_4 b_3), \\
c_8 &= a_4 b_4, \\
l_1 &= d_{011} + (\mu_0 + \alpha + \eta_0), \\
l_2 &= d_{012} + (\mu_0 + \alpha + \eta_0)d_{011} - \mu_0\gamma_{00} - \eta_0\gamma_{10},
\end{aligned}$$

$$\begin{aligned}
l_3 &= d_{013} + (\mu_0 + \alpha + \eta_0)d_{012} - d_{11}\mu_0\gamma_{00} - d_{01}\eta_0\gamma_{10} - \mu_0\gamma_{00}(\beta + \gamma_{01}) \\
&\quad -\eta_0\gamma_{10}(\beta + \gamma_{11}), \\
l_4 &= d_{014} + (\mu_0 + \alpha + \eta_0)d_{013} - d_{12}\mu_0\gamma_{00} - d_{02}\eta_0\gamma_{10} - d_{11}\mu_0\gamma_{00}(\beta + \gamma_{01}) \\
&\quad -d_{01}\eta_0\gamma_{10}(\beta + \gamma_{11}), \\
l_5 &= (\mu_0 + \alpha + \eta_0)d_{014} - d_{12}\mu_0\gamma_{00}(\beta + \gamma_{01}) - d_{02}\eta_0\gamma_{10}(\beta + \gamma_{11}), \\
m_1 &= d_{011} + (\mu_1 + \beta + \eta_1), \\
m_2 &= d_{012} + d_{011}(\mu_1 + \beta + \eta_1) - \mu_1\gamma_{01} - \eta_1\gamma_{11}, \\
m_3 &= d_{013} + d_{012}(\mu_1 + \beta + \eta_1) - d_{11}\mu_1\gamma_{01} - d_{01}\eta_1\gamma_{11} - \mu_1\gamma_{01}(\alpha + \gamma_{00}) \\
&\quad -\eta_1\gamma_{11}(\alpha + \gamma_{10}), \\
m_4 &= d_{014} + d_{013}(\mu_1 + \beta + \eta_1) - d_{12}\mu_1\gamma_{01} - d_{11}\mu_1\gamma_{01}(\alpha + \gamma_{00}) - d_{02}\eta_1\gamma_{11} \\
&\quad -d_{01}\eta_1\gamma_{11}(\alpha + \gamma_{10}), \\
m_5 &= d_{014}(\mu_1 + \beta + \eta_1) - d_{12}\mu_1\gamma_{01}(\alpha + \gamma_{00}) - d_{02}\eta_1\gamma_{11}(\alpha + \gamma_{10}),
\end{aligned}$$

$$\begin{aligned}
n_1 &= m_1 + l_1, \\
n_2 &= m_2 + l_1m_1 + l_2, \\
n_3 &= m_3 + l_1m_2 + l_2m_1 + l_3, \\
n_4 &= m_4 + l_1m_3 + l_2m_2 + l_3m_1 + l_4, \\
n_5 &= m_5 + l_1m_4 + l_2m_3 + l_3m_2 + l_4m_1 + l_5, \\
n_6 &= l_1m_5 + l_2m_4 + l_3m_3 + l_4m_2 + l_5m_1, \\
n_7 &= l_2m_5 + l_3m_4 + l_4m_3 + l_5m_2, \\
n_8 &= l_3m_5 + l_4m_4 + l_5m_3, \\
n_9 &= l_4m_5 + l_5m_4, \\
n_{10} &= l_5m_5,
\end{aligned}$$

$$\begin{array}{lllll}
r_1 = n_1, & r_2 = n_2 - \alpha\beta, & r_3 = n_3 - c_1, & r_4 = n_4 - c_2, & r_5 = n_5 - c_3, \\
r_6 = n_6 - c_4, & r_7 = n_7 - c_5, & r_8 = n_8 - c_6, & r_9 = n_9 - c_7, & r_{10} = n_{10} - c_8.
\end{array}$$

Splitting into partial fractions, (19)–(24) yield

$$P^*(0,0,s) = \frac{E_{000}}{s} + \sum_{j=1}^{9} \frac{E_{00j}}{(s-\xi_j)}, \qquad P^*(0,1,s) = \frac{E_{010}}{s} + \sum_{j=1}^{9} \frac{E_{01j}}{(s-\xi_j)}, \tag{25}$$

$$P^*(1,0,s) = \frac{E_{100}}{s} + \sum_{j=1}^{9} \frac{E_{10j}}{(s-\xi_j)}, \qquad P^*(1,1,s) = \frac{E_{110}}{s} + \sum_{j=1}^{9} \frac{E_{11j}}{(s-\xi_j)}, \tag{26}$$

$$P^*(2,0,s) = \frac{E_{200}}{s} + \sum_{j=1}^{9} \frac{E_{20j}}{(s-\xi_j)}, \qquad P^*(2,1,s) = \frac{E_{210}}{s} + \sum_{j=1}^{9} \frac{E_{21j}}{(s-\xi_j)}, \tag{27}$$

where

$$\begin{aligned}
E_{000} &= -\frac{\mu_0(\beta + \gamma_{01})A(0)D_1(0) + \mu_1\beta D_1(0)D_2(0)}{\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_7\xi_8\xi_9}, \\
E_{00j} &= \lim_{s\to\xi_j} \frac{(s-\xi_j)[\mu_0(s + \beta + \gamma_{01})A(s)D_1(s) + \mu_1\beta D_1(s)D_2(s)]}{D_2(s)D_3(s) - A(s)B(s)}, j = 1,2,\ldots,9, \\
E_{010} &= -\frac{\mu_0\alpha A(0)D_1(0) + \mu_1(\alpha + \gamma_{00})D_1(0)D_2(0)}{\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_7\xi_8\xi_9}, \\
E_{01j} &= \lim_{s\to\xi_j} \frac{(s-\xi_j)[\mu_0\alpha A(s)D_1(s) + \mu_1(s + \alpha + \gamma_{00})D_1(s)D_2(s)]}{D_2(s)D_3(s) - A(s)B(s)}, j = 1,2,\ldots,9, \\
E_{100} &= -\frac{\beta\eta_1 D_0(0)D_2(0) + \eta_0(\beta + \gamma_{11})A(0)D_0(0)}{\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_7\xi_8\xi_9}, \\
E_{10j} &= \lim_{s\to\xi_j} \frac{(s-\xi_j)[\beta\eta_1 D_0(s)D_2(s) + \eta_0(s + \beta + \gamma_{11})A(s)D_0(s)]}{D_2(s)D_3(s) - A(s)B(s)}, j = 1,2,\ldots,9,
\end{aligned}$$

$$\begin{aligned}
E_{110} &= -\frac{\alpha\eta_0 A(0)D_0(0)+\eta_1(\alpha+\gamma_{10})D_0(0)D_2(0)}{\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_7\xi_8\xi_9},\\
E_{11j} &= \lim_{s\to\xi_j}\frac{(s-\xi_j)[\alpha\eta_0 A(s)D_0(s)+\eta_1(s+\alpha+\gamma_{10})D_0(s)D_2(s)]}{D_2(s)D_3(s)-A(s)B(s)}, j=1,2,\ldots,9,\\
E_{200} &= -\frac{A(0)D_0(0)D_1(0)}{\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_7\xi_8\xi_9},\\
E_{20j} &= \lim_{s\to\xi_j}\frac{(s-\xi_j)[A(s)D_0(s)D_1(s)]}{D_2(s)D_3(s)-A(s)B(s)}, j=1,2,\ldots,9,\\
E_{210} &= -\frac{D_0(0)D_1(0)D_2(0)}{\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_7\xi_8\xi_9},\\
E_{21j} &= \lim_{s\to\xi_j}\frac{(s-\xi_j)[D_0(s)D_1(s)D_2(s)]}{D_2(s)D_3(s)-A(s)B(s)}, j=1,2,\ldots,9.
\end{aligned}$$

