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Conditional Coherent and Convex Risk Measures Under Uncertainty

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Abstract: In this paper, we take a new perspective to describe the model uncertainty, and thus propose two new classes of risk measures under model uncertainty. To be precise, we use an auxiliary random variable to describe model uncertainty. By proposing new sets of axioms under model uncertainty, we axiomatically introduce and characterize conditional coherent and convex risk measures under a random environment, respectively. As examples, we also discuss the connections of the introduced conditional coherent risk measures under random environments with two existing risk measures. This paper mainly gives some theoretical results, and it is expected to make meaningful complement to the study of coherent and convex risk measures under model uncertainty.

Keywords: coherent risk measures; convex risk measures; conditional risk measures; model uncertainty; representation

MSC: 91G70; 91B05

1. Introduction

Nowadays, risk measures have been widely used in both finance and insurance, such as capital requirement calculation and insurance pricing, etc. In their seminal work, ref. [1] axiomatically introduced and characterized coherent risk measures. Later, refs. [2,3] axiomatically introduced and characterized a broader class of convex risk measures. In the context of insurance, ref. [4] introduced an axiomatic approach to insurance pricing, which is also known as distortion premium principle. For more details about (univariate) risk measures, we refer to [5]. Mathematically, the random loss of a financial position is commonly described by a random variable defined on some measurable space $(\Omega, \mathscr{F})$. In practice, these classical risk measures usually require the acquirement of accurate distribution functions of the random variables. This requirement is mathematically equivalent to requiring an *accurate* probability measure on $(\Omega, \mathscr{F})$, which is sometimes known as a reference probability or a scenario in the literature. For instance, value at risk (VaR) and expected shortfall (ES) are the cases, which are two risk measures popularly used in practice.

From the practical point of view, it is usually difficult to accurately acquire the *true* distribution functions of the random variables, because the distribution functions are usually estimated from data (samples) or statistical hypothesis (simulation). In other words, it is usually hard to acquire an *accurate* (reference) probability measure on $(\Omega, \mathscr{F})$. Hence, the model uncertainty problem naturally arises. This kind of uncertainty could stem from the Knightian uncertainty; for instance, see [5]. There are two natural and important approaches to deal with model uncertainty. One approach is to consider random variables

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on some measurable space $(\Omega, \mathscr{F})$ without assuming a reference probability (scenario) on it; for instance, see [5]. Such models are also called *model-free* models; for instance, see [1]. Another approach is to consider multiple scenarios (reference probabilities) on $(\Omega, \mathscr{F})$, in which the set of scenarios describes model uncertainty; for instance, see [6–8] and the references therein. For more earlier studies on model uncertainty problem, we refer to [9–14], and the references therein. It should be mentioned that model uncertainty might have different disguises in different situations. For instance, ref. [15] studied background risk measures. Ref. [16] studied various systemic risk measures. Ref. [17] studied various VaR- and expectile-based systemic risk measures. Ref. [18] studied factor risk measures. Ref. [19] established an elegant conditional distortion risk measure. Various conditional risk measures also appear in [14,20,21] in different disguises, just name a few. In this paper, we also take into account the model uncertainty issue. Moreover, we adopt the way of considering *model-free* models.

In the present paper, we take a different perspective to describe model uncertainty. Specifically, we use an auxiliary random variable to describe model uncertainty. For example, for each value the auxiliary random variable takes, there corresponds a *scenario* (*reference probability*) on some measurable space. Our approach is to, respectively, measure the risk of a random loss provided that the auxiliary random variable takes different values first, and then aggregate these risk evaluations into a single value. In such a procedure of risk evaluation, the auxiliary random variable serves a role like a *condition* or *environment*. Hence, we regard it as a *random environment*, and refer to the resulting risk measures as *conditional risk measures under a random environment* later.

In this paper, by axiomatic approaches, we introduce conditional coherent and convex risk measures under a random environment, respectively. After studying their properties, we then axiomatically characterize them. The axioms are newly presented in the presence of model uncertainty, although they are motivated by those classical axioms. Finally, as examples, we also discuss the connections of the introduced conditional coherent risk measures under random environments with two existing risk measures. This paper mainly gives some theoretical results, and it is expected to make a meaningful complement to the study of coherent and convex risk measures under model uncertainty.

The rest of the paper is organized as follows. In Section 2, we state preliminaries. Section 3 is devoted to the main results of this paper. In Section 4, we provide two examples to discuss the connections of the proposed risk measures with the existing risk measures.

2. Preliminaries

In this paper, for any non-empty set S, we take the power set 2^S as the $\sigma-$algebra, making $(S, 2^S)$ a measurable space. The power set 2^S consists of all subsets of S. The reason that we use the power set 2^S as the σ-algebra on S is to avoid various tedious measurability consideration, because we will encounter integrals of various random variables with respect to non-additive measures in the sequel. For more about integral with respect to non-additive measures, we refer to [22]. We denote by $L^\infty(S)$ the linear space of all bounded measurable functions (i.e., random variables) on the measurable space $(S, 2^S)$.

Let $(\Omega, 2^\Omega)$ be a fixed measurable space, where the sample space Ω represents all possible states of the real world. In this paper, we will work on $L^\infty(\Omega)$ rather than the space of all essentially bounded random variables on certain probability space, so that we can deal with model uncertainty; for instance, see [5–7]. The random loss of a financial position is described by an element in $L^\infty(\Omega)$. The space $L^\infty(\Omega)$ will also serve as random environments. For the terminology convenience, for any random environment $Z \in L^\infty(\Omega)$, each value that Z takes is called a *state* of Z.

Given a non-empty set S, a mapping $\mu : 2^S \to [0, +\infty)$ is called a set function on 2^S, if $\mu(\emptyset) = 0$. A set function μ on 2^S is called normalized, if $\mu(S) = 1$; and called monotone, if $\mu(A) \leq \mu(B)$ for any $A, B \in 2^S$ with $A \subset B$. A normalized monotone set function μ on 2^S is called a finitely additive probability measure on 2^S, if $\mu(A \cup B) = \mu(A) + \mu(B)$ for any disjoint $A, B \in 2^S$. Note that a finitely additive probability measure on 2^S is a probability measure whenever S is a finite set (i.e., the number of elements of S is finite.). We denote by $\mathscr{M}_{1,f}(S)$ the set of all finitely additive probability measures on 2^S. Simply, we write $\mathscr{M}_{1,f}$ for $\mathscr{M}_{1,f}(\Omega)$. Given a $\mu \in \mathscr{M}_{1,f}(S)$, for any measurable function (i.e., random variable) V on $(S, 2^S)$, denote by $\mu(V) := E_\mu(V)$ the expectation of V with respect to μ. For more about the expectation under finitely additive probability measures, we refer to [5].

We denote by $\mathbb{R}$ the real numbers, and $\mathbb{R}_+$ the non-negative real numbers. For a non-empty set A, 1_A stands for the indicator function of A. By convention, $1_\emptyset := 0$. For any $X \in L^\infty(\Omega)$, Ran(X) stands for the range of X.

Give an environment $Z \in L^\infty(\Omega)$, for any state $z \in \text{Ran}(Z)$ and any random loss $X \in L^\infty(\Omega)$, we denote by X_z the restriction of X to the subset $\{Z = z\}$ of Ω, that is, $X_z : \{Z = z\} \to \mathbb{R}$,

$$X_z(\omega) := X(\omega), \quad \omega \in \{Z = z\}.$$

Notice that the domains of X_z and $X1_{\{Z=z\}}$ are different in general. If the random environment Z is degenerate, then X_z and X are the same.

