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On the Analysis of a System of Equations Containing a Parameter n and Describing a Special State of a Certain Table of Numbers

Dostonjon Numonjonovich Barotov

Department of Mathematics and Data Analysis, Faculty of Information Technology and Big Data Analysis, Financial University Under the Government of the Russian Federation, 4-th Veshnyakovsky Passage, 4, 109456 Moscow, Russia; dnbarotov@fa.ru

Abstract

In this paper, we study a system of 10 equations containing a parameter—a non-negative integer n—and associated with the problem of filling a special table with natural numbers. As a result, by developing a hybrid approach—first, by proving an a priori estimate for the solution of the system, which implies a finite set of solutions, thereby significantly narrowing the search space for a solution, and second, by performing a computer calculation of all remaining vectors satisfying the a priori estimate—we established that the system is not solvable for all values of the parameter n. We also prove a criterion for $(x_0, x_1, \ldots, x_9) \in \mathbb{N}^{10}$ to be a solution to the system for some value of the parameter n. Furthermore, we prove a fact that, in particular, implies that the set of values of the parameter n for which the system has at least 10 solutions is countable.

Keywords: system of equations in natural numbers; solvability of a system of equations; a priori estimate of a solution; magic table; mathematical puzzle

MSC: 11Y50; 11D45; 91A46; 91A68

1. Introduction

Since ancient times, mathematics has presented a variety of problems of varying complexity, with interesting formulations and properties, as well as game-like elements, which have attracted mathematicians and mathematics enthusiasts to study them [1–3]. And today, too, mathematical problems with interesting formulations and properties often arise in a wide variety of mathematical studies, in particular, in [4–10]. The interesting formulations and properties of such problems are often interpreted and called magical, which, in part, adds to their appeal, and these mathematical problems themselves are often called magical problems or puzzles. Such mathematical puzzles in their original formulation may not describe real applied problems, but they have enormous practical applications, developing logical thinking and problem-solving skills; specifically, research into computational complexity classes is sometimes advanced through the study of various puzzles [11]. In some cases, they are directly applicable to programming, engineering, finance, cryptography, and everyday life [3,12–14]. Many of their principles underlie algorithms and artificial intelligence, and today, they serve as a powerful motivation and basis for the development of engaging computer games and puzzles [15,16]. Another important significance of such problems, with interesting formulations and properties, i.e., such mathematical puzzles at the present time, is that, firstly, due to their interesting formulations and properties,

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they also serve, along with others, in particular those presented in [3,15–17], as a kind of bridge between the world of professional mathematics and the general public. Secondly, they are implemented on a computer and are generalized by capturing complexity and attractiveness; therefore, they represent a fascinating object of scientific research, including from the point of view of algorithms and computational complexity [1,15,18].

In this paper, for an arbitrary non-negative integer n, we study the solvability of a system of 10 equations of the following form:

$$\begin{cases} x_0 = 1 + \beta_0(n) + \beta_0(x_0) + \beta_0(x_1) + \ldots + \beta_0(x_9) \\ x_1 = 1 + \beta_1(n) + \beta_1(x_0) + \beta_1(x_1) + \ldots + \beta_1(x_9) \\ \ldots \\ x_9 = 1 + \beta_9(n) + \beta_9(x_0) + \beta_9(x_1) + \ldots + \beta_9(x_9) \end{cases}, \quad (1)$$

where $\beta_k(n)$ is the number of digits $k \in \{0, 1, \ldots, 9\}$ in the decimal notation of the number n, and $x_0, x_1, \ldots, x_9$ are the unknown natural numbers being sought. The above system (1) describes a special state, which will be called magic in this paper, of a table of numbers of the following form:

n	0	1	2	3	4	5	6	7	8	9
	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9

, (2)