From (25)–(27), we obtain the steady-state probabilities by applying the final value theorem of Laplace transform theory $\pi(i,j)=lim_{s\to 0}\, sP^*(i,j,s)$ as given below:

$$\pi(0,0)=E_{000},\qquad \pi(0,1)=E_{010}, \tag{28}$$
$$\pi(1,0)=E_{100},\qquad \pi(1,1)=E_{110}, \tag{29}$$
$$\pi(2,0)=E_{200},\qquad \pi(2,1)=E_{210}. \tag{30}$$

By taking inverse Laplace transform on both sides of (25)–(27), we obtain the transient probabilities as given below:

$$P(0,0,t) = E_{000}+\sum_{j=1}^{9}E_{00j}e^{\xi_j t},\qquad P(0,1,t)=E_{010}+\sum_{j=1}^{9}E_{01j}e^{\xi_j t}, \tag{31}$$
$$P(1,0,t) = E_{100}+\sum_{j=1}^{9}E_{10j}e^{\xi_j t},\qquad P(1,1,t)=E_{110}+\sum_{j=1}^{9}E_{11j}e^{\xi_j t}, \tag{32}$$
$$P(2,0,t) = E_{200}+\sum_{j=1}^{9}E_{20j}e^{\xi_j t},\qquad P(2,1,t)=E_{210}+\sum_{j=1}^{9}E_{21j}e^{\xi_j t}. \tag{33}$$

5. Availability Analysis

Let $A_{ij}(t)$ be the conditional probability that the system is available at time t given that the system started in the state $(i,j), i=0,1,2; j=0,1$ at time $t=0$. Using renewal-theoretic arguments, we obtain

$$A_{00}(t) = \alpha\int_0^t e^{-(\alpha+\gamma_{00})u}A_{01}(t-u)du+\gamma_{00}\int_0^t e^{-(\alpha+\gamma_{00})u}A_{20}(t-u)du, \tag{34}$$
$$A_{01}(t) = \beta\int_0^t e^{-(\beta+\gamma_{01})u}A_{00}(t-u)du+\gamma_{01}\int_0^t e^{-(\beta+\gamma_{01})u}A_{21}(t-u)du, \tag{35}$$
$$A_{10}(t) = \alpha\int_0^t e^{-(\alpha+\gamma_{10})u}A_{11}(t-u)du+\gamma_{10}\int_0^t e^{-(\alpha+\gamma_{10})u}A_{20}(t-u)du, \tag{36}$$
$$A_{11}(t) = \beta\int_0^t e^{-(\beta+\gamma_{11})u}A_{10}(t-u)du+\gamma_{11}\int_0^t e^{-(\beta+\gamma_{11})u}A_{21}(t-u)du, \tag{37}$$
$$\begin{aligned}A_{20}(t) &= e^{-(\mu_0+\alpha+\eta_0)t}+\alpha\int_0^t e^{-(\mu_0+\alpha+\eta_0)u}A_{21}(t-u)du\\ &\quad+\mu_0\int_0^t e^{-(\mu_0+\alpha+\eta_0)u}A_{00}(t-u)du+\eta_0\int_0^t e^{-(\mu_0+\alpha+\eta_0)u}A_{10}(t-u)du,\end{aligned} \tag{38}$$
$$\begin{aligned}A_{21}(t) &= e^{-(\mu_1+\beta+\eta_1)t}+\beta\int_0^t e^{-(\mu_1+\beta+\eta_1)u}A_{20}(t-u)du\\ &\quad+\mu_1\int_0^t e^{-(\mu_1+\beta+\eta_1)u}A_{01}(t-u)du+\eta_1\int_0^t e^{-(\mu_1+\beta+\eta_1)u}A_{11}(t-u)du.\end{aligned} \tag{39}$$

Taking Laplace transform on both sides of (34)–(39), we obtain

$$A_{00}^*(s) = \frac{\alpha}{(s+\alpha+\gamma_{00})} A_{01}^*(s) + \frac{\gamma_{00}}{(s+\alpha+\gamma_{00})} A_{20}^*(s), \tag{40}$$

$$A_{01}^*(s) = \frac{\beta}{(s+\beta+\gamma_{01})} A_{00}^*(s) + \frac{\gamma_{01}}{(s+\beta+\gamma_{01})} A_{21}^*(s), \tag{41}$$

$$A_{10}^*(s) = \frac{\alpha}{(s+\alpha+\gamma_{10})} A_{11}^*(s) + \frac{\gamma_{10}}{(s+\alpha+\gamma_{10})} A_{20}^*(s), \tag{42}$$

$$A_{11}^*(s) = \frac{\beta}{(s+\beta+\gamma_{11})} A_{10}^*(s) + \frac{\gamma_{11}}{(s+\beta+\gamma_{11})} A_{21}^*(s), \tag{43}$$

$$\begin{aligned} A_{20}^*(s) = {} & \frac{1}{(s+\mu_0+\alpha+\eta_0)} + \frac{\alpha}{(s+\mu_0+\alpha+\eta_0)} A_{21}^*(s) \\ & + \frac{\mu_0}{(s+\mu_0+\alpha+\eta_0)} A_{00}^*(s) + \frac{\eta_0}{(s+\mu_0+\alpha+\eta_0)} A_{10}^*(s), \end{aligned} \tag{44}$$

$$\begin{aligned} A_{21}^*(s) = {} & \frac{1}{(s+\mu_1+\beta+\eta_1)} + \frac{\beta}{(s+\mu_1+\beta+\eta_1)} A_{20}^*(s) \\ & + \frac{\mu_1}{(s+\mu_1+\beta+\eta_1)} A_{01}^*(s) + \frac{\eta_1}{(s+\mu_1+\beta+\eta_1)} A_{11}^*(s). \end{aligned} \tag{45}$$

