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Article

Stability Analysis of T-S Fuzzy Systems via Delay-Dependent Lyapunov–Krasovskii Functionals and Linear Switching Method

Chang-Ho Lee [1,2,†], Yeong-Jae Kim [1,†], Yong-Gwon Lee [1,3], Seung-Hoon Lee [4] and Oh-Min Kwon [1,*]

1 School of Electrical Engineering, Chungbuk National University, Cheongju 28644, Republic of Korea; ch90@cmcst.co.kr (C.-H.L.); cel2020@cbnu.ac.kr (Y.-J.K.); leeywkgg@chungbuk.ac.kr (Y.-G.L.)
2 R&D Center, Changmyung Control Technology Co., Ltd., Cheongju 28143, Republic of Korea
3 The Missile Systems Core Technology R&D Team, LIG Nex1 Co., Ltd., Seongnam 13488, Republic of Korea
4 Division of Electrical and Electronics Engineering, Korea Maritime and Ocean University, Busan 49112, Republic of Korea; shlee@kmou.ac.kr
* Correspondence: madwind@chungbuk.ac.kr; Tel.: +82-43-261-2422; Fax: +82-43-263-2419
† These authors contributed equally to this work.

Abstract

This paper investigates the problem of stability analysis for Takagi–Sugeno fuzzy systems with time-varying delays. By integrating an augmented delay-dependent Lyapunov–Krasovskii functional (LKF) structure, a refined LKF based on auxiliary function-based integral inequalities, and utilizing a linear switching method, this paper proposes less conservative stability criteria that effectively enhance fuzzy membership characteristics. The proposed stability criteria are formulated in the framework of linear matrix inequalities. Through three numerical examples, the effectiveness and superiority of the proposed approach are demonstrated by achieving significantly improved maximum delay bounds compared to the existing literature.

Keywords: linear matrix inequality; linear switching method; stability; time-varying delays; T-S fuzzy system

MSC: 34K20; 34K25

1. Introduction

The Takagi–Sugeno (T-S) fuzzy model is one of the widely used frameworks for representing nonlinear systems by interpolating a set of local linear models through membership functions. The modeling concept is straightforward, since nonlinear behavior is represented by several local linear models that are blended by membership functions to form the overall dynamics [1]. Because many analysis and synthesis conditions can be formulated as linear matrix inequalities (LMIs), the T-S framework provides a tractable bridge between nonlinear system modeling and convex optimization-based analysis [2]. However, stability conditions based on a Lyapunov function $x^T(t)Px(t)$ are often conservative. To reduce this conservatism, various Lyapunov constructions have been developed, including fuzzy weight-dependent, augmented, and asymmetric Lyapunov or Lyapunov–Krasovskii functionals (LKFs) [3]. Another important issue is the treatment of membership functions in Lyapunov analysis. In particular, when membership-dependent Lyapunov functions are employed, the time derivatives of the membership functions may appear explicitly and complicate the analysis. To overcome this difficulty, alternative fuzzy Lyapunov constructions have been proposed, among which the line-integral-based fuzzy Lyapunov approach

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is a representative example [4]. Along a related direction, membership-function-dependent Lyapunov constructions combined with switching ideas were also developed to exploit fuzzy weighting information more effectively and to further reduce conservatism [5].

Time delay is also unavoidable in practical systems where sensing, computation, communication, and actuator dynamics introduce delayed information. For T-S fuzzy systems with time-varying delays, delay-dependent stability is usually analyzed by constructing a LKF, whose effectiveness strongly depends on both the structure of the functional and the tightness of the inequalities used to bound its derivative. To reduce conservatism, various integral inequalities have been proposed to estimate the integral terms in the derivative of the LKF, including Jensen's, Wirtinger-based, Bessel–Legendre, auxiliary-function-based, free-weighting matrix integral inequalities [6–10]. In addition, when time-varying delays are considered, reciprocal delay-related terms often appear during the estimation of the LKF derivative. Reciprocally convex techniques are therefore commonly employed to preserve an LMI-based formulation [11,12].

Recent studies have continued to improve delay-dependent stability criteria for delayed T-S fuzzy systems by enriching the LKF and tightening the estimation of its derivative [12–27]. For example, augmented LKFs and fuzzy line-integral constructions have been introduced for T-S fuzzy systems with time-varying delay, while membership-function-dependent LKFs and switching schemes have been shown to enlarge the admissible delay bound by making more direct use of fuzzy weighting information [5,13]. Asymmetric LKFs were also proposed to relax the symmetry and positive-definiteness requirements imposed on internal matrices, thereby reducing computational complexity [14]. In addition, less conservative treatments of reciprocal convex terms and delay-dependent couplings have been developed through parameter-dependent reciprocally convex inequalities, quadratic-delay-product methods, and delay-product-dependent LKFs [15–17]. More recently, refined line-integral-type LKFs together with improved negative-determination conditions for high-order delay-dependent terms have also been reported to further enlarge admissible delay bounds [18]. Nevertheless, existing results may still leave room for improvement when multi-integral state components, delay-subinterval information, or richer couplings among current, delayed, and integral states are not explicitly embedded in the LKF [19,20].

Motivated by these observations, this paper investigates delay-dependent stability for continuous-time T–S fuzzy systems with time-varying delay. The main objective is to construct an LKF that explicitly incorporates multi-integral state components so that the relationships among the current, delayed, and integral states can be captured more effectively. By combining equality constraints, Finsler's lemma, and the Schur complement, the resulting stability condition is derived in an LMI form. The main contributions of this paper are summarized as follows.

- A delay-dependent stability criterion is constructed for continuous-time T-S fuzzy systems with time-varying delays satisfying $\dot{\tau}(t) \leq \tau_D$. To this end, a time-varying-delay-dependent LKF with explicit multi-integral state vector components is constructed, which leads to improved stability results for a relatively less explored delay condition.
- The term $\int_{t-\tau_M}^{t} x(v)\, dv - x(s)$ is included in the double integral term, so that more delay-related state information can be incorporated, leading to reduced conservatism.
- A membership-function-dependent switching scheme [5] is introduced to further improve the admissible delay bound in numerical computations.

The main stability conditions are derived, and numerical examples are presented to demonstrate the effectiveness of the proposed method.

Notation: $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ represents the n-dimensional Euclidean space and $m \times n$ real matrices. $X > 0$ and $X \geq 0$ signify that the matrix X is a real symmetric positive definite matrix and positive semi-definite. I_n denotes the $n \times n$ identity matrix. 0_n and

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$0_{m\times n}$ denotes the $n \times n$ and $m \times n$ zero matrix, respectively. $*$ represents the elements below the main diagonal of a symmetric matrix. $X_{[f(t)]} \in \mathbb{R}^{m\times n}$ indicates that $X_{[f(t)]}$ is the matrix with respect to $f(t)$, i.e., $X_{[f_0]} = X_{[f(t)=f_0]}$. $Sym\{X\}$ denotes $X + X^T$. $\odot$ denotes the quadratic form of the matrix. $X^{\perp}$ denotes a basis for the null space of X.

2. Problem Statements

Consider the following T-S fuzzy system with a time-varying delay.

Plant Rule i: If z_1(t) is $J_{i1}, \ldots$ and v_p(t) is J_{ip}, then

$$\dot{x}(t) = A_i x(t) + A_{d,i} x(t - \tau(t)),\ x(t) = \phi(t), t \in [-\tau_M, 0] \tag{1}$$

where $A_i \in \mathbb{R}^{n\times n}$ and $A_{d,i} \in \mathbb{R}^{n\times n}$ are system matrices, $x(t) \in \mathbb{R}^n$ is the state vector, and $\tau(t)$ is time-varying delay satisfying

$$0 \le \tau(t) \le \tau_M, \quad \dot{\tau}(t) \le \tau_D. \tag{2}$$

Also, $v_j(t)(j = 1, \ldots, p)$ denotes the premise variable, $J_{ij}(i = 1, \ldots, r; j = 1, \ldots, p)$ is fuzzy set associated with the jth premise variable in the ith rule and r is the number of fuzzy rules. Using the fuzzy rules, system (1) can be denoted as

$$\dot{x}(t) = \sum_{i=1}^{r} \nu_i(v(t))(A_i x(t) + A_{d,i} x(t - \tau(t))) \tag{3}$$

where

$$\nu_i(v(t)) = \frac{\omega(v(t))}{\sum_{i=1}^{r} \omega(v(t))}, \quad \omega_i(v(t)) = \prod_{j=1}^{p} J_{ij}(v_j(t)).$$

In the above, $\nu_i(v(t))$ is the normalized membership function and $J_{ij}(v_j(t))$ is the grade of membership of $v_j(t)$ in J_{ij}. Suppose $\omega_i(v_j(t)) \ge 0$ and $\sum_{i=1}^{r} \omega_i(v_j(t)) > 0$, then the membership function satisfy

$$\sum_{i=1}^{r} \nu_i(v(t)) = 1, \quad \sum_{i=1}^{r} \dot{\nu}_i(v(t)) = 0. \tag{4}$$

Before giving the main results, several useful lemmas are introduced.