Given a non-empty set S, in general, a risk measure on $L^\infty(S)$ can be defined as any functional $\rho : L^\infty(S) \to \mathbb{R}$. In particular, for any random variable $X \in L^\infty(S)$, the quantity $\rho(X)$ is called the risk measure of X. A risk measure $\rho : L^\infty(S) \to \mathbb{R}$ is called *monotone* if $\rho(X) \leq \rho(Y)$ for $X, Y \in L^\infty(S)$ with $X \leq Y$, *translation-invariant* if $\rho(X + a) = \rho(X) + a$ for $a \in \mathbb{R}$ and $X \in L^\infty(S)$, *positively homogeneous* if $\rho(\lambda X) = \lambda\rho(X)$ for $\lambda \in (0, +\infty)$ and $X \in L^\infty(S)$, *subadditive* if $\rho(X + Y) \leq \rho(X) + \rho(Y)$ for $X, Y \in L^\infty(S)$. A risk measure is called *coherent* if it is monotone, translation-invariant, positively homogeneous and subadditive, *convex* if it is monotone and translation-invariant, and satisfies that $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$ for $X, Y \in L^\infty(S)$ and $\lambda \in [0, 1]$. For a coherent risk measure ρ, both positive homogeneity and translation invariance imply that $\rho(a) = a$ for $a \in \mathbb{R}$. For more details about risk measures, we refer to [5].

We end this section by introducing the definitions of conditional risk measures (CRMs) under a fixed state and a random environment, respectively.

Definition 1. *Given a random environment $Z \in L^\infty(\Omega)$ and a state $z \in Ran(Z)$, a CRM under a fixed state z is defined as any functional $\rho_Z(\cdot; z) : L^\infty(\Omega) \to \mathbb{R}$. In particular, for any random loss $X \in L^\infty(\Omega)$, the quantity $\rho_Z(X; z)$ is called the risk measure of X under a fixed state z.*

Definition 2. *A CRM under a random environment is defined as any functional $\rho(\cdot; \cdot) : L^\infty(\Omega) \times L^\infty(\Omega) \to \mathbb{R}$. In particular, for any random loss $X \in L^\infty(\Omega)$ and any random environment $Z \in L^\infty(\Omega)$, the quantity $\rho(X; Z)$ is called the risk measure of X under a random environment Z.*

Note that there are two input arguments for a CRM under a random environment. The first input argument represents the random loss of a financial position. Compared with the first input argument, the second input argument serves only as a role of a kind of *condition* or *environment*. Different elements of $L^\infty(\Omega)$ represent different random environments. In the majority, we specify a *condition (environment)* $Z \in L^\infty(\Omega)$ first, and then measure the risk of random losses $X \in L^\infty(\Omega)$. Nevertheless, considering that the random environment Z can be any element of $L^\infty(\Omega)$, we think of the CRM under a random environment ρ as a *bivariate* functional on $L^\infty(\Omega) \times L^\infty(\Omega)$.

3. Main Results

In this section, we present the main results of this paper. By axiomatic approaches, we introduce and characterize conditional coherent and convex risk measures under a fixed state and random environment, respectively. Specifically, in the first subsection, we introduce and characterize conditional coherent risk measures under a fixed state and random environment, respectively. In the second subsection, we introduce and characterize conditional convex risk measures under a fixed state and random environment, respectively.

3.1. Conditional Coherent Risk Measures Under Random Environment

In this subsection, we list some axioms for CRMs under a random environment first. Then we construct conditional coherent risk measures under random environments and study their properties. Finally, we axiomatically characterize it.

Similar to the definition of classical coherent risk measures, we start with the definition of conditional coherent risk measures (CCRMs) under a fixed state.

Definition 3. *Given, arbitrarily, a random environment $Z \in L^\infty(\Omega)$ and a state $z \in Ran(Z)$, a CRM under a fixed state z $\rho_Z(\cdot;z) : L^\infty(\Omega) \to \mathbb{R}$ is called coherent if it satisfies the following properties (axioms):*

A1 Local indifference: *For $X, Y \in L^\infty(\Omega)$, if $X1_{\{Z=z\}} = Y1_{\{Z=z\}}$, then $\rho_Z(X;z) = \rho_Z(Y;z)$.*
A2 Local monotonicity: *For $X, Y \in L^\infty(\Omega)$, if $X1_{\{Z=z\}} \leq Y1_{\{Z=z\}}$, then $\rho_Z(X;z) \leq \rho_Z(Y;z)$.*
A3 Local translation invariance: *$\rho_Z(X+a;z) = \rho_Z(X;z) + a$ for any $X \in L^\infty(\Omega)$ and any $a \in \mathbb{R}$.*
A4 Local positive homogeneity: *$\rho_Z(aX;z) = a\rho_Z(X;z)$ for any $X \in L^\infty(\Omega)$ and any $a \in \mathbb{R}_+$.*
A5 Local sub-additivity: *$\rho_Z(X+Y;z) \leq \rho_Z(X;z) + \rho_Z(Y;z)$ for any $X, Y \in L^\infty(\Omega)$.*

Since the state of a given environment is pre-specified, the financial meanings of the above Axioms A1–A5 can be interpreted totally similarly to the classic setting; for instance, see [1,4–6,23].

Note that Axiom A2 implies Axiom A1. Now, we turn to introduce the definition of conditional coherent risk measures (CCRMs) under a random environment.

Definition 4. *A CRM under a random environment $\rho(X;Z) : L^\infty(\Omega) \times L^\infty(\Omega) \to \mathbb{R}$ is called coherent, if for any random environment $Z \in L^\infty(\Omega)$ and each state $z \in Ran(Z)$, there is a CCRM under a fixed state z $\rho_Z(\cdot;z) : L^\infty(\Omega) \to \mathbb{R}$ such that the following properties (axioms) hold:*

B1 Outcome indifference: *For any $X, Y \in L^\infty(\Omega)$, if $\rho_Z(X;z) = \rho_Z(Y;z)$ for any $z \in Ran(Z)$, then $\rho(X;Z) = \rho(Y;Z)$.*
B2 Global monotonicity: *For any $X, Y \in L^\infty(\Omega)$, if $\rho_Z(X;z) \leq \rho_Z(Y;z)$ for any $z \in Ran(Z)$, then $\rho(X;Z) \leq \rho(Y;Z)$.*
B3 Risk consistency: *For any $X, Y \in L^\infty(\Omega)$ and $a \in \mathbb{R}$, if $\rho_Z(Y;z) = \rho_Z(X;z) + a$ for any $z \in Ran(Z)$, then $\rho(Y;Z) = \rho(X;Z) + a$.*
B4 Multiplication consistency: *For any $X, Y \in L^\infty(\Omega)$ and $a \in \mathbb{R}_+$, if $\rho_Z(Y;z) = a\rho_Z(X;z)$ for any $z \in Ran(Z)$, then $\rho(Y;Z) = a\rho(X;Z)$.*
B5 Global sub-additivity: *For any $X, Y, W \in L^\infty(\Omega)$, if $\rho_Z(W;z) = \rho_Z(X;z) + \rho_Z(Y;z)$ for any $z \in Ran(Z)$, then $\rho(W;Z) \leq \rho(X;Z) + \rho(Y;Z)$.*

Axioms B1–B5 can be interpreted in the context of finance as follows. Note first that $\rho_Z(X;z)$ exactly represents the risk measure of random loss X under the condition that the environment Z takes the state z, as will be seen in the sequel. Axiom B1 means that for two random losses X_1 and X_2, if their state-wise riskiness $\rho_Z(X_1;\cdot)$ and $\rho_Z(X_1;\cdot)$ have the same distribution under the environment's probability distribution, then their overall riskiness

should be the same. This characteristic has some similarity to the law invariance in the classic setting. B2 says that if random loss X_1 is less risky than another random loss X_2 in an almost-all-state-wise sense, then the overall riskiness of X_1 should not exceed that of X_2. B3 says that for two random losses X and Y, if the riskiness of Y is the sum of that of X and a fixed amount of c in an almost-all-state-wise sense, then the overall riskiness of Y should also be the sum of the overall riskiness of X and the fixed amount c. This characteristic has some similarity to the translation invariance in the classic setting. B4 says that for two random losses X and Y, if the riskiness of Y is a multiple of that of X in an almost-all-state-wise sense, then the overall riskiness of Y should also be the same multiple of the overall riskiness of X. This characteristic has some similarity to the positive homogeneity in the classic setting. B5 says that for three random losses X, Y, and W, if the riskiness of W is the sum of that of X and Y in an almost-all-state-wise sense, then the overall riskiness of W should not exceed the sum of the overall riskiness of X and Y. This characteristic has some similarity to the subadditivity in the classic setting.

