

Arcs, Caps and Generalisations in a Finite Projective Space

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Abstract: Arcs and caps are fundamental structures in finite projective spaces. They can be generalised. Here, a survey is given of some important results on these objects, in particular on generalised ovals and generalised ovoids. This paper also contains recent results and several open problems.

Keywords: arc; cap; generalised oval; generalised ovoid; translation; finite projective space

MSC: 05B25; 51E12; 51E20; 51E21; 51E22

1. Introduction

A non-singular conic of the projective plane $\text{PG}(2, q)$ over the Galois field $\mathbb{F}_q$ consists of $q + 1$ points, no three of which are collinear. It is natural to ask if this non-collinearity condition for $q + 1$ points is sufficient for them to be a conic. In other words, does this combinatorial property characterise non-singular conics? For q odd, this question was affirmatively answered in 1954 by Segre [1,2].

Generalising, Segre considered sets of k points in the n -dimensional projective space $\text{PG}(n, q)$, $k \geq 3$ and $n \geq 2$, no three of which are collinear. For $n = 2$, such sets are k -arcs of $\text{PG}(2, q)$; for $n \geq 3$, these sets are k -caps of $\text{PG}(n, q)$. Further, Segre considers sets of k points in $\text{PG}(n, q)$, $k \geq n + 1$, no $n + 1$ of which lie in a hyperplane; these sets are k -arcs of $\text{PG}(n, q)$. There is a strong relation between arcs and algebraic curves, algebraic hypersurfaces and linear maximum-distance separable (MDS) codes.

Arcs and caps can be generalised by replacing their points with r -dimensional subspaces to obtain *generalised k -arcs* and *generalised k -caps* of $\text{PG}(m, q)$ [3]. These have strong connections to generalised quadrangles, projective planes, circle geometries, strongly regular graphs, finite groups and linear projective two-weight codes. In this survey, results and problems concerning these objects are discussed. The focus is on generalised ovals and generalised ovoids.

There is an enormous amount of material and results on arcs and caps in finite projective spaces. On these, just a few important definitions and results are mentioned. The emphasis is on some bounds, in particular on general bounds that hold for all values of the size q of the field, up to parity of q and a few exceptional small values. Some of these bounds, for example, Theorems 21 and 22, cannot be found in earlier surveys or books. Many interesting and strong results on arcs are contained in [4–6].

The main part of this paper is on generalised ovals and generalised ovoids. Many characterisations and classifications are given. The last part is focussed on the relations between certain generalised ovoids and finite translation generalised quadrangles of order (s, s^2) . Also, the relationship between Moufang generalised quadrangles, generalised ovoids



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and the theorem of Fong and Seitz on groups with a BN-pair of rank 2 is explained; see the recent paper [7].

Finally, several open problems are stated.

2. Arcs, Ovals and Hyperovals in $\text{PG}(2, q)$

- Definition 1.** (1) A k -arc in $\text{PG}(2, q)$ is a set of k points, with $k \geq 3$, such that no three of its points lie on a line.
 (2) An arc $\mathcal{K}$ is complete if it is not properly contained in a larger arc.
 (3) If $\mathcal{K} \cup \{P\}$ is an arc for a point P that is not in $\mathcal{K}$, then P extends $\mathcal{K}$.

Theorem 1 ([8] (Chapter 8)). Let $\mathcal{K}$ be a k -arc of $\text{PG}(2, q)$. Then,

- (i) $k \leq q + 2$;
- (ii) For q odd, $k \leq q + 1$;
- (iii) Any non-singular conic is a $(q + 1)$ -arc;
- (iv) For q even, a $(q + 1)$ -arc extends to a $(q + 2)$ -arc.

Definition 2. In $\text{PG}(2, q)$,

- (1) a $(q + 1)$ -arc is an oval;
- (2) a $(q + 2)$ -arc, q even, is a complete oval or hyperoval.

Remark 1. In [8] (Chapter 8), the definition of an oval differs slightly from the one given here. In [8], an oval of $\text{PG}(2, q)$ is a k -arc with $k = q + 1$ for q odd and $k = q + 2$ for q even.

Theorem 2 ([8–10]). In $\text{PG}(2, q)$, q odd, every oval is a non-singular conic.

Remark 2. For q even, a non-singular conic extends to a hyperoval $\mathcal{K}$. For $q \geq 8$, let $\mathcal{K} = C \cup \{P\}$, with C being a non-singular conic. If $P' \in C$, then $\mathcal{K} \setminus \{P'\}$ is an oval that is not a conic; this follows from the fact that two distinct non-singular conics have at most four points in common. Hence, for q even and $q \geq 8$, not every oval is a conic. Also, for q even and $q > 8$, there are many hyperovals that do not contain a conic; see [8].

Theorem 3 ([8–10]). (i) For q even, a k -arc with

$$k > q - \sqrt{q} + 1$$

extends to a hyperoval.

(ii) For q odd, a k -arc with

$$k > q - \frac{1}{4}\sqrt{q} + \frac{25}{16}$$

extends to an oval.

Open problem 1. Classify all ovals and hyperovals for q even.

3. Arcs in $\text{PG}(n, q)$, $n \geq 3$

Definition 3. (1) A k -arc in $\text{PG}(n, q)$ is a set $\mathcal{K}$ of k points, with $k \geq n + 1 \geq 3$, such that no $n + 1$ of its points lie in a hyperplane.

- (2) An arc $\mathcal{K}$ is complete if it is not properly contained in a larger arc.
- (3) Let $m(n, q)$ be the maximum size of a k -arc in $\text{PG}(n, q)$.
- (4) A normal rational curve of $\text{PG}(n, q)$, $n \geq 2$, is any set of points in $\text{PG}(n, q)$ that is projectively equivalent to

$$\{\mathbf{P}(t^n, t^{n-1}, \dots, t, 1) \mid t \in \mathbf{F}_q\} \cup \{\mathbf{P}(1, 0, \dots, 0)\}.$$

- Definition 4.** (1) With $V(m, q)$, the vector space of m dimensions over $\mathbf{F}_q$, a linear code C is a subspace of $V(m, q)$.
- (2) C is an $[m, k, d]$ or an $[m, k, d]_q$ code if it has dimension k and minimum distance d , where the distance between distinct vectors $(x_1, x_2, \dots, x_m)$ and $(y_1, y_2, \dots, y_m)$ of C is the number of indices i for which $x_i \neq y_i$ and with d the minimum of these distances; here, $d \leq m - k + 1$.
- (3) C is maximum distance separable (MDS) if $d = m - k + 1$.

Theorem 4. For $m \geq 3$, a MDS $[m, k, d]$ code C is equivalent to a k -arc of $\text{PG}(m - 1, q)$.

