mathematics

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A Class of Non-Hopf Bi-Frobenius Algebras Generated by n Elements

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Abstract: Bi-Frobenius algebras are a class of Frobenius algebras and Frobenius coalgebras with some compatible conditions. In this paper, we construct a class of bi-Frobenius algebras generated by n elements on graded algebra A. The comultiplication and counit are defined via a permutation π on A, such that A becomes a bi-Frobenius algebra. For any n, these bi-Frobenius algebras are neither Hopf algebras nor S-type bi-Frobenius algebras.

Keywords: Hopf algebras; Frobenius algebras; Frobenius coalgebras; bi-Frobenius algebras

MSC: 16T05; 16T15

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Academic Editor: Askar Tuganbaev

Received: 18 March 2025
Revised: 16 April 2025
Accepted: 20 April 2025
Published: 21 April 2025

Citation: Fa, Z.; Wang, Y. A Class of Non-Hopf Bi-Frobenius Algebras Generated by n Elements. *Mathematics* **2025**, *13*, 1357. https://doi.org/10.3390/math13081357

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1. Introduction

Finite-dimensional group algebras and finite-dimensional Hopf algebras are Frobenius algebras. As a generalization of finite-dimensional Hopf algebras, the concept of bi-Frobenius algebras was introduced by Doi and Takeuchi in [1]. Roughly speaking, this is a Frobenius algebra as well as a Frobenius coalgebra together with an antipode. Therefore, finite-dimensional Hopf algebras are bi-Frobenius algebras, but bi-Frobenius algebras are not necessarily finite-dimensional Hopf algebras. In [2], Haim showed that a bi-Frobenius algebra is a Hopf algebra if and only if it is a bialgebra. Consequently, constructing examples of bi-Frobenius algebras that are not Hopf algebras provides a meaningful approach to study bi-Frobenius algebras. The theory of bi-Frobenius algebras was further developed in [2–9]. So far, examples of bi-Frobenius algebras can be found in [1,2,4,6–9]. In [1], Doi and Takeuchi presented a class of bi-Frobenius algebras whose underlying algebras are given by $\mathbb{K}[x]/(x^n)$. In [2], Haim constructed a class of bi-Frobenius algebras on group-like algebras. Zhu and Tang presented a class of bi-Frobenius algebras by the extension of bialgebras in [9]. Additionally, by using quiver methods, a class of bi-Frobenius algebras was constructed in [6,7]. Jin and Zhang constructed a class of bi-Frobenius algebras on quantum complete intersections in [4,10]. In [8], a class of bi-Frobenius algebras generated by $2n$ elements was constructed.

In [11], an algebra A is Koszul AS regular if and only if the Koszul dual of A is Koszul and Frobenius. Based on the results in [12–14], the purpose of this paper is to construct a class bi-Frobenius algebras generated by n elements, which is a class of Yoneda algebras of Artin–Schelter regular algebras. Moreover, bi-Frobenius algebras in this article have a different Frobenius coalgebra structure from [8].

Let $\Lambda = (\lambda_{ij})_{n\times n}$ be a non-degenerate matrix and A be a $\mathbb{Z}$-graded algebra with $\mathbb{K}$-space $A = A_0 \oplus A_1 \oplus A_2$, where $A_0 = \mathbb{K}1_A$, $A_1 = \mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n$, $A_2 = \mathbb{K}y$. The

algebra structure of A is given by $x_ix_j = \lambda_{ij}y, x_iy = yx_i = y^2 = 0$, for $i = 1, 2, \cdots, n$. We endow a comultiplication on A by a permutation π (see Lemma 2), thus A is a pre-bi-Frobenius algebra. Then, we provide the necessary and sufficient condition fir A to become a bi-Frobenius algebra (See Theorem 1). Moreover, these bi-Frobenius algebras are not S-type bi-Frobenius algebras (see Remark 2). These bi-Frobenius algebras are different from bi-Frobenius algebras in [6–8] (see Remark 3). Furthermore, we show that the Nakayama and co-Nakayama automorphisms of the bi-Frobenius algebra A are the identity, and the antipode ψ satisfies $\psi^2 = \mathrm{id}$ (see Proposition 2). Ultimately, we establish the necessary conditions for the automorphism of A (see Proposition 3).