Remark 1.

(i) *Note that when arbitrarily given a random environment $Z \in L^\infty(\Omega)$, the existence of such a family $\{\rho_Z(\cdot;z); z \in Ran(Z)\}$ of CCRMs under a fixed state as in Definition 2.1 will be illustrated later; see (1) and Proposition 3.1 below. Given a family $\{\rho_Z(\cdot;z); z \in Ran(Z)\}$ of CCRMs under a fixed state such that B1-B5 hold, then we also say that the CRM under a random environment $\rho(\cdot;\cdot)$ is the corresponding CCRM under a random environment.*

(ii) *B4 implies that $\rho(0;Z) = 0$ for any random environment $Z \in L^\infty(\Omega)$. Both B4 and B3 imply that $\rho(a;Z) = a$ for $a \in \mathbb{R}$. Each of Axioms B2–B5 can imply Axiom B1.*

Now, we turn to construct CCRMs under a random environment. Given a random environment $Z \in L^\infty(\Omega)$ and a state $z \in \mathrm{Ran}(Z)$, let $\mathscr{Q}_z$ be a subset of $\mathscr{M}_{1,f}(\{Z = z\})$, and $\mathscr{Q}_Z$ be a subset of $\mathscr{M}_{1,f}(\mathrm{Ran}(Z))$. Then, we define CRMs under a fixed state and random environment $\rho_Z(\cdot;z)$ and $\rho(\cdot;\cdot)$, respectively, as follows:

$$\rho_Z(X;z) := \sup_{\mu \in \mathscr{Q}_z} E_\mu(X_z), \quad X \in L^\infty(\Omega), \tag{1}$$

and

$$\rho(X;Z) := \sup_{\nu \in \mathscr{Q}_Z} E_\nu(\rho_Z(X;\cdot)), \quad (X,Z) \in L^\infty(\Omega) \times L^\infty(\Omega), \tag{2}$$

where $\rho_Z(\cdot;z)$ is given by (1).

Remark 2.

(i) *For each $Q \in \mathscr{Q}_z$, it could be viewed as a plausible probability measure on $\{Z = z\}$, that is, the possible probability distribution $Q \circ X$ of X under the state z. When dealing with financial data, one should first confirm the random environment Z, for example, rate of interest, exchange rate, or the population mean, or so on, and then under each state of the random environment, use historical data in different time window or re-sampling to simulate the possible probability distributions of random loss X.*

(ii) *For each $\mu \in \mathscr{Q}_Z$, it could be regarded as a plausible probability distribution for Z. In practice, once the random environment Z is confirmed, one could use historical data in different time window or re-sampling to simulate the possible probability distributions of random environment Z.*

The next proposition shows the properties of the CRMs defined by (1) and (2).

Proposition 1.

(1) The CRM under a fixed state $\rho_Z(\cdot;z)$ defined by (1) is coherent, that is, it satisfies Axioms A1-A5.

(2) The CRM under a random environment ρ defined by (2) is coherent, that is, it satisfies Axioms B1-B5.

Proof. A1 and B1 are clear. A2–A5 and B2–B5 are straightforward since the expectation under the finitely additive probability measure is also linear with respect to the random variable. The proof is completed. □

Next, we proceed to characterize the CCRM under a random environment. The next theorem will show that any CCRM under a random environment is of a form as in (2), which is one of the main results of this paper.

Theorem 1. *Let $Z \in L^\infty(\Omega)$ be an arbitrarily given random environment. For any state $z \in Ran(Z)$, let $\rho_Z(\cdot;z) : L^\infty(\Omega) \to \mathbb{R}$ be a CCRM under a fixed state z, and $\rho(\cdot;\cdot) : L^\infty(\Omega) \times L^\infty(\Omega) \to \mathbb{R}$ be the corresponding CCRM under a random environment. Then,*

(1) $\rho_Z(\cdot;z)$ can be represented by

$$\rho_Z(X;z) = \sup_{\mu \in \mathscr{M}_z} E_\mu(X_z), \qquad X \in L^\infty(\Omega), \tag{3}$$

where $\mathscr{M}_z := \Big\{\mu \in \mathscr{M}_{1,f}(\{Z=z\}) : \ E_\mu(Y_z) \le \rho_Z(Y;z)$ for all $Y \in L^\infty(\Omega)\Big\}$.

(2) $\rho(\cdot;Z)$ can be represented by

$$\rho(X;Z) = \sup_{\nu \in \mathscr{M}_Z} E_\nu(\rho_Z(X;\cdot)), \qquad X \in L^\infty(\Omega), \tag{4}$$

where $\rho_Z(\cdot;z)$ is given by (3), and

$$\mathscr{M}_Z := \Big\{\nu \in \mathscr{M}_{1,f}(Ran(Z)) : \ E_\nu(\rho_Z(Y;\cdot)) \le \rho(Y;Z) \text{ for all } Y \in L^\infty(\Omega)\Big\}.$$

Proof.

(1) For any $X \in L^\infty(\{Z=z\})$, denote by X^* the natural extension of X from $\{Z=z\}$ to Ω, that is, $X^* : \ \Omega \to \mathbb{R}$ is defined by

$$X^*(\omega) := \begin{cases} X(\omega), & \text{if } \omega \in \{Z=z\}, \\ 0, & \text{if } \omega \notin \{Z=z\}. \end{cases}$$

Clearly, $X^* \in L^\infty(\Omega)$. Moreover, we define a risk measure $\rho : \ L^\infty(\{Z=z\}) \to \mathbb{R}$ by

$$\rho(X) := \rho_Z(X^*;z), \quad X \in L^\infty(\{Z=z\}). \tag{5}$$

First, we claim that the risk measure ρ defined by (5) is coherent. In fact, For any $X, Y \in L^\infty(\{Z=z\})$ with $X \le Y$, we know that $X^* \le Y^*$. Hence by Axiom A2, $\rho_Z(X^*;z) \le \rho_Z(Y^*;z)$. Thus, monotonicity of ρ follows from (5). Similarly, translation invariance of ρ follows from Axiom A3 and (5). Positive homogeneity of ρ follows from Axiom A4 and (5). Subadditivity of ρ follows from Axiom A5 and (5).

