

Article

New Properties and Determinantal Representations of Leonardo Finite Operator Polynomials

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Abstract

The aim of this paper is to introduce Leonardo finite operator polynomials and obtain some of their new properties. We first present the recurrence relation provided by Leonardo finite operator polynomials. Then, we give a Binet-like formula, generating function, exponential generating function, and a finite sum formula for Leonardo finite operator polynomials. We present a determinant representation for the n th term of Leonardo finite operator polynomials. Ultimately, by utilizing the generating function of the proposed polynomials, we establish generating relations for specific bilinear and bilateral polynomial families. This approach thus broadens the applicability of the finite operator framework to encompass a wider range of special functions.

Keywords: Leonardo polynomial; finite operator; determinant representation; generating function

MSC: 11B83; 11C20; 11Y55

1. Introduction and Preliminaries

Finite operator methods represent a significant area within algebraic combinatorics, offering operator-theoretic techniques for addressing recurrence relations, generating functions, and enumerative identities. Traditional methodologies, such as the finite difference operator, shift operators, and umbral calculus, have been extensively employed to convert discrete problems into algebraic manipulations. These tools are pivotal in the analysis of integer partitions, permutation statistics, graph enumeration, and the derivation of q -series expansions. Moreover, operator methods inherently connect with symbolic computation, algorithmic complexity, and asymptotic enumeration, rendering them essential in both theoretical and applied research. Recent developments have illustrated how operator-based frameworks not only consolidate existing results but also produce new combinatorial identities, thereby enriching the interaction between discrete mathematics, probability theory, and mathematical physics.

Finite operators hold significant importance within the theory of special functions. The action of such operators on various polynomial families has been extensively studied by Simsek. In his work, Simsek [1] introduced a new finite operator that generalizes some well-known operators in combinatorics such as the identity operator, the forward



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difference operator, the backward difference operator, the means operator, and the Gould operator as follows:

$$\mathbb{Y}_{\lambda,\beta}^{(i)}(f_n; s, t)(x) = \lambda E^s[f](x) + \beta E^t[f](x),$$

where s, t are real parameters and λ, β are real or complex parameters. Note that $E^s[f](x) = f(x + s)$. Utilizing this operator, Simsek investigated two novel categories of special polynomials and numbers. He further elucidated various associations between certain established polynomials and numbers (see also [2]). Consider s and t as integers, while λ and β are real-valued parameters. For any given polynomial sequence $f_n(x)$ and $i \geq 1$, the i th finite operator, denoted as $\mathbb{Y}_{\lambda,\beta}^{(i)}(f_n; s, t)(x)$ (or simply $\mathbb{Y}_{\lambda,\beta;s,t}^{(i)}(f_n(x))$), is defined by the following expression:

$$\mathbb{Y}_{\lambda,\beta;s,t}^{(i)}(f_n(x)) := \mathbb{Y}_{\lambda,\beta;s,t} \left[\mathbb{Y}_{\lambda,\beta;s,t}^{(i-1)}(f_n) \right](x), \quad i \geq 1,$$

with the initial condition

$$\mathbb{Y}_{\lambda,\beta;s,t}^{(0)}(f_n(x)) := f_n(x).$$

Consequently, for $i = 1$, we obtain

$$\mathbb{Y}_{\lambda,\beta;s,t}^{(1)}(f_n(x)) = \lambda f_n(x + s) + \beta f_n(x + t).$$

This finite operator acts as a broad extension of several well-established operators, including the identity operator, forward difference operator, backward difference operator, average operator, and Gould operator. Table 1 illustrates these specific instances of the finite operator in detail.

Table 1. Some special cases of the finite operator defined by Simsek.

λ	β	s	t	Operator
1	0	0	0	$\mathbb{Y}_{1,0}(f; 0, 0)(x) = I(f(x)) = f(x)$
1	−1	1	0	$\mathbb{Y}_{1,-1}(f; 1, 0)(x) = \Delta(f(x)) = f(x + 1) - f(x)$
1	−1	0	−1	$\mathbb{Y}_{1,-1}(f; 0, -1)(x) = \nabla(f(x)) = f(x) - f(x - 1)$
$\frac{1}{2}$	$-\frac{1}{2}$	1	0	$\mathbb{Y}_{\frac{1}{2}, -\frac{1}{2}}(f; 1, 0)(x) = M(f(x)) = \frac{1}{2}(f(x + 1) - f(x))$
1	−1	$s \rightarrow s + t$	$t \rightarrow s$	$\mathbb{Y}_{1,-1}(f; s + t, s)(x) = G_{st}(f(x)) = f(x + s + t) - f(x + s)$

The operators enumerated in Table 1 exhibit a broad spectrum of applications across engineering, physics, and applied mathematics. Furthermore, finite operators are extensively employed by researchers from various scientific disciplines in their analytical and computational studies.

By employing the finite operator as delineated in [1] to the Horadam sequence, Kızılateş [3] established the Horadam finite operator sequences. Furthermore, he explored several combinatorial properties associated with these sequences. Polatlı [4] derived various characteristics of (p, q) -Fibonacci finite operator polynomials through the application of the finite operator to (p, q) -Fibonacci polynomials. Terzioğlu and others [5] formulated numerous identities pertaining to Fibonacci finite operator quaternions by utilizing matrix representations. Yağmur [6] and Özimamoğlu [7] proposed additional extensions by applying the same operator framework.