Throughout, $\mathbb{K}$ will be a field with $\mathrm{char}\,\mathbb{K} = 0$. All algebras, coalgebras, and tensors are over $\mathbb{K}$. We use Sweedler's notation for the comultiplication of coalgebras.

2. Preliminaries

In this section, we review the definition and some basic properties of bi-Frobenius algebras.

Assume that A is a finite-dimensional algebra. Then, the dual space $A^* = \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$ has a natural A-A-bimodules structure given by $(af)(b) = f(ba)$, $(fa)(b) = f(ab)$, for any $f \in A^*$ and $a, b \in A$. We say that (A, ϕ) with $\phi \in A^*$ is a *Frobenius algebra* if $A^* = A\phi$ or $A^* = \phi A$. The homomorphism ϕ is called a *Frobenius basis*. For more about Frobenius algebras, see [1,14–16].

Dually, let (C, Δ, ϵ) be a finite-dimensional coalgebra with comultiplication Δ and counit ϵ. Then, C is a C^*-C^*-bimodule via $fc = \sum c_1 f(c_2)$, $cf = \sum f(c_1)c_2$, for any $f \in C^*$ and $c \in C$. A pair (C, t) with $t \in C$ is called a *Frobenius coalgebra* if $C = tC^*$ or equivalently $C = C^*t$. The element t is called a *coFrobenius basis* of C. We refer to [1] for the notion of Frobenius coalgebras.

Let V and W be vector spaces and $a \in V \otimes W$. By the *length* of $a \in V \otimes W$, we mean the smallest number l of all possible expression $a = \sum_{i=1}^{l} v_i \otimes w_i$. Then, $(C,\ t)$ is a Frobenius coalgebra if and only if the dimension of C is equal to the length of $\Delta(t)$ (see 1.3 in [1]).

Definition 1. *Let A be a finite-dimensional algebra and coalgebra with $\phi \in A^*$ and $t \in A$. We call the triple (A, ϕ, t) a pre-bi-Frobenius algebra if (A, ϕ) is a Frobenius algebra and (A, t) is a Frobenius coalgebra.*

If (A, ϕ, t) is a pre-bi-Frobenius algebra, then ϕ and t can be chosen, such that $\phi(t) = 1$. Define a linear map

$$\psi :\ A \longrightarrow A,\ \psi(a) = \sum \phi(t_1 a)t_2,\ a \in A. \tag{1}$$

Following Doi and Takeuchi, we recall the definition of bi-Frobenius algebras (see Definition 1.1 in [3] or 2.2 in [1]).

Definition 2. *Let (A, ϕ, t) be a pre-bi-Frobenius algebra. The quadruple $(A,\ \phi,\ t,\ \psi)$ is called a bi-Frobenius algebra if the following two conditions hold:*

(B1) *1_A is a group-like element and ϵ is an algebra map.*
(B2) *$\psi : A \longrightarrow A$ is an anti-algebra and anti-coalgebra map.*

The linear map ψ is called the antipode of bi-Frobenius algebra (A, ϕ, t, ψ). If in addition, the antipode ψ is the convolution inverse of the identity in $\mathrm{Hom}_{\mathbb{K}}(A, A)$, i.e., $\psi * id = u\epsilon = id * \psi$, then we say the bi-Frobenius algebra A is S-type.