namely, if $(x_0, x_1, \ldots, x_9)$ is a solution of System (1), then for any digit $k \in \{0, 1, \ldots, 9\}$ indicated in the first row of table (2), the number x_k indicated in the second row of table (2) and standing below it is equal to the number of digits k in the entire table (2). That is, it describes a puzzle associated with a table of numbers of the form (2), the essence of which, in fact, is that for a given non-negative integer n, it is necessary to select, if possible, such a vector of 10 natural numbers $(x_0, x_1, \ldots, x_9)$ so that a table of the form (2) with a given parameter n and a selected vector $(x_0, x_1, \ldots, x_9)$ has a magical state. It should be noted that in this work, the problem being analyzed is related to the problems of counting digits and autobiographical numbers, but the presence of the parameter n in it makes it complex, and its connection with the table of type (2) makes it even more attractive. The author's motivation for analyzing a system of the form (1), containing an arbitrary non-negative integer parameter n, is that, firstly, the results proposed in the work and their interpretations in the language of a magic table of the form (2) provide a new perspective on the problems of counting digits and, possibly, can serve, along with other works, in particular, those presented in [3,16,17], as a kind of bridge between the world of professional mathematics and the general public. Secondly, it can be applied and generalized in the analysis of similar problems, and it can complement similar results [19–24]. One can verify, for example, by trial and error, that System (1) for $n \in \{0, 1, 2\}$ is solvable. Indeed, for this, in view of the above-described connection of System (1) with table (2), it is sufficient to provide the following tables:

0	0	1	2	3	4	5	6	7	8	9
	2	6	4	1	2	1	2	1	1	1

,

1	0	1	2	3	4	5	6	7	8	9
	1	8	3	2	1	1	1	1	2	1

,

2	0	1	2	3	4	5	6	7	8	9
	1	7	3	3	1	1	1	2	1	1

,

2	0	1	2	3	4	5	6	7	8	9
	1	7	4	1	2	1	1	2	1	1

.

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We present a brief outline of this article. The second section presents the notation and definitions used. In the third section, an a priori estimate for a solution to System (1) is proved, which, in particular, implies that the number of solutions to System (1) for a fixed value of the parameter n is finite. In the fourth section, it is proved that System (1) for $n = 6$ is unsolvable. A criterion is also proved for when $(x_0, x_1, \ldots, x_9) \in \mathbb{N}^{10}$ will be a solution to System (1) for some value of the parameter n. In the fifth section, a fact is proved which, in particular, implies that the set of values of the parameter n for which System (1) has at least 10 solutions is countable.

2. Used Notations and Definitions

Let $\alpha(n) = \beta_0(n) + \beta_1(n) + \ldots + \beta_9(n) = \begin{cases} 1 + [\lg(n)], \text{ if } n \in \mathbb{N} \\ 1, \text{ if } n = 0 \end{cases}$ be the total number of digits in the decimal notation of the number n, where $\beta_k(n)$, as noted above, is the number of digits k in the decimal notation of the number n.

Let $\gamma(n) = \min\{x \in \mathbb{N} : 10^{x-1} - 10 \cdot x - \alpha(n) \geq 1\}$ be the minimal natural solution of the inequality $10^{x-1} - 10 \cdot x - \alpha(n) - 1 \geq 0$.

Definition 1. *We will say that table* (2)*, containing a non-negative parameter n and the corresponding vector of natural numbers* $(x_0, x_1, \ldots, x_9)$*, has a magic state if* $(x_0, x_1, \ldots, x_9)$ *is a solution of System* (1)*, i.e., for any digit* $k \in \{0, 1, \ldots, 9\}$ *indicated in the first row of table* (2)*, the number* x_k *indicated in the second row of table* (2) *and standing below it is equal to the number of occurrences of digit k in the entire table* (2)*.*

Remark 1. *We note that for any non-negative integer n, the natural number* $\gamma(n)$ *is well defined and also satisfies the following inequality:*

$$\max(2, 1 + \lg(\alpha(n))) < \gamma(n) < 4 + \lg(\alpha(n)).$$

Indeed, since, firstly, the following inequalities

$$\left(10^{x-1} - 10 \cdot x - \alpha(n) - 1\right)\Big|_{x=2} = 10 - 20 - \alpha(n) - 1 = -\alpha(n) - 11 \leq -1 - 11 < 0,$$

$$\left(10^{x-1} - 10 \cdot x - \alpha(n) - 1\right)\Big|_{x=1+\lg(\alpha(n))} = -11 - 10 \cdot \lg(\alpha(n)) \leq -11 - 10 \cdot 0 < 0,$$