Proof. Let C be an m -dimensional subspace of $V(k, q)$ and let G be an $m \times k$ generator matrix for C , that is, the rows of G form a basis for C . Then, C is a MDS if and only if any m columns of G are linearly independent; this property is preserved under multiplication of the columns by non-zero scalars. So, consider the columns of G as points $P_1, P_2, \dots, P_k$ of $\text{PG}(m - 1, q)$. It follows that C is a MDS if and only if $\{P_1, P_2, \dots, P_k\}$ is a k -arc of $\text{PG}(m - 1, q)$. $\square$

Theorem 5 ([9]; Kaneta and Maruta [11]). For $\text{PG}(n, q)$, take q odd and $n \geq 3$.

- (i) If $\mathcal{K}$ is a k -arc with $k > q - \frac{1}{4}\sqrt{q} + n - \frac{1}{4}$, then $\mathcal{K}$ lies on a unique normal rational curve.
- (ii) If $q > (4n - 5)^2$, every $(q + 1)$ -arc is a normal rational curve.
- (iii) If $q > (4n - 9)^2$, then $m(n, q) = q + 1$.

Theorem 6 ([10,12–14]). (i) For q even, $q \neq 2$, $n \geq 3$, if $\mathcal{K}$ is a k -arc in $\text{PG}(n, q)$ with

$$k > q - \frac{1}{2}\sqrt{q} + n - \frac{3}{4},$$

then $\mathcal{K}$ lies on a unique $(q + 1)$ -arc of $\text{PG}(n, q)$.

- (ii) A $(q + 1)$ -arc in $\text{PG}(n, q)$, q even and $n \geq 4$, with

$$q > (2n - \frac{7}{2})^2,$$

is a normal rational curve.

- (iii) If $\mathcal{K}$ is a k -arc in $\text{PG}(n, q)$, q even and $n \geq 4$, with

$$q > (2n - \frac{11}{2})^2,$$

then $k \leq q + 1$.

Remark 3. There are close relationships between k -arcs, algebraic curves and algebraic hypersurfaces.

4. Caps and Ovoids

Definition 5. (1) In $\text{PG}(n, q)$, $n \geq 3$, a set $\mathcal{K}$ of k points, no three of which are collinear, is a k -cap.

- (2) A k -cap is complete if it is not contained in a $(k + 1)$ -cap.
- (3) A line of $\text{PG}(n, q)$ is a secant, tangent or external line as it meets $\mathcal{K}$ in 2, 1 or 0 points.

4.1. Caps and Ovoids in $\text{PG}(3, q)$

Theorem 7 ([15] (Chapter 16); Bose [16]; Qvist [17]). (i) For a k -cap in $\text{PG}(3, q)$ with $q \neq 2$,

$$k \leq q^2 + 1.$$

- (ii) A k -cap in $\text{PG}(3, 2)$ has the bound $k \leq 8$; an 8-cap is the complement of a plane.
- (iii) Each elliptic quadric of $\text{PG}(3, q)$ is a $(q^2 + 1)$ -cap.

Definition 6. A $(q^2 + 1)$ -cap of $\text{PG}(3, q)$, $q \neq 2$, is an ovoid; for $q = 2$, an ovoid is a set of 5 points, no 4 of which are coplanar.

Theorem 8. In $\text{PG}(3, 2)$, a complete cap is either an ovoid, which is an elliptic quadric, or an 8-cap, which is the complement of a plane.

Theorem 9 ([15] (Chapter 16); Barlotti [18]; Panella [19]). In $\text{PG}(3, q)$, q odd, an ovoid is an elliptic quadric.

Theorem 10 (Brown [20]). In $\text{PG}(3, q)$, q even, an ovoid containing at least one conic section is an elliptic quadric.

Theorem 11 (Tits [21]). (i) For $q = 2^{2e+1}$, $e \geq 1$, the space $\text{PG}(3, q)$ has ovoids that are not elliptic quadrics. These are the Tits ovoids.

(ii) With $q = 2^{2e+1}$, the canonical form of a Tits ovoid $\mathcal{O}$ is the following:

$$\mathcal{O} = \{P(1, z, y, x) \mid z = xy + x^{\sigma+2} + y^{\sigma}\} \cup \{P(0, 1, 0, 0)\},$$

where σ is the automorphism $t \mapsto t^{2^{e+1}}$ of $\mathbb{F}_q$.

Remark 4. (1) For q even, the only ovoids known are the elliptic quadrics and the Tits ovoids.

(2) For $q = 4$, an ovoid of $\text{PG}(3, 4)$ is an elliptic quadric; see Barlotti [18] or [15].

(3) For $q = 8$, an ovoid is an elliptic quadric or a Tits ovoid; see Segre [22] and Fellegara [23].

(4) For $q = 16, 32$, all ovoids were determined by O’Keefe, Penttila and Royle; see [9,24–26].

(5) For $q = 64$, an ovoid of $\text{PG}(3, 64)$ is an elliptic quadric; see Penttila [27].

Remark 5. For the influence and many applications of the paper of Tits [21], see the recent paper [28] by Thas and Van Maldeghem.

Open problem 2. Determine all ovoids in $\text{PG}(3, q)$ for q even.

4.2. Caps in $\text{PG}(n, q)$, $n \geq 3$

Definition 7. $m_2(n, q)$ is the maximum size of a k -cap in $\text{PG}(n, q)$.

Theorem 12 (Hill [29]). (i) $m_2(n, q) \leq qm_2(n-1, q) - (q+1)$, for $n \geq 4, q > 2$.

(ii) $m_2(n, q) \leq q^{n-4}m_2(4, q) - q^{n-4} - 2\frac{q^{n-4}-1}{q-1} + 1$, for $n \geq 5, q > 2$.

Remark 6. These results were obtained using the theory of cap-codes.

Exact values of $m_2(n, q)$ are known in just a few cases.

Theorem 13. (i) (Bose [16]) $m_2(n, 2) = 2^n$; a 2^n -cap of $\text{PG}(n, 2)$ is the complement of a hyperplane.

(ii) (Pellegrino [30]) $m_2(4, 3) = 20$; there are nine projectively distinct 20-caps in $\text{PG}(4, 3)$.

(iii) (Hill [31]) $m_2(5, 3) = 56$; the 56-cap in $\text{PG}(5, 3)$ is projectively unique.

(iv) (Edel and Bierbrauer [32]) $m_2(4, 4) = 41$; there exist two projectively distinct 41-caps in $\text{PG}(4, 4)$.

Remark 7. No other values of $m_2(n, q)$, $n > 3$, are known.

Several bounds were obtained for the number k , for which there exist complete k -caps in $\text{PG}(3, q)$ that are not ovoids; these bounds are then used to determine bounds for

$m_2(n, q)$, with $n > 3$. Here are a few good bounds, without restrictions on q except for a few small cases.