Therefore, by the representation theorem for coherent risk measures (e.g., see [5]), we know that the risk measure ρ can be represented by

$$\rho(V) = \sup_{\mu \in \mathscr{M}_z} E_\mu(V), \quad V \in L^\infty(\{Z = z\}), \tag{6}$$

where

$$\begin{aligned}\mathscr{M}_z &:= \Big\{\mu \in \mathscr{M}_{1,f}(\{Z = z\}) : \ E_\mu(W) \le \rho(W) \text{ for all } W \in L^\infty(\{Z = z\})\Big\} \\ &= \Big\{\mu \in \mathscr{M}_{1,f}(\{Z = z\}) : \ E_\mu(Y_z) \le \rho_Z(Y1_{\{Z=z\}}; z)) \text{ for all } Y \in L^\infty(\Omega)\Big\} \\ &= \Big\{\mu \in \mathscr{M}_{1,f}(\{Z = z\}) : \ E_\mu(Y_z) \le \rho_Z(Y; z) \text{ for all } Y \in L^\infty(\Omega)\Big\},\end{aligned}$$

where the first equality is due to (5), and the second equality is due to Axiom A1. Now, we proceed to show (3). For any $X \in L^\infty(\Omega)$, write $X_z^* := (X_z)^*$, that is, $X_z^* := X1_{\{Z=z\}}$. Clearly, $X_z^* \in L^\infty(\Omega)$. Recall that $X_z \in L^\infty(\{Z = z\})$. Hence, from Axiom A1, (5) and the definition of X_z^*, it follows that

$$\rho_Z(X; z) = \rho_Z(X1_{\{Z=z\}}; z) = \rho_Z(X_z^*; z) = \rho(X_z),$$

which, together with (6), implies (3). The desired assertion holds.

(2) An arbitrarily fixed random environment $Z \in L^\infty(\Omega)$. Denote

$$\mathscr{H}_Z := \{\rho_Z(Y; \cdot) : \ \mathrm{Ran}(Z) \to \mathbb{R}; \ Y \in L^\infty(\Omega)\}.$$

Note that for each $h \in \mathscr{H}_Z$, there is some $X^h \in L^\infty(\Omega)$ depending on h so that

$$h(z) = \rho_Z(X^h; z), \quad z \in \mathrm{Ran}(Z).$$

Hence, we define a risk measure $\rho_Z : \ \mathscr{H}_Z \to \mathbb{R}$ by

$$\rho_Z(h) := \rho(X^h; Z), \quad h \in \mathscr{H}_Z, \tag{7}$$

where $h(z) = \rho_Z(X^h; z)$, $z \in \mathrm{Ran}(Z)$, with some $X^h \in L^\infty(\Omega)$.
First, we claim that the risk measure ρ_Z defined by (7) is well defined. In fact, if there are two random variables $X^h, Y^h \in L^\infty(\Omega)$ such that both

$$h(z) = \rho_Z(X^h; z), \quad z \in \mathrm{Ran}(Z),$$

and

$$h(z) = \rho_Z(Y^h; z), \quad z \in \mathrm{Ran}(Z),$$

then $\rho_Z(X^h; z) = \rho_Z(Y^h; z)$ for any $z \in \mathrm{Ran}(Z)$. Hence, by Axiom B1 we know that $\rho_Z(X^h; Z) = \rho_Z(Y^h; Z)$. Thus, ρ_Z as in (7) is well defined.
Now, we further conclude that

$$\mathscr{H}_Z = L^\infty(\mathrm{Ran}(Z)). \tag{8}$$

Indeed, for any $Y \in L^\infty(\Omega)$, there exists $M \in \mathbb{R}_+$ such that $\|Y\| \le M$. Since $\rho_Z(\cdot; z)$ is local translation-invariant and local monotone, $|\rho_Z(Y; z)| \le M$ for any $z \in \mathrm{Ran}(Z)$. Thus,

$$\mathscr{H}_Z \subset L^\infty(\mathrm{Ran}(Z)). \tag{9}$$

Conversely, for any $h \in L^\infty(\mathrm{Ran}(Z))$, we define a random variable $X^{(h)} : \Omega \to \mathbb{R}$ by

$$X^{(h)}(\omega) := h(z), \quad \omega \in \Omega, \text{ if } Z(\omega) = z,\ z \in \mathrm{Ran(Z)}. \tag{10}$$

Clearly, $X^{(h)} \in L^\infty(\Omega)$. Moreover, for each $z \in \mathrm{Ran}(Z)$, $X^{(h)}(\omega) = h(z)$ whenever $\omega \in \{Z = z\}$. Hence, by Axiom A1, we have that for any $z \in \mathrm{Ran}(Z)$,

$$\rho_Z(X^{(h)}; z) = \rho_Z(X^{(h)} 1_{\{Z=z\}}; z) = \rho_Z(h(z) 1_{\{Z=z\}}; z) = \rho_Z(h(z); z) = h(z), \tag{11}$$

which yields that

$$L^\infty(\mathrm{Ran}(Z)) \subset \mathscr{H}_Z. \tag{12}$$

Therefore, (8) follows from (9) and (12).

□

Next, we proceed to show that the risk measure ρ_Z on $L^\infty(\mathrm{Ran}(Z))$ defined by (7) is coherent. In fact, due to Axioms B2-B4, it is not hard to verify that ρ_Z is monotone, translation-invariant, and positively homogenous. Hence, it is sufficient for us to show the subadditivity. Indeed, given any $h, g \in L^\infty(\mathrm{Ran}(Z))$, similar to (10), we define $X^{(h)}, X^{(g)} \in L^\infty(\Omega)$. By (11), we know that

$$h(z) = \rho_Z(X^{(h)}; z), \quad z \in \mathrm{Ran}(Z),$$

and

$$g(z) = \rho_Z(X^{(g)}; z), \quad z \in \mathrm{Ran}(Z).$$

Denote $f := h + g$. Then, $f \in L^\infty(\mathrm{Ran}(Z))$. Similar to (10), we define $X^{(f)} \in L^\infty(\Omega)$, and have that

$$f(z) = \rho_Z(X^{(f)}; z), \quad z \in \mathrm{Ran}(Z).$$

Thus, for any $z \in \mathrm{Ran}(Z)$,

$$\rho_Z(X^{(f)}; z) = h(z) + g(z) = \rho_Z(X^{(h)}; z) + \rho_Z(X^{(g)}; z),$$

which, together with Axiom B5, gives rise to

$$\rho(X^{(f)}; Z) \le \rho(X^{(h)}; Z) + \rho(X^{(g)}; Z). \tag{13}$$

From (13) and (7), it follows that

$$\rho_Z(h + g) = \rho_Z(f) \le \rho_Z(h) + \rho_Z(g), \tag{14}$$

which means that ρ_Z is subadditive. Therefore, ρ_Z is a coherent risk measure.

Consequently, by the representation theorem for coherent risk measures (e.g., see [5]), we know that the risk measure ρ_Z can be represented by

$$\rho_Z(h) = \max_{\nu \in \mathscr{M}_Z} E_\nu(h), \quad h \in L^\infty(\mathrm{Ran}(Z)),$$

where $\mathscr{M}_Z := \left\{\nu \in \mathscr{M}_{1,f}(\mathrm{Ran}(Z)) : \ E_\nu(g) \le \rho_Z(g) \text{ for all } g \in L^\infty(\mathrm{Ran}(Z))\right\}$. By (8) and the definition of ρ_Z as in (7), $\rho(\cdot; Z)$ can be represented by

$$\rho(X; Z) = \sup_{\nu \in \mathscr{M}_Z} E_\nu(\rho_Z(X; \cdot)), \quad X \in L^\infty(\Omega),$$

where $\mathscr{M}_Z := \left\{\nu \in \mathscr{M}_{1,f}(\mathrm{Ran}(Z)) : \ E_\nu(\rho_Z(Y; \cdot)) \le \rho(Y; Z) \text{ for all } Y \in L^\infty(\Omega)\right\}$. Thus, (4) is proved. Theorem 3.1 is proved.

