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Partial Sums of the Hurwitz and Allied Functions and Their Special Values

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Abstract: We supplement the formulas for partial sums of the Hurwitz zeta-function and its derivatives, producing more integral representations and generic definitions of important constants. Then, these are used, coupled with the functional equation for the completed zeta-function to clarify the results of Choudhury, giving rise to closed expressions for the Riemann zeta-function and its derivatives.

Keywords: Riemann zeta-function; Hurwitz zeta-function; integral representation for partial sums; functional equation; special values

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1. Introduction and Some Examples

Let $s = \sigma + it$ be the complex variable, and let

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

denote the Riemann zeta-function, where $\sigma > 1$ in the first instance. $\zeta(s)$ is analytically continued to a meromorphic function over the whole plane with a simple pole at $s = 1$ with residue 1. Cf. [1].

Let $X(s) := \frac{1}{\zeta(s)}$, A be the Glaisher–Kinkelin constant defined by (5), and let A_2 be the second Glaisher-Kinkelin constant defined by (6), respectively. We have some interesting special values as

$$X(2) = \frac{6}{\pi^2}, \quad X'(2) = -\frac{6}{\pi^2}(\gamma + \log(2\pi) - 12\log A), \tag{1}$$
$$X''(2) = \frac{-\zeta''(2)\zeta^2(2) + 2\zeta(2)(\zeta'(2))^2}{\zeta^4(2)} = -\frac{\zeta''(2)}{\zeta^2(2)} + 2\frac{\zeta'(2)}{\zeta(2)}\frac{\zeta'(2)}{\zeta^2(2)}.$$

By the functional equation of the logarithmic derivative of $\zeta(s)$, we also have

$$\zeta''(2) = \frac{\pi^2}{6}\left((\gamma + \log 2\pi)^2 - 24(\gamma + \log 2\pi - 1)\log A - 12A_2 + \frac{\pi^2}{12} - \frac{3}{4}\right). \tag{2}$$

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Substituting

$$\frac{\zeta'}{\zeta}(2) = \gamma + \log 2\pi - 12\log A,$$

and (2) in (1), we deduce the closed form

$$X''(2) = -\frac{6}{\pi^2}\left((\gamma + \log 2\pi)^2 - 24(\gamma + \log 2\pi - 1)\log A - 12A_2 + \frac{\pi^2}{12} - \frac{3}{4}\right)$$
$$+ 2\left(\frac{6}{\pi^2}\right)(\gamma + \log 2\pi - 12\log A)^2.$$

Our method consists of considering

$$Z(s) = \frac{\zeta'}{\zeta}(s) + \frac{1}{s-1},$$

and using the functional equation

$$Z(s) = -Z(1-s) + \log \pi - \frac{1}{2}\left(\psi\left(\frac{s}{2} + 1\right) + \psi\left(\frac{3}{2} - \frac{s}{2}\right)\right), \tag{3}$$

where ψ is the Euler digamma-function defined by (22) in the following Lemma 2. The introduction of the function $Z(s)$ is only for dealing with the most essential case of $s \to 1$; for other arguments, there is no need to use it, and one can just compute normally. It can be seen that (3) with s, a bigger value (and $1 - s$ a small value) is the most appropriate to ensure the convergence of the arising integral. We illustrate the case $s = 2$ of (3).

$$Z(2) = -\left(\frac{\zeta'}{\zeta}(-1) - \frac{1}{2}\right) + \log \pi - \frac{1}{2}\left(\psi(1) + 1 + \psi\left(\frac{1}{2}\right)\right). \tag{4}$$

Substituting (5) in (4), and noting (23), we deduce the third equality in (1).

As has been shown above, we often omit the steps regarding the digamma-function hereafter. We gather some basic formulas in Lemma 2.

It is interesting to see how the Glaisher–Kinkelin constant comes in the formula for $X'(2)$. For instance, see (28) with $u = 1$, $l = 3$, and $a = 1$, and $x \in \mathbb{N}$ (for simplicity) reads

$$\begin{aligned} &-\zeta'(-1) + \frac{1}{12} \\ &= \lim_{x\to\infty}\left(\sum_{n\le x} n\log n - \frac{1}{2}\left(x^2 + x + \frac{1}{6}\right)\log x + \frac{x^2}{4}\right) \\ &=: \log A. \end{aligned} \tag{5}$$

Cf. (31) for a generalization.

This suggests the following. To find a closed form for $\zeta''(2)$, we may proceed as above to introduce a constant through a counterpart of (5) and apply the above procedure of using the functional equation.