Lemma 1 ([10]). *Let $x(\cdot) : [v, u] \to \mathbb{R}^n$ be a continuously differentiable function and augmented vector $\xi(t) \in \mathbb{R}^a$. For given scalars v, u with $v < u$, positive matrix $R \in \mathbb{R}^{n\times n}$, and any matrices L, M, $N \in \mathbb{R}^{an\times n}$, the following inequality holds:*

$$\begin{aligned} &-\int_v^u x^T(\alpha) R x(\alpha) d\alpha \le Sym\{\xi^T(t) L \varepsilon_1(v,u) + \xi^T(t) M \varepsilon_2(v,u) + \xi^T(t) N \varepsilon_3(v,u)\} \\ &+ (u - v)\xi^T(t) L R^{-1} L^T \xi(t) + \frac{(u - v)}{3} \xi^T(t) M R^{-1} M^T \xi(t) + \frac{(u - v)}{5} \xi^T(t) N R^{-1} N \xi(t) \end{aligned} \tag{5}$$

where

$$\begin{aligned} \varepsilon_1(v,u) &= \int_v^u x(\alpha) d\alpha, \\ \varepsilon_2(v,u) &= -\int_v^u x(\alpha) d\alpha + \frac{2}{(u - v)} \int_v^u \int_\alpha^u x(\beta) d\beta d\alpha, \\ \varepsilon_3(v,u) &= \int_v^u x(\alpha) d\alpha - \frac{6}{(u - v)} \int_v^u \int_\alpha^u x(\beta) d\beta d\alpha + \frac{12}{(u - v)^2} \int_v^u \int_\alpha^u \int_\beta^u x(\gamma) d\gamma d\beta d\alpha. \end{aligned}$$

Lemma 2 ([5])**.** *For matrices $K_\nu = \sum_{i=1}^{r} \nu_i(v(t))K_i$ and $J_\nu = \sum_{i=1}^{r} \nu_i(v(t))J_i$ with $K_i > 0$ and $J_i > 0$. We have $\dot{K}_\nu \le 0$ and $\dot{J}_\nu \le 0$ if the following inequalities are satisfied:*

$$\dot{\nu}_j(v(t)) < 0 : K_j < K_r,\ J_j < J_r, \tag{6}$$
$$\dot{\nu}_j(v(t)) \ge 0 : K_j \ge K_r,\ J_j \ge J_r \tag{7}$$

where $j = 1, 2, \ldots, r-1$.

Lemma 3 (Finsler's lemma [28])**.** *Let $v \in \mathbb{R}^n$, $F = F^T \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{m\times n}$. The following formulas are equivalent:*

$$\begin{aligned}
(i) &\quad v^T F v < 0,\ \forall Gv = 0,\ v \neq 0,\\
(ii) &\quad \exists E \in \mathbb{R}^{n\times m} : F + EG + G^T E^T < 0,\\
(iii) &\quad \left(G^\perp\right)^T F G^\perp < 0.
\end{aligned}$$

3. Main Results

In this section, the main delay-dependent stability conditions for the considered T-S fuzzy system with time-varying delay are presented. For simplicity, some expressions are listed as follows:

$$\begin{aligned}
R_{k,\nu} &= \sum_{i=1}^{r} \nu_i(v(t))R_{k,i}(k=1,2,3), N_\nu = \sum_{i=1}^{r} \nu_i(v(t))N_i, G_\nu = \sum_{i=1}^{r} \nu_i(v(t))G_i, W_\nu = \sum_{i=1}^{r} \nu_i(v(t))W_i,\\
q_0 &= 0_{17n\times n} \in \mathbb{R}^{17n\times n},\\
q_i &= [0_{n\times(i-1)n}, I_n, 0_{n\times(17-i)n}]^T \in \mathbb{R}^{17n\times n}(i = 1,2,\ldots,17),\\
\varphi(x(t)) &= \left[x^T(t), \dot{x}^T(t)\right]^T,\\
\varpi_1(t) &= \left[x^T(t), x^T(t-\tau_M), \int_{t-\tau_M}^{t} x^T(s)ds, \int_{t-\tau_M}^{t}\int_{s}^{t} x^T(u)duds\right]^T,\\
\varpi_2(t,s) &= \left[\varphi^T(s), \int_{s}^{t} \varphi^T(u)du, x^T(t), x^T(t-\tau_M)\right]^T,\\
\varpi_3(t,s) &= \left[x^T(s), \int_{s}^{t} \varphi^T(u)du, x^T(t), x^T(t-\tau_M)\right]^T,\\
\varpi_4(t,s) &= \left[\dot{x}^T(s), x^T(s), \int_{s}^{t} \dot{x}^T(v)dv, \int_{t-\tau_M}^{t} x(v)dv - x(s)\right]^T,\\
\xi(t) &= \Big[x^T(t), x^T(t-\tau(t)), x^T(t-\tau_M), \dot{x}^T(t), \dot{x}^T(t-\tau_M), \int_{t-\tau(t)}^{t} x^T(s)ds, \int_{t-\tau_M}^{t-\tau(t)} x^T(s)ds,\\
&\quad \frac{1}{\tau(t)}\int_{t-\tau(t)}^{t}\int_{s}^{t} x^T(u)duds, \frac{1}{\tau_M-\tau(t)}\int_{t-\tau_M}^{t-\tau(t)}\int_{s}^{t-\tau(t)} x^T(u)duds,\\
&\quad \frac{1}{\tau(t)}\int_{t-\tau(t)}^{t} x^T(s)ds, \frac{1}{\tau_M-\tau(t)}\int_{t-\tau_M}^{t-\tau(t)} x^T(s)ds,\\
&\quad \frac{1}{\tau^2(t)}\int_{t-\tau(t)}^{t}\int_{s}^{t}\int_{u}^{t} x^T(v)dvduds, \frac{1}{(\tau_M-\tau(t))^2}\int_{t-\tau_M}^{t-\tau(t)}\int_{s}^{t-\tau(t)}\int_{u}^{t-\tau(t)} x^T(v)dvduds,\\
&\quad \frac{1}{\tau^2(t)}\int_{t-\tau(t)}^{t}\int_{s}^{t} x^T(u)duds, \frac{1}{(\tau_M-\tau(t))^2}\int_{t-\tau_M}^{t-\tau(t)}\int_{s}^{t-\tau(t)} x^T(u)duds,\\
&\quad \int_{t-\tau_M}^{t} x^T(s)ds, \int_{t-\tau_M}^{t}\int_{s}^{t} x^T(u)duds\Big]^T,
\end{aligned}$$