$$\left(10^{x-1} - 10 \cdot x - \alpha(n) - 1\right)\Big|_{x=3+\lg(\alpha(n))} = 99 \cdot \alpha(n) - 10 \cdot \lg(\alpha(n)) - 31 \geq 99 \cdot \alpha(n) - 10 \cdot \alpha(n) -$$
$$-31 = 89 \cdot \alpha(n) - 31 = 89 \cdot (\alpha(n) - 1) + 58 \geq 58 > 0,$$

are satisfied and, secondly,

$$0 > \left(10^{k-1} - 10 \cdot k - \alpha(n) - 1\right)\Big|_{k\in\{1,2\}} < \left(10^{k-1} - 10 \cdot k - \alpha(n) - 1\right)\Big|_{k=3} <$$
$$< \left(10^{k-1} - 10 \cdot k - \alpha(n) - 1\right)\Big|_{k=4} < \ldots < \left(10^{k-1} - 10 \cdot k - \alpha(n) - 1\right)\Big|_{k=+\infty} = +\infty.$$

3. A Priori Estimate of the Solution to the General System

We begin the exposition by formulating and justifying the a priori estimate of the solution to System (1).

Lemma 1. *If* $(x_0^*, x_1^*, \ldots, x_9^*) \in \mathbb{N}^{10}$ *is a solution of System* (1)*, then the following estimate holds:*

$$x_k^* < 10 \cdot \gamma(n) + \alpha(n) - 9 \quad \forall k \in \{0, 1, \ldots, 9\}. \tag{3}$$

Proof. First, note that there exists $i \in \{0, 1, \ldots, 9\}$ such that $x_i^* \geq x_j^*$ for all $j \in \{0, 1, \ldots, 9\}$; in other words, $x_i^* = \max(x_0^*, x_1^*, \ldots, x_9^*)$. Let $x_i^* = \overline{a_{s-1}a_{s-2}\ldots a_1 a_0}$ be an s-digit number, where $a_{s-1} \in \{1, 2, \ldots, 9\}$. We will begin the proof by justifying an auxiliary a priori estimate of the form

$$\max(x_0^*, x_1^*, \ldots, x_9^*) < 10^{\gamma(n)-1}. \tag{4}$$

We will reason by contradiction: suppose that the following inequality holds:

$$x_i^* = \max(x_0^*, x_1^*, \ldots, x_9^*) \geq 10^{\gamma(n)-1}. \tag{5}$$

Then, on the one hand, in view of $s, \gamma(n) \in \mathbb{N}$, we obtain a chain of inequalities

$$x_i^* \geq 10^{s-1} \geq 10^{\gamma(n)-1}, \tag{6}$$

and on the other hand, due to the $(i+1)$-th equation of System (1), we have

$$x_i^* = 1 + \beta_i(n) + \beta_i(x_0^*) + \beta_i(x_1^*) + \cdots + \beta_i(x_9^*) < 1 + \alpha(n) + 10 \cdot s. \tag{7}$$

From inequalities (6) and (7), we obtain

$$10^{s-1} \leq x_i^* < 10 \cdot s + \alpha(n) + 1. \tag{8}$$

However, this leads to a contradiction, since based on the right-hand side of (6) and the definition of $\gamma(n)$, we have

$$10^{s-1} \geq 10 \cdot s + \alpha(n) + 1. \tag{9}$$

The validity of the auxiliary a priori estimate (4) is thus established.

Now, based on (4), we justify a more precise estimate of the form (3). Indeed, due to (4), we have that $x_i^* = \max(x_0^*, x_1^*, \ldots, x_9^*)$ is at most a $(\gamma(n)-1)$-digit number. Then, based on (7), we obtain

$$x_i^* < 1 + \alpha(n) + 10 \cdot s \leq 10 \cdot (\gamma(n) - 1) + \alpha(n) + 1 = 10 \cdot \gamma(n) + \alpha(n) - 9. \tag{10}$$

The estimate obtained in (10) completes the proof. Lemma 1 is completely proven. □

4. Establishing the Unsolvability of a Specific System

In this section, we prove that for $n = 6$, a particular system of equations of the form (1) is unsolvable.

Theorem 1. *The system of equations*

$$\begin{cases} x_0 = 1 + \beta_0(6) + \beta_0(x_0) + \beta_0(x_1) + \cdots + \beta_0(x_9) \\ x_1 = 1 + \beta_1(6) + \beta_1(x_0) + \beta_1(x_1) + \cdots + \beta_1(x_9) \\ \ldots \\ x_9 = 1 + \beta_9(6) + \beta_9(x_0) + \beta_9(x_1) + \cdots + \beta_9(x_9) \end{cases} \tag{11}$$

in natural numbers is unsolvable.