3.2. Conditional Convex Risk Measures Under Random Environment

In this subsection, we extend the previous class of CCRMs under a random environment to a broader class of conditional convex risk measures under a random environment. We construct and axiomatically characterize conditional convex risk measures under a random environment.

Similar to the definition of classical convex risk measures, we start with the definition of conditional convex risk measures (CConRMs) under a fixed state.

Definition 5. *Given, arbitrarily, a random environment $Z \in L^\infty(\Omega)$ and a state $z \in Ran(Z)$, a CRM under a fixed state z $\rho_Z(\cdot; z) : L^\infty(\Omega) \to \mathbb{R}$ is called convex if it satisfies Axioms A1–A3 and A6 below:*

A6 Local convexity: *$\rho_Z(aX + (1-a)Y; z) \le a\rho_Z(X; z) + (1-a)\rho_Z(Y; z)$ for any $X, Y \in L^\infty(\Omega)$ and $a \in (0, 1)$.*

Similar to the explanation of the financial meanings of Axioms A1–A5 above, since the state of a given environment is pre-specified, the financial meanings of Axiom A6 above can be interpreted totally similarly to the classic setting; for instance, see [2,3,5].

Now, we turn to introduce the definition of conditional convex risk measures (CConRMs) under a random environment.

Definition 6. *A CRM under a random environment $\rho(X; Z) : L^\infty(\Omega) \times L^\infty(\Omega) \to \mathbb{R}$ is called convex If, for any random environment $Z \in L^\infty(\Omega)$ and each state $z \in Ran(Z)$, there is a CConRM under a fixed state z $\rho_Z(\cdot; z) : L^\infty(\Omega) \to \mathbb{R}$ such that Axioms B1–B3, B0 and B6 below hold:*

B0 Standard: *$\sup_{z \in Ran(Z)} |\rho_Z(0; z)| < \infty$.*

B6 Global convexity: *For any $X, Y, W \in L^\infty(\Omega)$ and $a \in (0, 1)$, if $\rho_Z(W; z) = a\rho_Z(X; z) + (1-a)\rho_Z(Y; z)$ for any $z \in Ran(Z)$, then $\rho(W; Z) \le a\rho(X; Z) + (1-a)\rho(Y; Z)$.*

Axioms B0 and B6 can be interpreted in the context of finance as follows. Note first that $\rho_Z(X; z)$ exactly represents the risk measure of random loss X under the condition that the environment Z takes the state z, as will be seen in the sequel. B0 can be viewed as a kind of technical condition. Nevertheless, B0 could be interpreted as follows: keeping global monotonicity B2 and risk consistency B3 in mind, B0 says that for any bounded random loss X, its riskiness should be uniformly finite with respect to almost all states of the random environment. B6 says that for three random losses X, Y, and W, if the riskiness of W is a certain convex combination of the riskiness of X and Y in an almost-all-state-wise sense, then the overall riskiness of W should not exceed the same combination of the

overall riskiness of X and Y. This characteristic has some similarity to the convexity in the classic setting.

Throughout this subsection, for a given random environment $Z \in L^\infty(\Omega)$ and any state $z \in \mathrm{Ran}(Z)$, let α_z be a functional from $\mathscr{M}_{1,f}(\{Z = z\})$ to $\mathbb{R} \cup \{+\infty\}$ with

$$\sup_{z \in \mathrm{Ran}(Z)} |\inf_{\mu \in \mathscr{M}_{1,f}(\{Z=z\})} \alpha_z(\mu)| < \infty.$$

Let α_Z be a functional from $\mathscr{M}_{1,f}(\mathrm{Ran}(Z))$ to $\mathbb{R} \cup \{+\infty\}$ with $|\inf_{\mu \in \mathscr{M}_{1,f}(\mathrm{Ran}(Z))} \alpha_Z(\mu)| < \infty$.

Now, we turn to construct CConRMs under a random environment. Given a random environment $Z \in L^\infty(\Omega)$ and a state $z \in \mathrm{Ran}(Z)$, we define CRMs under a fixed state and random environment $\rho_Z(\cdot; z)$ and $\rho(\cdot; \cdot)$, respectively, as follows:

$$\rho_Z(X; z) := \sup_{\mu \in \mathscr{M}_{1,f}(\{Z=z\})} (E_\mu(X_z) - \alpha_z(\mu)), \quad X \in L^\infty(\Omega), \tag{15}$$

and

$$\rho(X; Z) := \sup_{\nu \in \mathscr{M}_{1,f}(\mathrm{Ran}(Z))} (E_\nu(\rho_Z(X; \cdot)) - \alpha_Z(\nu)), \tag{16}$$

$(X, Z) \in L^\infty(\Omega) \times L^\infty(\Omega)$, where $\rho_Z(\cdot; z)$ is given by (15).

The next proposition shows the properties of the CRMs defined by (15) and (16).

Proposition 2.

(1) The CRM under a fixed state $\rho_Z(\cdot; z)$ defined by (15) is convex, that is, it satisfies Axioms A1-A3 and A6.

(2) The CRM under a random environment ρ defined by (16) is convex, that is, it satisfies Axioms B0-B3 and B6.

Proof. For any $X \in L^\infty(\Omega)$, there exists $M \in \mathbb{R}_+$ such that $|X| \leq M$. Note that the expectation under finitely additive probability measure is monotone, not only are both A2 and B2 obvious, but also it holds that

$$\rho_Z(-M; z) \leq \rho_Z(X; z) \leq \rho_Z(M; z).$$

Since the expectation under finitely additive probability measure is also translation-invariant, A3 and B3 are clear. Moreover,

$$\rho_Z(0; z) - M \leq \rho_Z(X; z) \leq \rho_Z(0; z) + M. \tag{17}$$

Meanwhile, from $\sup_{z \in \mathrm{Ran}(Z)} |\inf_{\mu \in \mathscr{M}_{1,f}} \alpha_z(\mu)| < \infty$, we can obtain B0. Together with (17), it holds that there exists $N \in \mathbb{R}_+$ such that

$$|\rho_Z(X; z)| \leq N, \quad z \in \mathrm{Ran}(Z).$$

Thus, by B2, B3 and $|\inf_{\mu \in \mathscr{M}_{1,f}(\mathrm{Ran}(Z))} \alpha_Z(\mu)| < \infty$, we have that $|\rho(X; Z)| < \infty$.

For any $X, Y \in L^\infty(\Omega)$, we have that

$$\begin{aligned}\rho_Z(aX+(1-a)Y;z) &= \sup_{\mu\in\mathscr{M}_{1,f}(\{Z=z\})}(E_\mu((aX+(1-a)Y)_z)-\alpha_z(\mu))\\ &= \sup_{\mu\in\mathscr{M}_{1,f}(\{Z=z\})}(aE_\mu(X_z)+(1-a)E_\mu(Y_z)-\alpha_z(\mu))\\ &\le a\sup_{\mu\in\mathscr{M}_{1,f}(\{Z=z\})}(E_\mu(X_z)-\alpha_z(\mu))\\ &\quad+(1-a)\sup_{\mu\in\mathscr{M}_{1,f}(\{Z=z\})}(E_\mu(Y_z)-\alpha_z(\mu))\\ &= a\rho_Z(X;z)+(1-a)\rho_Z(Y;z) \end{aligned} \tag{18}$$

for any $a \in (0,1)$. Hence, A6 is proved. □

For any $X, Y \in L^\infty(\Omega)$, by (18), for any $a \in (0,1)$, we have that

$$\begin{aligned}\rho(aX+(1-a)Y;Z) &= \sup_{\nu\in\mathscr{M}_{1,f}(\mathrm{Ran}(Z))}(E_\nu(\rho_Z(aX+(1-a)Y;z))-\alpha_Z(\nu))\\ &\le \sup_{\nu\in\mathscr{M}_{1,f}(\mathrm{Ran}(Z))}(E_\nu(a\rho_Z(X;z)+(1-a)\rho_Z(Y;z))-\alpha_Z(\nu))\\ &\le a\rho(X;Z)+(1-a)\rho(Y;Z).\end{aligned}$$

Therefore, B6 is proved. The proposition is proved.