$$
\begin{aligned}
\Theta_1 &= \begin{bmatrix} 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ I_n & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & I_n & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & I_n \end{bmatrix}, \Theta_2 = \begin{bmatrix} I_n & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ -I_n & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ -I_n & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \end{bmatrix}, \\
\Gamma_l &= [A_l, A_{d,l}, 0_{n\times n}, -I_n, 0_{13n\times n}], \\
\Xi_{1,l[\tau(t)]} &= Sym\{[q_1, q_3, q_{16}, q_{17}](R_{1,l} + \tau(t)R_{2,l} + (\tau_M - \tau(t))R_{3,l})[q_4, q_5, q_1 - q_3, \tau_M q_1 - q_{16}]^T\} \\
&\quad + [q_1, q_3, q_{16}, q_{17}](\tau_D(R_{2,l} - R_{3,l}))[q_1, q_3, q_{16}, q_{17}]^T, \\
\Xi_{2,l} &= [q_1, q_4, q_0, q_0, q_1, q_3]N_l[q_1, q_4, q_0, q_0, q_1, q_3]^T \\
&\quad - [q_3, q_5, q_{16}, q_1 - q_3, q_1, q_3]N_l[q_3, q_5, q_{16}, q_1 - q_3, q_1, q_3]^T \\
&\quad + Sym\{[q_{16}, q_1 - q_3, q_{17}, \tau_M q_1 - q_{16}, \tau_M q_1 \tau_M q_3]N_l[q_0, q_0, q_1, q_4, q_4, q_5]^T, \\
\Xi_{31,l} &= [q_1, q_0, q_0, q_1, q_3]G_l[q_1, q_0, q_0, q_1, q_3]^T \\
&\quad - (1 - \tau_D)[q_2, q_6, q_1 - q_2, q_1, q_3]G_l[q_2, q_6, q_1 - q_2, q_1, q_3]^T, \\
\Xi_{32,l[\tau(t)]} &= Sym\{[q_6, \tau(t)q_8, \tau(t)q_1 - q_6, \tau(t)q_1, \tau(t)q_3]G_l[q_0, q_1, q_4, q_4, q_5]^T\}, \\
\Xi_4 &= \tau_M[q_4, q_1, q_0, q_{16} - q_1]Q[q_4, q_1, q_0, q_{16} - q_1]^T \\
&\quad + Sym\{\left[\tau_M q_1 - q_{16}, q_{17}, (\tau_M^2/2)q_1 - q_{17}, (\tau_M^2/2)q_{16} - q_{17}\right]Q[q_0, q_0, q_4, q_1 - q_3]^T\}, \\
\Xi_{41[h(t)]} &= Sym\{L_1[q_1 - q_2, q_6, \tau(t)q_1 - q_6, \tau(t)q_{16} - q_6]^T\} \\
&\quad + Sym\{M_1[q_1 + q_2 - 2q_{10}, -q_6 + 2q_8, q_6 - 2q_8, q_6 - 2q_8]^T\} \\
&\quad + Sym\{N_1[q_1 - q_2 + 6q_{10} - 12q_{14}, q_6 - 6q_8 + 12q_{12}, -q_6 + 6q_8 - 12q_{12}, \\
&\qquad -q_6 + 6q_8 - 12q_{12}]^T\}, \\
\Xi_{42[h(t)]} &= Sym\{L_2[q_2 - q_3, q_7, (\tau_M - \tau(t))q_1 - q_7, (\tau_M - \tau(t))q_{16} - q_7]^T\} \\
&\quad + Sym\{M_2[q_2 + q_3 - 2q_{11}, -q_7 + 2q_9, q_7 - 2q_9, q_7 - 2q_9]^T\} \\
&\quad + Sym\{N_2[q_2 - q_3 + 6q_{11} - 12q_{15}, q_7 - 6q_9 + 12q_{13}, -q_7 + 6q_9 - 12q_{13}, \\
&\qquad -q_7 + 6q_9 - 12q_{13}]^T\}, \\
\Xi_5 &= [q_1, q_0, q_{16} - q_1]P_1[q_1, q_0, q_{16} - q_1]^T + [q_2, q_1 - q_2, q_{16} - q_2](P_2 - P_1)[q_2, q_1 - q_2, q_{16} - q_2]^T \\
&\quad - [q_3, q_1 - q_3, q_{16} - q_3]P_2[q_3, q_1 - q_3, q_{16} - q_3]^T, \\
\Xi_{6[h(t)]} &= Sym\{(\tau(t)q_{10} - q_6)\Psi_1\} + Sym\{((\tau_M - \tau(t))q_{11} - q_7)\Psi_2\} \\
&\quad + Sym\{(\tau(t)q_{14} - q_8)\Psi_3\} + Sym\{((\tau_M - \tau(t))q_{15} - q_9)\Psi_4\} \\
&\quad + Sym\{(q_{16} - q_6 - q_7)((\tau(t)/\tau_M)\Psi_5 + ((\tau_M - \tau(t))/\tau_M)\Psi_6)\} \\
&\quad + Sym\{(q_{17} - \tau(t)q_8 - (\tau_M - \tau(t))q_6 - (\tau_M - \tau(t))q_9)\Psi_7\}, \\
Q_{aug1} &= Q + Sym\{\Theta_1 P_1 \Theta_2\}, Q_{aug2} = Q + Sym\{\Theta_1 P_2 \Theta_2\}, \\
\Xi_{7,l} &= \tau_M q_4 W_l q_4^T - \tau_M q_5 W_l q_5^T - Sym\{(q_1 - q_3)W_l(q_4 - q_5)^T\} \\
&\quad - 3Sym\{(-q_1 - q_3 + (2/\tau_M)q_{16})W_l(-q_4 - q_5 + (2/\tau_M)(q_1 - q_3))^T\} \\
&\quad - 5Sym\{(q_1 - q_3 + (6/\tau_M)q_{16} - (12/\tau_M^2)q_{17})W_l \\
&\qquad \times (q_4 - q_5 + (6/\tau_M)(q_1 - q_3) - 6q_1 + (12/\tau_M^2)q_{16})^T\}, \\
\Xi_{l[\tau(t)]} &= \Xi_{1,l[\tau(t)]} + \Xi_{2,l} + \Xi_{31,l} + \Xi_{32,l[\tau(t)]} + \Xi_4 + \Xi_{41[\tau(t)]} + \Xi_{42[\tau(t)]} + \Xi_5 + \Xi_{6[h(t)]} + \Xi_{7,l}.
\end{aligned}
$$

Theorem 1. *For given constants $\tau_M > 0$ and τ_D, the time-delay T-S fuzzy system (3) is asymptotically stable for all delays satisfying $0 \le \tau(t) \le \tau_M$, and $\dot{\tau_M}(t) \le \tau_D$, if there exist positive definite matrices $N_v \in \mathbb{R}^{6n\times 6n}, G_v \in \mathbb{R}^{5n\times 5n}, Q \in \mathbb{R}^{4n\times 4n}, W_v \in \mathbb{R}^{n\times n}$, symmetric matrices*

$R_{i,\nu} \in \mathbb{R}^{4n\times 4n}(i = 1,2,3)$, $P_1, P_2 \in \mathbb{R}^{3n\times 3n}$, *and any matrices* $\Psi_i \in \mathbb{R}^{n\times 17n}(i = 1,\cdots,7)$, $L_1, L_2, M_1, M_2, N_1, N_2 \in \mathbb{R}^{17n\times 4n}$ *such that the following LMIs hold for* $l = 1,\cdots,r$:

$$\dot{R}_{1,\nu} \leq 0, \quad \dot{R}_{2,\nu} \leq 0, \quad \dot{R}_{3,\nu} \leq 0, \quad \dot{N}_{\nu} \leq 0, \quad \dot{G}_{\nu} \leq 0, \quad \dot{W}_{\nu} \leq 0, \tag{8}$$

$$R_{1,l} + R_{2,l} > 0, \tag{9}$$

$$R_{1,l} + R_{3,l} > 0, \tag{10}$$

$$R_{2,l} - R_{3,l} > 0, \tag{11}$$

$$\begin{bmatrix} (\Gamma_l^{\perp})^T(\Xi_{l[\tau(t)=\tau_M]})(\Gamma_l^{\perp}) & (\Gamma_l^{\perp})^T(\tau_M L_1) & (\Gamma_l^{\perp})^T(\tau_M M_1) & (\Gamma_l^{\perp})^T(\tau_M N_1) \\ * & -\tau_M Q_{aug1} & 0_{4n\times 4n} & 0_{4n\times 4n} \\ * & * & -3\tau_M Q_{aug1} & 0_{4n\times 4n} \\ * & * & * & -5\tau_M Q_{aug1} \end{bmatrix} < 0, \tag{12}$$

$$\begin{bmatrix} (\Gamma_l^{\perp})^T(\Xi_{l[\tau(t)=0]})(\Gamma_l^{\perp}) & (\Gamma_l^{\perp})^T(\tau_M L_2) & (\Gamma_l^{\perp})^T(\tau_M M_2) & (\Gamma_l^{\perp})^T(\tau_M N_2) \\ * & -\tau_M Q_{aug2} & 0_{4n\times 4n} & 0_{4n\times 4n} \\ * & * & -3\tau_M Q_{aug2} & 0_{4n\times 4n} \\ * & * & * & -5\tau_M Q_{aug2} \end{bmatrix} < 0. \tag{13}$$

Proof. Let us consider the LKF candidate as follows:

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t)$$

where

$$\begin{aligned}
V_1(t) &= \varpi_1^T(t)(R_{1,\nu} + \tau(t)R_{2,\nu} + (\tau_M - \tau(t))R_{3,\nu})(\odot), \\
V_2(t) &= \int_{t-\tau_M}^{t} \varpi_2^T(t,s)N_{\nu}(\odot)ds, \\
V_3(t) &= \int_{t-\tau(t)}^{t} \varpi_3^T(t,s)G_{\nu}(\odot)ds, \\
V_4(t) &= \int_{t-\tau_M}^{t}\int_{s}^{t} \varpi_4^T(t,u)Q(\odot)duds, \\
V_5(t) &= h\int_{t-\tau_M}^{t} \dot{x}^T(s)W_{\nu}\dot{x}(s)ds - \left(\int_{t-\tau_M}^{t}\dot{x}(s)ds\right)^T W_{\nu}(\odot) \\
&\quad - 3\left(\int_{t-\tau_M}^{t}\dot{x}(s)ds - \frac{2}{\tau_M}\int_{t-\tau_M}^{t}\int_{s}^{t}\dot{x}(u)duds\right)^T W_{\nu}(\odot) \\
&\quad - 5\left(\int_{t-\tau_M}^{t}\dot{x}(s)ds - \frac{6}{\tau_M}\int_{t-\tau_M}^{t}\int_{s}^{t}\dot{x}(u)duds + \frac{12}{\tau_M^2}\int_{t-\tau_M}^{t}\int_{s}^{t}\int_{u}^{t}\dot{x}(v)dvduds\right)^T W_{\nu}(\odot).
\end{aligned}$$