Proof. First, we prove that if $(x_0^*, x_1^*, \dots, x_9^*) \in \mathbb{N}^{10}$ is a solution to System (11), then the following estimate holds:

$$x_k^* \le 9 \quad \forall k \in \{2,3,4,5,7,8,9\}. \tag{12}$$

According to the definition above, we have $\gamma(n) = \gamma(6) = 3$. Therefore, in view of (3), if $(x_0^*, x_1^*, \dots, x_9^*) \in \mathbb{N}^{10}$ is a solution to System (11), then we get the following inequality:

$$1 \le x_j^* < 10 \cdot \gamma(6) + \alpha(6) - 9 = 22 \quad \forall j \in \{0,1,\dots,9\} \tag{13}$$

Based on (13), we prove (12) by contradiction. Suppose $x_k^* \ge 10$ for some $k \in \{2,3,4,5,7,8,9\}$. Then, based on the $(k+1)$-th equation of System (11), the estimate (13), and the inclusion $k \in \{2,3,4,5,7,8,9\}$, we obtain

$$10 \le x_k^* = 1 + \beta_k(6) + \beta_k(x_0^*) + \beta_k(x_1^*) + \dots + \beta_k(x_9^*) \le 1 + 0 + 1 + 1 + \dots + 1 = 11. \tag{14}$$

Based on (14) regarding the possible value of x_k^*, we consider two cases.

Case 1. Let $x_k^* = 11$. Then, by virtue of the $(k+1)$-th equation of System (11), the estimate (13), and the inclusion $k \in \{2,3,4,5,7,8,9\}$, we obtain

$$11 = x_k^* = 1 + \beta_k(6) + \beta_k(11) + \sum_{\substack{j=0 \\ j \ne k}}^{9} \beta_k(x_j^*) \le 1 + 0 + 0 + \sum_{\substack{j=0 \\ j \ne k}}^{9} 1 = 10. \tag{15}$$

A contradiction arose in (15).

Case 2. Let $x_k^* = 10$. Then, by virtue of the $(k+1)$-th equation of System (11), the estimate (13), and the inclusion $k \in \{2,3,4,5,7,8,9\}$, we obtain

$$x_j^* \in \{k, 10+k\} \quad \forall j \in \{0,1,\dots,9\} \setminus \{k\}. \tag{16}$$

Based on (16) and the first equation of System (11), we obtain

$$\begin{cases} x_0^* \in \{k, 10+k\} \\ x_0^* = 1 + \beta_0(6) + \beta_0(10) + \sum\limits_{\substack{j=0 \\ j \ne k}}^{9} \beta_0(x_j^*) = 2 \end{cases}. \tag{17}$$

If $k \ne 2$, then the result in (17) contradicts $k \ne 2$, and if $k = 2$, then in the 4-th equation of System (11), a contradiction of the following form arises:

$$\{2, 12\} \ni x_3^* = 1 + \beta_3(6) + \beta_3(10) + \sum_{\substack{j=0 \\ j \ne k}}^{9} \beta_3(x_j^*) = 1. \tag{18}$$

The contradiction in (18) completes the justification of Estimate (12).

Now, using the proved Estimate (12) and similar reasoning, it is easy to prove that if $(x_0^*, x_1^*, \dots, x_9^*) \in \mathbb{N}^{10}$ is a solution of System (11), then the following estimate holds:

$$1 + \beta_k(6) \le x_k^* \le 9 \quad \forall k \in \{0,1,6\}. \tag{19}$$

Combining the estimates (12) and (19), we obtain

$$1 + \beta_k(6) \le x_k^* \le 9 \quad \forall k \in \{0,1,\dots,9\}. \tag{20}$$

Thus, on the one hand, based on a priori estimates, we conclude that if System (11) has a solution, then Inequality (20) must be satisfied. On the other hand, having checked on a computer all possible variants satisfying Inequality (20), which could be solutions (since there are few of them left), i.e., substituting into System (11) each element of the set $\prod_{k=0}^{5}\{1,2,\ldots,9\}\times\{2,3,\ldots,9\}\times\prod_{k=7}^{9}\{1,2,\ldots,9\}$, which has $9^6\cdot 8\cdot 9^3$ elements, it is easy to see in a matter of minutes that System (11) is unsolvable. Theorem 1 is completely proven. □

Now let us consider this problem from a slightly different point of view; namely, let us determine when a specific vector of natural numbers $(x_0^*, x_1^*, \ldots, x_9^*)$ is a solution to System (1) for some n.