From (40)–(45), we obtain

$$A_{00}^*(s) = \frac{\gamma_{00}(s+\beta+\gamma_{01})}{D_0(s)} A_{20}^*(s) + \frac{\alpha\gamma_{01}}{D_0(s)} A_{21}^*(s), \tag{46}$$

$$A_{01}^*(s) = \frac{\beta\gamma_{00}}{D_0(s)} A_{20}^*(s) + \frac{\gamma_{01}(s+\alpha+\gamma_{00})}{D_0(s)} A_{21}^*(s), \tag{47}$$

$$A_{10}^*(s) = \frac{\gamma_{10}(s+\beta+\gamma_{11})}{D_1(s)} A_{20}^*(s) + \frac{\alpha\gamma_{11}}{D_1(s)} A_{21}^*(s), \tag{48}$$

$$A_{11}^*(s) = \frac{\beta\gamma_{10}}{D_1(s)} A_{20}^*(s) + \frac{\gamma_{11}(s+\alpha+\gamma_{10})}{D_1(s)} A_{21}^*(s), \tag{49}$$

$$A_{20}^*(s) = \frac{D_0(s)D_1(s)}{D_2(s)} + \frac{\alpha D_0(s)D_1(s) + \mu_0\alpha\gamma_{01}D_1(s) + \alpha\gamma_{11}\eta_0 D_0(s)}{D_2(s)} A_{21}^*(s), \tag{50}$$

$$A_{21}^*(s) = \frac{D_0(s)D_1(s)}{D_3(s)} + \frac{\beta D_0(s)D_1(s) + \mu_1\beta\gamma_{00}D_1(s) + \beta\gamma_{10}\eta_1 D_0(s)}{D_3(s)} A_{20}^*(s). \tag{51}$$

Using (50) and (51), we get

$$A_{20}^*(s) = \frac{A_{200}(s)}{D_2(s)D_3(s) - A(s)B(s)}, \tag{52}$$

$$A_{21}^*(s) = \frac{A_{210}(s)}{D_2(s)D_3(s) - A(s)B(s)}, \tag{53}$$

where

$$\begin{aligned} A_{200}(s) &= D_0(s)D_1(s)[D_3(s) + \alpha D_0(s)D_1(s) + \mu_0\alpha\gamma_{01}D_1(s) + \alpha\gamma_{11}\eta_0 D_0(s)], \\ A_{210}(s) &= D_0(s)D_1(s)[D_2(s) + \beta D_0(s)D_1(s) + \mu_1\beta\gamma_{00}D_1(s) + \beta\gamma_{10}\eta_1 D_0(s)]. \end{aligned}$$

Using (52) and (53) in (46)–(49), we get

$$A_{00}^*(s) = \frac{A_{001}(s) + A_{002}(s)}{D_2(s)D_3(s) - A(s)B(s)}, \tag{54}$$

$$A_{01}^*(s) = \frac{A_{011}(s) + A_{012}(s)}{D_2(s)D_3(s) - A(s)B(s)}, \tag{55}$$

$$A_{10}^*(s) = \frac{A_{101}(s) + A_{102}(s)}{D_2(s)D_3(s) - A(s)B(s)}, \tag{56}$$

$$A_{11}^*(s) = \frac{A_{111}(s) + A_{112}(s)}{D_2(s)D_3(s) - A(s)B(s)}, \tag{57}$$

where

$$
\begin{aligned}
A_{001}(s) &= \gamma_{00}(s+\beta+\gamma_{01})D_1(s)[D_3(s)+\alpha D_0(s)D_1(s)+\mu_0\alpha\gamma_{01}D_1(s)+\alpha\gamma_{11}\eta_0 D_0(s)],\\
A_{002}(s) &= \alpha\gamma_{01}D_1(s)[D_2(s)+\beta D_0(s)D_1(s)+\mu_1\beta\gamma_{00}D_1(s)+\beta\gamma_{10}\eta_1 D_0(s)],\\
A_{011}(s) &= \beta\gamma_{00}D_1(s)[D_3(s)+\alpha D_0(s)D_1(s)+\mu_0\alpha\gamma_{01}D_1(s)+\alpha\gamma_{11}\eta_0 D_0(s)],\\
A_{012}(s) &= \gamma_{01}(s+\alpha+\gamma_{00})D_1(s)[D_2(s)+\beta D_0(s)D_1(s)+\mu_1\beta\gamma_{00}D_1(s)+\beta\gamma_{10}\eta_1 D_0(s)],\\
A_{101}(s) &= \gamma_{10}(s+\beta+\gamma_{11})D_0(s)[D_3(s)+\alpha D_0(s)D_1(s)+\mu_0\alpha\gamma_{01}D_1(s)+\alpha\gamma_{11}\eta_0 D_0(s)],\\
A_{102}(s) &= \alpha\gamma_{11}D_0(s)[D_2(s)+\beta D_0(s)D_1(s)+\mu_1\beta\gamma_{00}D_1(s)+\beta\gamma_{10}\eta_1 D_0(s)],\\
A_{111}(s) &= \beta\gamma_{10}D_0(s)[D_3(s)+\alpha D_0(s)D_1(s)+\mu_0\alpha\gamma_{01}D_1(s)+\alpha\gamma_{11}\eta_0 D_0(s)],\\
A_{112}(s) &= \gamma_{11}(s+\alpha+\gamma_{10})D_0(s)[D_2(s)+\beta D_0(s)D_1(s)+\mu_1\beta\gamma_{00}D_1(s)+\beta\gamma_{10}\eta_1 D_0(s)].
\end{aligned}
$$

Inverting (52)–(57), we obtain the system availability functions as follows:

$$A_{00}(t) = F_{000} + \sum_{j=1}^{9} F_{00j}e^{\xi_j t}, \qquad A_{01}(t) = F_{010} + \sum_{j=1}^{9} F_{01j}e^{\xi_j t}, \tag{58}$$

$$A_{10}(t) = F_{100} + \sum_{j=1}^{9} F_{10j}e^{\xi_j t}, \qquad A_{11}(t) = F_{110} + \sum_{j=1}^{9} F_{11j}e^{\xi_j t}, \tag{59}$$

$$A_{20}(t) = F_{200} + \sum_{j=1}^{9} F_{20j}e^{\xi_j t}, \qquad A_{21}(t) = F_{210} + \sum_{j=1}^{9} F_{21j}e^{\xi_j t}, \tag{60}$$

where

$$
\begin{aligned}
F_{000} &= -\frac{A_{001}(0)+A_{002}(0)}{\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_7\xi_8\xi_9}, \quad F_{00j} = \lim_{s\to\xi_j}\frac{(s-\xi_j)[A_{001}(s)+A_{002}(s)]}{D_2(s)D_3(s)-A(s)B(s)},\ j=1,2,\ldots,9,\\
F_{010} &= -\frac{A_{011}(0)+A_{012}(0)}{\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_7\xi_8\xi_9}, \quad F_{01j} = \lim_{s\to\xi_j}\frac{(s-\xi_j)[A_{011}(s)+A_{012}(s)]}{D_2(s)D_3(s)-A(s)B(s)},\ j=1,2,\ldots,9,\\
F_{100} &= -\frac{A_{101}(0)+A_{102}(0)}{\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_7\xi_8\xi_9}, \quad F_{10j} = \lim_{s\to\xi_j}\frac{(s-\xi_j)[A_{101}(s)+A_{102}(s)]}{D_2(s)D_3(s)-A(s)B(s)},\ j=1,2,\ldots,9,\\
F_{110} &= -\frac{A_{111}(0)+A_{112}(0)}{\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_7\xi_8\xi_9}, \quad F_{11j} = \lim_{s\to\xi_j}\frac{(s-\xi_j)[A_{111}(s)+A_{112}(s)]}{D_2(s)D_3(s)-A(s)B(s)},\ j=1,2,\ldots,9,\\
F_{200} &= -\frac{A_{200}(0)}{\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_7\xi_8\xi_9}, \quad F_{20j} = \lim_{s\to\xi_j}\frac{(s-\xi_j)A_{200}(s)}{D_2(s)D_3(s)-A(s)B(s)},\ j=1,2,\ldots,9,\\
F_{210} &= -\frac{A_{210}(0)}{\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_7\xi_8\xi_9}, \quad F_{21j} = \lim_{s\to\xi_j}\frac{(s-\xi_j)A_{210}(s)}{D_2(s)D_3(s)-A(s)B(s)},\ j=1,2,\ldots,9.
\end{aligned}
$$