Theorem 14 (Meshulam [33]). For $n \geq 4$, $q = p^h$ and p is an odd prime,

$$m_2(n, q) \leq \frac{nh+1}{(nh)^2} q^n + m_2(n-1, q). \quad (1)$$

Theorem 15 ([15] (Chapter 18)). In $\text{PG}(3, q)$, q odd and $q \geq 67$, if $\mathcal{K}$ is a complete k -cap that is not an elliptic quadric, then

$$k < q^2 - \frac{1}{4}q^{\frac{3}{2}} + 2q.$$

More precisely,

$$k \leq q^2 - \frac{1}{4}q^{\frac{3}{2}} + R(q),$$

where

$$R(q) = \frac{(31q + 14\sqrt{q} - 53)}{16}.$$

Definition 8. Let $m'_2(2, q)$ be the size of the second largest complete arc of $\text{PG}(2, q)$ and let $m'_2(3, q)$ be the size of the second largest cap of $\text{PG}(3, q)$.

Nagy and Szőnyi [34] follow more or less the line of the proof of Theorem 15, and derive a bound for $m'_2(3, q)$ in terms of $m'_2(2, q)$. Their bound involves a more careful enumeration of certain plane sections of a large cap; so it yields an improvement on the bounds in Theorem 15.

Theorem 16 (Nagy and Szőnyi [34]). If, for q odd, $m'_2(2, q) \geq (5q + 19)/6$, then

$$m'_2(3, q) < qm'_2(2, q) + \frac{3}{4}(q + \frac{10}{3} - m'_2(2, q))^2 - q - 1.$$

Theorem 17 ([35]). In $\text{PG}(3, q)$, q even and $q \geq 8$, if $\mathcal{K}$ is a complete k -cap that is not an ovoid, then

$$k < q^2 - (\sqrt{5} - 1)q + 5.$$

Remark 8. Combining the previous theorem with the main theorem of Storme and Szőnyi [36], there is an improvement in the previous result. This important remark is due to Szőnyi.

Theorem 18 ([35]). In $\text{PG}(3, q)$, q even and $q \geq 2048$, if $\mathcal{K}$ is a complete k -cap that is not an ovoid, then,

$$k < q^2 - 2q + 3\sqrt{q} + 2.$$

Relying on Theorems 12 and 15, the following result is obtained.

Theorem 19 ([9,37]). In $\text{PG}(n, q)$, $n \geq 4$, $q \geq 197$ and odd,

$$m_2(n, q) < q^{n-1} - \frac{1}{4}q^{n-\frac{3}{2}} + 2q^{n-2}.$$

For $n \geq 4$, $q \geq 67$ and odd,

$$m_2(n, q) < q^{n-1} - q^{n-\frac{3}{2}} + \frac{1}{16}(31q^{n-2} + 22q^{n-\frac{5}{2}} - 112q^{n-3} - 14q^{n-\frac{7}{2}} + 69q^{n-4}) - 2(q^{n-5} + q^{n-6} + \dots + q + 1),$$

where there is no term $-2(q^{n-5} + q^{n-6} + \dots + q + 1)$ for $n = 4$.

Relying on Nagy and Szőnyi [34], the following improvement of Theorem 19 is obtained.

Theorem 20 (Storme, Thas and Vereecke [38]). *If, for q odd,*

$$m'_2(2, q) \geq \frac{5q + 25}{6} \text{ and } m_2(4, q) > \frac{41q^3 + 202q^2 - 47q}{48},$$

then

$$m_2(4, q) < (q + 1)(qm'_2(2, q) + \frac{3}{4}(q + \frac{10}{3} - m'_2(2, q))^2 - q - 1 - m'_2(2, q)) + m'_2(2, q).$$

Bounds for $m_2(n, q)$, $n > 4$ and q odd, can now be calculated using Hill's Theorem 12.

Remark 9. In [38], small improvements of Theorem 12 are obtained.

Relying on Theorems 12 and 18, the following results are obtained.

Theorem 21 ([39]). (i) $m_2(4, 8) \leq 479$.

(ii) $m_2(4, q) < q^3 - q^2 + 2\sqrt{5}q - 8$, q even, $q > 8$.

(iii) $m_2(4, q) < q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 2$, q even, $q \geq 2048$.

Theorem 22 ([39]). *For q even, $q > 2$, $n \geq 5$,*

(i) $m_2(n, 4) \leq \frac{118}{3}4^{n-4} + \frac{5}{3}$;

(ii) $m_2(n, 8) \leq 478.8^{n-4} - 2(8^{n-5} + 8^{n-6} + \dots + 8 + 1) + 1$;

(iii) $m_2(n, q) < q^{n-1} - q^{n-2} + 2\sqrt{5}q^{n-3} - 9q^{n-4} - 2(q^{n-5} + q^{n-6} + \dots + 1) + 1$, for $q > 8$;

(iv) $m_2(n, q) < q^{n-1} - 2q^{n-2} + 3q^{n-3}\sqrt{q} + 8q^{n-3} - 9q^{n-4}\sqrt{q} - 7q^{n-4} - 2(q^{n-5} + q^{n-6} + \dots + q + 1) + 1$,
for $q \geq 2048$.

Remark 10. For a survey on caps, see Hirschfeld and Storme [40].

5. Generalised Ovals

5.1. Introduction

Arcs, ovals and hyperovals can be generalised by replacing their points with $(n - 1)$ -dimensional subspaces, $n \geq 1$, to get *generalised k -arcs*, *pseudo-ovals* and *pseudo-hyperovals*.

These objects were defined in 1971 by Thas [3]. In 1973 [41,42], the relation between pseudo-ovals and generalised quadrangles was discovered. In 1974, Thas [41,42] showed that these pseudo-ovals play a key role in the theory of translation generalised quadrangles with the same number of points and lines.

5.2. Generalised k -Arcs

Definition 9. (1) A generalised k -arc in $\text{PG}(3n - 1, q)$ is a set $\mathcal{K}$ of $(n - 1)$ -dimensional subspaces, with $|\mathcal{K}| = k \geq 3$, such that no three of its elements lie in a hyperplane.

(2) $\mathcal{K}$ is complete if it is not properly contained in a larger generalised arc.

Example 1. (1) For $n = 1$, then the k -arcs of $\text{PG}(2, q)$ arise.

(2) For $n = 2$, then $\mathcal{K}$ is a set of k lines in $\text{PG}(5, q)$ such that every three generate the space.

Theorem 23 ([3]). (i) For every generalised k -arc in $\text{PG}(3n - 1, q)$,

(a) $k \leq q^n + 2$;

(b) $k \leq q^n + 1$ when q is odd.

(ii) In $\text{PG}(3n - 1, q)$, there exist $(q^n + 1)$ -arcs for every q ; for q even, there exist $(q^n + 2)$ -arcs.

- (iii) If O is a generalised $(q^n + 1)$ -arc in $\text{PG}(3n - 1, q)$, then each element π_i of O is contained in exactly one $(2n - 1)$ -dimensional subspace τ_i that is disjoint from all elements of $O \setminus \{\pi_i\}$; here, τ_i is the tangent space of O at π_i .
- (iv) For q even, all tangent spaces of a generalised $(q^n + 1)$ -arc O of $\text{PG}(3n - 1, q)$ contain a common $(n - 1)$ -dimensional space π , the nucleus of O . Hence, O is not complete and extends to a generalised $(q^n + 2)$ -arc by adding its nucleus.