In parallel to (5), we may prove a counterpart of (28), whose special case with $u = 1$, $l = 3$, and $a = 1$, and $x \in \mathbb{N}$ (for simplicity) reads

$$\begin{aligned} &\zeta''(-1) \\ &= \lim_{x\to\infty}\left(\sum_{n\le x} n\log^2 n - \frac{1}{2}x^2\left(\log^2 x - \log x + \frac{1}{4}\right) - \frac{1}{2}x(\log^2 x + x) - \frac{1}{12}(\log^2 x + 2\log x)\right) \\ &=: A_2, \end{aligned} \tag{6}$$

the second Glaisher–Kinkelin constant.

Then, we apply the differentiated form of (3)

$$\begin{aligned} Z'(s) &= \left(\frac{\zeta'}{\zeta}(s)\right)' - \frac{1}{(s-1)^2} \\ &= Z'(1-s) - \frac{1}{4}\left(\psi'\left(\frac{s}{2}+1\right) - \psi'\left(\frac{3}{2}-\frac{s}{2}\right)\right), \end{aligned} \tag{7}$$

to deduce (2).

2. Partial Sums for Higher Derivatives of the Hurwitz Zeta-Function

Let

$$\zeta(s,a) = \sum_{n=0}^{\infty} (n+a)^{-s}, \quad \sigma > 1, 0 < a \leq 1,$$

be the Hurwitz zeta-function, where we write $s = \sigma + it$ throughout. We consider the asymptotic formula as well as the integral representation for the partial sum

$$L_u(x,a) = \sum_{0 \leq n \leq x} (n+a)^u, \quad 0 < a \leq 1,$$

and its derivatives. The following results can be found in [2] (Chapter 3). Let

$$\Gamma(s) = \int_0^{\infty} e^{-s} x^{s-1} dx, \sigma > 0,$$

be the gamma function, let $\psi(z) = \frac{\Gamma'}{\Gamma}(z)$ be the Euler digamma function in Lemma 2, and let $\bar{B}_r(x) = B_r(x - [x])$ denote the *rth periodic Bernoulli polynomial*, where the rth Bernoulli polynomial is defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{r=0}^{\infty} \frac{B_r(x)}{r!} t^r \quad (|t| < 2\pi).$$

Subsequent results depend on the *Euler–Maclaurin sum formula* [2] (p. 201, Theorem B.5),

$$\begin{aligned} \sum_{\alpha < n \leq x} f(n) = &\int_{\alpha}^{x} f(t)\,\mathrm{d}t + \sum_{r=1}^{l} \frac{(-1)^r}{r!} [\bar{B}_r(t)\, f^{(r-1)}(t)]_{\alpha}^{x} \\ &+ \frac{(-1)^{l+1}}{l!} \int_{\alpha}^{x} \bar{B}_l(t)\, f^{(l)}(t)\,\mathrm{d}t, \end{aligned} \tag{8}$$

which follows by repeated application of integration by parts to Euler's summation formula in view of

$$\frac{\mathrm{d}}{\mathrm{d}t} \bar{B}_r(t) = r \bar{B}_{r-1}(t).$$

The Euler–Maclaurin 'constant' $C = C(f)$ (EM constant for short) is defined by ($\alpha \geq 0$)

$$C \sim \sum_{\alpha \leq n \leq x} f(n) - \int^{x} f(t)\,\mathrm{d}t - \sum_{r=1}^{l} \frac{(-1)^r}{r!} \bar{B}_r(x)\, f^{(r-1)}(x),$$

where $\int^x$ means the lower limit is removed, and l may be ∞ as the case may be, cf. [3] (8). In the case of the sums of powers, this corresponds to the constant term in (10). The following theorem and its corollaries ([2] (pp. 55–61)) are far-reaching, but, in the literature, it went unnoticed. A prototype can be found in [4]. We will show its effect by interpreting the results in [3].