Alongside these developments, the Leonardo sequence introduced by Catarino and Borges [8] has attracted increasing interest. It is defined recursively by

$$\text{Le}_n = \text{Le}_{n-1} + \text{Le}_{n-2} + 1, \quad \text{Le}_0 = \text{Le}_1 = 1,$$

yielding the sequence 1, 1, 3, 5, 9, 15, 25, 41, 67, 109, 177, 287, 465, ... The Leonardo sequence has inspired numerous generalizations and variations in the literature [9–17], highlighting its significance in number theory and combinatorial analysis. Building upon this concept, Prasad and Kumari [18] introduced the n th Leonardo polynomial $\text{Le}_n(x)$ via the recurrence relation:

$$\text{Le}_{n+2}(x) = x\text{Le}_{n+1}(x) + \text{Le}_n(x) + x,$$

where $\text{Le}_0(x) = 1$ and $\text{Le}_1(x) = 2x - 1$. They obtained the Binet-like formula of the Leonardo polynomial $\text{Le}_n(x)$ as follows:

$$\text{Le}_n(x) = 2 \left(\frac{s_1^{n+1}(x) - s_2^{n+1}(x)}{\sqrt{x^2 + 4}} \right) - 1,$$

where $s_1(x) = (x + \sqrt{x^2 + 4})/2$ and $s_2(x) = (x - \sqrt{x^2 + 4})/2$. They also showed that Leonardo polynomials form a class of irreducible polynomials. Moreover, they examined derivatives of Leonardo polynomials and their explicit expressions.

Despite the significant contributions of Simsek's generalized finite operator [1,2], its interaction with Leonardo-type polynomials remains unexplored. This study addresses this gap by introducing the Leonardo finite operator polynomials (or shortly, LFOPs), which serve to extend both the finite operator and Leonardo frameworks. Through the application of operator calculus, we derive the following:

- Recurrence relations and Binet-like formulas;
- (Exponential and Poisson) generating functions;
- Binomial and finite sum identities;
- A determinant representation;
- Some generating functions.

This study introduces and examines a novel class of polynomials termed Leonardo finite operator polynomials. By employing Simsek's generalized finite operator on the Leonardo polynomial sequence, we derive some combinatorial properties, including recurrence relations, Binet-like and generating functions, exponential generating and finite sum representations, as well as a determinant formulation. By employing the generating function of the proposed polynomials, we derive generating relations for particular families of bilinear and bilateral polynomials.

2. Some Properties of Leonardo Finite Operator Polynomials

In this section, we first apply the finite operator, based on the definitions in [1–3], to recently obtained Leonardo polynomials [18]. After that, we obtain the recurrence relation provided by Leonardo finite operator polynomials. Then, we give various combinatorial properties involving Leonardo finite operator polynomials.

Let s and t be integers and let λ and β be real parameters. For any polynomial sequence $\{p_n(x)\}_{n=0}^{\infty}$ and $i \geq 1$, the i th finite operator $\mathbb{Y}_{\lambda,\beta;s,t}^{(i)}(p_n(x))$ or, briefly, $p_n^{(i)}(x)$ is defined by

$$\mathbb{Y}_{\lambda,\beta;s,t}^{(i)}(p_n(x)) = p_n^{(i)}(x) = \lambda \mathbb{Y}_{\lambda,\beta;s,t}^{(i-1)}(p_{n+s}(x)) + \beta \mathbb{Y}_{\lambda,\beta;s,t}^{(i-1)}(p_{n+t}(x))$$

where $\mathbb{Y}_{\lambda,\beta;s,t}^{(1)}(p_n(x)) = p_n^{(1)}(x) = \lambda p_{n+s}(x) + \beta p_{n+t}(x)$.

Now, we start to apply the finite operator to Leonardo polynomials. Using (2), we obtain

$$\mathbb{Y}_{\lambda,\beta;s,t}^{(1)}(\mathbf{Le}_{n+1}(x)) = \mathbf{Le}_{n+1}^{(1)}(x) = \lambda \mathbf{Le}_{n+1+s}(x) + \beta \mathbf{Le}_{n+1+t}(x) \quad (1)$$

where $\mathbb{Y}_{\lambda,\beta;s,t}^{(1)}(\mathbf{Le}_{n+1}(x))$ is the first finite operator of $\mathbf{Le}_{n+1}(x)$.

If we apply the finite operator to (1) again, then the second finite operator of $\mathbf{Le}_{n+1}(x)$ is obtained by

$$\mathbb{Y}_{\lambda,\beta;s,t}^{(2)}(\mathbf{Le}_{n+1}(x)) = \mathbf{Le}_{n+1}^{(2)}(x) = \lambda^2 \mathbf{Le}_{n+1+2s}(x) + 2\lambda\beta \mathbf{Le}_{n+1+s+t}(x) + \beta^2 \mathbf{Le}_{n+1+2t}(x).$$