Next, we proceed to characterize the CConRM under a random environment. The next theorem will show that any CConRM under a random environment is of a form as in (16), which is another one of the main results of this paper.

Theorem 2. *Let $Z \in L^\infty(\Omega)$ be an arbitrarily given random environment. For any state $z \in Ran(Z)$, let $\rho_Z(\cdot;z) : L^\infty(\Omega) \to \mathbb{R}$ be a CConRM under a fixed state z, and $\rho(\cdot;\cdot) : L^\infty(\Omega) \times L^\infty(\Omega) \to \mathbb{R}$ be the corresponding CConRM under a random environment. Then,*

(1) $\rho_Z(\cdot;z)$ can be represented by

$$\rho_Z(X;z) = \sup_{\mu\in\mathscr{M}_{1,f}(\{Z=z\})}(E_\mu(X_z)-\alpha_z(\mu)),\quad X\in L^\infty(\Omega), \tag{19}$$

where

$$\alpha_z(\mu) := \sup_{X\in L^\infty(\Omega):\ \rho_Z(X;z)\le 0} E_\mu(X_z),\quad \mu\in\mathscr{M}_{1,f}(\{Z=z\}).$$

(2) $\rho(\cdot;Z)$ can be represented by

$$\rho(X;Z) = \sup_{\nu\in\mathscr{M}_{1,f}(Ran(Z))}(E_\nu(\rho_Z(X;\cdot))-\alpha_Z(\nu)),\quad X\in L^\infty(\Omega), \tag{20}$$

where $\rho_Z(\cdot;z)$ is given by (19), and

$$\alpha_Z(\nu) := \sup_{X\in L^\infty(\Omega):\ \rho(X;Z)\le 0} E_\nu(\rho_Z(X;\cdot)),\quad \nu\in\mathscr{M}_{1,f}(Ran(Z)).$$

Proof.

(1) For any given $z \in \mathrm{Ran}(Z)$, we define the risk measure $\rho(\cdot) : L^\infty(\{Z = z\}) \to \mathbb{R}$ as in (5), that is,

$$\rho(X) := \rho_Z(X^*; z), \quad X \in L^\infty(\{Z = z\}),$$

where $X^* \in L^\infty(\Omega)$ is the natural extension of X from $\{Z = z\}$ to Ω. Similar to the proof of Theorem 3.1(1), we can steadily show that ρ is a convex risk measure on $L^\infty(\{Z = z\})$.
Therefore, by the representation theorem of convex risk measure (e.g., see [5]), we know that the risk measure ρ can be represented as

$$\rho(V) = \max_{\mu \in \mathscr{M}_{1,f}(\{Z=z\})} (E_\mu(V) - \alpha(\mu)), \quad V \in L^\infty(\{Z = z\}),$$

where

$$\alpha(\mu) := \sup_{V \in L^\infty(\{Z=z\}):\, \rho(V) \le 0} E_\mu(V), \quad \mu \in \mathscr{M}_{1,f}(\{Z = z\}).$$

Consequently, by the definition of ρ, we obtain the desired assertion (19).

(2) Basically, the argument is quite similar to the proof of Theorem 1(2). Specifically, an arbitrarily fixed random environment $Z \in L^\infty(\Omega)$. Similarly, denote

$$\mathscr{H}_Z := \{\rho_Z(Y; \cdot) :\ \mathrm{Ran}(Z) \to \mathbb{R};\ Y \in L^\infty(\Omega)\}.$$

Note that for each $h \in \mathscr{H}_Z$, there is some $X^h \in L^\infty(\Omega)$ depending on h so that

$$h(z) = \rho_Z(X^h; z), \quad z \in \mathrm{Ran}(Z).$$

Similar to (7), we well define a risk measure $\rho_Z :\ \mathscr{H}_Z \to \mathbb{R}$ by

$$\rho_Z(h) := \rho(X^h; Z), \quad h \in \mathscr{H}_Z, \tag{21}$$

where $h(z) = \rho_Z(X^h; z)$, $z \in \mathrm{Ran}(Z)$, with some $X^h \in L^\infty(\Omega)$.

□

Now, we claim that

$$\mathscr{H}_Z = L^\infty(\mathrm{Ran}(Z)). \tag{22}$$

In fact, for any $Y \in L^\infty(\Omega)$, there exists $M \in \mathbb{R}_+$ such that $|Y| \le M$. For any $z \in \mathrm{Ran}(Z)$, by the local monotonicity and local translation invariance of $\rho_Z(\cdot; z)$,

$$\rho_Z(0; z) - M = \rho_Z(-M; z) \le \rho_Z(Y; z) \le \rho_Z(M; z) = \rho_Z(0; z) + M, \tag{23}$$

which, togetrher with Axiom B0, yields that $|\rho_Z(Y; z)| < \infty$ for any $z \in \mathrm{Ran}(Z)$. Therefore,

$$\mathscr{H}_Z \subset L^\infty(\mathrm{Ran}(Z)). \tag{24}$$

Conversely, for any $h \in L^\infty(\mathrm{Ran}(Z))$, similar to (10), we define a random variable $X^{(h)} : \Omega \to \mathbb{R}$ by

$$X^{(h)}(\omega) := h(z) - \rho_Z(0; z), \quad \omega \in \Omega, \text{ if } Z(\omega) = z, z \in \mathrm{Ran(Z)}. \tag{25}$$

Clearly, $X^{(h)} \in L^\infty(\Omega)$. Moreover, for each $z \in \mathrm{Ran}(Z)$, $X^{(h)}(\omega) = h(z) - \rho_Z(0;z)$ whenever $\omega \in \{Z = z\}$. Hence, by Axiom A1 and the local translation invariance of $\rho_Z(\cdot;z)$, we have that for any $z \in \mathrm{Ran}(Z)$,

$$\rho_Z(X^{(h)};z) = \rho_Z(X^{(h)}1_{\{Z=z\}};z) = \rho_Z(h(z);z) - \rho_Z(0;z) = [h(z) + \rho_Z(0;z)] - \rho_Z(0;z) = h(z),$$

which implies that

$$L^\infty(\mathrm{Ran}(Z)) \subset \mathscr{H}_Z. \tag{26}$$

Therefore, (22) follows from (24) and (26).

Next, we proceed to show that the risk measure ρ_Z on $L^\infty(\mathrm{Ran}(Z))$ defined by (21) is convex. Indeed, from Axioms B2 and B3, it follows that ρ_Z is monotone and translation-invariant. Moreover, with the help of Axiom B6, by an argument similar to (14), we can show the convexity of ρ_Z. Therefore, the risk measure ρ_Z is convex.