Lemma 1 (Integral Representations [2]). *For every $u \in \mathbb{C}$ and any $l \in \mathbb{N}$ with $l > \operatorname{Re} u + 1$, we have the integral representation (for any $x \geq 0$)*

$$\begin{aligned} L_u(x,a) = &\sum_{r=1}^{l} \frac{\Gamma(u+1)}{\Gamma(u+2-r)} \frac{(-1)^r}{r!} \bar{B}_r(x)\,(x+a)^{u-r+1} \\ &+ \frac{(-1)^l}{l!} \frac{\Gamma(u+1)}{\Gamma(u+1-l)} \int_x^{\infty} \bar{B}_l(t)\,(t+a)^{u-l}\,\mathrm{d}t \\ &+ \begin{cases} \dfrac{1}{u+1}\,(x+a)^{u+1} + \zeta(-u,a), & u \neq -1 \\ \log(x+a) - \psi(a), & u = -1 \end{cases}. \end{aligned} \tag{9}$$

Also, the asymptotic formula

$$\begin{aligned} L_u(x,a) = &\sum_{r=1}^{l} \frac{(-1)^r}{r} \binom{u}{r-1} \bar{B}_r(x)(x+a)^{u-r+1} + O\Big(x^{\operatorname{Re} u - l}\Big) \\ &+ \begin{cases} \dfrac{1}{u+1}\,(x+a)^{u+1} + \zeta(-u,a), & u \neq -1 \\ \log(x+a) - \psi(a), & u = -1 \end{cases}. \end{aligned} \tag{10}$$

holds true as $x \to \infty$, where $\zeta(-u,a)$ is to be interpreted as (11).

Furthermore, the integral representation

$$\begin{aligned} \zeta(-u,a) = a^u - \frac{1}{u+1}\,a^{u+1} - &\sum_{r=1}^{l} \frac{(-1)^r}{r} \binom{u}{r-1} B_r\, a^{u-r+1} \\ &+ (-1)^{l+1} \binom{u}{l} \int_0^{\infty} \bar{B}_l(t)(t+a)^{u-l}\mathrm{d}t, \end{aligned} \tag{11}$$

holds true for all complex $u \neq -1$, $\operatorname{Re} u < l - 1$, which follows from (9) by putting $x = 0$. For $u = -1$, we have the integral representation for the digamma function

$$\psi(a) = \log a - \frac{1}{2}\,a^{-1} + \int_0^{\infty} \bar{B}_1(t)\,(t+a)^{-2}\,\mathrm{d}t.$$

Proof. By the Euler–Maclaurin summation Formula (8), we have

$$\begin{aligned} L_u(x,a) = &(-1)^{l+1} \binom{u}{l} \int_0^{\infty} \bar{B}_l(t)\,(t+a)^{u-l}\mathrm{d}t \\ &- \sum_{r=1}^{l} \binom{u}{r-1} \frac{1}{r}\,(-1)^r\, B_r\, a^{u-r+1} + a^u + \int_0^x (t+a)^u\,\mathrm{d}t \\ &+ \sum_{r=1}^{l} \binom{u}{r-1} \frac{1}{r}\,(-1)^r\, \bar{B}_r(x)\,(x+a)^{u-r+1} + (-1)^l \binom{u}{l} \int_x^{\infty} \bar{B}_l(t)\,(t+a)^{u-l}\mathrm{d}t, \end{aligned} \tag{12}$$

where

$$\int_0^x (t+a)^u\,\mathrm{d}t = \begin{cases} \frac{1}{u+1}\big((x+a)^{u+1} - a^{u+1}\big), & u \neq -1 \\ \log(x+a) - \log a, & u = -1 \end{cases}.$$

In the case $\operatorname{Re} u < -1$, we let $x \to \infty$ in (12) to see that it amounts to (11) for $\operatorname{Re} u < -1$. Since the integral in (11) is absolutely convergent for $\operatorname{Re} u < \ell - 1$ and ℓ can be as large as we wish, (12) gives a meromorphic continuation of $\zeta(s,a)$ over the whole plane with a simple pole at $u = -1$. In the case $u = -1$, (12) reads for x large

$$L_{-1}(x,a) - \log(x+a) = \sum_{0 \leq n \leq x} \frac{1}{n+a} - \log(x+a) = f(a) + O\Big(x^{-1}\Big),$$

where $f(a)$ is a function in a only. Since this is the generic definition (22) of $\psi(a)$, we must have $f(a) = -\psi(a)$, which determines the constant term in the asymptotic Formula (10). □

For $u = n \in \mathbb{N} \cup \{0\}$ and $a = 1$, (11) gives

$$\zeta(-n) = -\frac{1}{n+1}\sum_{r=0}^{n+1}(-1)^r\binom{n+1}{r}B_r = -\frac{B_{n+1}}{n+1}, \quad \zeta(0) = B_1,$$

and

$$\zeta(-2n) = 0 \quad (n > 0), \quad \zeta(0) = -\frac{1}{2}.$$

Since the integrals appearing in Theorem 1 are analytic in the region $\Re u < 1 - l$, we can differentiate (9) and (11) in u. The results can be found in [2] (pp. 54–60). The case $u = -1$ needs to be treated separately as in the proof of Theorem 1 above. In the following, we give some illustrating examples of derivatives.