By continuing the process in this manner, we obtain the i th finite operator of $\mathbf{Le}_{n+1}(x)$

$$\mathbb{Y}_{\lambda,\beta;s,t}^{(i)}(\mathbf{Le}_{n+1}(x)) = \mathbf{Le}_{n+1}^{(i)}(x) = \lambda \mathbb{Y}_{\lambda,\beta;s,t}^{(i-1)}(\mathbf{Le}_{n+1+s}(x)) + \beta \mathbb{Y}_{\lambda,\beta;s,t}^{(i-1)}(\mathbf{Le}_{n+1+t}(x))$$

or alternatively,

$$\mathbb{Y}_{\lambda,\beta;s,t}^{(i)}(\mathbf{Le}_{n+1}(x)) = \mathbf{Le}_{n+1}^{(i)}(x) = \sum_{k=0}^i \binom{i}{k} \lambda^{i-k} \beta^k \mathbf{Le}_{n+tk+(i-k)s+1}(x)$$

which is termed as Leonardo finite operator polynomials. The initial conditions of these operator polynomial sequences are given as follows:

$$\mathbf{Le}_0^{(i)}(x) = \sum_{k=0}^i \binom{i}{k} \lambda^{i-k} \beta^k \mathbf{Le}_{tk+(i-k)s}(x),$$

and

$$\mathbf{Le}_1^{(i)}(x) = \sum_{k=0}^i \binom{i}{k} \lambda^{i-k} \beta^k \mathbf{Le}_{tk+(i-k)s+1}(x).$$

We observe that special instances of the Leonardo finite operator polynomials $\mathbf{Le}_{n+1}^{(i)}(x)$ follow from particular selections of the parameters $(\lambda, \beta; s, t)$. The choices $(1, 0; 0, 0)$, $(1, -1; 1, 0)$, $(1, -1; 0, -1)$, $(\frac{1}{2}, -\frac{1}{2}; 1, 0)$, and $(1, -1; s+t, s)$ yield the identity, forward difference, backward difference, means-type, and Leonardo–Gould operator polynomial sequences, respectively, as derived below.

Here, we note that some special cases of the Leonardo finite operator polynomials $\mathbf{Le}_{n+1}^{(i)}(x)$ can be derived from the operator

$$\mathbb{Y}_{\lambda,\beta;s,t}^{(i)}(\mathbf{Le}_{n+1}(x)) = \mathbf{Le}_{n+1}^{(i)}(x) = \lambda \mathbb{Y}_{\lambda,\beta;s,t}^{(i-1)}(\mathbf{Le}_{n+1+s}(x)) + \beta \mathbb{Y}_{\lambda,\beta;s,t}^{(i-1)}(\mathbf{Le}_{n+1+t}(x)), \quad i \geq 1,$$

with the initial condition

$$\mathbb{Y}_{\lambda,\beta;s,t}^{(0)}(\mathbf{Le}_{n+1}(x)) = \mathbf{Le}_{n+1}(x).$$

In particular, for $i = 1$, we obtain

$$I(\mathbf{Le}_{n+1}(x)) = \mathbb{Y}_{1,0;0,0}^{(1)}(\mathbf{Le}_{n+1}(x)) = \mathbf{Le}_{n+1}(x),$$

that is, the identity operator polynomial sequence.

$$\Delta(\mathbf{Le}_{n+1}(x)) = \mathbb{Y}_{1,-1;1,0}^{(1)}(\mathbf{Le}_{n+1}(x)) = \mathbf{Le}_{n+2}(x) - \mathbf{Le}_{n+1}(x),$$

which corresponds to the forward difference operator polynomial sequence.

$$\nabla(\mathbf{Le}_{n+1}(x)) = \mathbb{Y}_{1,-1;0,-1}^{(1)}(\mathbf{Le}_{n+1}(x)) = \mathbf{Le}_{n+1}(x) - \mathbf{Le}_n(x),$$

which corresponds to the backward difference operator polynomial sequence.

$$M(\text{Le}_{n+1}(x)) = \mathbb{Y}_{\frac{1}{2}, -\frac{1}{2}; 1, 0}^{(1)}(\text{Le}_{n+1}(x)) = \frac{1}{2}(\text{Le}_{n+2}(x) - \text{Le}_{n+1}(x)),$$

which gives the means-type operator polynomial sequence.

$$G_{s,t}(\text{Le}_{n+1}(x)) = \mathbb{Y}_{1,-1;s+t,s}^{(1)}(\text{Le}_{n+1}(x)) = \text{Le}_{n+1+s+t}(x) - \text{Le}_{n+1+s}(x), \quad st \neq 0,$$

which yields the Leonardo–Gould type operator polynomial sequence.

Remark 1. If we take $x = 1$ in the above operator polynomial sequences, we obtain the corresponding finite Leonardo operator number sequences. In other words, each operator acting on $\text{Le}_{n+1}(x)$ reduces to its numerical counterpart when the polynomial variable is fixed at $x = 1$.

Now, we present our main results. We first give the recurrence relation satisfied by the sequence $\text{Le}_n^{(i)}(x)$. Namely, by extending the recurrence structure of Leonardo polynomials through the application of a generalized operator, we derive the following recurrence relation.

Theorem 1. Leonardo finite operator polynomials satisfy the following recurrence relation:

$$\text{Le}_{n+1}^{(i)}(x) = x\text{Le}_n^{(i)}(x) + \text{Le}_{n-1}^{(i)}(x) + (\lambda + \beta)^i x, \quad (n \geq 1) \quad (2)$$

with initial conditions $\text{Le}_0^{(i)}(x)$ and $\text{Le}_1^{(i)}(x)$.