If A is a bi-Frobenius algebra, then there exists the following linear isommorphisms:

$$\theta : A \cong A^*, \quad a \mapsto \phi \leftharpoonup a, \quad \text{and} \quad \kappa : A^* \cong A, \quad f \mapsto t \leftharpoonup f.$$

$$\theta' : A \cong A^*, \quad a \mapsto a \rightharpoonup \phi, \quad \text{and} \quad \kappa' : A^* \cong A, \quad f \mapsto f \rightharpoonup t.$$

Then we can obtain

$$\psi = \kappa \circ \theta', \quad \mathcal{N} = \theta'^{-1} \circ \theta, \quad \text{and} \quad {}^c\mathcal{N} = \kappa' \circ \kappa^{-1}.$$

Here, $\mathcal{N}$ is the Nakayama automorphism of (A, ϕ), satisfying $\phi(xy) = \phi(y\mathcal{N}(x))$. Meanwhile, ${}^c\mathcal{N}$ is the coNakayama automorphism of (A, t), satisfying $\sum_c \mathcal{N}(t_2) \otimes t_1 = \sum t_1 \otimes t_2$.

3. The Construction

Let n be a positive integer and $A = A_0 \oplus A_1 \oplus A_2 = \mathbb{K}1_A \oplus (\mathbb{K}x_1 \oplus \mathbb{K}x_2 \oplus \cdots \oplus \mathbb{K}x_n) \oplus \mathbb{K}y$ a $\mathbb{Z}$-graded vector space. Let $\Lambda = (\lambda_{ij})_{n\times n}$ be a non-degenerate matrix and let Λ^T denote its transpose.

Define a multiplication on A as follows:

$$x_i x_j = \lambda_{ij} y, \;\; x_i y = y x_i = y^2 = 0, \tag{2}$$

for $i, j = 1, \cdots, n$. It is obvious that the multiplication of A is associative. We call Λ the *structure matrix* of A.

Lemma 1. *The algebra A is a Frobenius algebra.*

Proof. As $\Lambda = (\lambda_{ij})_{n\times n}$ is a non-degenerate matrix, one has a unique non-degenerate associative bilinear form $\chi : A \times A \longrightarrow \mathbb{K}$, such that $\chi(x_i, x_j) = \lambda_{ij}$, $\chi(1, y) = \chi(y, 1) = 1$ and $\chi(1, x_i) = \phi(x_i, 1) = \phi(y, x_i) = \phi(x_i, y) = 0$. Let ϕ be $\chi(1, -) = \chi(-, 1) \in A^*$. Then, (A, ϕ) is a Frobenius algebra with a Frobenius basis ϕ. Clearly, $\phi(y) = 1$ and $\phi(1) = \phi(x_i) = 0$ for $i \in \{1, 2, \cdots, n\}$. □

Let $\tau_k = (12\cdots k)$, and $P_{\tau_k} = \begin{pmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 1 & & & & \end{pmatrix}$ be the permutation matrix of τ_k. Generally, let $\pi_k = (i_1 i_2 \cdots i_k)$, and P_{π_k} be the permutation matrix of π_k. Then, $P_{\tau_k} + P_{\tau_k{}^{-1}}$ is similiar to $P_{\pi_k} + P_{\pi_k{}^{-1}}$.

Let $\pi \in S_n$, $i_j \in \{1, 2, \cdots, n\}$ and $\pi_{k_j} = (i_j \pi(i_j) \cdots \pi^{k_j - 1}(i_j))$, then

$$\pi = \pi_{k_1} \pi_{k_2} \cdots \pi_{k_s}, \tag{3}$$

for any $1 \leq s \leq n$ and $k_1 + k_2 + \cdots + k_s = n$. It is obvious that $\pi_{k_1}, \pi_{k_2}, \cdots, \pi_{k_s}$ are disjoint permutations.

In the following, denote $[\cdot]$ as the floor function.