Theorem 1. *Suppose* $\operatorname{Re} u < 1$. *Then, we have asymptotic formulas*

$$\begin{aligned}\frac{\partial^k}{\partial^k u}L_u(x,a) &= -\bar{B}_1(x)\,(x+a)^u\log^k(x+a) + O(x^{\operatorname{Re}u-1}\log^k(x+a))\\ &+\begin{cases}(x+a)^{u+1}\sum_{r=0}^{k}(-1)^r r!\binom{k}{r}(u+1)^{-1-r}\log^{k-r}(x+a) + (-1)^k\zeta^{(k)}(-u,a), & u \neq -1\\ \frac{1}{k+1}\log^{k+1}(x+a) + (-1)^k\gamma_k(a), & u = -1\end{cases}.\end{aligned} \tag{13}$$

for $\operatorname{Re} u < 0$ *and*

$$\begin{aligned}\frac{\partial^k}{\partial^k u}L_u(x,a) &= -\bar{B}_1(x)\,(x+a)^u\log^k(x+a) + \frac{\bar{B}_2(x)}{2}\sum_{r=0}^{1}\binom{k}{r}u^{(r)}(x+a)^{u-1}\log^{k-r}(x+a)\\ &+(x+a)^{u+1}\sum_{r=0}^{k}(-1)^r r!\binom{k}{r}(u+1)^{-1-r}\log^{k-r}(x+a) + (-1)^k\zeta^{(k)}(-u,a)\\ &+O(x^{\operatorname{Re}u-2}\log^k(x+a)),\end{aligned} \tag{14}$$

for $0 \le \operatorname{Re} u < 1$.

The EM constant for the function $f(t) = (t+a)^u\log^k(t+a)$ *(for* $\operatorname{Re} u < 1$*) is* $(-1)^k\zeta^{(k)}(-u,a)$.

Proof. We distinguish two cases $\operatorname{Re} u < 0$ and $0 \le \operatorname{Re} u < 1$ and consider the former case in which we may take $l = 1$. Then, (9) reads

$$\begin{aligned}L_u(x,a) &= -\bar{B}_1(x)\,(x+a)^u - u\int_x^\infty \bar{B}_1(t)\,(t+a)^{u-1}\,\mathrm{d}t\\ &+\begin{cases}\frac{1}{u+1}\,(x+a)^{u+1} + \zeta(-u,a), & u \neq -1\\ \log(x+a) - \psi(a), & u = -1\end{cases}.\end{aligned}$$

For $0 \le \operatorname{Re} u < 1$, we may choose $l = 2$, and (9) reads

$$\begin{aligned}L_u(x,a) &= -\bar{B}_1(x)\,(x+a)^u + \frac{u}{2}\bar{B}_2(x)\,(x+a)^{u-1} + \frac{1}{u+1}\,(x+a)^{u+1} + \zeta(-u,a)\\ &+\frac{u(u-1)}{2}\int_x^\infty \bar{B}_2(t)\,(t+a)^{u-2}\,\mathrm{d}t.\end{aligned}$$

□

Example 1. (i) *We state the case $k = 1$ of* (14). *For a generalization of* (14), *cf.* (28) *whose special case is* (30), *we state* (14).

The case $k = 1$ of (14) *leads to the asymptotic formula*

$$\begin{aligned}&-\zeta'(-u,a)\\&=\sum_{0\le n\le x}(n+a)^u\log(n+a)-(u+1)^{-1}(x+a)^{u+1}\log(x+a)+(u+1)^{-2}(x+a)^{u+1}\\&+\bar{B}_1(x)(x+a)^u\log(x+a)-\frac{1}{2}\bar{B}_2(x)(x+a)^{u-1}(1+u\log(x+a))\\&+O(x^{\operatorname{Re}u-2}).\end{aligned}$$

Hence, as a generic definition ($N \in \mathbb{N}$)

$$\begin{aligned}&-\zeta'(0,a)\\&=\lim_{N\to\infty}\Big(\sum_{0\le n\le N}\log(n+a)-(N+a)\log(N+a)+N+a+B_1\log(N+a)\Big).\end{aligned}\tag{15}$$

The generic definition (15) *plays an essential role in the proof of Lerch's formula [2] (p. 82, Theorem 5.1)*

$$\zeta'(0,a)=\log\frac{\Gamma(a)}{\sqrt{2\pi}},\tag{16}$$

and the Weierstrass product for the gamma function [2] (p. 88, Exercise 5.2). Hence, in particular,

$$\zeta'(0)=-\log\sqrt{2\pi}.\tag{17}$$

Substituting this and data in Lemma 2 in (3), *we conclude that*

$$\lim_{s\to1}\left(\frac{\zeta'}{\zeta}+\frac{1}{s-1}\right)=\gamma.\tag{18}$$

Applying (18), (17) *was deduced by substituting* (36) *in* (3).