Proof. We use induction on i for the proof. It is clear that (2) holds for $i = 1$. Now, suppose that (2) is true for $i > 1$, that is,

$$\text{Le}_{n+1}^{(i)}(x) = x\text{Le}_n^{(i)}(x) + \text{Le}_{n-1}^{(i)}(x) + (\lambda + \beta)^i x.$$

Then, for $i + 1$, we have

$$\begin{aligned} \text{Le}_{n+1}^{(i+1)}(x) &= \lambda \text{Le}_{n+1+s}^{(i)}(x) + \beta \text{Le}_{n+1+t}^{(i)}(x) \\ &= \lambda [x\text{Le}_{n+s}^{(i)}(x) + \text{Le}_{n+s-1}^{(i)}(x) + (\lambda + \beta)^i x] \\ &\quad + \beta [x\text{Le}_{n+t}^{(i)}(x) + \text{Le}_{n+t-1}^{(i)}(x) + (\lambda + \beta)^i x] \\ &= x [\lambda \text{Le}_{n+s}^{(i)}(x) + \beta \text{Le}_{n+t}^{(i)}(x)] + \lambda \text{Le}_{n+s-1}^{(i)}(x) + \beta \text{Le}_{n+t-1}^{(i)}(x) + (\lambda + \beta)^{i+1} x \\ &= x\text{Le}_n^{(i+1)}(x) + \text{Le}_{n-1}^{(i+1)}(x) + (\lambda + \beta)^{i+1} x. \end{aligned}$$

Thus, (2) is held for $i + 1$. This completes the induction. $\square$

Theorem 2. The Binet-like formula of Leonardo finite operator polynomials is of the form

$$\begin{aligned} \text{Le}_n^{(i)}(x) &= \left(\frac{\text{Le}_1^{(i)}(x) - s_2(x)\text{Le}_0^{(i)}(x) + (1 - s_2(x))(\lambda + \beta)^i}{s_1(x) - s_2(x)} \right) s_1^n(x) \\ &\quad + \left(\frac{s_1(x)\text{Le}_0^{(i)}(x) - \text{Le}_1^{(i)}(x) + (s_1(x) - 1)(\lambda + \beta)^i}{s_1(x) - s_2(x)} \right) s_2^n(x) - (\lambda + \beta)^i, \quad (3) \end{aligned}$$

where $s_1(x) = (x + \sqrt{x^2 + 4})/2$ and $s_2(x) = (x - \sqrt{x^2 + 4})/2$.

Proof. As the recurrence (2) is a nonhomogeneous difference equation, we consider the solutions to the homogeneous and nonhomogeneous parts separately. First, we consider the homogeneous part of (2), that is,

$$\left[\text{Le}_{n+1}^{(i)}(x) \right]_h = x \text{Le}_n^{(i)}(x) + \text{Le}_{n-1}^{(i)}(x). \quad (4)$$

Note that the recurrence (4) is a second-order linear difference equation, and its characteristic equation is

$$s^2 - xs - 1 = 0.$$

Roots of this characteristic equation are

$$s_1(x) = (x + \sqrt{x^2 + 4})/2$$

and

$$s_2(x) = (x - \sqrt{x^2 + 4})/2.$$

So, we have $s_1(x) - s_2(x) = \sqrt{x^2 + 4}$.

The general solution of the homogeneous part is

$$\left[\text{Le}_n^{(i)}(x) \right]_h = F s_1^n(x) + G s_2^n(x),$$

where F and G are functions depending on x .

Now, consider the nonhomogeneous term $(\lambda + \beta)^i x$ of (2). Since

$$D = xD + D + (\lambda + \beta)^i x \implies D = -(\lambda + \beta)^i = \left[\text{Le}_n^{(i)}(x) \right]_p,$$

where D is a constant, the general solution of (2) is of the form

$$\text{Le}_n^{(i)}(x) = F s_1^n(x) + G s_2^n(x) - (\lambda + \beta)^i. \quad (5)$$

If (5) is taken together with the initial conditions of (2), then the following system of equations is obtained:

$$\begin{aligned} \text{Le}_0^{(i)}(x) &= F + G - (\lambda + \beta)^i, \\ \text{Le}_1^{(i)}(x) &= F s_1(x) + G s_2(x) - (\lambda + \beta)^i. \end{aligned}$$

After performing some basic calculations, we obtain

$$F = \frac{\text{Le}_1^{(i)}(x) - s_2(x) \text{Le}_0^{(i)}(x) + (1 - s_2(x))(\lambda + \beta)^i}{s_1(x) - s_2(x)}$$

and

$$G = \frac{s_1(x) \text{Le}_0^{(i)}(x) - \text{Le}_1^{(i)}(x) + (s_1(x) - 1)(\lambda + \beta)^i}{s_1(x) - s_2(x)}.$$

Therefore,

$$\begin{aligned} \text{Le}_n^{(i)}(x) &= \left(\frac{\text{Le}_1^{(i)}(x) - s_2(x) \text{Le}_0^{(i)}(x) + (1 - s_2(x))(\lambda + \beta)^i}{s_1(x) - s_2(x)} \right) s_1^n(x) \\ &\quad + \left(\frac{s_1(x) \text{Le}_0^{(i)}(x) - \text{Le}_1^{(i)}(x) + (s_1(x) - 1)(\lambda + \beta)^i}{s_1(x) - s_2(x)} \right) s_2^n(x) - (\lambda + \beta)^i. \end{aligned}$$

Thus, the statement is proven. $\square$

Theorem 3. The generating function of Leonardo finite operator polynomials is expressed by the following formula:

$$L^{(i)}(t, x) = \frac{\text{Le}_0^{(i)}(x) + [\text{Le}_1^{(i)}(x) - (1+x)\text{Le}_0^{(i)}(x)]t + [x\text{Le}_0^{(i)}(x) - \text{Le}_1^{(i)}(x) + (\lambda + \beta)^i x]t^2}{1 - (1+x)t - (1-x)t^2 + t^3}. \quad (6)$$