5.3. Pseudo-Ovals and Pseudo-Hyperovals

- Definition 10.** (1) A generalised $(q^n + 1)$ -arc of $\text{PG}(3n - 1, q)$ is a generalised oval or pseudo-oval or $[n - 1]$ -oval of $\text{PG}(3n - 1, q)$. For $n = 1$, a pseudo-oval is just an oval of $\text{PG}(2, q)$.
- (2) With q even, a generalised $(q^n + 2)$ -arc of $\text{PG}(3n - 1, q)$ is a generalised hyperoval or pseudo-hyperoval or $[n - 1]$ -hyperoval of $\text{PG}(3n - 1, q)$. For $n = 1$, a pseudo-hyperoval is just a hyperoval of $\text{PG}(2, q)$.

- Theorem 24** (Payne and Thas [42]). (i) In $\text{PG}(3n - 1, q)$, each hyperplane not containing a tangent space of the pseudo-oval O contains either 0 or 2 elements of O . When q is even, each hyperplane contains either 0 or 2 elements of a pseudo-hyperoval.
- (ii) For q odd, each point of $\text{PG}(3n - 1, q)$ not contained in an element of the pseudo-oval O belongs to either 0 or 2 tangent spaces of O .

5.4. Regular Pseudo-Ovals and Regular Pseudo-Hyperovals

In the extension $\text{PG}(3n - 1, q^n)$ of the space $\text{PG}(3n - 1, q)$, take n planes ζ_i , $i = 1, 2, \dots, n$, which are conjugate for the extension $\mathbb{F}_{q^n}$ of $\mathbb{F}_q$ and which span $\text{PG}(3n - 1, q^n)$. Thus, they form an orbit of the Galois group corresponding to the extension and span $\text{PG}(3n - 1, q^n)$. In ζ_1 , consider an oval

$$O_1 = \{P_0^{(1)}, P_1^{(1)}, \dots, P_{q^n}^{(1)}\}$$

or a hyperoval

$$O'_1 = \{P_0^{(1)}, P_1^{(1)}, \dots, P_{q^n+1}^{(1)}\}.$$

Next, for $i = 0, 1, \dots, q^n$ or $i = 0, 1, \dots, q^n + 1$, let $P_i^{(1)}, P_i^{(2)}, \dots, P_i^{(n)}$ be conjugate in $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. These points define an $(n - 1)$ -dimensional subspace π_i over $\mathbb{F}_q$. Consequently, $O = \{\pi_0, \pi_1, \dots, \pi_{q^n}\}$ is a pseudo-oval and $O' = \{\pi_0, \pi_1, \dots, \pi_{q^n+1}\}$ is a pseudo-hyperoval of $\text{PG}(3n - 1, q^n)$.

These are the *regular* or *elementary* pseudo-ovals and the *regular* or *elementary* pseudo-hyperovals of $\text{PG}(3n - 1, q)$. If O_1 is a conic in $\text{PG}(2, q^n)$, then the corresponding pseudo-oval is a *classical pseudo-oval* or *pseudo-conic*.

Alternatively, let V be the vector space over $\mathbb{F}_{q^n}$ underlying the projective plane $\text{PG}(2, q^n)$. If V is considered as an $\mathbb{F}_q$ -vector space, each point of $\text{PG}(2, q^n)$ becomes an $(n - 1)$ -dimensional subspace of $\text{PG}(3n - 1, q)$. If O_1 is an oval or hyperoval of $\text{PG}(2, q^n)$, then here it becomes a regular pseudo-oval or regular pseudo-hyperoval of $\text{PG}(3n - 1, q)$.

Remark 11. Every known pseudo-oval and pseudo-hyperoval is regular. By Segre's theorem, for q odd every regular pseudo-oval is a pseudo-conic.

Open problem 3. Is every pseudo-oval regular? Is every pseudo-hyperoval regular?

Theorem 25 (Payne and Thas [42]). For q odd, the tangent spaces of a pseudo-oval O in $\text{PG}(3n - 1, q)$ form a pseudo-oval O^* in the dual space of $\text{PG}(3n - 1, q)$.

Definition 11. The pseudo-oval O^* is the translation dual of the pseudo-oval O .

Open problem 4. For q odd, is every pseudo-oval O isomorphic to its translation dual?

6. Characterisations

6.1. Pseudo-Ovals, Pseudo-Hyperovals and Spreads

Let $O = \{\pi_0, \pi_1, \dots, \pi_{q^n}\}$ be a pseudo-oval in $\text{PG}(3n-1, q)$. The tangent space of O at π_i is τ_i . Choose $i \in \{0, 1, \dots, q^n\}$ and let Π_{2n-1} be skew to π_i . Further, let $\tau_i \cap \Pi_{2n-1} = \eta_i$ and $\langle \pi_i, \pi_j \rangle \cap \Pi_{2n-1} = \eta_j$ for $j \neq i$; here, $\langle \pi_i, \pi_j \rangle$ is the subspace generated by π_i and π_j . Then, $\{\eta_0, \eta_1, \dots, \eta_{q^n}\} = \Delta_i$ is an $(n-1)$ -spread of Π_{2n-1} , that is, the elements of Δ_i partition Π_{2n-1} .

Now, let q be even and let π be the nucleus of O . Let $\Pi_{2n-1} \subset \text{PG}(3n-1, q)$ be skew to π . If $\xi_i = \Pi_{2n-1} \cap \langle \pi, \pi_i \rangle$, then $\{\xi_0, \xi_1, \dots, \xi_{q^n}\} = \Delta$ is an $(n-1)$ -spread of Π_{2n-1} .

Next, let q be odd. Choose τ_i for $i \in \{0, 1, \dots, q^n\}$. If $\tau_i \cap \tau_j = \delta_j$ with $j \neq i$, then

$$\{\delta_0, \delta_1, \dots, \delta_{i-1}, \pi_i, \delta_{i+1}, \dots, \delta_{q^n}\} = \Delta_i^*$$

is an $(n-1)$ -spread of τ_i .

Definition 12. Let V be the 2-dimensional vector space that defines the projective line $\text{PG}(1, q^n)$. Considering V as an $\mathbb{F}_q$ -vector space, each point of $\text{PG}(1, q^n)$ becomes an $(n-1)$ -dimensional subspace of $\text{PG}(2n-1, q)$. The $(n-1)$ -spread of $\text{PG}(2n-1, q)$ consisting of these $q^n + 1$ subspaces is a regular spread of $\text{PG}(2n-1, q)$.

Theorem 26 (Casse, Thas and Wild [41,42]). Let O be a pseudo-oval of $\text{PG}(3n-1, q)$, with q odd. Then, at least one of the $(n-1)$ -spreads

$$\Delta_0, \Delta_1, \dots, \Delta_{q^n}, \Delta_0^*, \Delta_1^*, \dots, \Delta_{q^n}^*$$

is regular if and only if O is regular, that is, if and only if O is a pseudo-conic.