(ii) *We state the case $k = 2$ of* (14).

Equation (14) *leads to the asymptotic formula*

$$\begin{aligned}&\zeta''(-u,a)=\sum_{0\le n\le x}(n+a)^u\log^2(n+a)\\&-(x+a)^{u+1}((u+1)^{-1}\log^2(x+a)-2(u+1)^{-2}\log(x+a)+(u+1)^{-3})\\&+\bar{B}_1(x)(x+a)^u\log^2(x+a)-\frac{1}{2}\bar{B}_2(x)(x+a)^{u-1}(u\log^2(x+a)+2\log(x+a))\\&+O(x^{\operatorname{Re}u-2}\log x).\end{aligned}$$

Hence, we have an asymptotic formula ($N \in \mathbb{N}$)

$$\begin{aligned}\zeta''(0,a)=\sum_{n=0}^{N}\log^2(n+a)-(N+a)\Big\{\log^2(N+a)-2\log(N+a)+1\Big\}-\frac{1}{2}\log^2(N+a)\\-\int_N^\infty\frac{\bar{B}_1(t)}{t+a}\log^2(t+a)\mathrm{d}t,\end{aligned}$$

and the generic definition

$$\begin{aligned}&-\zeta''(0,a)\\&=\lim_{N\to\infty}\Big(\sum_{n=0}^{N}\log^2(n+a)-(N+a)\Big\{\log^2(N+a)-2\log(N+a)+1\Big\}-\frac{1}{2}\log^2(N+a)\Big).\end{aligned}\tag{19}$$

See that (19) *plays an essential role in the theory of Deninger's R-function [2] (p. 92, (5.36))*

$$R(a) = -\zeta''(0, a). \tag{20}$$

In particular, (19) *contains [5] (Theorem 2.3) involving the Gaussian representation and the Weierstrass representation [2] (pp. 89–91).*

Lerch's Formula (16) *and Deninger's Formula* (20) *are principal solutions to the difference equation furnished by the Dufrenoy–Pisot theorem; cf. [6]. As* (17) *is proved on the ground of Stirling's formula, the closed-form expression*

$$-R(1) = \zeta''(0) = -\frac{1}{2}(\log 2\pi)^2 + \gamma_1 + \frac{1}{2}\gamma^2 - \frac{\zeta(2)}{4}. \tag{21}$$

depends on an extraneous factor of the logarithmic differentiation, cf. Section 3 below.

For a systematic study on the values $\zeta^{(k)}(0)$*, we refer to [7].*

Lemma 2. *Under the definition of the Euler constant*

$$\gamma = \lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n} - \log(N+z)\right).$$

for any z other than negative integers, the two definitions for the Euler digamma function ψ *are equivalent: the generic definition*

$$\psi(z) = -\lim_{N\to\infty}\left(\sum_{n=0}^{N}\frac{1}{n+z} - \log(N+z)\right), \tag{22}$$

where z is not a non-positive integer, and the Gaussian representation

$$\psi(z) + \gamma = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{z+n-1}\right),$$

We assemble here some well-known formulas for ψ*:*

$$\psi(z+1) = \psi(z) + \frac{1}{z} \quad \psi'(z+1) - \psi'(z) = -\frac{1}{z^2},$$

$$\psi\left(\frac{1}{2}\right) = -\gamma - 2\log 2, \quad \psi'(1) = \zeta(2), \quad \psi'\left(\frac{1}{2}\right) = 3\zeta(2). \tag{23}$$

Example 2. *The kth Laurent coefficient of the Hurwitz zeta-function (at* $s = 1$*) is given by* $\frac{(-1)^k}{k!}\gamma_k(a)$*, where*

$$\gamma_k(a) = \lim_{x\to\infty}\left(\sum_{0\le n\le x}\frac{\log^k(n+a)}{n+a} - \frac{\log^{k+1}(x+a)}{k+1}\right), \tag{24}$$

is sometimes called the Stieltjes or generalized Euler constant. $\gamma_k(a)$ *admits the integral representation*

$$\begin{aligned}\gamma_k(a) = {} & \frac{1}{2a}\log^k a - \frac{1}{k+1}\log^{k+1} a \\ & - \int_0^{\infty}\frac{\bar{B}_1(t)}{(t+a)^2}\Big(\log^k(t+a) - k\log^{k-1}(t+a)\Big)\mathrm{d}t.\end{aligned} \tag{25}$$