Proof. Let us represent

$$L^{(i)}(t, x) = \sum_{n=0}^{\infty} \text{Le}_n^{(i)}(x)t^n$$

as the generating function of Leonardo finite operator polynomials. Then, thanks to (2), we have

$$\begin{aligned} L^{(i)}(t, x) - \text{Le}_0^{(i)}(x) - t\text{Le}_1^{(i)}(x) &= \sum_{n=1}^{\infty} \text{Le}_{n+1}^{(i)}(x)t^{n+1} \\ &= x \sum_{n=1}^{\infty} \text{Le}_n^{(i)}(x)t^{n+1} + \sum_{n=1}^{\infty} \text{Le}_{n-1}^{(i)}(x)t^{n+1} + \sum_{n=1}^{\infty} (\lambda + \beta)^i xt^{n+1} \\ &= xt \sum_{n=1}^{\infty} \text{Le}_n^{(i)}(x)t^n + t^2 \sum_{n=1}^{\infty} \text{Le}_{n-1}^{(i)}(x)t^{n-1} + \frac{(\lambda + \beta)^i xt^2}{1-t} \\ &= xt(L^{(i)}(t, x) - \text{Le}_0^{(i)}(x)) + t^2 L^{(i)}(t, x) + \frac{(\lambda + \beta)^i xt^2}{1-t}. \end{aligned}$$

If $L^{(i)}(t, x)$ is left alone in the above equation as a result of basic algebraic calculations, then we obtain

$$L^{(i)}(t, x) = \frac{\text{Le}_0^{(i)}(x) + [\text{Le}_1^{(i)}(x) - (1+x)\text{Le}_0^{(i)}(x)]t + [x\text{Le}_0^{(i)}(x) - \text{Le}_1^{(i)}(x) + (\lambda + \beta)^i x]t^2}{1 - (1+x)t - (1-x)t^2 + t^3}.$$

This completes the proof. $\square$

Theorem 4. The exponential generating function of Leonardo finite operator polynomials is given by the following formula:

$$E^{(i)}(t, x) = \frac{p(x)e^{ts_1(x)} + q(x)e^{ts_2(x)} - \sqrt{x^2 + 4}(\lambda + \beta)^i e^t}{\sqrt{x^2 + 4}},$$

where

$$p(x) = \text{Le}_1^{(i)}(x) - s_2(x)\text{Le}_0^{(i)}(x) + (1 - s_2(x))(\lambda + \beta)^i$$

and

$$q(x) = s_1(x)\text{Le}_0^{(i)}(x) - \text{Le}_1^{(i)}(x) + (s_1(x) - 1)(\lambda + \beta)^i.$$

Proof. If we take

$$p(x) = \text{Le}_1^{(i)}(x) - s_2(x)\text{Le}_0^{(i)}(x) + (1 - s_2(x))(\lambda + \beta)^i$$

and

$$q(x) = s_1(x)\text{Le}_0^{(i)}(x) - \text{Le}_1^{(i)}(x) + (s_1(x) - 1)(\lambda + \beta)^i$$

in (3), then we obtain

$$\begin{aligned} E^{(i)}(t, x) &= \sum_{n=0}^{\infty} \left[\frac{p(x)s_1^n(x) + q(x)s_2^n(x)}{s_1(x) - s_2(x)} - (\lambda + \beta)^i \right] \frac{t^n}{n!} \\ &= \frac{p(x)}{s_1(x) - s_2(x)} \sum_{n=0}^{\infty} \frac{(ts_1(x))^n}{n!} + \frac{q(x)}{s_1(x) - s_2(x)} \sum_{n=0}^{\infty} \frac{(ts_2(x))^n}{n!} - (\lambda + \beta)^i \sum_{n=0}^{\infty} \frac{t^n}{n!} \\ &= \frac{p(x)e^{ts_1(x)} + q(x)e^{ts_2(x)} - \sqrt{x^2 + 4}(\lambda + \beta)^i e^t}{\sqrt{x^2 + 4}}. \end{aligned}$$

Thus, the proof is completed. $\square$

Corollary 1. The Poisson generating function of Leonardo finite operator polynomials is expressed by the following formula:

$$E_p^{(i)}(t, x) = \frac{p(x)e^{t(s_1(x)-1)} + q(x)e^{t(s_2(x)-1)}}{\sqrt{x^2 + 4}} - (\lambda + \beta)^i.$$

Proof. The proof follows from the relation $E_p^{(i)}(t, x) = e^{-t}E^{(i)}(t, x)$. $\square$

Theorem 5. The following binomial sum formula is provided:

$$\sum_{j=0}^n \binom{n}{j} x^j \text{Le}_j^{(i)}(x) = \text{Le}_{2n}^{(i)}(x) + (\lambda + \beta)^i (1 - (1+x)^n).$$