The time derivatives of $V_1(t)$ are induced as

$$\begin{aligned}
\dot{V}_1(t) &= 2\varpi_1^T(t)(R_{1,\nu} + \tau(t)R_{2,\nu} + (\tau_M - \tau(t))R_{3,\nu})\begin{bmatrix} \dot{x}(t) \\ \dot{x}(t-\tau_M) \\ x(t) - x(t-\tau_M) \\ \tau_M x(t) - \int_{t-\tau_M}^{t} x(s)ds \end{bmatrix} \\
&\quad + \varpi_1^T(t)\big(\dot{\tau}(t)(R_{2,\nu} - R_{3,\nu}) + \dot{R}_{1,\nu} + \tau(t)\dot{R}_{2,\nu} + (\tau_M - \tau(t))\dot{R}_{3,\nu}\big)(\odot).
\end{aligned} \tag{14}$$

If LMI (11) is satisfied, then (14) can be derived as follows.

$$\dot{V}_1(t) \le \sum_{i=1}^{r} \nu_i(v(t))\xi^T(t)\Xi_{1,i[\tau(t)]}\xi(t) + \sum_{i=1}^{r} \nu_i(v(t))\varpi_1^T\big(\dot{R}_{1,i} + \tau(t)\dot{R}_{2,i} + (\tau_M - \tau(t))\dot{R}_{3,i}\big)(\odot). \tag{15}$$

The time derivatives of $V_2(t)$–$V_4(t)$ are given as follows.

$$\begin{aligned}\dot{V}_2(t) &= 2\begin{bmatrix} \varphi(t) \\ 0_{2n\times 1} \\ x(t) \\ x(t-\tau_M) \end{bmatrix}^T N_\nu(\odot) - \begin{bmatrix} \varphi(t-\tau_M) \\ \int_{t-\tau_M}^{t}\varphi(u)du \\ x(t) \\ x(t-\tau_M) \end{bmatrix}^T N_\nu(\odot) \\ &\quad + 2\begin{bmatrix} \int_{t-\tau_M}^{t}\varphi(s)ds \\ \int_{t-\tau_M}^{t}\int_s^t\varphi(u)duds \\ \tau_M x(t) \\ \tau_M x(t-\tau_M) \end{bmatrix}^T N_\nu \begin{bmatrix} 0_{2n\times 1} \\ \varphi(t) \\ \dot{x}(t) \\ \dot{x}(t-\tau_M) \end{bmatrix} + \int_{t-\tau_M}^{t}\varpi_2^T(t,s)\dot{N}_\nu(\odot)ds \\ &= \sum_{i=1}^{r}\nu_i(v(t))\xi^T(t)\Xi_{2,i}\xi(t) + \int_{t-\tau_M}^{t}\varpi_2^T(t,s)\dot{N}_i(\odot)ds, \end{aligned} \tag{16}$$

$$\begin{aligned}\dot{V}_3(t) &= \begin{bmatrix} x(t) \\ 0_{2n\times 1} \\ x(t) \\ x(t-\tau_M) \end{bmatrix}^T G_\nu(\odot) - (1-\dot{\tau}(t))\begin{bmatrix} x(t-\tau(t)) \\ \int_{t-\tau(t)}^{t}\varphi(s)ds \\ x(t) \\ x(t-\tau_M) \end{bmatrix}^T G_\nu(\odot) \\ &\quad + 2\begin{bmatrix} \int_{t-\tau(t)}^{t}x(s)ds \\ \int_{t-\tau(t)}^{t}\int_s^t\varphi(u)duds \\ \tau_M(t)x(t) \\ \tau(t)x(t-\tau_M) \end{bmatrix}^T G_\nu \begin{bmatrix} 0_{n\times 1} \\ \varphi(t) \\ \dot{x}(t) \\ \dot{x}(t-\tau_M) \end{bmatrix} + \int_{t-\tau(t)}^{t}\varpi_3^T(t,s)\dot{G}_\nu(\odot)ds \\ &\le \sum_{i=1}^{r}\nu_i(v(t))\xi^T(t)(\Xi_{31,i} + \Xi_{32,i[\tau(t)]})\xi(t) + \int_{t-\tau(t)}^{t}\varpi_3^T(t,s)\dot{G}_i(\odot)ds, \end{aligned} \tag{17}$$

$$\begin{aligned}\dot{V}_4(t) &= \tau_M\varpi_4^T(t,s)Q(\odot) - \int_{t-\tau_M}^{t}\varpi_4^T(t,s)Q(\odot)ds \\ &\quad + 2\begin{bmatrix} \int_{t-\tau_M}^{t}\int_s^t\dot{x}(u)duds \\ \int_{t-\tau_M}^{t}\int_s^t x(u)duds \\ \int_{t-\tau_M}^{t}\int_s^t\int_u^t x(v)dvduds \\ \int_{t-\tau_M}^{t}\int_s^t\left(\int_u^t x(v)dv - x(u)\right)duds \end{bmatrix}^T Q\begin{bmatrix} 0_{n\times 1} \\ 0_{n\times 1} \\ \dot{x}(t) \\ x(t)-x(t-\tau_M) \end{bmatrix} \\ &= \xi^T(t)\Xi_4\xi(t) - \int_{t-\tau_M}^{t}\varpi_4^T(t,s)Q(\odot)ds. \end{aligned} \tag{18}$$

Let us consider the following two augmented zero equalities with $P_1, P_2 \in \mathbb{R}^{3n\times 3n}$.

$$\begin{aligned}\chi_1 : 0 &= \begin{bmatrix} x(s) \\ 0_{n\times 1} \\ \int_{t-\tau_M}^{t}x(s)ds - x(t) \end{bmatrix}^T P_1(\odot) - \begin{bmatrix} x(t-\tau(t)) \\ x(t)-x(t-\tau(t)) \\ \int_{t-\tau_M}^{t}x(s)ds - x(t-\tau(t)) \end{bmatrix}^T P_1(\odot) \\ &\quad - 2\int_{t-\tau(t)}^{t}\varpi_4^T(t,s)\Theta_1 P_1\Theta_2(\odot)ds, \end{aligned} \tag{19}$$

$$\chi_2 : 0 = \begin{bmatrix} x(t-\tau(t)) \\ x(t) - x(t-\tau(t)) \\ \int_{t-\tau_M}^{t} x(s)ds - x(t-\tau(t)) \end{bmatrix}^T P_2(\odot) - \begin{bmatrix} x(t-\tau_M) \\ x(t) - x(t-\tau_M) \\ \int_{t-\tau_M}^{t} x(s)ds - x(t-\tau_M) \end{bmatrix}^T P_2(\odot)$$
$$- 2\int_{t-\tau_M}^{t-\tau(t)} \varpi_4^T(t,s)\Theta_1 P_2 \Theta_2(\odot)ds. \tag{20}$$

By adding the following zero equality to (18), it follows that

$$\dot{V}_4(t) + \chi_1 + \chi_2 = \xi^T(t)\Xi_5\xi(t) - \int_{t-\tau(t)}^{t} \varpi_4^T(t,s)Q_{aug1}(\odot)ds$$
$$- \int_{t-\tau_M}^{t-\tau(t)} \varpi_4^T(t,s)Q_{aug2}(\odot)ds. \tag{21}$$

Then, by applying Lemma 1 to the integral terms in (21), the following results are obtained.