Lemma 2. *The matrix P is similiar to the block matrix*

$$\begin{pmatrix} P_{\tau_{k_1}} + P_{\tau_{k_1}^{-1}} & & & \\ & P_{\tau_{k_2}} + P_{\tau_{k_2}^{-1}} & & \\ & & \ddots & \\ & & & P_{\tau_{k_s}} + P_{\tau_{k_s}^{-1}} \end{pmatrix}.$$

By applying the algorithm proposed in [17], we compute that when k_j is a multiple of 4, $|P_{\tau_{k_j}} + P_{\tau_{k_j}^{-1}}| = 0$, where $|\cdot|$ evaluates the determinant. In what follows, we provide a proof of this fact.

Lemma 3. *Let π be a permutation in S_n. Then, $|P| = 0$ if and only if there exists some j, such that $k_j = 4m$ for any positive integer $m \leq [\frac{n}{4}]$.*

Proof. Let $m \leq n$ be a positive integer. Note that P is similiar to the block matrix that contains a $4m$-th block

$$P_{\tau_{4m}} + P_{\tau_{4m}^{-1}} = \begin{pmatrix} 0 & 1 & & & & 1 \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 \\ 1 & & & & 1 & 0 \end{pmatrix},$$

then, the eigenvalues of $P_{\tau_{4m}} + P_{\tau_{4m}^{-1}}$ are $\lambda_j = \omega_j + \omega_j^{4m-1}$, where ω_j is a primitive $4m$-th root of unity with $0 \leq j \leq 4m-1$. Particularly,

$$\lambda_m = \omega_m + \omega_m^{4m-1} = e^{\frac{2\pi i m}{4m}} + e^{\frac{2\pi i(4m-1)}{4m}} = e^{\frac{\pi i}{2}} + e^{-\frac{\pi i}{2}} = 0,$$

then $|P_{\tau_{4m}} + P_{\tau_{4m}^{-1}}| = 0$.

On the contrary, if $|P| = 0$, there exists an eigenvalue λ of P, such that $\lambda = 0$. Then, P is similiar to the block matrix that contains a $4m$-th block. □

In the framework of π, we assume that $4 \nmid k_j$ for any index j.

Lemma 4. *Let π be a permutation. Then, (A, y) is a Frobenius coalgebra with comultiplication and counit as follows:*

$$\begin{aligned} &\Delta(1) = 1 \otimes 1,\ \Delta(x_i) = 1 \otimes x_i + x_i \otimes 1, \\ &\Delta(y) = 1 \otimes y + y \otimes 1 + \sum_{i=1}^{n} x_i \otimes x_{\pi(i)} + \sum_{i=1}^{n} x_{\pi(i)} \otimes x_i, \\ &\quad = 1 \otimes y + y \otimes 1 + \sum_{i=1}^{n} x_i \otimes (x_{\pi(i)} + x_{\pi^{-1}(i)}), \\ &\epsilon(1) = 1,\ \epsilon(x_i) = \epsilon(y) = 0. \end{aligned}$$

Proof. First, we prove that (A, Δ) is a coalgebra. It is obvious that coassociativity and counit axiom hold for 1 and x_i. Moreover, we have

$$\begin{aligned} (\Delta \otimes 1)\Delta(y) \quad = \quad & 1 \otimes 1 \otimes y + 1 \otimes y \otimes 1 + y \otimes 1 \otimes 1 + \sum_{k=1}^{n} x_k \otimes (x_{\pi(k)} + x_{\pi^{-1}(k)}) \otimes 1 \\ & + \sum_{k=1}^{n} 1 \otimes x_k \otimes (x_{\pi(k)} + x_{\pi^{-1}(k)}) + \sum_{k=1}^{n} x_k \otimes 1 \otimes (x_{\pi(k)} + x_{\pi^{-1}(k)}) \end{aligned}$$

and

$$\begin{aligned}(1\otimes\Delta)\Delta(y) &= 1\otimes 1\otimes y+1\otimes y\otimes 1+\sum_{k=1}^{n}1\otimes x_k\otimes(x_{\pi(k)}+x_{\pi^{-1}(k)})+y\otimes 1\otimes 1\\ &\quad+\sum_{k=1}^{n}x_k\otimes 1\otimes(x_{\pi(k)}+x_{\pi^{-1}(k)})+\sum_{k=1}^{n}x_k\otimes(x_{\pi(k)}+x_{\pi^{-1}(k)})\otimes 1.\end{aligned}$$

Therefore, coassociativity also holds for y. A direct calculation also shows that $(\epsilon\otimes id)\Delta(y)=y=(id\otimes\epsilon)\Delta(y)$. Thus, (A,Δ,ϵ) is a coalgebra.