Proof. By virtue of (3) and binomial theorem, we have

$$\begin{aligned} \sum_{j=0}^n \binom{n}{j} x^j \text{Le}_j^{(i)}(x) &= \sum_{j=0}^n \binom{n}{j} x^j \left(\frac{p(x)s_1^j(x) + q(x)s_2^j(x)}{s_1(x) - s_2(x)} - (\lambda + \beta)^i \right) \\ &= \frac{p(x)}{s_1(x) - s_2(x)} \sum_{j=0}^n \binom{n}{j} (xs_1(x))^j + \frac{q(x)}{s_1(x) - s_2(x)} \sum_{j=0}^n \binom{n}{j} (xs_2(x))^j \\ &\quad - (\lambda + \beta)^i \sum_{j=0}^n \binom{n}{j} x^j \\ &= \frac{p(x)(1 + xs_1(x))^n + q(x)(1 + xs_2(x))^n}{s_1(x) - s_2(x)} - (\lambda + \beta)^i (1+x)^n \\ &= \frac{p(x)s_1^{2n}(x) + q(x)s_2^{2n}(x)}{s_1(x) - s_2(x)} - (\lambda + \beta)^i (1+x)^n \\ &= \text{Le}_{2n}^{(i)}(x) + (\lambda + \beta)^i (1 - (1+x)^n) \end{aligned}$$

as desired. $\square$

Theorem 6. For $n \geq 0$, the following finite sum formula is provided:

$$x \sum_{j=0}^n \text{Le}_j^{(i)}(x) = \text{Le}_{n+2}^{(i)}(x) + (1-x)\text{Le}_{n+1}^{(i)}(x) - \text{Le}_1^{(i)}(x) + (x-1)\text{Le}_0^{(i)}(x) - (n+1)(\lambda + \beta)^i x.$$

Proof. With the help of (2), we have the following equalities:

$$\begin{aligned} \text{Le}_0^{(i)}(x) &= \text{Le}_2^{(i)}(x) - x\text{Le}_1^{(i)}(x) - (\lambda + \beta)^i x, \\ \text{Le}_1^{(i)}(x) &= \text{Le}_3^{(i)}(x) - x\text{Le}_2^{(i)}(x) - (\lambda + \beta)^i x, \\ \text{Le}_2^{(i)}(x) &= \text{Le}_4^{(i)}(x) - x\text{Le}_3^{(i)}(x) - (\lambda + \beta)^i x, \\ &\vdots \\ \text{Le}_{n-1}^{(i)}(x) &= \text{Le}_{n+1}^{(i)}(x) - x\text{Le}_n^{(i)}(x) - (\lambda + \beta)^i x, \\ \text{Le}_n^{(i)}(x) &= \text{Le}_{n+2}^{(i)}(x) - x\text{Le}_{n+1}^{(i)}(x) - (\lambda + \beta)^i x. \end{aligned}$$

If we add these equations side by side and make the necessary simplifications, then we obtain

$$x \sum_{j=0}^n \text{Le}_j^{(i)}(x) = \text{Le}_{n+2}^{(i)}(x) + (1-x)\text{Le}_{n+1}^{(i)}(x) - \text{Le}_1^{(i)}(x) + (x-1)\text{Le}_0^{(i)}(x) - (n+1)(\lambda + \beta)^i x.$$

This completes the proof. $\square$

3. Determinant Representations of Leonardo Finite Operator Polynomials

Determinant representations are instrumental for both numerical evaluation and symbolic computation, offering closed forms that can be effectively implemented within computer algebra systems. In this section, we will present a determinantal expression for

the n th term of Leonardo finite operator polynomials. In order to obtain the determinant representation, we first need the following lemma given in [19].

Lemma 1. Let $\mu(t)$ and $\omega(t) \neq 0$ be differentiable functions, $P_{(n+1) \times 1}(t)$ be an $(n+1) \times 1$ matrix whose entries are $p_{k,1}(t) = \mu^{(k-1)}(t)$ for $1 \leq k \leq n+1$, and $Q_{(n+1) \times n}(t)$ be an $(n+1) \times n$ matrix whose entries are

$$q_{i,j}(t) = \begin{cases} \binom{i-1}{j-1} \omega^{(i-j)}(t), & i \geq j \\ 0, & i < j \end{cases}$$

for $1 \leq i \leq n+1$ and $1 \leq j \leq n$. Let $|S_{(n+1) \times (n+1)}(t)|$ be the lower Hessenberg determinant of the $(n+1) \times (n+1)$ lower Hessenberg matrix

$$S_{(n+1) \times (n+1)}(t) = \begin{bmatrix} P_{(n+1) \times 1}(t) & Q_{(n+1) \times n}(t) \end{bmatrix}.$$

Then, the n th derivative of $\mu(t)/\omega(t)$ can be computed by

$$\frac{d^n}{dt^n} \left[\frac{\mu(t)}{\omega(t)} \right] = (-1)^n \frac{|S_{(n+1) \times (n+1)}(t)|}{\omega^{n+1}(t)}.$$

For interested readers, we note that many researchers use this lemma in their calculations. For more information and details, see [20,21] and related references.

Theorem 7. For $n \geq 0$, a determinant representation of Leonardo finite operator polynomials is given by

$$\begin{aligned} & \text{Le}_n^{(i)}(x) \\ &= \frac{1}{n!} \begin{vmatrix} \text{Le}_0^{(i)}(x) & -1 & 0 & 0 \\ \text{Le}_1^{(i)}(x) - (1+x)\text{Le}_0^{(i)}(x) & \binom{1}{0}(1+x) & -1 & 0 \\ 2(x\text{Le}_0^{(i)}(x) - \text{Le}_1^{(i)}(x) + (\lambda + \beta)^i x) & 2\binom{2}{0}(1-x) & \binom{2}{1}(1+x) & -1 \\ 0 & -6\binom{3}{0} & 2\binom{3}{1}(1-x) & \binom{3}{2}(1+x) \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 \\ \ddots & 0 & 0 & 0 \\ \ddots & \ddots & \vdots & \vdots \\ \cdots & \binom{n-2}{n-3}(1+x) & -1 & 0 \\ \cdots & 2\binom{n-1}{n-3}(1-x) & \binom{n-1}{n-2}(1+x) & -1 \\ \cdots & -6\binom{n}{n-3} & 2\binom{n}{n-2}(1-x) & \binom{n}{n-1}(1+x) \end{vmatrix}. \end{aligned} \quad (7)$$