$$\begin{aligned}
&- \int_{t-\tau(t)}^{t} \varpi_4^T(t,s)Q_{aug1}(\odot)ds \\
&\leq Sym\Big\{\xi^T(t)L_1\eta_1(t) + \xi^T(t)M_1\eta_2(t) + \xi^T(t)N_1\eta_3(t)\Big\} \\
&\quad + \tau_M(t)\xi^T(t)\left(L_1 Q_{aug1}^{-1} L_1^T + \frac{1}{3}M_1 Q_{aug1}^{-1} M_1^T + \frac{1}{5}N_1 Q_{aug1}^{-1} N_1^T\right)^T \xi(t) \\
&= \xi^T(t)\left(\Xi_{41[\tau(t)]} + \tau_M(t)\left(L_1 Q_{aug1}^{-1} L_1^T + \frac{1}{3}M_1 Q_{aug1}^{-1} M_1^T + \frac{1}{5}N_1 Q_{aug1}^{-1} N_1^T\right)^T\right)\xi(t),
\end{aligned} \tag{22}$$

$$\begin{aligned}
&- \int_{t-\tau_M}^{t-\tau(t)} \varpi_4^T(t,s)Q_{aug2}(\odot)ds \\
&\leq Sym\{\xi^T(t)L_2\eta_4(t) + \xi^T(t)M_2\eta_5(t) + \xi^T(t)N_2\eta_6(t)\} \\
&\quad + (\tau_M - \tau(t))\xi^T(t)\left(L_2 Q_{aug2}^{-1} L_2^T + \frac{1}{3}M_2 Q_{aug2}^{-1} M_2^T + \frac{1}{5}N_2 Q_{aug2}^{-1} N_2^T\right)^T \xi(t) \\
&= \xi^T(t)\left(\Xi_{42[\tau(t)]} + (\tau_M - \tau(t))\left(L_2 Q_{aug2}^{-1} L_2^T + \frac{1}{3}M_2 Q_{aug2}^{-1} M_2^T + \frac{1}{5}N_2 Q_{aug2}^{-1} N_2^T\right)^T\right)\xi(t)
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
\eta_1(t) &= [q_1 - q_2, q_6, \tau_M(t)q_1 - q_6, \tau_M(t)q_{16} - q_6]^T\xi(t), \\
\eta_2(t) &= [q_1 + q_2 - 2q_{10}, -q_6 + 2q_8, q_6 - 2q_8, q_6 - 2q_8]^T\xi(t), \\
\eta_3(t) &= [q_1 - q_2 + 6q_{10} - 12q_{14}, q_6 - 6q_8 + 12q_{12}, -q_6 + 6q_8 - 12q_{12}, -q_6 + 6q_8 - 12q_{12}]^T\xi(t), \\
\eta_4(t) &= [q_2 - q_3, q_7, (\tau_M - \tau(t))q_1 - q_7, (\tau_M - \tau(t))q_{16} - q_7]^T\xi(t), \\
\eta_5(t) &= [q_2 + q_3 - 2q_{11}, -q_7 + 2q_9, q_7 - 2q_9, q_7 - 2q_9]^T\xi(t), \\
\eta_6(t) &= [q_2 - q_3 + 6q_{11} - 12e_{15}, q_7 - 6q_9 + 12q_{13}, -q_7 + 6q_9 - 12q_{13}, -q_7 + 6q_9 - 12q_{13}]^T\xi(t).
\end{aligned}$$

So, a novel bound of $\dot{V}_4(t) + \chi_1 + \chi_2$ is

$$\begin{aligned}
\dot{V}_4(t) + \chi_1 + \chi_2 \leq \xi^T(t)\Big(&\Xi_4 + \Xi_{41[\tau(t)]} + \Xi_{42[\tau(t)]} + \Xi_5 \\
&+ \tau_M(t)\left(L_1 Q_{aug1}^{-1} L_1^T + \frac{1}{3}M_1 Q_{aug1}^{-1} M_1^T + \frac{1}{5}N_1 Q_{aug1}^{-1} N_1^T\right)^T \\
&+ (\tau_M - \tau(t))\left(L_2 Q_{aug2}^{-1} L_2^T + \frac{1}{3}M_2 Q_{aug2}^{-1} M_2^T + \frac{1}{5}N_2 Q_{aug2}^{-1} N_2^T\right)^T\Big)\xi(t).
\end{aligned} \tag{24}$$

From the relationship among the elements of $\xi(t)$, the following zero equalities are introduced with $\Psi_i(i = 1, \cdots, 7) \in \mathbb{R}^{n\times 17n}$.

$$\chi_3 : 0 = \xi^T(t)Sym\{(\tau(t)q_{10} - q_6)\Psi_1\}\xi(t), \tag{25}$$
$$\chi_4 : 0 = \xi^T(t)Sym\{((\tau_M - \tau(t))q_{11} - q_7)\Psi_2\}\xi(t), \tag{26}$$
$$\chi_5 : 0 = \xi^T(t)Sym\{(\tau(t)q_{14} - q_8)\Psi_3\}\xi(t), \tag{27}$$
$$\chi_6 : 0 = \xi^T(t)Sym\{((\tau_M - \tau(t))q_{15} - q_9)\Psi_4\}\xi(t), \tag{28}$$
$$\chi_7 : 0 = \xi^T(t)Sym\{(q_{16} - q_6 - q_7)((\tau(t)/\tau_M)\Psi_5 + ((\tau_M - \tau(t))/\tau_M)\Psi_6)\}\xi(t), \tag{29}$$
$$\chi_8 : 0 = \xi^T(t)Sym\{(q_{17} - \tau(t)q_8 - (\tau_M - \tau(t))q_6 - (\tau_M - \tau(t))q_9)\Psi_7\}\xi(t). \tag{30}$$

By combining the zero equalities (25)–(30), the obtained equation can be written as

$$0 = \xi^T(t)\Xi_{6[\tau(t)]}\xi(t). \tag{31}$$

The time derivative of $V_5(t)$ is written by

$$\begin{aligned}
\dot{V}_5(t) &= \tau_M\dot{x}^T(t)W_v\dot{x}(t) - \tau_M\dot{x}^T(t-\tau_M)W_v\dot{x}(t-\tau_M) \\
&\quad - 2(x(t) - x(t-\tau_M))^T W_v(\dot{x}(t) - \dot{x}(t-\tau_M)) \\
&\quad - 6\left(-x(t) - x(t-\tau_M) + \frac{2}{h}\int_{t-\tau_M}^{t} x(s)ds\right)^T W_v\left(-\dot{x}(t) - \dot{x}(t-\tau_M) + \frac{2}{h}(x(t) - x(t-\tau_M))\right) \\
&\quad - 10\left(x(t) - x(t-\tau_M) + \frac{6}{h}\int_{t-\tau_M}^{t} x(s)ds - \frac{12}{\tau_M^2}\int_{t-\tau_M}^{t}\int_s^t x(u)duds\right)^T W_v \\
&\quad \times\left(\dot{x}(t) - \dot{x}(t-\tau_M) + \frac{6}{h}(x(t) - x(t-\tau_M)) - \frac{12}{\tau_M}x(t) + \frac{12}{\tau_M^2}\int_{t-\tau_M}^{t} x(s)ds\right) \\
&= \sum_{i=1}^{r}\nu_i(v(t))\xi^T(t)\Xi_{7,i}\xi(t).
\end{aligned} \tag{32}$$

If LMI (8) is satisfied, then it follows from (15)–(32) that the following stability condition holds.

$$\begin{aligned}
\dot{V}(t) \le \sum_{i=1}^{r}\nu_i(v(t))\xi^T(t)(\Xi_{i[\tau(t)]} &+ \tau(t)\left(L_1Q_{aug1}^{-1}L_1^T + \frac{1}{3}M_1Q_{aug1}^{-1}M_1^T + \frac{1}{5}N_1Q_{aug1}^{-1}N_1^T\right) \\
&+ (\tau_M - \tau(t))\left(L_2Q_{aug2}^{-1}L_2^T + \frac{1}{3}M_2Q_{aug2}^{-1}M_2^T + \frac{1}{5}N_2Q_{aug2}^{-1}N_2^T\right))\xi(t).
\end{aligned}$$

By using Lemma 3 with the equality constraint $\sum_{i=1}^{r}\nu_i(v(t))\Gamma_h\xi(t) = 0$, the following is obtained:

$$\begin{aligned}
&\sum_{i=1}^{r}\nu_i(v(t))(\Gamma_i^{\perp})^T(\Xi_{i[\tau(t)]} + \tau(t)\left(L_1Q_{aug1}^{-1}L_1^T + \frac{1}{3}M_1Q_{aug1}^{-1}M_1^T + \frac{1}{5}N_1Q_{aug1}^{-1}N_1^T\right) \\
&+ (\tau_M - \tau(t))\left(L_2Q_{aug2}^{-1}L_2^T + \frac{1}{3}M_2Q_{aug2}^{-1}M_2^T + \frac{1}{5}N_2Q_{aug2}^{-1}N_2^T\right))(\Gamma_i^{\perp}) < 0.
\end{aligned} \tag{33}$$