Note that

$$\begin{vmatrix}0 & 1 & \\ 1 & 0 & \\ & & P\end{vmatrix}\neq 0,$$

then $A=A^*y$. □

Note that Frobenius coalgebra (A,y) is different from Frobenius coalgebra in [8] for n is even. For example, $A=\mathbb{K}1_A\oplus\mathbb{K}x_1\oplus\mathbb{K}x_2\oplus\mathbb{K}y$, the comultiplication of element y in [8] is $\Delta(y)=1\otimes y+y\otimes 1+x_1\otimes x_2+x_2\otimes x_1$, but the comultiplication of y in the previous conclusion is $\Delta(y)=1\otimes y+y\otimes 1+2x_1\otimes x_2+2x_2\otimes x_1$.

In what follows, we refer to the Frobenius algebra (A,ϕ) and Frobenius coalgebra (A,y) defined above as the triple (A,ϕ,y). Combining Lemma 1 and Lemma 4, we obtain the following corollary.

Proposition 1. *Let π be a permutation. Then, (A,ϕ,y) is a pre-bi-Frobenius algebra.*

Let (A,ϕ,y) be a pre-bi-Frobenius algebra. We define a linear map $\psi:A\to A$ as in (1) by

$$\psi(a)=\sum\phi(y_1a)y_2,\forall a\in A,$$

where $\Delta(y)=\sum y_1\otimes y_2$. Then, we have

$$\begin{aligned}&\psi(1_A)=1_A,\\ &\psi(x_i)=\sum_{k=1}^{n}(\lambda_{\pi^{-1}(k)i}+\lambda_{\pi(k)i})x_k,\\ &\psi(y)=y,\end{aligned}$$

for $i=1,2,\cdots,n$.

Now we will give the main theorem of this paper.

Theorem 1. *Let π be a permutation, P_π be the permutation matrix of π, and P is defined as $P_\pi+P_{\pi^{-1}}$. Then, the pre-bi-Frobenius algebra (A,ϕ,y) is a bi-Frobenius algebra with antipode ψ if and only if Λ and π satisfy Equations (4) and (5):*

$$(P\Lambda)^2=I_n \tag{4}$$

$$\Lambda=\Lambda^T \tag{5}$$

Proof. We first prove the "if" part. If (A,ϕ,y) is a bi-Frobenius algebra with antipode ψ, then ψ is an anti-algebra map. This means that $\psi(x_ix_j)=\psi(x_j)\psi(x_i)$. Now, we have

$$\psi(x_ix_j)=\psi(\lambda_{ij}y)=\lambda_{ij}y$$

and

$$\begin{aligned}
& \psi(x_j)\psi(x_i) \\
= & \sum_{k=1}^{n}(\lambda_{\pi^{-1}(k)j}+\lambda_{\pi(k)j})x_k\sum_{s=1}^{n}(\lambda_{\pi^{-1}(s)i}+\lambda_{\pi(s)i})x_s \\
= & \sum_{k=1}^{n}\sum_{s=1}^{n}(\lambda_{\pi^{-1}(k)j}\lambda_{\pi^{-1}(s)i}+\lambda_{\pi^{-1}(k)j}\lambda_{\pi(s)i}+\lambda_{\pi(k)j}\lambda_{\pi^{-1}(s)i}+\lambda_{\pi(k)j}\lambda_{\pi(s)i})x_kx_s \\
= & \sum_{k=1}^{n}\sum_{s=1}^{n}(\lambda_{\pi^{-1}(k)j}\lambda_{\pi^{-1}(s)i}+\lambda_{\pi^{-1}(k)j}\lambda_{\pi(s)i}+\lambda_{\pi(k)j}\lambda_{\pi^{-1}(s)i}+\lambda_{\pi(k)j}\lambda_{\pi(s)i})\lambda_{ks}y \\
= & \sum_{k=1}^{n}\sum_{s=1}^{n}(\lambda_{\pi^{-1}(k)j}\lambda_{ks}\lambda_{\pi^{-1}(s)i}+\lambda_{\pi^{-1}(k)j}\lambda_{ks}\lambda_{\pi(s)i} \\
& +\lambda_{\pi(k)j}\lambda_{ks}\lambda_{\pi^{-1}(s)i}+\lambda_{\pi(k)j}\lambda_{ks}\lambda_{\pi(s)i})y.
\end{aligned}$$