6. Reliability Analysis

Let $R_{2j}(t)$ be the probability that the system has not visited the down state until time t given that the system was in upstate and the environment level was $j, j = 0, 1$ at time $t = 0$. Then, we obtain the following integral equations:

$$R_{20}(t) = e^{-(\mu_0+\alpha+\eta_0)t} + \alpha\int_0^t e^{-(\mu_0+\alpha+\eta_0)u}R_{21}(t-u)du, \tag{61}$$

$$R_{21}(t) = e^{-(\mu_1+\beta+\eta_1)t} + \beta\int_0^t e^{-(\mu_1+\beta+\eta_1)u}R_{20}(t-u)du. \tag{62}$$

Taking Laplace transform on both sides of (61) and (62), we get

$$(s+\mu_0+\alpha+\eta_0)R^*_{20}(s) = 1+\alpha R^*_{21}(s), \tag{63}$$

$$(s+\mu_1+\beta+\eta_1)R^*_{21}(s) = 1+\beta R^*_{20}(s). \tag{64}$$

Solving (63) and (64), we get

$$R^*_{20}(s) = \frac{(s+\mu_1+\alpha+\beta+\eta_1)}{(s+\mu_0+\alpha+\eta_0)(s+\mu_1+\beta+\eta_1)-\alpha\beta}, \tag{65}$$

$$R^*_{21}(s) = \frac{(s+\mu_0+\alpha+\beta+\eta_0)}{(s+\mu_0+\alpha+\eta_0)(s+\mu_1+\beta+\eta_1)-\alpha\beta}. \tag{66}$$

Inverting (65) and (66), we get

$$R_{20}(t) = \frac{(\Gamma_1+\mu_1+\alpha+\beta+\eta_1)}{(\Gamma_1-\Gamma_2)}e^{\Gamma_1 t} - \frac{(\Gamma_2+\mu_1+\alpha+\beta+\eta_1)}{(\Gamma_1-\Gamma_2)}e^{\Gamma_2 t}, \tag{67}$$

$$R_{21}(t) = \frac{(\Gamma_1+\mu_0+\alpha+\beta+\eta_0)}{(\Gamma_1-\Gamma_2)}e^{\Gamma_1 t} - \frac{(\Gamma_2+\mu_0+\alpha+\beta+\eta_0)}{(\Gamma_1-\Gamma_2)}e^{\Gamma_2 t}, \tag{68}$$

where Γ_1 and Γ_2 are the zeros of $(s+\mu_0+\alpha+\eta_0)(s+\mu_1+\beta+\eta_1)-\alpha\beta$. Both Γ_1 and Γ_2 are negative.

7. Mean Time to Failure

An important measure of system performance is the mean time to failure of the system. We obtain an expression for the same.

Case (i) We assume that, at time $t=0$, the load of the environment is less and the unit is just put online. Then, $V(0)=(2,0)$. Let T_0 be the random time such that the unit has not failed in the interval $(0,T_0]$ and enters into failure state in the interval $(T_0, T_0+\Delta]$. Let $f_0(t)$ be the probability density function of T_0. Then, we have

$$\begin{aligned} f_0(t)\Delta + o(\Delta) &= Pr(t < T_0 \le t+\Delta) \\ &= Pr(T_0 \le t+\Delta) - Pr(T_0 \le t) \\ &= Pr(T_0 > t) - Pr(T_0 > t+\Delta) \\ &= R_{20}(t) - R_{20}(t+\Delta). \end{aligned}$$

Consequently, we obtain

$$f_0(t) = -\frac{dR_{20}(t)}{dt}. \tag{69}$$

Hence the mean time to failure is given by

$$\begin{aligned} E(T_0) &= \int_0^\infty t f_0(t)dt \\ &= -\int_0^\infty t dR_{20}(t) \\ &= \int_0^\infty R_{20}(t)dt \end{aligned}$$

$$E(T_0) = \frac{(\mu_1+\alpha+\beta+\eta_1)}{\Gamma_1\Gamma_2}. \tag{70}$$

Case (ii) We assume that, at time $t=0$, the load of the environment is heavy and the unit is just put online. Then, $V(0)=(2,1)$. Let T_1 be the random time such that the unit has not failed in the interval $(0, T_1]$ and enters into failure state in the interval $(T_1, T_1 + \Delta]$. Let $f_1(t)$ be the probability density function of T_1. Then, we have

$$\begin{aligned} f_1(t)\Delta + o(\Delta) &= Pr(t < T_1 \leq t + \Delta) \\ &= Pr(T_1 \leq t + \Delta) - Pr(T_1 \leq t) \\ &= Pr(T_1 > t) - Pr(T_1 > t + \Delta) \\ &= R_{21}(t) - R_{21}(t + \Delta). \end{aligned}$$

Consequently, we obtain

$$f_1(t) = -\frac{dR_{21}(t)}{dt}. \tag{71}$$

Hence the mean time to failure is given by

$$\begin{aligned} E(T_1) &= \int_0^\infty t f_1(t) dt \\ &= -\int_0^\infty t dR_{21}(t) \\ &= \int_0^\infty R_{21}(t) dt \end{aligned}$$

$$E(T_1) = \frac{(\mu_0 + \alpha + \beta + \eta_0)}{\Gamma_1 \Gamma_2}. \tag{72}$$

8. Efficiency of the System

Another important measure of system performance is the mean value of system down time in a given interval. To obtain an expression for the measure, we define a Bernoulli random variable

$$J(t) = \begin{cases} 0, & \text{if V(t)} \in \{\text{(2,k)} \mid \text{k=0, 1}\} \\ 1, & \text{otherwise} \end{cases}$$

Let the initial condition be that at time $t = 0$, the load of the environment is heavy and repair of unit is just completed so that $V(0) = (2, 1)$. Let $D(t)$ be the random variable representing the total down time of the system in the interval $[0, t]$. Then, $D(t)$ can be expressed as a stochastic integral

$$D(t) = \int_0^t J(u) du. \tag{73}$$

Consequently, the mean down time of the system in $[0, t]$ is given by

$$\begin{aligned} E[D(t)] &= \int_0^t Pr[J(u) = 1] du \\ &= \int_0^t [1 - Pr(J(u) = 0)] du \end{aligned}$$

$$E[D(t)] = \int_0^t [1 - P(2, 0, u) - P(2, 1, u)] du. \tag{74}$$

Using (33) in (74), we obtain

$$E[D(t)] = t - (E_{200} + E_{210})t - \sum_{j=1}^{9} (E_{20j} + E_{21j}) \left(\frac{e^{\xi_j t} - 1}{\xi_j} \right). \tag{75}$$

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The efficiency of the system at any given time is given by the measure

$$E_{ff}(t) = 1 - \frac{E[D(t)]}{t}.$$

9. A Numerical Illustration

For the purpose of illustration, we assume the following values of the system parameters:

$$\begin{array}{lllll} \alpha = 0.8; & \mu_0 = 1.0; & \gamma_{00} = 1.1; & \gamma_{01} = 2.1; & \eta_0 = 0.1; \\ \beta = 0.6; & \mu_1 = 2.0; & \gamma_{10} = 1.2; & \gamma_{11} = 1.3; & \eta_1 = 0.2. \end{array}$$

These parameters are chosen only for illustrative purposes; in practical applications, they would be estimated from observed failure and repair logs of the system under study. We have computed the time-dependent state probabilities and plotted them as Figures 7–12.