Theorem 27 (Rottey, Van de Voorde [43,44]). Let O be a pseudo-oval in $\text{PG}(3n-1, q)$, with $q = 2^h$, $h > 1$, n prime. Then, all the $(n-1)$ -spreads $\Delta_0, \Delta_1, \dots, \Delta_{q^n}$ are regular if and only if O is regular.

Open problem 5. From this theorem, the following questions arise.

- I. What happens when $q = 2$?
- II. What happens when n is not prime?
- III. What happens when not all spreads Δ_i are regular?
- IV. What happens when at least one of the spreads Δ_i is regular?

Remark 12. In [45], a shorter proof of Theorem 27 is given and a slightly stronger result is obtained. Metsch and Van de Voorde [46] used the considerations in [45] to prove that it is sufficient to assume that at least $q^n - q + 3$ spreads are regular.

Definition 13. In $\text{PG}(3n-1, q)$, let π_1, π_2, π_3 be mutually skew $(n-1)$ -dimensional subspaces. Also, let τ_i be a $(2n-1)$ -dimensional subspace containing π_i but skew to π_j and π_k , and let $\tau_i \cap \tau_j = \eta_k$ with $\{i, j, k\} = \{1, 2, 3\}$. The subspace generated by η_i and π_i is ζ_i . If the $(2n-1)$ -dimensional spaces $\zeta_1, \zeta_2, \zeta_3$ have an $(n-1)$ -dimensional subspace in common, then $\{\pi_1, \pi_2, \pi_3\}$ and $\{\tau_1, \tau_2, \tau_3\}$ are in perspective.

Theorem 28 ([47]). Consider a pseudo-oval $O = \{\pi_0, \pi_1, \dots, \pi_{q^n}\}$ of $\text{PG}(3n-1, q)$, q odd, and let τ_i be the tangent space of O at π_i for each i . If, for any three distinct i, j, k , the triples $\{\pi_i, \pi_j, \pi_k\}$ and $\{\tau_i, \tau_j, \tau_k\}$ are in perspective, then O is a pseudo-conic. The converse also holds.

Remark 13. By Segre [1,2], for $n = 1$ and q odd, the triples $\{\pi_i, \pi_j, \pi_k\}$ and $\{\tau_i, \tau_j, \tau_k\}$ are always in perspective, and so O is a conic. Hence, for q odd, every oval is a conic. To prove that the two triples are in perspective, Segre uses his famous Lemma of Tangents; see Lemma 8.11 of [8]. What happens for $n > 1$?

6.2. The Case $n = 2$

For $n = 2$, a pseudo-oval O consists of $q^2 + 1$ lines of $\text{PG}(5, q)$, every three of which generate the space.

Theorem 29 (Shult and Thas [48]). If the pseudo-oval O is contained in a non-singular hyperbolic quadric $\mathcal{H}(5, q)$, with q odd, then O is a pseudo-conic.

Let O be a pseudo-oval contained in a non-singular elliptic quadric $\mathcal{E}(5, q)$ of $\text{PG}(5, q)$ with q odd. It can be shown that O is equivalent to a set of $q^2 + 1$ points on the non-singular Hermitian variety $\mathcal{U}(3, q^2)$ of $\text{PG}(3, q^2)$, with the property that no three of them are in a common tangent plane of the variety; see, for example, Payne and Thas [42] or Shult [49].

Any pseudo-conic O of $\text{PG}(5, q)$, q odd, is the intersection of a non-singular hyperbolic quadric $\mathcal{H}(5, q)$ and a non-singular elliptic quadric $\mathcal{E}(5, q)$.

Bamberg, Monzillo and Siciliano [50] showed that a pseudo-oval on $\mathcal{E}(5, q)$, q odd, is a subset of a five-class association scheme, defined on certain line sets of $\mathcal{E}(5, q)$. Pseudo-ovals and pseudo-conics are analysed in terms of these association schemes.

Remark 14. For q even, a pseudo-oval O of $\text{PG}(5, q)$ is never contained in a non-singular quadric, since all tangent spaces of O contain a common line; see Shult and Thas [48] and Thas [47].

Open problem 6. Is each pseudo-oval on $\mathcal{E}(5, q)$, q odd, a pseudo-conic?

7. Generalised Ovoids

7.1. Introduction

Ovoids can be generalised by replacing their points with $(n-1)$ -dimensional spaces, $n \geq 1$, to obtain *generalised ovoids*. This generalisation was first considered in 1971 by Thas [3]. However, this generalisation was too restricted and, in 1974 [41], it was shown that these generalised ovoids are always of a very particular kind. The appropriate definition of a generalised ovoid appeared in *Finite Generalized Quadrangles* by Payne and Thas [42].

7.2. Pseudo-Ovoids

In $\Omega = \text{PG}(4n-1, q)$, let O be a set of $(n-1)$ -dimensional subspaces π_i , $i = 0, 1, \dots, q^{2n}$, such that

- (a) every three generate a Π_{3n-1} ;
- (b) for every $i \in \{0, 1, \dots, q^{2n}\}$, there is a $(3n-1)$ -dimensional subspace τ_i that contains π_i and is disjoint from π_j for $j \neq i$.

The space τ_i is the *tangent space* of O at π_i ; it is uniquely defined by O and π_i .

Definition 14. The set O is a *generalised ovoid* or a *pseudo-ovoid* or an *egg* or an $[n-1]$ -ovoid of $\text{PG}(4n-1, q)$.

Example 2. (1) When $n = 1$, the ovoids of $\text{PG}(3, q)$ arise; the tangent spaces are planes.

- (2) When $n = 2$, a pseudo-ovoid of $\text{PG}(7, q)$ contains $q^4 + 1$ lines; the tangent spaces are 5-dimensional.

Theorem 30 (Payne and Thas [42]). (i) Each hyperplane of $\text{PG}(4n - 1, q)$ not containing a tangent space of the pseudo-ovoid O contains exactly $q^n + 1$ elements of O .

- (ii) Each point which is not contained in an element of O is contained in exactly $q^n + 1$ tangent spaces.

Corollary 1. (i) Let $\tilde{O}$ be the union of all elements of a pseudo-ovoid O in the space $\text{PG}(4n - 1, q)$ and let Π be any hyperplane. Then $|\tilde{O} \cap \Pi| \in \{\gamma_1, \gamma_2\}$, with

$$\gamma_1 = \frac{(q^n - 1)(q^{2n-1} + 1)}{q - 1}, \quad \gamma_1 - \gamma_2 = q^{2n-1}.$$

- (ii) (Delsarte [51]) Hence $\tilde{O}$ defines a projective 2-weight linear code and a strongly regular graph.