In particular, $\gamma_0(a) = -\psi(a)$*.*

Proof. We prove the integral representation (27) corresponding to (9), which also establishes (13) with $u = -1$. Theorem 1 with $l = 1$ and $-s$ ($s \neq 1, \sigma > 0$) for u

$$L_{-s}(x,a) = \frac{(x+a)^{1-s}}{1-s} + \zeta(s,a) - \frac{\bar{B}_1(x)}{(x+a)^s} + s\int_x^\infty \frac{\bar{B}_1(t)}{(t+a)^{s+1}}\mathrm{d}t. \tag{26}$$

Since both sides of (26) are analytic in $\sigma > 0$, we may compute the kth Taylor coefficients around $s = 1$. Equating them, we conclude that

$$\begin{aligned}\gamma_k(a) = &\sum_{0\le n\le x} (n+a)^{-1}\log^k(n+a)\\ &- \frac{\log^{k+1}(x+a)}{k+1} + \frac{\bar{B}_1(x)}{x+a}\log^k(x+a)\\ &- \int_x^\infty \frac{\bar{B}_1(t)}{(t+a)^2}\Big(\log^k(t+a) - k\log^{k-1}(t+a)\Big)\mathrm{d}t.\end{aligned} \tag{27}$$

As in Theorem 1, letting $x \to \infty$ and $x = 0$ implies (24) and (25), respectively. □

Example 3. *As Corollary 1 establishes [3] Theorems 2 and 3, (28) gives [3] Theorem 1 to the effect that the EM constant is $-\zeta'(u,a)$ in general and in the special case of m being an even positive integer that*

$$\zeta'(-m) = \frac{B_{m+1}}{m+1}\sum_{j=1}^{m}\frac{1}{j}.$$

We state special cases of the Lemma 1. In the case of $u = m$, $m \in \mathbb{N}\cup\{0\}$. For $\mathbb{N} \ni l > m+1$, we have the generic definition

$$\begin{aligned}-\zeta'(-m,a) = \lim_{N\to\infty}\Bigg(&\sum_{n=0}^{N}(n+a)^m\log(n+a)\\ &- \frac{1}{m+1}(N+a)^{m+1}\log(N+a) + \frac{1}{(m+1)^2}(N+a)^{m+1}\\ &- \frac{1}{2}(N+a)^m\log(N+a) - \sum_{r=2}^{m+1}\binom{m}{r-1}\frac{B_r}{r!}\\ &\cdot\left(\frac{1}{m} + \cdots + \frac{1}{m-r+2} + \log(N+a)\right)(N+a)^{m-r+1}\Bigg),\end{aligned} \tag{28}$$

where an empty sum is to be interpreted as 0 and the integral representation

$$\begin{aligned}&\zeta'(-m,a)\\ &= \frac{1}{m+1}a^{m+1}\log a - \frac{1}{(m+1)^2}a^{m+1} - \frac{1}{2}a^m\log a + \frac{1}{12}a^{m-1}\log a\\ &+ \sum_{r=4}^{m+1}\frac{B_r}{r}\left(\sum_{j=0}^{r-2}(-1)^j\binom{m}{j}\frac{1}{r-1-j} + \binom{m}{r-1}\log a\right)a^{m-r+1}\\ &+ \frac{1}{m+1}\sum_{r=m+2}^{l}B_r\left(\sum_{j=0}^{r-1}(-1)^j\binom{r-m-2}{j}\frac{1}{r-j}\right)a^{m-r+1}\\ &+ (-1)^{l+1}\int_0^\infty\left(\sum_{j=0}^{l-1}(-1)^j\binom{l-m-1}{j}\frac{1}{l-j}\bar{B}_l(t)(t+a)^{m-l}\right)\mathrm{d}t,\end{aligned} \tag{29}$$

where (28) and (29) correspond to (10) and (11), respectively. Additionally, (28) implies Example 3.