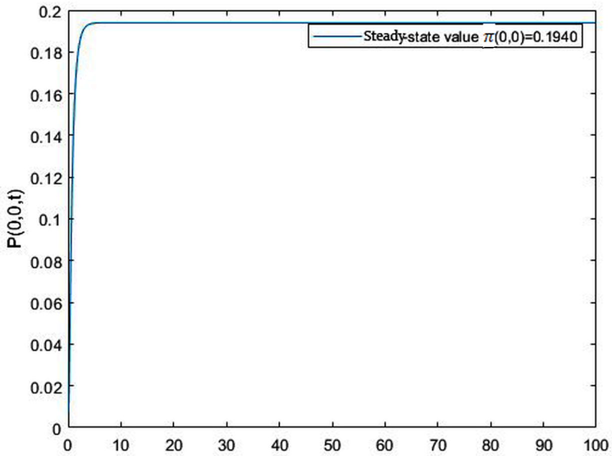

Figure 7. $P(0,0,t)$ as a function of t.

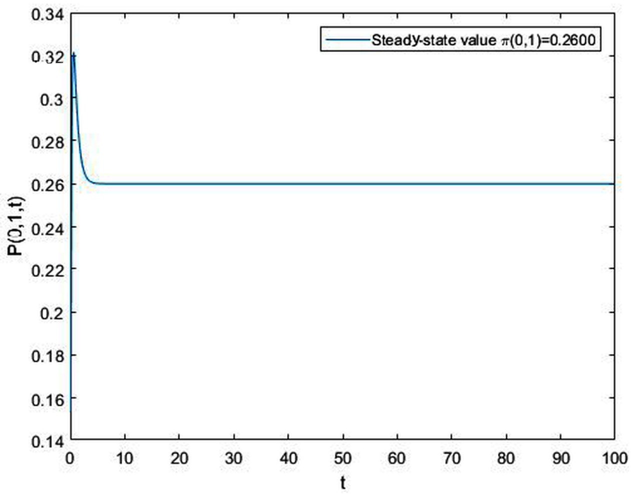

Figure 8. $P(0,1,t)$ as a function of t.

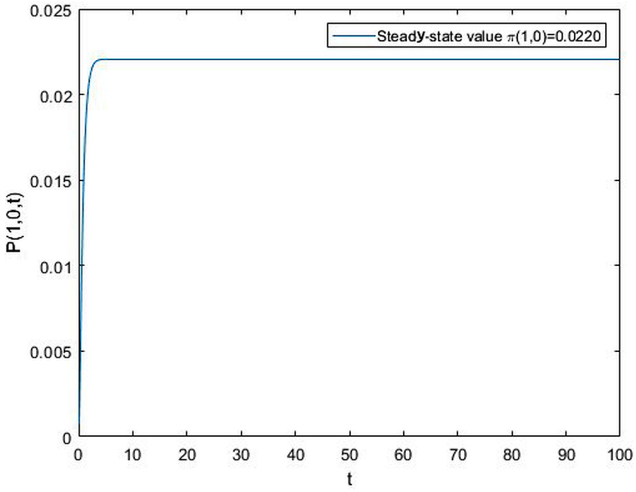

Figure 9. $P(1, 0, t)$ as a function of t.

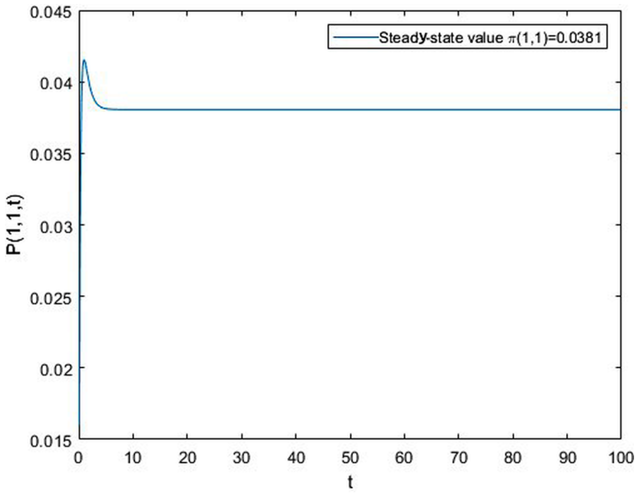

Figure 10. $P(1, 1, t)$ as a function of t.

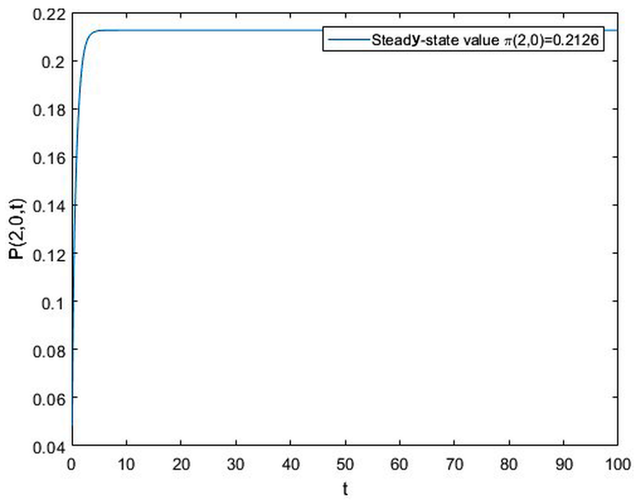

Figure 11. $P(2, 0, t)$ as a function of t.

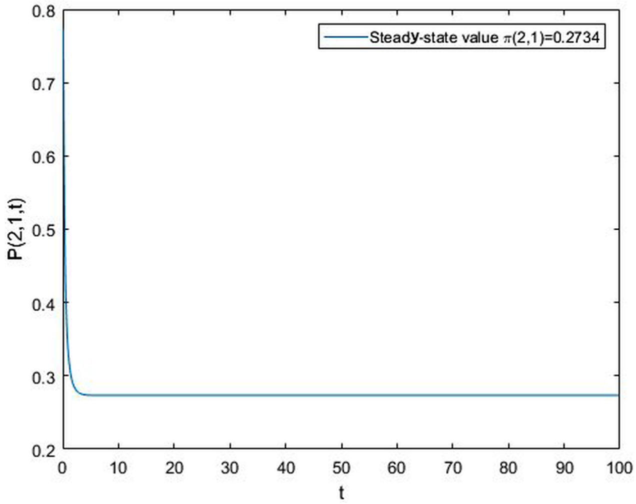

Figure 12. $P(2,1,t)$ as a function of t.