Finally, by applying the Schur complement, the derived stability condition is equivalently transformed into the LMIs (12) and (13) in Theorem 1. Hence, if the LMIs (12) and (13) are satisfied for all $i = 1, \cdots, r$ and for all admissible membership functions $\nu_i(v(t))$, then the T-S fuzzy system (3) is asymptotically stable. This completes the proof. □

Remark 1. *Different from existing works, where the delay-dependent term in $V_1(t)$ is constructed in the form of $\varpi_1^T(t)R_{1,v}(\odot) + \tau(t)x^T(t)R_{2,v}(\odot)$ and only the current state vector $x(t)$ is used* *in the $\tau(t)$-dependent part, the proposed LKF extends this term by using the augmented vector $\varpi_1(t)$. This enriched structure enables the LKF to capture more detailed coupling relationships among the current state, delayed state, and integral states over the delay interval. Moreover, not only the $\tau(t)$-dependent term but also the $(\tau_M - \tau(t))$-dependent term is constructed with $\varpi_1(t)$. In addition, to guarantee the $V_1(t) > 0$ for all $\tau(t) \in [0, \tau_M]$, the LMI conditions $R_{1,l} + \tau(t)R_{2,l} + (\tau_M - \tau(t))R_{3,l} > 0\,(l = 1, \cdots, r)$ is imposed.*

Remark 2. *In the proposed Lyapunov–Krasovskii functional, the fourth vector component in $V_4(t)$ is explicitly incorporated into the stability analysis of the T-S fuzzy system. Compared with existing results for T-S fuzzy systems with time-varying delays, this construction makes a more direct use of multi-integral state information. As a result, additional cross-coupling terms are introduced into the resulting LMIs, which helps reduce the conservatism of the derived stability criterion.*

Remark 3. *The proposed LKF $V_5(t)$ is constructed by using the auxiliary-function-based integral inequality (AUXII) so that additional information on the integral terms of the augmented vector can be included in the analysis. This AUXII-based construction allows more delay-related information to be retained in the functional.*

Remark 4. *In Theorem 1, membership-function-dependent LKFs are employed in $V_1(t)$, $V_2(t)$, $V_3(t)$, and $V_5(t)$. Since their time derivatives generate additional terms related to the membership functions, LMI (8) is required to guarantee these terms. To guarantee LMI (8), which is derived from the membership-function-dependent LKF, the linear switching method in Lemma 2 is employed. In Theorem 1, the matrices $R_{k,\nu}$ $(k = 1, 2, 3)$, N_ν, G_ν, and W_ν are constructed as membership-function-dependent matrices, and their time derivatives include the terms related to $\dot{v}_i(v(t))$. The sign of $\dot{v}_j(v(t))$ is associated with the corresponding matrix-ordering conditions, and the derivative terms of the membership-function-dependent matrices are guaranteed to satisfy LMI (8).*

To verify the effect of the proposed LKF, the following corollary is derived from Theorem 1 by removing the delay-dependent terms and $V_5(t)$.

Corollary 1. *For given constants $\tau_M > 0$ and τ_D, the time-delay T-S fuzzy system (3) is asymptotically stable for all delays satisfying $0 \le \tau(t) \le \tau_M$, and $\dot{\tau_M}(t) \le \tau_D$, if there exist positive definite matrices $R_{1,\nu} \in \mathbb{R}^{4n\times 4n}$, $N_\nu \in \mathbb{R}^{6n\times 6n}$, $G_\nu \in \mathbb{R}^{5n\times 5n}$, $Q \in \mathbb{R}^{4n\times 4n}$, symmetric matrices $P_1, P_2 \in \mathbb{R}^{3n\times 3n}$, and any matrices $\Psi_i \in \mathbb{R}^{n\times 17n}(i = 1, \cdots, 7)$, $L_1, L_2, M_1, M_2, N_1, N_2 \in \mathbb{R}^{17n\times 4n}$ such that the following LMIs hold for $l = 1, \cdots, r$:*

$$\dot{R}_{1,\nu} \le 0, \quad \dot{N}_\nu \le 0, \quad \dot{G}_\nu \le 0, \quad \dot{W}_\nu \le 0, \tag{34}$$

$$\begin{bmatrix} (\Gamma_l^\perp)^T(\bar{\Xi}_{l[\tau(t)=\tau_M]})(\Gamma_l^\perp) & (\Gamma_l^\perp)^T(\tau_M L_1) & (\Gamma_l^\perp)^T(\tau_M M_1) & (\Gamma_l^\perp)^T(\tau_M N_1) \\ * & -\tau_M Q_{aug1} & 0_{4n\times 4n} & 0_{4n\times 4n} \\ * & * & -3\tau_M Q_{aug1} & 0_{4n\times 4n} \\ * & * & * & -5\tau_M Q_{aug1} \end{bmatrix} < 0, \tag{35}$$

$$\begin{bmatrix} (\Gamma_l^\perp)^T(\bar{\Xi}_{l[\tau(t)=0]})(\Gamma_l^\perp) & (\Gamma_l^\perp)^T(\tau_M L_2) & (\Gamma_l^\perp)^T(\tau_M M_2) & (\Gamma_l^\perp)^T(\tau_M N_2) \\ * & -\tau_M Q_{aug2} & 0_{4n\times 4n} & 0_{4n\times 4n} \\ * & * & -3\tau_M Q_{aug2} & 0_{4n\times 4n} \\ * & * & * & -5\tau_M Q_{aug2} \end{bmatrix} < 0, \tag{36}$$

where

$$\begin{aligned} \bar{\Xi} = &\, Sym\{[q_1, q_3, q_{16}, q_{17}]R_{1,i}[q_4, q_5, q_1 - q_3, \tau_M q_1 - q_{16}]^T\} \\ &+ \Xi_{2,i} + \Xi_{31,i} + \Xi_{32,i[\tau(t)]} + \Xi_4 + +\Xi_{41[\tau(t)]} + \Xi_{42[\tau(t)]} + \Xi_5 + \Xi_{6[\tau(t)]}. \end{aligned} \tag{37}$$

Proof. Let us consider the LKF candidate as follows:

$$\bar{V}(t) = \varpi_1^T(t)R_{1,\nu}(\odot) + V_2(t) + V_3(t) + V_4(t).$$

The other process is very similar to the proof of Theorem 1, so it is omitted. □

4. Numerical Examples

The superiority of the proposed method is illustrated through three representative numerical examples.

Example 1. *Consider the following T-S fuzzy system with time-varying delay:*

$$\dot{x}(t) = \sum_{i=1}^{2} \nu_i(x(t))(A_i x(t) + A_{d,i}x(t-\tau(t))) \tag{38}$$

where

$$A_1 = \begin{bmatrix} -2.1 & 0.1 \\ -0.2 & -0.9 \end{bmatrix}, \quad A_{d,1} = \begin{bmatrix} -1.1 & 0.1 \\ -0.8 & -0.9 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1.9 & 0 \\ -0.2 & -1.1 \end{bmatrix}, \quad A_{d,2} = \begin{bmatrix} -0.9 & 0 \\ -1.1 & -1.2 \end{bmatrix},$$
$$\nu_1(t) = \frac{1}{1+e^{-2x_1(t)}}, \quad \nu_2(t) = 1 - \nu_1(t).$$

By applying the proposed LMI-based stability criteria in Theorem 1 and Corollary 1 to Example 1, the maximum admissible delay upper bound τ_M is computed for several values of τ_D. The obtained results are summarized in Table 1, where a direct comparison with existing approaches [13,21–23] is provided. It can be observed that the proposed method provides larger delay bounds, which indicates a reduced conservatism.

Table 1. Maximum sampling intervals τ_M for system (Example 1).

Methods	0	0.1	0.5	Condition	NoDVs*
Theorem 1 (m = 3) [21]	4.37	3.41	1.95	$\dot{\tau}(t) \leq \tau_D$	$15.5n^2 + 6.5n$
Theorem 1 (m = 4) [22]	4.61	-	-	$\dot{\tau}(t) = 0$	$9.5n^2 + 4.5n$
Theorem 1 [23]	5.58	-	-	$\dot{\tau}(t) = 0$	$4.5n^2 + 2.5n$
Theorem 1 [13]	5.5826	4.2044	2.0685	$\dot{\tau}(t) \leq \tau_D$	$209n^2 + 9n$
Corollary 1	5.6897	4.2662	2.1735	$\dot{\tau}(t) \leq \tau_D$	$693n^2 + 20n$
Theorem 1	5.6897	4.2652	2.1785	$\dot{\tau}(t) \leq \tau_D$	$726n^2 + 29n$

NoDVs*: The number of decision variables.