So, (28) with $m = 1$ leads to a generalization of (5):

$$-\zeta'(-1, a) = \log A(a) - \frac{B_2}{2}, \tag{30}$$

where

$$\begin{aligned} &\log A(a) \\ &= \lim_{N\to\infty} \left(\sum_{n=0}^{N} (n+a)\log(n+a) - \frac{1}{2}(N+a)^2 \log(N+a) + \frac{1}{2^2}(N+a)^2 \right. \\ &\left. - \frac{1}{2}(N+a)\log(N+a) - \frac{B_2}{2}\log(N+a) \right). \end{aligned} \tag{31}$$

3. Special Values of the Hurwitz Zeta-Function and Its Derivatives

As we have shown in Example 1 above, it is the relations between the derivatives of the Hurwitz zeta-function and the allied gamma-functions ($\log\Gamma(a)$, $R(x)$). However, the special values of the zeta- and gamma-allied functions are of interest. Equation (21) is a typical example in which one knows the closed form for the value $R(1)$, which is of importance as with the values $\Gamma(1) = 1$.

The other value that can be readily obtained is

$$R\left(\frac{1}{2}\right) = -\zeta''\left(0, \frac{1}{2}\right) = \frac{1}{2}\log^2 2 - 2\log 2 \log\sqrt{2\pi}.$$

This follows by differentiating

$$\zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\zeta(s), \tag{32}$$

twice

$$\zeta''\left(0, \frac{1}{2}\right) = 2^s(\log 2)^2\zeta(s) + 2^{s+1}\log 2\zeta'(s) + (2^s - 1)\zeta''(s).$$

Example 4 ([8], Part I). *For the alternating series, we have, similar to* (32),

$$\ell_s\left(\frac{1}{2}\right) = (2^{1-s} - 1)\zeta(s), \tag{33}$$

where

$$\ell_s(x) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n^s},$$

indicates the Lerch zeta-function. The method of logarithmic differentiation seems to be first used by de la Vallée-Poussin. Applying it to (33), *we deduce that*

$$\frac{\zeta'}{\zeta}(s) = \frac{\ell_s'}{\ell_s}\left(\frac{1}{2}\right) + \frac{\log 2}{2^{1-s} - 1}. \tag{34}$$

It suffices to find the Laurent constant at $s = 1$, and so below all asymptotic equations mean the ones near $a = 1$. Since $\frac{\log 2}{2^{1-s}-1} = -\frac{1}{s-1} + \frac{1}{2}\log 2 + O(s-1)$, (34) *implies that*

$$\frac{\zeta'}{\zeta}(s) + \frac{1}{s-1} = \frac{\ell_s'}{\ell_s}\left(\frac{1}{2}\right) + \frac{1}{2}\log^2 2 + O(s-1).$$

Substituting the Laurent expansion

$$\ell_s'\left(\frac{1}{2}\right) = -\gamma \log 2 + \frac{1}{2}\log 2 + O(s-1), \tag{35}$$

we conclude

$$\lim_{s\to 1}\left(\frac{\zeta'}{\zeta}(s) + \frac{1}{s-1}\right) = \gamma. \tag{36}$$

To prove (35)*, we differentiate both sides of* (33) *to obtain*

$$\ell_s'\left(\frac{1}{2}\right) = (2^{1-s}-1)\zeta'(s) - 2^{1-s}(\log 2)\zeta(s).$$

Substituting the Laurent expansions

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1), \quad \zeta'(s) = -\frac{1}{(s-1)^2} + \gamma_1 + O(s-1),$$

and

$$2^{1-s} = 1 - (\log 2)(s-1) + \frac{\log^2 2}{2}(s-1)^2 + O((s-1)^2),$$

we deduce (35) *after some manipulations. Combining*

$$\ell_1\left(\frac{1}{2}\right) = -\log 2,$$

and (35)*, we obtain*

$$\frac{\ell_1'}{\ell_1}\left(\frac{1}{2}\right) = \gamma - \frac{1}{2}\log 2.$$

Remark 1. *Example 4 is stated by de la Vallée-Poussin [8], available as [9] (pp. 63–64) and reproduced in [10] (pp. 216–217). de la Vallée-Poussin uses the alternating property of* $\ell_1\left(\frac{1}{2}\right)$ *and appeals to the generic definition of* γ *without using the integral representation.*

Now, we briefly describe a modification of Song's method [11] of logarithmic differentiation and non-trivial zeros. As described in Section 1, our method depends on the functional equation and integral representation furnished by Theorem 1. We rewrite (11) with $u = -s$, $l = 1$, and $a = 1$ as

$$(s-1)\zeta(s) = 1 + J(s), \quad J(s) = (s-1)\left(\frac{1}{2} - s\int_1^\infty \bar{B}_1(t)t^{-s-1}\,\mathrm{d}t\right),$$

where $\sigma > 0$.