We observe that the state probabilities are non-decreasing in less load environment and non-increasing in heavy load environment. They approach the steady values

$$\pi(0,0) = 0.1940, \pi(1,0) = 0.0220, \pi(2,0) = 0.2126,$$
$$\pi(0,1) = 0.2600, \pi(1,1) = 0.0381, \pi(2,1) = 0.2734.$$

These values are consistent with the model parameters since

(i) The repair rate of intrinsic failure is higher than that of shock failure in heavy load ($\gamma_{01} = 2.1$ and $\gamma_{11} = 1.3$); in less load the two rates are comparable ($\gamma_{00} = 1.1 \simeq \gamma_{10} = 1.2$);
(ii) The rate of arrival of shocks in the less load environment is lower than that in the heavy load environment ($\eta_0 = 0.1$ and $\eta_1 = 0.2$);
(iii) The intrinsic failure rate dominates the shock failure rate in both environments ($\mu_0 = 1.0, \mu_1 = 2.0$ versus $\eta_0 = 0.1, \eta_1 = 0.2$), so that the steady-state probability mass in the life-failure states $(0,.)$ is roughly an order of magnitude greater than in the shock-failure states $(1,.)$;
(iv) The switch-over rate from less load environment to heavy load environment exceeds that from heavy load environment to less load environment ($\alpha = 0.8$ and $\beta = 0.6$), so that the system spends a larger fraction of time, $\frac{\alpha}{\alpha+\beta} = \frac{4}{7} \simeq 0.571$, in the heavy-load regime which is reflected in the heavy-load steady-state probabilities $\pi(.,1)$, each of which exceeds the corresponding less-load probability $\pi(.,0)$; numerically, $\pi(0,1) = 0.2600 > \pi(0,0) = 0.1940$, $\pi(1,1) = 0.0381 > \pi(1,0) = 0.0220$, and $\pi(2,1) = 0.2734 > \pi(2,0) = 0.2126$.

The influence of $(\mu_0, \mu_1, \eta_0, \eta_1)$ on the steady-state distribution can be read directly from the governing equations. An increase in μ_0 raises $\pi(0,0)$ and lowers $\pi(2,0)$: more intrinsic failures occur in less-load periods, and probability mass is shifted from the operational state in level 0 to the corresponding repair state. The effect of μ_1 on $\pi(0,1)$ and $\pi(2,1)$ is symmetric. An increase in η_0 raises $\pi(1,0)$ at the expense of $\pi(2,0)$; η_1 does the same for $\pi(1,1)$ and $\pi(2,1)$. Since the steady-state availability is $A(\infty) = \pi(2,0) + \pi(2,1)$, it is monotonically decreasing in each of $\mu_0, \mu_1, \eta_0, \eta_1$ and monotonically increasing in each of $\gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{11}$. These monotonicity properties translate into immediate design guidance: any reduction in the failure or shock rates, and any reduction in the mean repair time, increases the long-run availability.

The graphs in Figures 7–12 show a brief boundary-layer adjustment from the initial condition $V(0) = (2,1)$, followed by rapid stabilization within roughly $t \simeq 5$ time units.

Next, we have computed system availabilities $A_{ij}(t), i = 0, 1, 2; j = 0, 1$ as functions of time t and plotted them as Figures 13–18.

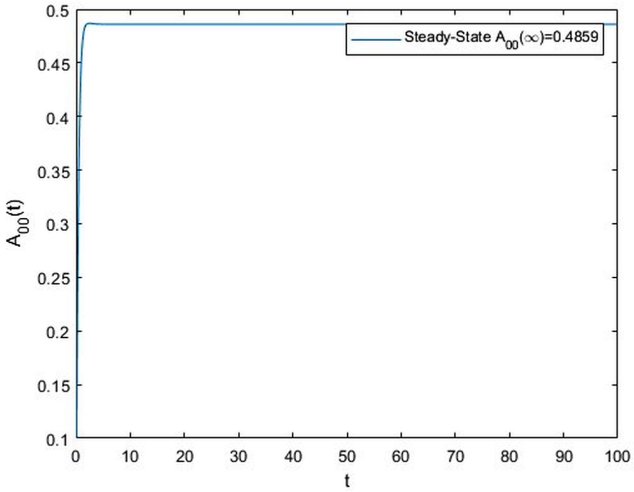

Figure 13. $A_{00}(t)$ as a function of t.

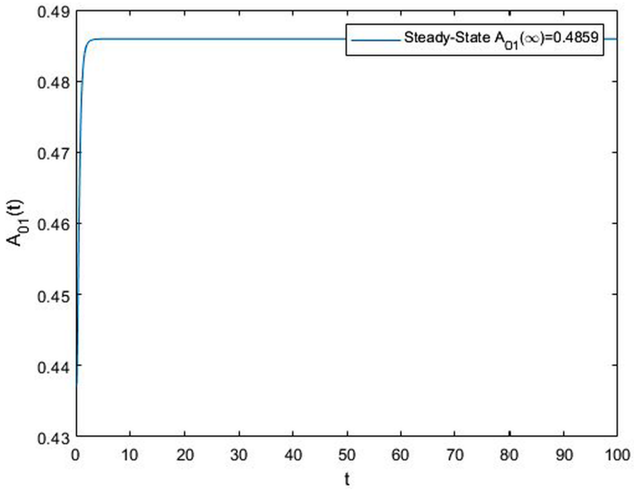

Figure 14. $A_{01}(t)$ as a function of t.

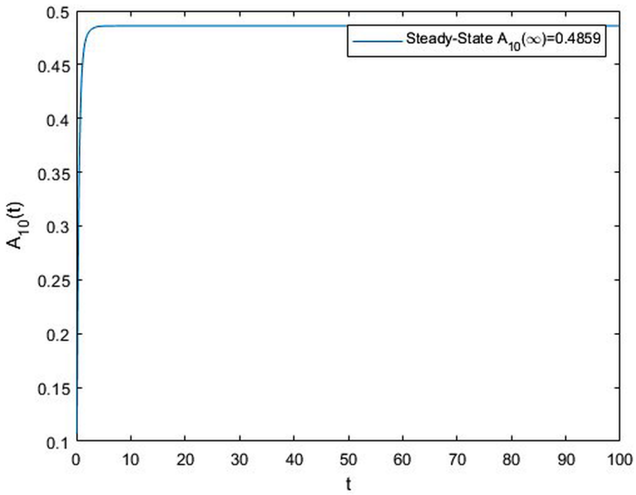

Figure 15. $A_{10}(t)$ as a function of t.

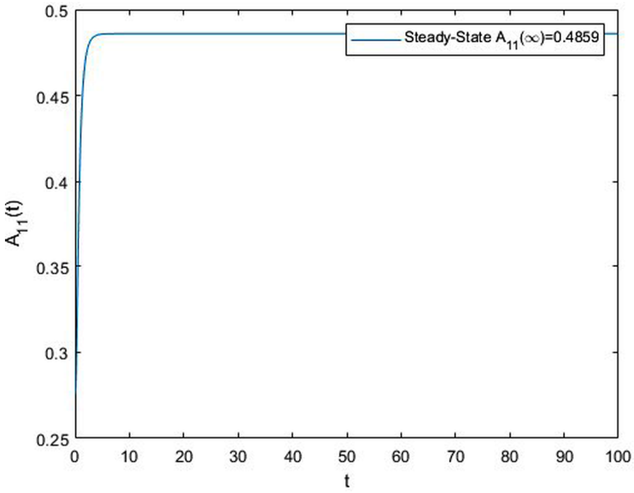

Figure 16. $A_{11}(t)$ as a function of t.