7.3. Regular Pseudo-Ovoids

In the extension $\text{PG}(4n - 1, q^n)$ of $\text{PG}(4n - 1, q)$, consider n solids ξ_i , $i = 1, 2, \dots, n$, that are conjugate in the extension $\mathbb{F}_{q^n}$ of $\mathbb{F}_q$ and which span $\text{PG}(4n - 1, q)$. This means that they form an orbit of the Galois group corresponding to this extension and span $\text{PG}(4n - 1, q^n)$.

In the space ξ_1 , take an ovoid $O_1 = \{P_0^{(1)}, P_1^{(1)}, \dots, P_{q^{2n}-1}^{(1)}\}$. Next, let $P_i^{(1)}, P_i^{(2)}, \dots, P_i^{(n)}$, $i = 0, 1, \dots, q^{2n}$, be conjugate in $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. The points $P_i^{(1)}, P_i^{(2)}, \dots, P_i^{(n)}$ now define an $(n - 1)$ -dimensional subspace π_i over $\mathbb{F}_q$ for each $i = 0, 1, \dots, q^{2n}$. It follows that the ovoid $O = \{\pi_0, \pi_1, \dots, \pi_{q^{2n}}\}$ is a pseudo-ovoid of $\text{PG}(4n - 1, q)$.

These are the *regular* or *elementary* pseudo-ovoids. If O_1 is an elliptic quadric over $\mathbb{F}_{q^n}$, the corresponding pseudo-ovoid is *classical* or a *pseudo-quadric*.

Alternatively, let V be the 4-dimensional vector space underlying the projective space $\text{PG}(3, q^n)$. Considering V as an $\mathbb{F}_q$ -vector space, each point of $\text{PG}(3, q^n)$ becomes an $(n - 1)$ -dimensional subspace of $\text{PG}(4n - 1, q)$. If O_1 is an ovoid of $\text{PG}(3, q^n)$, then O_1 becomes a regular pseudo-ovoid of $\text{PG}(4n - 1, q)$.

Remark 15. For q even, every known pseudo-ovoid is regular. For q odd, there are pseudo-ovoids that are not regular. By the theorem of Barlotti and Panella, for q odd, every regular pseudo-ovoid is a pseudo-quadric.

Open problem 7. Is every pseudo-ovoid of $\text{PG}(4n - 1, q)$, q even, regular?

7.4. Translation Duals

Theorem 31 (Payne and Thas [42]). The tangent spaces of a pseudo-ovoid O of the space $\text{PG}(4n - 1, q)$ form a pseudo-ovoid O^* in the dual space of $\text{PG}(4n - 1, q)$.

Definition 15. The pseudo-ovoid O^* is the translation dual of the pseudo-ovoid O .

Open problem 8. For q even, is every pseudo-ovoid O of $\text{PG}(4n - 1, q)$ isomorphic to its translation dual?

Remark 16. For q odd, there are pseudo-ovoids that are not isomorphic to their translation dual.

7.5. Characterisations

Theorem 32 (Payne and Thas [42]). *The pseudo-ovoid O of $\text{PG}(4n - 1, q)$ is regular if and only if one of the following holds:*

- (i) *For any point P not contained in an element of O , the $q^n + 1$ tangent spaces containing P have exactly $(q^n - 1)/(q - 1)$ points in common;*
- (ii) *Each Π_{3n-1} that contains at least three elements of O contains exactly $q^n + 1$ elements of O .*

Theorem 33 (Brown and Lavrauw [52]). *A pseudo-ovoid O in $\text{PG}(4n - 1, q)$, q even, contains a pseudo-conic if and only if it is a pseudo-quadric.*

Open problem 9. Is a pseudo-ovoid of $\text{PG}(4n - 1, q)$, q even, containing a regular pseudo-oval always regular?

7.6. Good Pseudo-Ovoids

- Definition 16.** (i) *The pseudo-ovoid O in $\text{PG}(4n - 1, q)$ is good at its element π if any Π_{3n-1} containing π and at least two other elements of O contains exactly $q^n + 1$ elements of O .*
- (ii) *In this case, π is a good element of O , and O is also said to be good.*

A regular pseudo-ovoid is good at each of its elements.

Remark 17. Every known pseudo-ovoid or its translation dual is good.

Open problem 10. Is every pseudo-ovoid or its translation dual good?

Theorem 34 ([41]). *For q even, if the pseudo-ovoid O is good at the element π , then the translation dual O^* is good at the tangent space of O at π .*

Remark 18. For q odd, this theorem is not true.

Open problem 11. For q even, is every good pseudo-ovoid O of $\text{PG}(4n - 1, q)$ regular?

Theorem 35 ([41]). *Let O be a pseudo-ovoid of $\text{PG}(4n - 1, q)$, q even, that is good at $\pi \in O$. If the $q^{2n} + q^n$ pseudo-ovals on O containing π are regular, then O is regular.*

Now, the case that q is odd is considered.

Theorem 36 ([41]). *Let the pseudo-ovoid O of $\text{PG}(4n - 1, q)$, q odd, be good at π . Then, π is contained in exactly $q^{2n} + q^n$ pseudo-conics lying on O .*

Veronese surfaces play a key role in the theory of pseudo-ovoids.

Definition 17. In $\text{PG}(5, K)$, with K any field, in a suitable reference system, a Veronese surface $\mathcal{V}_2^4$ consists of the points

$$\mathbf{P}(x_0^2, x_1^2, x_2^2, x_1x_2, x_2x_0, x_0x_1)$$

with $x_0, x_1, x_2 \in K$ and not all zero.

- Theorem 37.** (i) *When $K = \mathbf{F}_q$, the surface $\mathcal{V}_2^4$ contains $q^2 + q + 1$ points and conics.*
- (ii) *Any two of the points are contained in just one of these conics.*

The planes containing the conics are the conic planes of $\mathcal{V}_2^4$. For more on Veronese surfaces, see Hirschfeld and Thas [37] (Chapter 4).

The classification of good pseudo-ovals in $\text{PG}(4n - 1, q)$, q odd, is now given.

Theorem 38 ([41]). Let $O = \{\pi, \pi_1, \pi_2, \dots, \pi_{q^{2n}}\}$ be a pseudo-oval in the space $\text{PG}(4n - 1, q)$, q odd, that is good at π . Then, there are three separate cases.