Comparing the coefficient of y in $\psi(x_ix_j)$ and $\psi(x_j)\psi(x_i)$, we have

$$\begin{aligned}
\lambda_{ij} = & \sum_{k=1}^{n}\sum_{s=1}^{n}(\lambda_{\pi^{-1}(k)j}\lambda_{ks}\lambda_{\pi^{-1}(s)i}+\lambda_{\pi^{-1}(k)j}\lambda_{ks}\lambda_{\pi(s)i} \\
& +\lambda_{\pi(k)j}\lambda_{ks}\lambda_{\pi^{-1}(s)i}+\lambda_{\pi(k)j}\lambda_{ks}\lambda_{\pi(s)i}) \\
= & \sum_{k=1}^{n}\sum_{s=1}^{n}(\lambda_{\pi^{-1}(s)i}\lambda_{ks}\lambda_{\pi^{-1}(k)j}+\lambda_{\pi(s)i}\lambda_{ks}\lambda_{\pi^{-1}(k)j} \\
& +\lambda_{\pi^{-1}(s)i}\lambda_{ks}\lambda_{\pi(k)j}+\lambda_{\pi(s)i}\lambda_{ks}\lambda_{\pi(k)j}).
\end{aligned}$$

This means

$$\begin{aligned}
\Lambda = & \Lambda^TP_{\pi^{-1}}\Lambda^TP_{\pi^{-1}}\Lambda+\Lambda^TP_{\pi}\Lambda^TP_{\pi^{-1}}\Lambda+\Lambda^TP_{\pi^{-1}}\Lambda^TP_{\pi}\Lambda+\Lambda^TP_{\pi}\Lambda^TP_{\pi}\Lambda \\
= & \Lambda^T(P_{\pi}+P_{\pi^{-1}})\Lambda^T(P_{\pi}+P_{\pi^{-1}})\Lambda \\
= & \Lambda^TP\Lambda^TP\Lambda.
\end{aligned}$$

As Λ is non-degenerate, the above equality yields

$$\Lambda^TP\Lambda^TP = I_n, \tag{6}$$

where I_n the identity matrix of $n\times n$. Recall that $P_{\pi}^{-1}=P_{\pi^{-1}}=P_{\pi}^T$, so $(P_{\pi}+P_{\pi^{-1}})^T = P_{\pi}+P_{\pi}^{-1}$, i.e., $P^T=P$. Take the transposition of (6),

$$P\Lambda P\Lambda = I_n.$$

As a consequence,

$$(P\Lambda)^2 = I_n.$$

In addition, ψ is also an anti-coalgebra map, i.e., $(\psi\otimes\psi)\Delta^{cop}=\Delta\psi$. We have

$$\begin{aligned}
(\psi\otimes\psi)\Delta^{cop}(x_i) = & (\psi\otimes\psi)(1\otimes x_i+x_i\otimes 1) \\
= & 1\otimes\sum_{k=1}^{n}(\lambda_{\pi(k)i}+\lambda_{\pi^{-1}(k)i})x_k+\sum_{k=1}^{n}(\lambda_{\pi(k)i}+\lambda_{\pi^{-1}(k)i})x_k\otimes 1 \\
= & \sum_{k=1}^{n}(\lambda_{\pi(k)i}+\lambda_{\pi^{-1}(k)i})(1\otimes x_k+x_k\otimes 1) \\
= & \sum_{k=1}^{n}(\lambda_{\pi(k)i}+\lambda_{\pi^{-1}(k)i})\Delta(x_k) \\
= & \Delta\psi(x_i).
\end{aligned}$$