Proof. For computational simplicity in the proof, if we take

$$\alpha(x) = \text{Le}_1^{(i)}(x) - (1+x)\text{Le}_0^{(i)}(x)$$

and

$$\beta(x) = x\text{Le}_0^{(i)}(x) - \text{Le}_1^{(i)}(x) + (\lambda + \beta)^i x$$

in the generating function of Leonardo finite operator polynomials,

$$\mu(t) = \text{Le}_0^{(i)}(x) + \alpha(x)t + \beta(x)t^2$$

as well as

$$\omega(t) = 1 - (1+x)t - (1-x)t^2 + t^3$$

in Lemma 1, then we obtain

$$\begin{aligned} & \frac{d^n}{dt^n} \left[\frac{\text{Le}_0^{(i)}(x) + \alpha(x)t + \beta(x)t^2}{1 - (1+x)t - (1-x)t^2 + t^3} \right] \\ &= \frac{(-1)^n}{(1 - (1+x)t - (1-x)t^2 + t^3)^{n+1}} \times \\ & \quad \begin{vmatrix} \text{Le}_0^{(i)}(x) + \alpha(x)t + \beta(x)t^2 & \omega(t) & 0 & 0 \\ \alpha(x) + 2\beta(x)t & \binom{1}{0}\omega'(t) & \omega(t) & 0 \\ 2\beta(x) & \binom{2}{0}\omega''(t) & \binom{2}{1}\omega'(t) & \omega(t) \\ 0 & 6\binom{3}{0} & \binom{3}{1}\omega''(t) & \binom{3}{2}\omega'(t) \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 \\ \ddots & 0 & 0 & 0 \\ \ddots & \ddots & \vdots & \vdots \\ \cdots & \binom{n-2}{n-3}\omega'(t) & \omega(t) & 0 \\ \cdots & \binom{n-1}{n-3}\omega''(t) & \binom{n-1}{n-2}\omega'(t) & \omega(t) \\ \cdots & 6\binom{n}{n-3} & \binom{n}{n-2}\omega''(t) & \binom{n}{n-1}\omega'(t) \end{vmatrix}. \end{aligned}$$

If we take $t \rightarrow 0$ in the above equation, for $n \in \mathbb{N}$, then we obtain

$$\text{Le}_n^{(i)}(x) = \frac{1}{n!} \lim_{t \rightarrow 0} \frac{d^n}{dt^n} \left[\frac{\text{Le}_0^{(i)}(x) + \alpha(x)t + \beta(x)t^2}{1 - (1+x)t - (1-x)t^2 + t^3} \right]$$

which gives (7). $\square$

Remark 2. The determinant expansion presented in Theorem 7 can alternatively be derived by employing Equation (6). For elegant and concise proofs, the reader is referred to [22].

4. Some Applications for Generating Functions

In this section of our paper, we present a theorem encompassing various families of generating functions associated with the Leonardo finite operator polynomials. Some of the methods employed in the present analysis are closely related to those previously considered by Erkus and Srivastava [23], Erkus-Duman and Tuglu [24]; see also the related developments in Srivastava and Manocha [25].

Throughout this section, let $n \in \mathbb{N}_0$, $r \in \mathbb{N}$; $\rho, \sigma \in \mathbb{C}$; $a_k \in \mathbb{C} \setminus \{0\}$ ($k \in \mathbb{N}_0$). Assume that

$$\Theta_{\rho, \sigma}[u_1, \dots, u_s; \gamma] := \sum_{k=0}^{\infty} a_k \Psi_{\rho + \sigma k}(u_1, \dots, u_s) \gamma^k$$

and

$$\Psi_{\rho} : \mathbb{C}^s \rightarrow \mathbb{C} \setminus \{0\}$$

denotes a bounded function.

Theorem 8. Define

$$\Upsilon_{n,r}^{\rho,\sigma}(x; u_1, \dots, u_s; y) := \sum_{k=0}^{[n/r]} a_k \text{Le}_{n-rk}^{(i)}(x) \Psi_{\rho+\sigma k}(u_1, \dots, u_s) y^k. \quad (8)$$

Then, the following generating relation holds:

$$\sum_{n=0}^{\infty} \Upsilon_{n,r}^{\rho,\sigma} \left(x; u_1, \dots, u_s; \frac{\gamma}{t^r} \right) t^n = \frac{\text{Le}_0^{(i)}(x) + [\text{Le}_1^{(i)}(x) - (1+x)\text{Le}_0^{(i)}(x)]t + [x\text{Le}_0^{(i)}(x) - \text{Le}_1^{(i)}(x) + (\lambda + \beta)^i x]t^2}{1 - (1+x)t - (1-x)t^2 + t^3} \Theta_{\rho,\sigma}[u_1, \dots, u_s; \gamma]. \quad (9)$$

Proof. Denote by $\mathcal{U}$ the left-hand side of identity (9). Substituting relation (8) into it gives

$$\mathcal{U} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/r]} a_k \text{Le}_{n-rk}^{(i)}(x) \Psi_{\rho+\sigma k}(u_1, \dots, u_s) \gamma^k t^{n-rk}.$$