- (a) There exists Π_3 in $\text{PG}(4n - 1, q^n)$ which has exactly one point in common with the extension $\bar{\pi}$ of π to $\mathbb{F}_{q^n}$ and with the extensions $\bar{\pi}_i$ of π_i to $\mathbb{F}_{q^n}$ for $i = 1, 2, \dots, q^{2n}$. These $q^{2n} + 1$ points form an elliptic quadric in Π_3 and O is a pseudo-quadric.
- (b) There exists Π_4 in $\text{PG}(4n - 1, q^n)$ that intersects the extension $\bar{\pi}$ of π to $\mathbb{F}_{q^n}$ in a line ℓ and which has exactly one point R_i in common with each $\bar{\pi}_i$, $i = 1, 2, \dots, q^{2n}$. Further, let $\mathcal{W} = \{R_i \mid i = 1, 2, \dots, q^{2n}\}$ and let $\mathcal{M}$ be the set of all common points of ℓ and the conics that contain exactly q^n points of $\mathcal{W}$. Then, the set $\mathcal{M} \cup \mathcal{W}$ is the projection of a Veronese surface $\mathcal{V}_2^4$ from a point P in a conic plane θ of $\mathcal{V}_2^4$ onto a hyperplane Π_4 of the $\text{PG}(5, q^n)$ containing $\mathcal{V}_2^4$; the point P is an exterior point of the conic $\mathcal{V}_2^4 \cap \theta$. Here, O is a non-classical Kantor-Knuth pseudo-oval.
- (c) There exists Π_5 in $\text{PG}(4n - 1, q^n)$ that intersects the extension $\bar{\pi}$ of π in a plane μ and which has exactly one point R_i in common with each $\bar{\pi}_i$, $i = 1, 2, \dots, q^{2n}$. Let $\mathcal{W} = \{R_i \mid i = 1, 2, \dots, q^{2n}\}$ and let $\mathcal{C}$ be the set of all common points of μ and the conics that contain exactly q^n points of $\mathcal{W}$. In this case, $\mathcal{W} \cup \mathcal{C}$ is a Veronese surface in Π_5 .

Remark 19. (1) For more on Kantor-Knuth pseudo-ovals, see [41,53].

- (2) A Kantor-Knuth pseudo-oval is isomorphic to its translation dual. Conversely, when q is odd and the good pseudo-oval O is isomorphic to its translation dual, then O is classical or of Kantor-Knuth type; see Sections 4.12.4 and 5.1.4 of [41].
- (3) Each known example of Class (c) has $q = 3^h$.
- (4) Good pseudo-ovals play a key role in the theory of translation generalised quadrangles. They also give rise to new results for particular point-sets in classical polar spaces, as well as to the construction of new projective planes, new flocks of quadratic cones in $\text{PG}(3, q^n)$ and new semifields. For a detailed study of the relation between these objects, see Lunardon [54,55].

Open problem 12. If the pseudo-oval O of $\text{PG}(4n - 1, q)$, q odd, is good, but neither classical nor of Kantor-Knuth type, is q necessarily a power of 3? It is more difficult to classify all pseudo-ovals in $\text{PG}(4n - 1, q)$, q odd, in Case (c).

Theorem 39 (Blokhuis, Lavrauw and Ball [56]). Suppose that the good pseudo-oval O of $\text{PG}(4n - 1, q)$, q odd, satisfies the inequality

$$q \geq 4n^2 - 8n + 2.$$

Then, O is classical or of Kantor-Knuth type.

Open problem 13. Improve the inequality in this theorem.

The known examples of pseudo-ovals for q odd are the following [41].

- (i) Kantor-Knuth pseudo-ovals, including the classical ones.
- (ii) Ganley and Roman pseudo-ovals, for $q = 3^h$, $h \geq 1$. They are translation dual to each other; the Roman ones are due to Payne [57].
- (iii) The Penttila-Williams-Bader-Lunardon-Pinneri pseudo-oval and its translation dual, for $q = 3^5$.

8. Pseudo-Ovoids in $\text{PG}(2n + m - 1, q)$, $m \neq n$ and $n \geq 1$

Definition 18. In $\Omega = \text{PG}(2n + m - 1, q)$, $m \neq n$ and $n \geq 1$, define a set $O = O(n, m, q)$ of subspaces as follows. O is a set of $(n - 1)$ -dimensional subspaces π_i , $i = 0, 1, \dots, q^m$, such that the following applies:

- (i) Every three generate a Π_{3n-1} ;
- (ii) For every $i \in \{0, 1, \dots, q^m\}$, there is an $(m + n - 1)$ -dimensional subspace τ_i that contains π_i and is disjoint from π_j for $j \neq i$.

The space τ_i is the *tangent space* of O at π_i , and is uniquely defined by O and π_i .

Definition 19. The set O is a generalised ovoid or a pseudo-ovoid or an egg or an $[n - 1]$ -ovoid of $\text{PG}(2n + m - 1, q)$.

Remark 20. For $m = 2n$, the pseudo-ovoids of Section 7.2 are obtained.

Theorem 40 (Payne and Thas [42]). (i) $n < m \leq 2n$ for any $O(n, m, q)$.

(ii) $n(c + 1) = mc$, with $c \geq 1$ odd.

(iii) $m = 2n$ for q even.

Definition 20. $O(n, n, q)$ is a pseudo-oval in $\text{PG}(3n - 1, q)$.

Open problem 14. Does there exist an egg $O(n, m, q)$ for q odd and $m \neq 2n$?

9. Generalised Quadrangles and the Sets $O(n, m, q)$

In this section, the equivalence between translation generalised quadrangles and the sets $O(n, m, q)$ is explained.

Definition 21. A finite generalised quadrangle (GQ) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathbf{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint non-empty sets of points and lines for which $\mathbf{I}$ is a symmetric point-line incidence relation satisfying the following axioms:

- (i) Each point is incident with $t + 1$ lines, $t \geq 1$, and two distinct points are incident with at most one line;
- (ii) Each line is incident with $s + 1$ points, $s \geq 1$, and two distinct lines are incident with at most one point;
- (iii) If P is a point and ℓ is a line not incident with P , then there is always a unique pair $(Q, m) \subseteq \mathcal{P} \times \mathcal{B}$ for which $P \mathbf{I} m \mathbf{I} Q \mathbf{I} \ell$.

The integers s and t are the parameters of $\mathcal{S}$, which has order (s, t) ; if $s = t$, then $\mathcal{S}$ has order s .

There is a point-line duality for generalised quadrangles. This means that, in any definition or theorem, interchanging ‘point’ and ‘line’ as well as the parameters s and t gives a valid result.

Given two, not necessarily distinct, points P and Q of the generalised quadrangle $\mathcal{S}$, write $P \sim Q$ and say that P and Q are *collinear*, provided that there is some line ℓ for which $P \mathbf{I} \ell \mathbf{I} Q$; also, $P \not\sim Q$ means that P and Q are not collinear. Dually, for $\ell, m \in \mathcal{B}$, write $\ell \sim m$ or $\ell \not\sim m$ as ℓ and m are concurrent or not.

For $P \in \mathcal{P}$, put $P^\perp = \{Q \in \mathcal{P} \mid Q \sim P\}$; note that $P \in P^\perp$.

For terminology, notation and results on GQs, see the monograph [42] of Payne and Thas.

- Theorem 41.** (i) Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathbf{I})$ be a GQ of order (s, t) . If $|\mathcal{P}| = v$ and $|\mathcal{B}| = b$, then $v = (s + 1)(st + 1)$ and $b = (t + 1)(st + 1)$; see 1.2.1 of Chapter 1 in [42]. Also, $s + t$ divides $st(s + 1)(t + 1)$; see 1.2.2 of Chapter 1 in [42].
- (ii) If $s > 1$ and $t > 1$, then $t \leq s^2$ and dually $s \leq t^2$; see 1.2.3 of Chapter 1 in [42].