Now, apply the logarithmic differentiation to obtain

$$Z(s) = \frac{\zeta'}{\zeta}(s) + \frac{1}{s-1} = \frac{J'(s)}{1+J(s)}, \tag{37}$$

where

$$J'(s) = \frac{1}{2} + (1-2s)\int_1^\infty \bar{B}_1(t)t^{-s-1}\,\mathrm{d}t + s(s-1)\int_1^\infty \bar{B}_1(t)t^{-s-1}\log t\,\mathrm{d}t. \tag{38}$$

Hence, $J'(s) = \frac{1}{2} - s\int_1^\infty \bar{B}_1(t)t^{-s-1}\,\mathrm{d}t + O(s-1)$ and (36) followed by (25).

One more differentiation of (37) gives us

$$\begin{aligned} &\frac{\zeta''}{\zeta}(s) - \left(\frac{\zeta'}{\zeta}(s)\right)^2 - \frac{1}{(s-1)^2} \\ &= Z'(s) = \left(\frac{\zeta'}{\zeta}(s) + \frac{1}{s-1}\right)' = \frac{J''(s)(1+J(s)) - (J'(s))^2}{(1+J(s))^2}. \end{aligned} \tag{39}$$

From (38), we have

$$\begin{aligned} J''(s) = &-2\int_1^\infty \bar{B}_1(t)t^{-s-1}\,\mathrm{d}t + 2(2s-1)\int_1^\infty \bar{B}_1(t)t^{-s-1}\log t\,\mathrm{d}t \\ &- s(s-1)\int_1^\infty \bar{B}_1(t)t^{-s-1}\log^2 t\,\mathrm{d}t. \end{aligned} \tag{40}$$

We take the limit as $s \to 1$ of $Z'(1-s)$ for which we put $s = 1$ in (40). We have

$$J''(1) = -2\int_1^\infty \bar{B}_1(t)t^{-2}\,\mathrm{d}t + 2\int_1^\infty \bar{B}_1(t)t^{-2}\log t\,\mathrm{d}t. \tag{41}$$

Substituting (25) in (41), we conclude that

$$J''(1) = -2\gamma_1.$$

Substituting the data (23) in the far-right side of (39), we obtain

$$\lim_{s\to 1}\left(\frac{\zeta'}{\zeta}(s) + \frac{1}{s-1}\right)' = \frac{J''(1)(1+J(1)) - (J'(1))^2}{(1+J(1))^2} = -2\gamma_1 - \gamma^2.$$

Substituting the above data and $\psi'\left(\frac{3}{2}\right) = 3\zeta(2) - 4$ into (7), we deduce (21).

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References

1. Titchmarsh, E.C. *The Theory of the Riemann Zeta-Function*; Oxford University Press: Oxford, UK, 1986.
2. Kanemitsu, S.; Tsukada, H. *Vistas of Special Functions*; World Scientific: Singapore, 2007.
3. Choudhury, B.K. The Riemann zeta-function and its derivatives. *Proc. R. Soc. Lond. A* **1995**, *450*, 477–499.
4. Mellin, H. Die Dirchletschen Reihen, die zahlentheoretischen Funktionen und die unendlichen Produkte von endlichem Geschlecht. *Acta Soc. Sci. Fenn.* **1902**, *31*, 1–48.
5. Deninger, C. On the analogue of Chowla-Selberg formula for real quadratic fields. *J. Reine Angew. Math.* **1984**, *351*, 171–191.
6. Chakraborty, K.; Kanemitsu, S.; Kuzumaki, T. Finite expressions for higher derivatives of the Dirichlet *L*-function and the Deninger *R*-function. *Hardy-Ramanujan J.* **2009**, *32*, 38–53. [CrossRef]
7. Apostol, T.M. Formulas for higher derivatives of the Riemann zeta function. *Math. Comp.* **1985**, *44*, 223–232. [CrossRef]
8. de la Vallée Poussin, C.-J. Recherches analytiques de la théorie des nombres premiers. *Ann. Soc. Sci. Brux.* **1896**, *20*, 183–256. 281–397.

9. de la Vallée Poussin, C.-J. *Recherches Analytiques de la Théorie des Nombres Premiers, Parts 1–5*; Wentworth Press: Bruxelles, Belgium, 1897.
10. Narkiewicz, W. *Development of Prime Number Theory—From Euclid to Hardy and Littlewood*; Bd II; Springer: Berlin/Heidelberg, Germany, 2000.
11. Song, Z.-Q. On the values of $\zeta^{(n)}(0)$. 2025, *to be submitted*.

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