More precisely, the membership-function-dependent inequalities give rise to two complementary switching cases, which can be described by the following matrix-ordering patterns:

Case 1: $\{R_{1,1} < R_{1,2},\ R_{2,1} < R_{2,2},\ R_{3,1} < R_{3,2},\ N_1 < N_2,\ G_1 < G_2,\ W_1 < W_2\}$,
Case 2: $\{R_{1,1} \geq R_{1,2},\ R_{2,1} \geq R_{2,2},\ R_{3,1} \geq R_{3,2},\ N_1 \geq N_2,\ G_1 \geq G_2,\ W_1 \geq W_2\}$.

For each case, the corresponding LMIs are solved independently, and the final admissible delay bound is determined by taking the minimum value that guarantees feasibility over all admissible switching scenarios. For instance, in Example 1 with $\tau_D = 0.1$, Theorem 1 gives $\tau_M = 4.2690$ for Case 1 and $\tau_M = 4.2672$ for Case 2. Hence, the final bound is selected as $\tau_M = \min\{4.2690, 4.2672\} = 4.2672$.

To numerically verify the stability of the system under the computed admissible delay bound, the simulation is carried out with the initial condition $x(0) = [1, -1]^T$ and the time-varying delay $\tau(t) = 1.0893 + 1.0893sin(0.5t/1.0893)$, which satisfies $0 \leq \tau(t) \leq \tau_M$ with $\tau_M = 2.1786$ and $\tau_D = 0.5$. The resulting state responses are shown in Figure 1.

From the figure, the responses converge to the origin, which provides numerical evidence that the system remains stable even when the delay varies up to the computed maximum allowable bound.

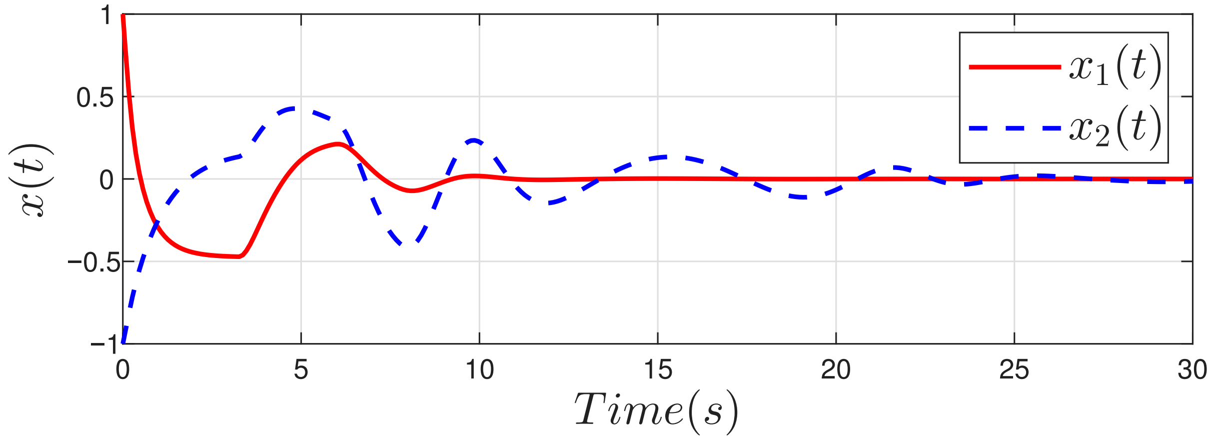

Figure 1. The state responses of the system (38).

Example 2. *Consider the following T-S fuzzy system with time-varying delay described by:*

$$\dot{x}(t) = \sum_{i=1}^{2} \nu_i(x(t))(A_i x(t) + A_{d,i} x(t - \tau(t))) \tag{39}$$

where

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_{d,1} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & 0 \\ 0.5 & -1 \end{bmatrix}, \quad A_{d,2} = \begin{bmatrix} -1 & 0 \\ 0.1 & -1 \end{bmatrix},$$
$$\nu_1(t) = \left(1 - \frac{1}{1 + e^{-5(x_1(t) - \pi/6)}}\right) \frac{1}{1 + e^{-5(x_1(t) - \pi/6)}},$$
$$\nu_2(t) = 1 - \nu_1(t).$$

Table 2 shows the admissible delay bounds obtained by applying Theorem 1 and Corollary 1, along with those reported in [14,16–19]. It can be seen that the proposed method significantly enlarges the admissible delay bounds for $\tau_D \in \{0, 0.1, 0.4, 0.9\}$. Although several compared results are derived under the constraint $|\dot{\tau}(t)| \leq \tau_D$, the proposed condition under $\dot{\tau}(t) \leq \tau_D$ still yields improved bounds, which demonstrates the effectiveness of the proposed construction.

Table 2. Maximum sampling intervals τ_M for system (Example 2).

Methods	0	0.1	0.4	0.9	Condition	NoDVs
Theorem 1 [14]	3.6167	-	-	-	$\dot{\tau}(t) = 0$	$13n^2 + 5n$
Theorem 1 [16]	3.6446	3.2464	-	-	$\lvert\dot{\tau}(t)\rvert \leq \tau_D$	$315.5n^2 + 50.5n$
Theorem 1 ($\sigma_a \leq \frac{ln4.5}{0.01}$) [19]	-	-	2.0925	1.7950	$\lvert\dot{\tau}(t)\rvert \leq \tau_D$	$266n^2 + 22n$
Theorem 1($N = 2, C = 0.4$) [17]	-	-	2.1281	1.8742	$\lvert\dot{\tau}(t)\rvert \leq \tau_D$	$230.5n^2 + 25.5n$
Theorem 1($N = 1, C = 0.9$) [18]	-	-	2.1994	1.9101	$\lvert\dot{\tau}(t)\rvert \leq \tau_D$	$243.5n^2 + 28.5n$
Corollary 1	6.1686	4.7878	2.7355	1.7787	$\dot{\tau}(t) \leq \tau_D$	$693n^2 + 20n$
Theorem 1	6.1686	4.7910	2.7453	1.7865	$\dot{\tau}(t) \leq \tau_D$	$726n^2 + 29n$

To numerically verify the stability under the computed admissible delay bound, a time-domain simulation is performed with the initial condition $x(0) = [1, -1]^T$ and the time-varying delay $\tau(t) = 0.8932 + 0.8932sin(0.9t/0.8932)$ with $\tau_M = 1.7865$ and $\tau_D = 0.9$. The resulting state responses are shown in Figure 2.

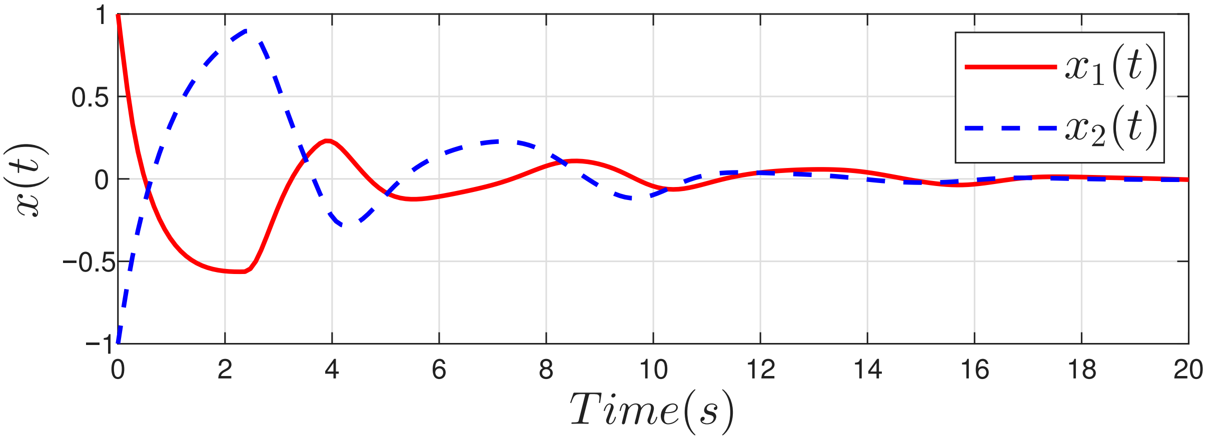

Figure 2. The state responses of the system (39).