Theorem 2. *The vector of natural numbers $(x_0^*, x_1^*, \ldots, x_9^*)$ will be a solution of System (1) for some n if and only if the following two conditions are satisfied:*

(1) for each $i\in\{0,1,\ldots,9\}$, the inequality $m_i^*\geq 0$ holds, where $m_i^* = x_i^* - 1 - \sum_{k=0}^{9}\beta_i(x_k^*)$,

(2) either the inequality $(m_1^*,\ldots,m_9^*)\neq(0,\ldots,0)$ holds, or the equality $(m_0^*, m_1^*,\ldots,m_9^*) = (1,0,\ldots,0)$ holds.

Proof. Necessity. Let $(x_0^*, x_1^*, \ldots, x_9^*)$ be a solution to System (1) for some $n^*\in\{0,1,2,\ldots\}$. Then, in view of the definition of $\beta_k(n)$ and the $(i+1)$-th equation of System (1), for each $i\in\{0,1,\ldots,9\}$, we have

$$\beta_i(n^*) = x_i^* - 1 - \sum_{k=0}^{9}\beta_i(x_k^*) = m_i^*\geq 0.$$

To justify condition (2), consider two cases.

Case 1. Let $n^* = 0$. Then, we obtain

$$(m_0^*, m_1^*,\ldots,m_9^*) = (\beta_0(0),\beta_0(1),\ldots,\beta_0(9)) = (1,0,\ldots,0).$$

Case 2. Let $n^*\in\mathbb{N}$. Then, there exists $j\in\{1,2,\ldots,9\}$ such that $\beta_j(n^*)\geq 1$, and therefore,

$$(m_1^*,\ldots,m_9^*) = (\beta_1(n^*),\ldots,\beta_9(n^*))\neq(0,\ldots,0).$$

Sufficiency. Let $(x_0^*, x_1^*, \ldots, x_9^*)$ satisfy Conditions (1) and (2). Then, by constructing the number

$$n^* = \overline{\underbrace{999\ldots9}_{m_9^*\text{ times}}\underbrace{888\ldots8}_{m_8^*\text{ times}}\ldots\underbrace{111\ldots1}_{m_1^*\text{ times}}\underbrace{000\ldots0}_{m_0^*\text{ times}}}, \tag{21}$$

due to conditions (1) and (2), the number n^* is correctly defined (written), and we justify that $(x_0^*, x_1^*, \ldots, x_9^*)$ is a solution to System (1) for $n = n^*$. Indeed, due to (21), for each $i\in\{0,1,\ldots,9\}$, we have

$$1+\beta_i(n^*)+\sum_{k=0}^{9}\beta_i(x_k^*) = 1+m_i^*+\sum_{k=0}^{9}\beta_i(x_k^*) = 1+x_i^*-1 = x_i^*.$$

Theorem 2 is completely proven. □

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5. The Cardinality of the Set of Parameter Values n for Which the System Is Solvable

In this section, we prove that the set of parameter values n for which table (2) has a magic state is countable.

Theorem 3. *The set of non-negative integer values of the parameter n for which System (1) is solvable is countable.*

Proof. Let S be the set of non-negative integer values of the parameter n for which System (1) is solvable. Then, the following embedding holds:

$$S \subset \{0, 1, 2, \ldots\} = \{0\} \cup \mathbb{N} \tag{22}$$

Now, we construct the following special set of the form

$$S^* = \left\{ \underbrace{111\ldots11}_{\left(\frac{1}{9}(10^{m+2}-1)-m-12\right)\ \text{times}} \quad : \quad m \in \mathbb{N} \right\} \tag{23}$$