Applying the relation

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/r]} R(n, k) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} R(n + rk, k),$$

and using (6), this expression can be rearranged as

$$\begin{aligned} \mathcal{U} &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k \text{Le}_n^{(i)}(x) \Psi_{\rho+\sigma k}(u_1, \dots, u_s) \gamma^k t^n \\ &= \sum_{n=0}^{\infty} \text{Le}_n^{(i)}(x) t^n \sum_{k=0}^{\infty} a_k \Psi_{\rho+\sigma k}(u_1, \dots, u_s) \gamma^k \\ &= \frac{\text{Le}_0^{(i)}(x) + [\text{Le}_1^{(i)}(x) - (1+x)\text{Le}_0^{(i)}(x)]t + [x\text{Le}_0^{(i)}(x) - \text{Le}_1^{(i)}(x) + (\lambda + \beta)^i x]t^2}{1 - (1+x)t - (1-x)t^2 + t^3} \\ &\quad \times \Theta_{\rho,\sigma}[u_1, \dots, u_s; \gamma]. \end{aligned}$$

This last statement reduces exactly to the right-hand side of (9), and the claim follows. $\square$

Theorem 8 offers numerous applications through the selection of specific forms of the multivariable function $\Psi_{\rho+\sigma k}(u_1, \dots, u_s)$. Due to the function's definition within a broad and general framework, it allows for the derivation of various specific identities as particular cases. To illustrate this, we present two examples.

Example 1. The generating function relation of the bivariate Leonardo polynomials is (see [26])

$$\frac{1 + (x-2)t + yt^2}{1 - (x+1)t - (y-x)t^2 + yt^3} = \sum_{k=0}^{\infty} \mathcal{L}_k(x, y) t^k. \quad (10)$$

If we take $s = 2$, $u_1 = x$, $u_2 = y$, $a_k = 1$, $\rho = 0$, $\sigma = 1$ and replace the function $\Psi_{\rho+\sigma k}$ in Theorem 8 with the bivariate Leonardo polynomials, using the relation (10) and Theorem 8, we obtain

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/r]} \text{Le}_{n-rk}^{(i)}(x) \mathcal{L}_k(x, y) \gamma^k t^{n-rk}$$

$$= \frac{1 + (x-2)\gamma + y\gamma^2}{1 - (x+1)\gamma - (y-x)\gamma^2 + y\gamma^3} \times \frac{\text{Le}_0^{(i)}(x) + [\text{Le}_1^{(i)}(x) - (1+x)\text{Le}_0^{(i)}(x)]t + [x\text{Le}_0^{(i)}(x) - \text{Le}_1^{(i)}(x) + (\lambda + \beta)^i x]t^2}{1 - (1+x)t - (1-x)t^2 + t^3},$$

which is a class of bilateral generating functions for the bivariate Leonardo polynomials and the Leonardo finite operator polynomials.

Example 2. Taking $s = 1$, $u_1 = u$, $a_k = 1$, $\rho = 0$, $\sigma = 1$ and taking the Leonardo finite operator polynomials instead of the function $\Psi_{\rho+\sigma k}$ in Theorem 8 and also using (6), we obtain the following class of bilinear generating functions for the Leonardo finite operator polynomials:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/r]} \text{Le}_{n-rk}^{(i)}(x) \text{Le}_k^{(i)}(x) \gamma^k t^{n-rk}$$

$$= \frac{\text{Le}_0^{(i)}(x) + [\text{Le}_1^{(i)}(x) - (1+x)\text{Le}_0^{(i)}(x)]t + [x\text{Le}_0^{(i)}(x) - \text{Le}_1^{(i)}(x) + (\lambda + \beta)^i x]t^2}{1 - (1+x)t - (1-x)t^2 + t^3} \times \frac{\text{Le}_0^{(i)}(x) + [\text{Le}_1^{(i)}(x) - (1+x)\text{Le}_0^{(i)}(x)]\gamma + [x\text{Le}_0^{(i)}(x) - \text{Le}_1^{(i)}(x) + (\lambda + \beta)^i x]\gamma^2}{1 - (1+x)\gamma - (1-x)\gamma^2 + \gamma^3}.$$

5. Conclusions

In this research, we have introduced and conducted a systematic analysis of a new category of special polynomials, known as Leonardo finite operator polynomials (LFOPs). These polynomials are characterized by the integration of the structural features of the Leonardo polynomial sequence with the generalized finite operator developed by Simsek. Through the application of this operator framework, we have identified several fundamental properties that contribute to a deeper algebraic and combinatorial understanding of Leonardo-type sequences.

In particular, since the Leonardo numbers can be expressed as Fibonacci numbers, the same operator formalism applied to the Leonardo sequence naturally yields a Fibonacci-type operator structure. This connection provides a new algebraic bridge between the Leonardo and Fibonacci operator polynomial families, allowing the derivation of analogous identities, recurrence relations, and generating functions within a unified framework.

In subsequent research, the generalized finite operator introduced herein may be applied to other established polynomial families to develop novel classes of finite operator polynomials. This approach allows for a systematic examination of their analytical, algebraic, and combinatorial properties, including generating functions, recurrence relations, and determinant representations. Such extensions are anticipated to further unify operator-based polynomial theory and create new opportunities for exploration in discrete mathematics and computational analysis, with broad and sustainable application prospects in the future.

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