Consequently, by the representation theorem for convex risk measures (e.g., see [5]), we know that the risk measure ρ_Z can be represented by

$$\rho_Z(h) = \sup_{\nu \in \mathscr{M}_{1,f}(\mathrm{Ran}(Z))} (E_\nu(h) - \alpha_Z(\nu)), \quad h \in L^\infty(\mathrm{Ran}(Z)),$$

where

$$\alpha_Z(\nu) := \sup_{h \in L^\infty(\mathrm{Ran}(Z)):\, \rho_Z(h) \le 0} E_\nu(h), \quad \nu \in \mathscr{M}_{1,f}(\mathrm{Ran}(Z)).$$

By (22) and the definition of ρ_Z as in (21), $\rho(\cdot;Z)$ can be represented by

$$\rho(X;Z) = \sup_{\nu \in \mathscr{M}_{1,f}(\mathrm{Ran}(Z))} (E_\nu(\rho_Z(X;\cdot)) - \alpha_Z(\nu)), \quad X \in L^\infty(\Omega),$$

where $\alpha_Z(\nu) := \sup_{X \in L^\infty(\Omega):\, \rho(X;Z) \le 0} E_\nu(\rho_Z(X;\cdot))$, $\nu \in \mathscr{M}_{1,f}(\mathrm{Ran}(Z))$. The desired assertion (20) is proved. The proof of Theorem 3.2 is completed.

4. Examples

In this section, through two examples, we discuss the connections between our axiomatic approach and some existing risk measures in the literature. The first example examines the connection with the factor-separable risk measures introduced by [24]. The second example shows that the random environment can describe the model uncertainty discussed by [7].

Example 1 (Factor-separable risk measures). *The factor-separable risk measures are introduced by [24]. Let $(\Omega, \mathscr{F})$ be certain measurable space. We denote by $\mathscr{X}$ the space of all measurable functions (i.e., random variables) on $(\Omega, \mathscr{F})$. In the construction of factor-separable risk measures, assume that a risk (random) factor Z takes finitely many values $\{z_1, \ldots, z_m\}$, and Ω is partitioned into (disjoint) subsets $\Omega = \cup_{i=1}^m \Omega_i$, where $\Omega_i := \{\omega \in \Omega : Z(\omega) = z_i\}$, $i = 1, \ldots, m$. For the risk factor Z, a factor-separable risk measure ρ_Z is then defined as*

$$\rho_Z(X) := \rho\left(\sum_{i=1}^m \rho_i(X) 1_{\Omega_i}\right), \quad X \in \mathscr{X}, \tag{27}$$

where ρ and ρ_i, $i = 1, \ldots, m$, are coherent risk measures. For more details, we refer to [24].

Next, for our discussion convenience, we set $\mathscr{F} = 2^{\Omega}$ and $\mathscr{X} = L^{\infty}(\Omega)$. Compared with the random loss $X \in L^{\infty}(\Omega)$, we view the risk factor $Z \in L^{\infty}(\Omega)$ as a *random environment*. Since ρ is coherent, by the representation for coherent risk measures (e.g., see [5]), there is a subset $\mathscr{Q}$ of $\mathscr{M}_{1,f}(\Omega)$ so that

$$\rho\left(\sum_{i=1}^{m}\rho_i(X)1_{\Omega_i}\right) = \sup_{\nu\in\mathscr{Q}} E_{\mu}\left(\sum_{i=1}^{m}\rho_i(X)1_{\Omega_i}\right), \quad X \in L^{\infty}(\Omega), \tag{28}$$

Indeed, the set $\mathscr{Q}$ above can be chosen as

$$\mathscr{Q} := \{\mu \in \mathscr{M}_{1,f}(\Omega) : \ E_{\mu}(Y) \le \rho(Y) \text{ for all } Y \in L^{\infty}(\Omega)\}.$$

Notice that for any $\mu \in \mathscr{Q}$, by the linearity of expectation,

$$E_{\mu}\left(\sum_{i=1}^{m}\rho_i(X)1_{\Omega_i}\right) = \sum_{i=1}^{m}\rho_i(X)\mu(\Omega_i), \quad X \in L^{\infty}(\Omega). \tag{29}$$

For any given random loss $X \in L^{\infty}(\Omega)$, we define a random variable $\rho_Z(X;\cdot)$: $\mathrm{Ran}(Z) \to \mathbb{R}$ by

$$\rho_Z(X; z_i) := \rho_i(X), \quad z_i \in \mathrm{Ran}(Z).$$

Since ρ_i is coherent, for each $z_i \in \mathrm{Ran}(Z)$, $\rho_Z(\cdot; z_i)$ is a CCRM under a fixed state z_i. Furthermore, for any $\mu \in \mathscr{Q}$, we define a probability measure ν_{μ} on $2^{\mathrm{Ran}(Z)}$ as follows:

$$\nu_{\mu}(\{z_i\}) := \mu(\Omega_i), \quad z_i \in \mathrm{Ran}(Z),$$

and

$$\nu_{\mu}(A) := \sum_{i:\ z_i\in A} \mu(\{z_i\}) = \sum_{i:\ z_i\in A} \mu(\Omega_i), \quad A \in 2^{\mathrm{Ran}(Z)}.$$

We endow the random variable $\rho_Z(X;\cdot)$ a probability mass function under ν_{μ} as follows:

$$\nu_{\mu}(\rho_Z(X;\cdot) = \rho_i(X)) := \mu(\Omega_i), \quad i \in \{1, \cdots, m\}.$$

Then,

$$\sum_{i=1}^{m}\rho_i(X)\mu(\Omega_i) = E_{\nu_{\mu}}(\rho_Z(X;\cdot)). \tag{30}$$

Thus, from (27)–(30), it follows that the factor-separable risk measure ρ_Z defined by (27) can be expressed by

$$\rho_Z(X) = \sup_{\nu\in\mathscr{Q}_Z} E_{\nu}(\rho_Z(X;\cdot)), \quad X \in L^{\infty}(\Omega),$$

where $\mathscr{Q}_Z := \{\nu_{\mu} : \ \mu \in \mathscr{Q}\} \subset \mathscr{M}_{1,f}(\mathrm{Ran}(Z))$, which is in accordance with the CCRM $\rho(\cdot; Z)$ under a random environment Z as in (2).

In the next example, we will show that the random environment can describe the model uncertainty.

Example 2 (Scenario-based risk measures). *Ref. [7] introduced the $\mathscr{Q}$-mixture of ES. Let $\mathscr{Q} := \{Q_1, \cdots, Q_n\}$ be a set of probability measures on some measurable space $(\Omega, \mathscr{F})$. We denote by $\hat{\rho}$ the $\mathscr{Q}$-mixture of ES, which is defined by*

$$\hat{\rho}(X) := \sum_{i=1}^{n} w_i \int_0^1 ES_p^{Q_i}(X) dh_i(p), \quad X \in L^\infty(\Omega, \mathscr{F}),$$

for some $\mathbf{w} = (w_1, \cdots, w_n) \in [0,1]^n$ with $\sum_{i=1}^n w_i = 1$ and distribution functions $h_1, \cdots, h_n$ on $[0,1]$, where $ES_p^{Q_i}(X)$ stands for the expected shortfall (ES) of X under probability measure Q_i with probability level $p \in (0,1)$. For more details, we refer to [7].

Next, for our discussion convenience, we set $\mathscr{F} = 2^\Omega$. We will discuss the connection of the CCRMs under a random environment with the $\mathscr{Q}$-mixture of ES.