and argue that the embedding holds

$$S^* \subset S. \tag{24}$$

Let n^* be an arbitrary element of the set S^*. Then, there exists a unique natural number m^* such that the equality $n^* = \underbrace{111\ldots11}_{\left(\frac{1}{9}(10^{m^*+2}-1)-m^*-12\right)\ \text{times}}$ holds. It is sufficient to show that for $n = n^*$, the vector $\left(1, \frac{1}{9}\left(10^{m^*+2}-1\right), 1, 1, 1, 1, 1, 1, 1\right)$ is a solution to System (1). Indeed, since

$$1 + \beta_k(n^*) + \beta_k(x_0) + \beta_k(x_1) + \cdots + \beta_k(x_9) =$$

$$= 1 + \beta_k\left(\underbrace{111\ldots11}_{\left(\frac{1}{9}(10^{m^*+2}-1)-m^*-12\right)\ \text{times}}\right) + \beta_k(1) + \beta_k\left(\frac{1}{9}\left(10^{m^*+2}-1\right)\right) + \cdots + \beta_k(1) =$$

$$= 1 + 0 + 0 + 0 + \cdots + 0 = 1 = x_k \quad \forall k \in \{0, 1, \ldots, 9\} \setminus \{1\},$$

$$1 + \beta_1(n^*) + \beta_1(x_0) + \beta_1(x_1) + \cdots + \beta_1(x_9) =$$

$$= 1 + \beta_1\left(\underbrace{111\ldots11}_{\left(\frac{1}{9}(10^{m^*+2}-1)-m^*-12\right)\ \text{times}}\right) + \beta_1(1) + \beta_1\left(\frac{1}{9}\left(10^{m^*+2}-1\right)\right) + \cdots + \beta_1(1) =$$

$$= 1 + \frac{1}{9}\left(10^{m^*+2}-1\right) - m^* - 12 + 1 + (m^*+2) + 8 = \frac{1}{9}\left(10^{m^*+2}-1\right) = x_1.$$

Thus, due to the countability of the sets S^* and $\{0\} \cup \mathbb{N}$ and the embeddings (22) and (24), we obtain that the set S is countable. Theorem 3 is completely proven. □

At the end of this paper, we prove one more statement that strengthens (refines) Theorem 3.

Theorem 4. *Let $(s_{k,0}, s_{k,1}, \ldots, s_{k,9})_{k=1}^{m}$ be m different solutions to System (1) for $n = n^* \in \mathbb{N}$ such that $s_{1,i} = s_{2,i} = \ldots = s_{m,i} = s_i$ for some $i \in \{0, 1, \ldots, 9\}$. Then, System (1) for $n = n^* \cdot 10^{9s_i+i-1} + \frac{i}{9}(10^{9s_i+i-1} - 1)$ has at least m different solutions.*

Proof. It is sufficient to prove that for each $k \in \{1,2,\ldots,m\}$, a vector of 10 natural numbers of the form $(s_{k,0},\ldots,s_{k,i-1},10s_i+i,s_{k,i+1},\ldots,s_{k,9})$ is a solution of System (1) for $n = n^* \cdot 10^{9s_i+i-1} + \frac{i}{9}(10^{9s_i+i-1}-1)$. Indeed, if $j \in \{0,1,\ldots,9\} \setminus \{i\}$, then due to the fact that $(s_{k,0},\ldots,s_{k,i-1},s_i,s_{k,i+1},\ldots,s_{k,9})$ is a solution of System (1) for $n = n^*$, the following equality holds:

$$1+\beta_j\left(n^* \cdot 10^{9s_i+i-1} + \frac{i}{9}(10^{9s_i+i-1}-1)\right) + \beta_j(10s_i+i) + \sum_{\substack{l=0 \\ l\neq i}}^{9} \beta_j(s_{k,l}) =$$

$$= 1+\beta_j(n^*) + \beta_j(s_i) + \sum_{\substack{l=0 \\ l\neq i}}^{9} \beta_j(s_{k,l}) = 1+\beta_j(n^*) + \sum_{l=0}^{9} \beta_j(s_{k,l}) = s_{k,j}, \tag{25}$$

and if $j = i$, then due to the fact that $(s_{k,0},\ldots,s_{k,i-1},s_i,s_{k,i+1},\ldots,s_{k,9})$ is a solution of System (1) for $n = n^*$, the following equality holds:

$$1+\beta_i\left(n^* \cdot 10^{9s_i+i-1} + \frac{i}{9}(10^{9s_i+i-1}-1)\right) + \beta_i(10s_i+i) + \sum_{\substack{l=0 \\ l\neq i}}^{9} \beta_i(s_{k,l}) =$$

$$= 1+\beta_i(n^*) + 9s_i + i - 1 + \beta_i(s_i) + 1 + \sum_{\substack{l=0 \\ l\neq i}}^{9} \beta_i(s_{k,l}) =$$

$$= 9s_i + i + \left(1+\beta_i(n^*) + \sum_{l=0}^{9} \beta_i(s_{k,l})\right) = 9s_i + i + s_i = 10s_i + i. \tag{26}$$