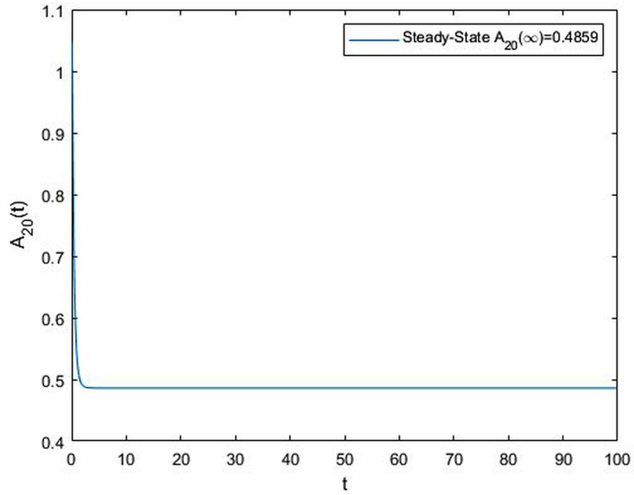

Figure 17. $A_{20}(t)$ as a function of t.

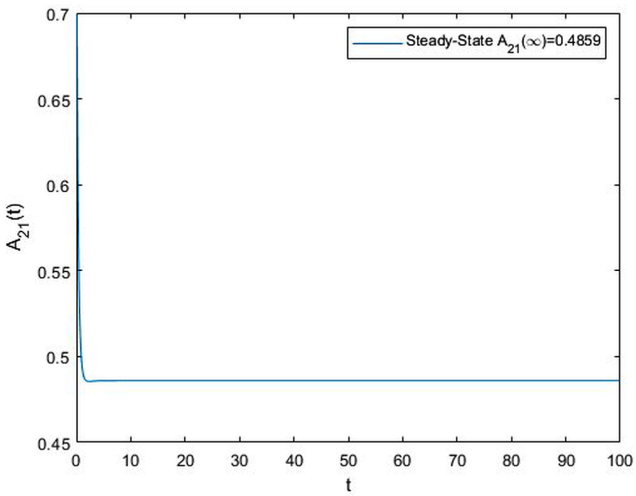

Figure 18. $A_{21}(t)$ as a function of t.

We observe that availabilities are non-decreasing if we start with failed unit; that is, $A_{ij}(t), i = 0, 1; j = 0, 1$ are non-decreasing. On the other hand, availabilities are non-increasing if we start with working unit; that is, $A_{ij}(t), i = 2; j = 0, 1$ are non-increasing. It is interesting to note that whatever be the initial condition, all availability functions reach the same steady-state value 0.4859.

We have computed the values of the reliability functions $R_{20}(t)$ and $R_{21}(t)$ for various values of t and presented them in Table 1.

Table 1. Reliability comparison table.

t	$R_{20}(t)$	$R_{21}(t)$	t	$R_{20}(t)$	$R_{21}(t)$
0.1	0.8922040	0.8051423	4.1	0.0024304	0.0011438
0.2	0.7905155	0.6523885	4.2	0.0020869	0.0009821
0.3	0.6965552	0.5318249	4.3	0.0017920	0.0008432
0.4	0.6110390	0.4360175	4.4	0.0015387	0.0007239
0.5	0.5340895	0.3593676	4.5	0.0013213	0.0006216
0.6	0.4654538	0.2976392	4.6	0.0011345	0.0005337
0.7	0.4046541	0.2476112	4.7	0.0009742	0.0004582
0.8	0.3510905	0.2068199	4.8	0.0008365	0.0003934
0.9	0.3041094	0.1733706	4.9	0.0007183	0.0003378
1.0	0.2630496	0.1457965	5.0	0.0006168	0.0002901
1.1	0.2272697	0.1229551	5.1	0.0005296	0.0002490
1.2	0.1961660	0.1039501	5.2	0.0004547	0.0002138
1.3	0.1691812	0.0880738	5.3	0.0003905	0.0001836
1.4	0.1458086	0.0747636	5.4	0.0003353	0.0001577
1.5	0.1255926	0.0635691	5.5	0.0002879	0.0001354
1.6	0.1081271	0.0541276	5.6	0.0002472	0.0001162
1.7	0.0930523	0.0461447	5.7	0.0002123	0.0000998
1.8	0.0800515	0.0393806	5.8	0.0001823	0.0000857
1.9	0.0688471	0.0336384	5.9	0.0001565	0.0000736
2.0	0.0591963	0.0287557	6.0	0.0001344	0.0000632
2.1	0.0508878	0.0245980	6.1	0.0001154	0.0000543
2.2	0.0437377	0.0210533	6.2	0.0000991	0.0000466
2.3	0.0375866	0.0180282	6.3	0.0000851	0.0000400
2.4	0.0322965	0.0154440	6.4	0.0000730	0.0000343
2.5	0.0277480	0.0132349	6.5	0.0000627	0.0000295
2.6	0.0238380	0.0113452	6.6	0.0000539	0.0000253
2.7	0.0204773	0.0097278	6.7	0.0000462	0.0000217
2.8	0.0175893	0.0083427	6.8	0.0000397	0.0000187
2.9	0.0151078	0.0071562	6.9	0.0000341	0.0000160
3.0	0.0129757	0.0061393	7.0	0.0000293	0.0000138
3.1	0.0111442	0.0052677	7.1	0.0000251	0.0000118
3.2	0.0095708	0.0045203	7.2	0.0000216	0.0000101
3.3	0.0082193	0.0038793	7.3	0.0000185	0.0000087
3.4	0.0070585	0.0033295	7.4	0.0000159	0.0000075
3.5	0.0060615	0.0028578	7.5	0.0000137	0.0000064
3.6	0.0052052	0.0024531	7.6	0.0000117	0.0000055
3.7	0.0044699	0.0021058	7.7	0.0000101	0.0000047
3.8	0.0038384	0.0018077	7.8	0.0000087	0.0000041
3.9	0.0032960	0.0015519	7.9	0.0000074	0.0000035
4.0	0.0028303	0.0013323	8.0	0.0000064	0.0000030

Both reliability functions decay nearly exponentially, with decay rate close to $|\Gamma_1|$, the largest root (closest to zero) of $(s+\mu_0+\alpha+\eta_0)(s+\mu_1+\beta+\eta_1)-\alpha\beta$, which determines the long-term rate at which the survival probability tends to zero. The slightly slower decay of $R_{20}(t)$ compared with $R_{21}(t)$ reflects the lower failure pressure experienced when the system starts in the less-load environment.

Figure 19 provides a comparative picture of $R_{20}(t)$ and $R_{21}(t)$.

We find that for the chosen values of the parameters, $R_{21}(t) < R_{20}(t)$ for all t.

By using (70) and (72), we computed the mean time to failure (MTTF) of the system. The MTTF is 0.7438 if the system is put on line in less load environment at time $t = 0$ and 0.5165 if the system is put on line in heavy load environment at time $t = 0$.

We have also computed the system efficiency over time for the model and exhibited the same in Figure 20.