As an important case studied by [7], we assume that the probability measures $Q_1, \cdots, Q_n$ are mutually singular, that is, there exist finitely many $A_1, \cdots, A_n \in \mathscr{F}$, such that $\bigcup_{i=1}^n A_i = \Omega$, $A_i A_j = \emptyset$ for any $i \neq j$ and $Q_i(A_i) = 1$, $i = 1, \cdots, n$. In this situation, we define a random environment $Z : \Omega \to \mathbb{R}$ by $Z(\omega) = i$, if $\omega \in A_i$ for some (unique) $i \in \{1, \cdots, n\}$. Clearly, $\mathrm{Ran}(Z) = \{1, \cdots, n\}$. Moreover, for each $i \in \mathrm{Ran}(Z)$, we define a distortion function g_i by

$$g_i(x) := \int_0^x \int_{(t,1]} s^{-1} dh_i(s) dt, \quad x \in [0,1], \tag{31}$$

where h_i is some distribution function on $[0,1]$. By [5], we know that the distortion risk measure of a random loss X with the distortion function g_i under the probability measure Q_i is a weighted value at risk (WVaR), that is,

$$\int X dg_i \circ Q_i = \int_0^1 \mathrm{ES}_p^{Q_i}(X) dh_i(p).$$

For each $i \in \mathrm{Ran}(Z)$, we define a risk measure $\rho_Z(\cdot; i) : L^\infty(\Omega) \to \mathbb{R}$ by

$$\rho_Z(X; i) := \int X dg_i \circ Q_i = \int_0^1 \mathrm{ES}_p^{Q_i}(X) dh_i(p), \quad X \in L^\infty(\Omega).$$

Then, it is not hard to verify that $\rho_Z(\cdot; i)$ is a CCRM under a fixed state i. Furthermore, by (2), for any random loss $X \in L^\infty(\Omega)$, the corresponding CCRM of X under a random environment Z is defined by

$$\rho(X; Z) := \sup_{(w_1, \cdots, w_n) \in \mathscr{W}_Z} \sum_{i=1}^{n} w_i \rho_Z(X; i),$$

where $\mathscr{W}_Z$ is some subset of $\mathscr{M}_{1,f}(\mathrm{Ran}(Z)) = \mathscr{W} := \{(w_1, \cdots, w_n) \in [0,1]^n : \sum_{i=1}^n w_i = 1\}$. Hence,

$$\rho(X; Z) = \sup_{(w_1, \cdots, w_n) \in \mathscr{W}_Z} \sum_{i=1}^{n} w_i \int_0^1 \mathrm{ES}_p^{Q_i}(X) dh_i(p), \quad X \in L^\infty(\Omega).$$

Thus, by Proposition 1 we can see that a CCRM $\rho(\cdot; Z)$ under a random environment Z becomes a supremum of $\mathscr{Q}$-mixtures of ES.

5. Empirical Study

In this section, we take data from the stock market to calculate the risk measure for daily returns with respect to six stocks: Tesla, Netflix, Apple, Microsoft, P&G, Johnson &

Johnson. The random environment is taken as the risk-free interest rate. We categorize the levels of the risk-free interest rate into three levels: high, medium, and low. We use the re-sampling method to simulate different potential distributions of the risk-free interest rate. Each re-sampling of the distribution can be regarded as a ν in (2). Under each level of the risk-free interest rate, we also use the re-sampling method to calculate the potential distributions of the daily return of these six stocks and take the average of the worst 10% to obtain $E_\mu(X_z)$ in (1). Then, by taking the maximum of all the re-samplings for $E_\mu(X_z)$, we obtain the $\rho_Z(X;z)$ as in (1). Furthermore, by taking the maximum of the weighted sum of $\rho_Z(X;z)$ with respect to the distribution of ν, we finally achieve the risk measure for these six stocks.

The data for risk-free interest rate are shown in Figure 1. The data for daily stock price is shown are Figure 2. The data for daily returns of all the six stocks are shown in Figure 3. One can notice that the range of the daily returns for Tesla and Netflix is the most drastic, the range for Apple and Mircosoft is less drastic, and the performance for P&G and Johnson & Johnson is much more stable. Also, from the risk measures for daily stock returns, which are listed in Table 1, one can also observe the same results.



Figure 1. Risk-free interest rate.

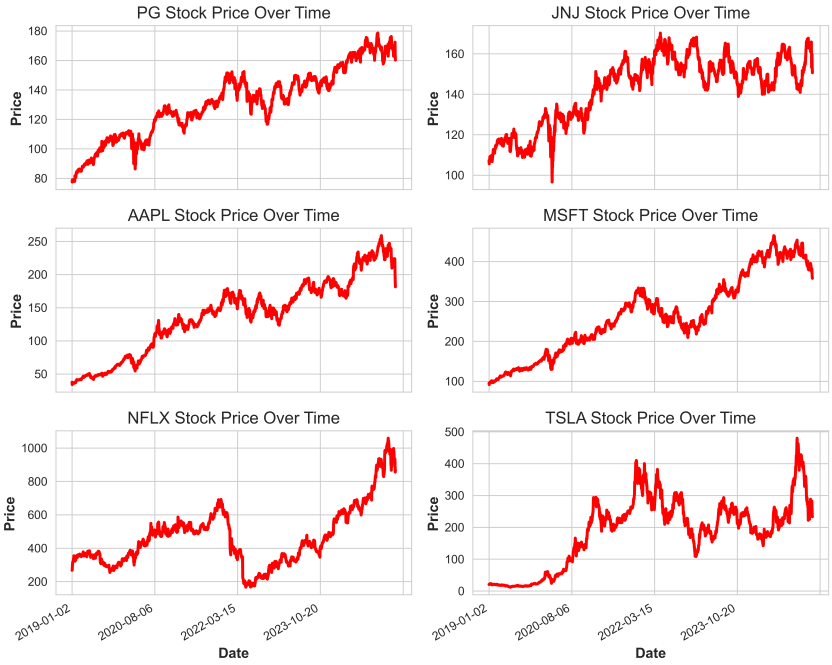

Figure 2. Daily stock price.

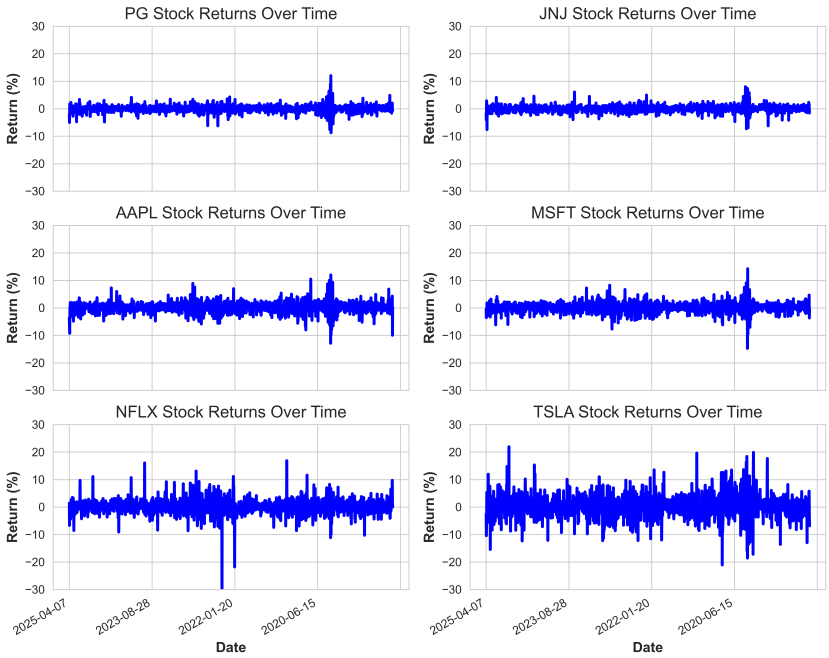

Figure 3. Daily stock return.

Table 1. Risk measures for daily stock returns.

PG	−2.1696566
JNJ	−2.047720133
AAPL	−3.521605
MSFT	−3.1755596
NFLX	−4.402695
TSLA	−7.143867533

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