Remark 5. *Most of the existing works dealing with the condition $\dot{\tau}(t) < \tau_D$ are relatively old, so they are not enough for a full comparison with recent results. Therefore, this example also compares the proposed method with recent criteria developed under the condition $|\dot{\tau}(t)| < \tau_D$. Since $|\dot{\tau}(t)| < \tau_D$ is less restrictive than $\dot{\tau}(t) < \tau_D$, it usually gives larger admissible delay bounds. Nevertheless, the proposed method still yields better results for $\tau_D = 0.4$.*

Example 3. *The system matrices for Example 3 are given as:*

$$\dot{x}(t) = \sum_{i=1}^{2} \nu_i(x(t))(A_i x(t) + A_{d,i}x(t - h(t))) \tag{40}$$

where

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_{d,1} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1.5 & 1 \\ 0 & -0.75 \end{bmatrix}, \quad A_{d,2} = \begin{bmatrix} -1 & 0 \\ 1 & -0.85 \end{bmatrix},$$
$$\nu_1(t) = \left(1 - \frac{1}{1 + e^{-5(x_1(t) - \pi/6)}}\right) \frac{1}{1 + e^{-5(x_1(t) - \pi/6)}},$$
$$\nu_2(t) = 1 - \nu_1(t).$$

Table 3 provides a comparison of the maximum allowable delay bounds obtained from Theorem 1 and Corollary 1 with several recent criteria [12,15,20,25–27]. It is clearly observed that the proposed method yields substantially larger admissible bounds for all tested values of τ_D, which demonstrates a remarkable reduction in conservatism.

Table 3. Maximum sampling intervals τ_M for system (Example 3).

Methods	0.2	0.4	0.6	Condition	NoDVs
Corollary 1 [25]	1.6409	1.4588	1.3905	$\lvert\dot{\tau}(t)\rvert \leq \tau_D$	$10.5n^2 + 3.5n$
Corollary 2 [26]	1.7805	1.5339	1.4082	$\lvert\dot{\tau}(t)\rvert \leq \tau_D$	$113n^2 + 17n$
Theorem 1 [27]	1.8728	1.6517	1.5205	$\lvert\dot{\tau}(t)\rvert \leq \tau_D$	$137n^2 + 31n$
Theorem 1 [12]	2.0101	1.8487	1.7349	$\lvert\dot{\tau}(t)\rvert \leq \tau_D$	$309n^2 + 31n$
Theorem 3.1 ($c_1 = 0.1, c_2 = 0.9$) [15]	2.1406	1.9132	1.7597	$\lvert\dot{\tau}(t)\rvert \leq \tau_D$	$84.5n^2 + 7.5n$
Theorem 1 [20] ($N = 3, \beta_1 = 0.5h_M, \beta_2 = 0.9h_M$)	2.2401	2.0811	1.9475	$\lvert\dot{\tau}(t)\rvert \leq \tau_D$	$94.5n^2 + 20.5n$
Corollary 1	3.9202	2.8412	2.2540	$\dot{\tau}(t) \leq \tau_D$	$693n^2 + 20n$
Theorem 1	3.9162	2.8421	2.2732	$\dot{\tau}(t) \leq \tau_D$	$726n^2 + 29n$

To numerically verify the stability under the computed admissible delay bound, a time-domain simulation is carried out with $x(0) = [1, -1]^T$ and $\tau(t) = 1.1366 + 1.1366sin(0.6t/1.1366)$ with $\tau_M = 2.2732$ and $\tau_D = 0.6$. The corresponding state responses under this worst-case delay setting are presented in Figure 3.

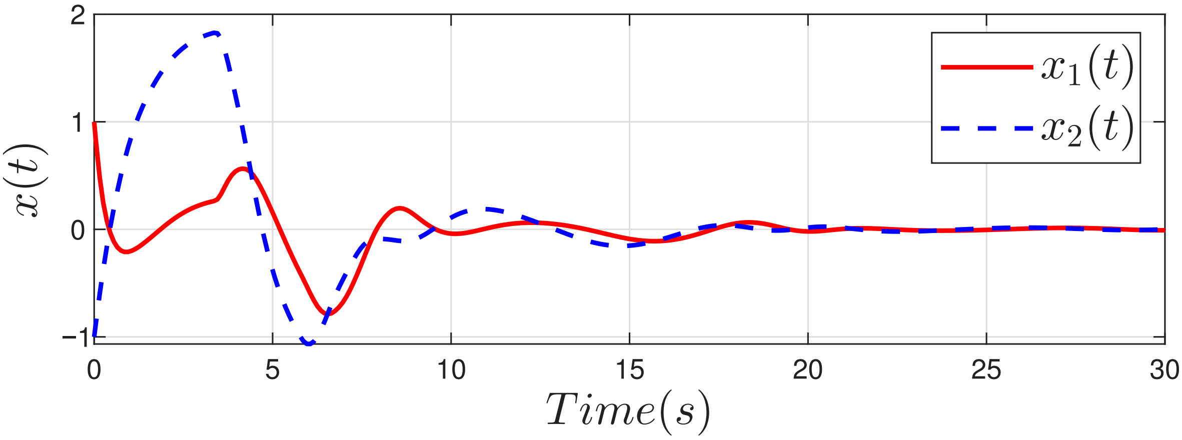

Figure 3. The state responses of the system (40).

Example 4. *This example deals with a truck-trailersystem introduced in [13]. The system parameters are*

$$\bar{A}_1 = \begin{bmatrix} -c\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ c\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ c\frac{v^2(\bar{t})^2}{2Lt_0} & \frac{v\bar{t}}{t_0} & 0 \end{bmatrix}, \quad A_{d,1} = \begin{bmatrix} -(1-c)\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ (1-c)\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ (1-c)\frac{v^2(\bar{t})^2}{2Lt_0} & \frac{v\bar{t}}{t_0} & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} \frac{v\bar{t}}{lt_0} \\ 0 \\ 0 \end{bmatrix},$$
$$\bar{A}_2 = \begin{bmatrix} -c\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ c\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ c\frac{v^2(\bar{t})^2}{2Lt_0} & \frac{dv\bar{t}}{t_0} & 0 \end{bmatrix}, \quad A_{d,2} = \begin{bmatrix} -(1-c)\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ (1-c)\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ (1-c)\frac{v^2(\bar{t})^2}{2Lt_0} & \frac{v\bar{t}}{t_0} & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \frac{v\bar{t}}{lt_0} \\ 0 \\ 0 \end{bmatrix},$$
$$\nu_1(x(t)) = \frac{1}{1 + e^{x_1(t)+0.5}}, \quad \nu_2(x(t)) = 1 - \nu_1(x(t)). \tag{41}$$

where $L = 0.5$, $\bar{t} = 2.0$, $l = 2.8$, $t_0 = 0.5$, $c = 0.7$, and $d = 10t_0/\pi$. By fixing the control gains $K_1 = [20.6936, -51.9608, 0.5275]$ and $K_2 = [20.6902, -51.9170, 0.5256]$, the T-S fuzzy system can be expressed as follows with $A_1 = \bar{A}_1 + B_1K_1$, $A_2 = \bar{A}_2 + B_2K_2$.

$$\dot{x}(t) = \sum_{i=1}^{2} \nu_i(x(t))(A_i x(t) + A_{d,i}x(t-h(t))) \tag{42}$$

Table 4 provides a comparison of the maximum allowable delay bounds obtained from Theorem 1 and Corollary 1 with recent criteria [13]. It can be observed that Corollary 1 gives a smaller admissible delay bound than the previous results, whereas Theorem 1 gives a larger one.

Table 4. Maximum sampling intervals τ_M for system (Example 4).

Methods	τ_D = 0.9	Condition
Theorem 1 [13]	9.3915	$\dot{\tau}(t) \leq \tau_D$
Corollary 1	9.2125	$\dot{\tau}(t) \leq \tau_D$
Theorem 1	9.4344	$\dot{\tau}(t) \leq \tau_D$

Remark 6. *From Example 1 with $\tau_D = 0.1$ and Example 3 with $\tau_D = 0.2$, it is observed that Corollary 1 yields larger admissible delay bounds than Theorem 1. On the other hand, Theorem 1 is*

derived from a delay-dependent LKF, so that more information on h_D is included in the resulting LMIs. Hence, as τ_D becomes larger, Theorem 1 tends to provide larger admissible delay bounds.

5. Conclusions

This paper has addressed the delay-dependent stability problem for continuous-time T-S fuzzy systems with time-varying delay. To reduce conservatism, improved LKFs were constructed by explicitly incorporating multi-integral state components. Moreover, a time-varying-delay-dependent augmented term was introduced to capture more detailed coupling information among the current state, delayed state, and integral states over the delay interval. Furthermore, by applying the AXUII, the proposed LKF was constructed in a more effective form, and a membership-function-dependent switching scheme was adopted to further reduce conservatism and enlarge the admissible delay bounds. Based on these developments, the delay-dependent stability conditions were derived in the form of LMIs. Finally, numerical examples were provided to show the effectiveness of the proposed method.

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