It is observed that the system efficiency decreases over time. The stationary efficiency of the system is found to be 0.4859421.

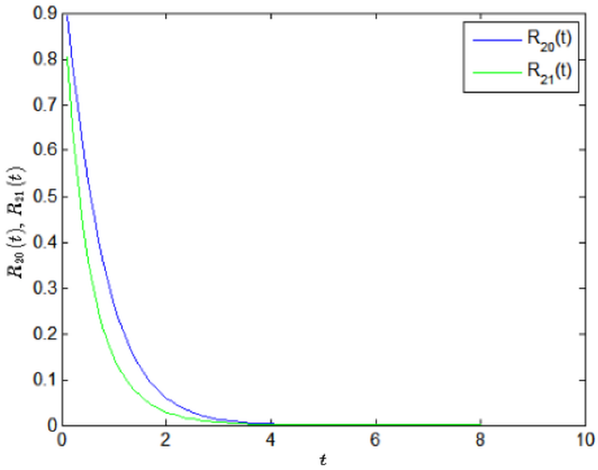

Figure 19. Comparison of $R_{20}(t)$ with $R_{21}(t)$.

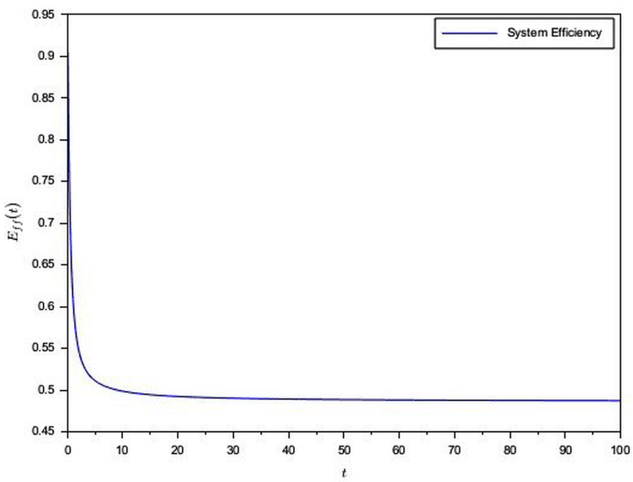

Figure 20. System Efficiency against time.

9.1. Application to a Power-Distribution Transformer

The above model may be applied to a regional power-distribution transformer under a daily duty cycle. The two environment levels correspond to off-peak and peak demand periods (night versus evening). Intrinsic failures represent insulation and winding degradation, and hence $\mu_1 > \mu_0$ because heating accelerates ageing during peak load. Shocks correspond to lightning strikes, feeder short-circuits and switching transients, and thus $\eta_1 > \eta_0$ for the same period. A single utility maintenance crew (the repairman) handles all outages, and repair duration depends on whether the failure is degradation-driven or shock-driven, and on whether crews and parts are mobilized during a day shift (level 1) or a night shift (level 0). In this scenario, $\pi(2,0) + \pi(2,1)$ is the long-run availability used in service-level agreements, $E(T_0)$ and $E(T_1)$ inform spare-transformer inventory sizing, and the system efficiency curve in Figure 20 supports preventive-maintenance scheduling.

9.2. Parameter Sensitivity

From the steady-state distributions derived in Sections 4 and 5, we observe that three groupings of parameters are of practical interest:

First, the failure-rate parameters $(\mu_0, \mu_1, \eta_0, \eta_1)$ all act as out-flow rates from the operational states $(2, 0)$ and $(2, 1)$; the steady-state availability $A(\infty) = \pi(2, 0) + \pi(2, 1)$ is therefore strictly decreasing in each of them, and μ_1, and η_1 exert stronger influence than μ_0 and η_0 whenever the environment spends a larger fraction of time in heavy load—that is, whenever $\frac{\alpha}{\alpha+\beta}$ is large.

Second, the repair-rate parameters $(\gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{11})$ act as in-flow rates to the operational states; $A(\infty)$ is strictly increasing in each, and the γ_{01} and γ_{11} terms dominate when heavy-load occupancy is high.

Third, the environment-switching rates (α, β) reshape the stationary distribution of $Y(t)$ without directly affecting failure or repair, but they govern the relative weight given to level-0 versus level-1 contributions; the ratio $\frac{\beta}{\alpha}$ is therefore the single most influential design factor when the heavy-load regime is significantly harsher than the less-load regime.

From Section 7, for the mean time to failure, Equations (70) and (72) show that $E(T_j)$ is inversely proportional to the product $\Gamma_1\Gamma_2$, which is itself a polynomial in $(\mu_0, \mu_1, \eta_0, \eta_1, \alpha, \beta)$; the MTTF is consequently most sensitive to whichever of these rates is currently smallest, since reductions in the dominant time-scale produce the largest relative gain.

Hence, we observe that when shock arrival rates are externally fixed (for instance, by the ambient lightning frequency in the transformer application of Section 9.1), the most cost-effective improvement is typically to reduce repair times in the heavy-load regime—that is, to lower $\frac{1}{\gamma_{01}}$ and $\frac{1}{\gamma_{11}}$.

10. Conclusions

In this paper we have introduced a single-unit repairable reliability model operating in a doubly stochastic environment, in which a two-state Markov environment process simultaneously governs the intrinsic failure rate, the shock arrival rate and the type-dependent repair rate. The state space comprises six states obtained by considering three unit conditions (intrinsic failure, shock failure, operational) with two environment levels. We have derived the Kolmogorov equations, obtained the Laplace transforms of the transient state probabilities and inverted them in closed form, derived the steady-state probabilities through the final-value theorem, and developed explicit expressions for the availability functions $A_{ij}(t)$, the reliability functions $R_{2j}(t)$, the mean time to failure $E(T_j)$ and the system efficiency $E_{ff}(t)$.

The major contribution of this paper is the joint treatment of environment-modulated failure, shock arrival and repair within a single tractable Markov framework, together with the dichotomous classification of failures into intrinsic and shock-induced types with type and environment-dependent repair. The closed-form availability and reliability expressions extend the single-environment results of [12,13,15] to the doubly stochastic setting.

The numerical illustration showed that, regardless of the initial condition, the availability functions converge to a common steady-state value (0.4859 for the chosen parameters), and that the mean time to failure is higher when the system is put online in a less-load environment (0.7438) than in a heavy-load environment (0.5165). These outputs translate directly into design and operational guidance for power-distribution systems as discussed in Section 9.1.

The limitations of the model such as the assumption of exponential life times and exponential repair times, a single-unit configuration, instantaneous fatal shocks (every shock leads to immediate failure), and a perfect repairman who is unaffected by shocks and never engaged in any other task may be relaxed individually or in a combined manner which will lead to analytical complexities. Further, the two-level environment itself may be extended to incorporate a more complex load process.

This work may be extended to (i) generalizing the life-time and repair-time distributions to phase-type or general distributions through a semi-Markov or supplementary-variable formulation; (ii) multi-unit systems with redundancy (cold, warm or hot standby) and a shared repair facility; (iii) introducing cumulative damage from non-fatal shocks together with a damage-threshold failure rule; and (iv) relaxing the two-state environment to a finite-state, or even diffusion-modulated, environment.

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Conflicts of Interest: The authors declare no conflicts of interest.

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