Since the number $k \in \{1,2,\ldots,m\}$ in the reasoned Equalities (25) and (26) is arbitrary, we obtain that $(s_{k,0},\ldots,s_{k,i-1},10s_i+i,s_{k,i+1},\ldots,s_{k,9})$ is a solution of System (1) for $n = n^* \cdot 10^{9s_i+i-1} + \frac{i}{9}(10^{9s_i+i-1}-1)$ for any $k \in \{1,2,\ldots,m\}$. Theorem 4 is completely proven. □

Corollary 1. *If System (1) has m different solutions with a general i-th, where $i \in \{0,1,\ldots,9\}$, coordinate for some $n = n^*$, then the set of values of the parameter n for which System (1) has at least m different solutions with a general i-th coordinate is countable.*

Indeed, we note that the *i*-th coordinates of all the new *m* different solutions presented in the proof of Theorem 4 are equal to each other; therefore, each time, starting from the currently achieved set of *m* different solutions corresponding to n^*, with a common *i*-th coordinate, repeating the proof (process), we present the next natural number greater than n^* and also the corresponding subsequent new *m* different solutions with a common *i*-th coordinate.

Remark 2. *Combining Theorem 4 and Corollary 1, given that*

$$(19,4,5,3,3,2,2,16,2,2), \quad (19,5,4,3,3,2,1,17,2,2),$$

$$(19,5,5,2,2,3,1,17,2,2), \quad (19,5,6,1,2,2,2,17,2,2),$$

$$(19,6,4,1,4,1,2,17,2,2), \quad (20,4,5,3,3,2,2,16,2,1),$$

$$(20,5,4,3,3,2,1,17,2,1), \quad (20,5,5,2,2,3,1,17,2,1),$$

$$(20,5,6,1,2,2,2,17,2,1), \quad (20,6,4,1,4,1,2,17,2,1),$$

having the 9-th common coordinate equal to 2, are solutions of System (1) for

$$n = \overline{148\underbrace{77\ldots7}_{15\ times}\underbrace{00\ldots0}_{18\ times}},$$

is a strengthening of Theorem 3; namely, it follows that the set of values of the parameter n for which System (1) has at least 10 solutions is countable.

6. Conclusions

This study established that System (1) is solvable for some values of the non-negative integer parameter n, but not for all values of the non-negative integer parameter n. Specifically, firstly, an important a priori estimate of its solution was proven, which significantly narrows the search space to a finite domain, thereby, in particular, proving the finiteness of the set of solutions of the original system. Secondly, based on computer enumeration of all remaining variants satisfying the a priori estimate, it was proven that for $n = 6$, it is unsolvable. It has also been proven that the set of values of the parameter n for which System (1) has at least 10 solutions is countable. The results presented in this article can be used in the study of similar problems, the analysis of similar mathematical puzzles [2,16,19,25], or the development of computer games, and they can also be useful in the study of some financial problems [14,26,27] that are reducible to puzzles or games from a mathematical point of view.

Also, System (1), due to its connection with table (2), can be an object of further study. Therefore, the following non-trivial problems related to it can be noted as suitable for further study:

1. Finding the set of values of the parameter n for which System (1) is unsolvable, i.e., if the value of the parameter n is taken from this set, then it is pointless to play a game associated with table (2), as in the game "15" [16]—it is pointless to select a vector of 10 natural numbers $(x_0, x_1, \ldots, x_9)$ such that when this vector is substituted into table (2), this table will have a magical state.
2. Investigation of the possibility of realizing each natural number m as a common number of solutions of System (1), i.e., for each natural number m, the existence of a corresponding number n_m such that System (1) with $n = n_m$ will have exactly m solutions.

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