Definition 22. Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathbf{I})$ be a GQ of order (s, t) . If $P \sim Q$, $P \neq Q$, or if $P \not\sim Q$ and $|\{P, Q\}^{\perp\perp}| = t + 1$, then the pair $\{P, Q\}$ is regular. The point P is regular if $\{P, Q\}$ is regular for all $Q \in \mathcal{P}$, $Q \neq P$.

Theorem 42. If $\mathcal{S}$ contains a regular pair $\{P, Q\}$, then either $s = 1$ or $s \geq t$; see 1.3.6 of Chapter 1 in [42].

- Definition 23.** (i) Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathbf{I})$ be a GQ of order (s, t) , with $s \neq 1, t \neq 1$. A collineation θ in $\mathcal{S}$ is an elation about the point P if $\theta = I$ or if θ fixes all lines incident with P and fixes no point of $\mathcal{P} \setminus P^\perp$. If there is a group $\mathcal{H}$ of elations about P acting regularly on $\mathcal{P} \setminus P^\perp$, then $\mathcal{S}$ is an elation generalised quadrangle (EGQ) with elation group $\mathcal{H}$ and base point P .
- (ii) If the elation group $\mathcal{H}$ is abelian, then the EGQ is a translation generalised quadrangle (TGQ) with base point or translation point P and translation group $\mathcal{H}$. In this case, $\mathcal{H}$ is the set of all elations about P ; see 8.6.4 of Chapter 8 in [42]. For any TGQ, each line incident with the base point is regular; so $t \geq s$.

For a detailed study of TGQs, see the monograph [41] by Thas, Thas and Van Maldeghem.

Definition 24. The kernel $\mathbf{K}$ of the TGQ $\mathcal{S}$ is the field with multiplicative group isomorphic to the group of all collineations of $\mathcal{S}$ fixing line-wise the translation point P and any given point $Q \not\sim P$. From [42], $|\mathbf{K}| \leq s$.

Theorem 43. Let $\mathcal{S}$ be a TGQ of order (s, t) , $s \neq 1, t \neq 1$, with kernel $\mathbf{K}$. If $\mathbb{F}_q$ is a subfield of $\mathbf{K}$, then $\mathcal{S}$ corresponds to a set $O(n, m, q)$ with $s = q^n$, $t = q^m$. Conversely, to each set $O(n, m, q)$ corresponds a TGQ $\mathcal{S}$ of order (s, t) , with $s = q^n$, $t = q^m$, where $\mathbb{F}_q$ is a subfield of the kernel.

- Corollary 2.** (i) The theory of finite TGQs is equivalent to the theory of the sets $O(n, m, q)$.
- (ii) If $\mathcal{S}$ is a TGQ of order (s, t) , with $s \neq 1, t \neq 1, s = q^n, t = q^m$ and $n \neq m$, then, by Theorem 40, $n < m \leq 2n$, $n(c + 1) = mc$ with $c \geq 1$ odd, and $2n = m$ for q even.

10. Pseudo-Ovoids, Moufang Quadrangles and Fong-Seitz

This section contains recent developments on the relation between TGQs, sets $O(n, m, q)$, and finite groups.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathbf{I})$ be a GQ of order (s, t) . Let $P \in \mathcal{P}$ be fixed and define the following condition $(M)_P$.

- Definition 25.** (i) $(M)_P$: For any two distinct lines ℓ, ℓ' of $\mathcal{S}$ incident with P , the group of collineations of $\mathcal{S}$ fixing ℓ and ℓ' point-wise and P line-wise is transitive on the set of lines distinct from ℓ and incident with a given point Q on ℓ , with $Q \neq P$.
- (ii) (M) : $\mathcal{S}$ satisfies (M) if it satisfies $(M)_P$ for all $P \in \mathcal{P}$.
- (iii) $\mathcal{S}$ is a Moufang GQ if and only if it satisfies (M) and its dual (M') .

Theorem 44 (Tits [58]). Every finite Moufang GQ is classical or dual classical, and conversely.

The classical and dual classical GQs are those arising from a quadric, a Hermitian variety or a symplectic polarity.

Remark 21. Tits observes that this result follows from the classification in [59,60] of all finite groups with a BN-pair of rank 2 having a Weyl group D_8 .

Remark 22. In their monograph [42], Payne and Thas made an almost successful attempt to prove the Moufang theorem of Tits, in the finite case, in a geometric way. To complete a geometric proof of this result, it would suffice to show geometrically that each GQ of order (s, s^2) , $s \neq 1$, for which each point is a translation point, is necessarily the GQ arising from an elliptic quadric in $PG(5, s)$. As a corollary, a purely geometric proof of a large part of the theorem of Fong and Seitz would follow, namely in the case where the Weyl group is D_8 .

In [7], the following stronger result on GQs of order (s, s^2) is obtained.

Theorem 45 ([7]). Let S be a TGQ of order (s, s^2) , $s > 1$, having a regular line not incident with the translation point and having $\mathcal{O} = O(n, 2n, q)$ as corresponding pseudo-ovoid in $PG(4n - 1, q)$, $q \neq 2$. Then, $\mathcal{O}$ is good.

Then, relying on results of Brown, Lavrauw, Lunardon, Payne and Thas, the next theorem is obtained.

Theorem 46 ([7]). Let S be a TGQ of order (s, s^2) , $s > 1$, with translation point P and kernel $K \neq F_2$. If S has a regular line not incident with P , then the following hold:

- (i) If q is odd, then S is the point-line dual of the translation dual of a semifield flock TGQ;
- (ii) If q is even, then S is classical.

Remark 23 ([7]). Theorems 45 and 46 have many implications. For example, for $K \neq F_2$, the ‘missing part’ in [42], a purely geometric proof of the theorem of Tits and a large part of the theorem of Fong and Seitz.

Open problem 16. What happens in the case $K = F_2$?

11. Weak Generalised Ovoids

Definition 26. A weak generalised ovoid of $PG(4n - 1, q)$ is a set of $(n - 1)$ -dimensional subspaces, $q^{2n} + 1$ in number, such that any three generate a Π_{3n-1} .

Open problem 17. Is every weak generalised ovoid a generalised ovoid?

For results on weak generalised ovoids, see Rottey and Van de Voorde [43,44].

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Notations

$\mathbf{F}_q$	the finite field of order q
$\text{PG}(n, q)$	the projective space of n dimensions over $\mathbf{F}_q$
$\mathbf{P}(x_0, x_1, \dots, x_n)$	the point of $\text{PG}(n, q)$ with coordinate vector $(x_0, x_1, \dots, x_n)$
Π_r	a subspace of dimension r in $\text{PG}(n, q)$
$V(k, q)$	the vector space of k dimensions over $\mathbf{F}_q$

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