This means that $(\psi \otimes \psi)\Delta^{cop}(x_i) = \Delta\psi(x_i)$ is automatically true for $i = 1, 2, \cdots n$.

Moreover,

$$\begin{aligned}(\psi \otimes \psi)\Delta^{cop}(y) &= (\psi \otimes \psi)(1 \otimes y + y \otimes 1 + \sum_{i=1}^{n} x_i \otimes (x_{\pi(i)} + x_{\pi^{-1}(i)})) \\ &= 1 \otimes y + y \otimes 1 + \sum_{i=1}^{n}\sum_{k=1}^{n}(\lambda_{\pi(k)i} + \lambda_{\pi^{-1}(k)i})x_k \\ &\quad \otimes \sum_{s=1}^{n}(\lambda_{\pi(s)\pi(i)} + \lambda_{\pi^{-1}(s)\pi(i)} + \lambda_{\pi(s)\pi^{-1}(i)} + \lambda_{\pi^{-1}(s)\pi^{-1}(i)})x_s, \\ \Delta\psi(y) &= 1 \otimes y + y \otimes 1 + \sum_{i=1}^{n}(x_{\pi(i)} + x_{\pi^{-1}(i)}) \otimes x_i.\end{aligned}$$

Comparing the coefficient of $x_k \otimes x_s$ in $(\psi \otimes \psi)\Delta^{cop}(y)$ and $\Delta\psi(y)$, we have

$$P\Lambda P\Lambda^T P = P.$$

As A is a Frobenius coalgebra, then

$$P\Lambda P\Lambda^T = I_n.$$

By Lemma 3 and Equation (4)

$$\Lambda = \Lambda^T.$$

Next, we prove the "only if" part. If (A, ϕ, y) is a pre-bi-Frobenius algebra, it is easy to show that the linear map $\psi : A \longrightarrow A$ is an anti-algebra map if (4) holds. By the proof of "if" part, ψ is also an anti-coalgebra map if (5) holds. Moreover, one observes that 1 is a group-like element and ϵ is an algebra map. Thus, (A, ϕ, y, ψ) is a bi-Frobenius algebra. □

Remark 1. *The bi-Frobenius algebras (A, ϕ, y, ψ) are not a Hopf algebra. In fact, we have*

$$\Delta(x_i x_j) = \Delta(\lambda_{ij} y) = \lambda_{ij}(1 \otimes y + y \otimes 1 + \sum_{i=1}^{n} x_i \otimes (x_{\pi(i)} + x_{\pi^{-1}(i)})),$$

but

$$\begin{aligned}\Delta(x_i)\Delta(x_j) &= (1 \otimes x_i + x_i \otimes 1)(1 \otimes x_j + x_j \otimes 1) \\ &= \lambda_{ij}(1 \otimes y + y \otimes 1) + x_i \otimes x_j + x_j \otimes x_i.\end{aligned}$$

Thus, $\Delta(x_i x_j) \neq \Delta(x_i)\Delta(x_j)$ for $i = 1 \cdots n$, which says that the comultiplication Δ is not an algebra map. So the bi-Frobenius algebra (A, ϕ, y, ψ) is not a Hopf algebra.

Remark 2. *The bi-Frobenius algebra (A, ϕ, y, ψ) is not S-type as the antipode ψ is not the inverse of identity.*

Remark 3. *The bi-Frobenius algebras are different from bi-Frobenius algebras constructed from quivers in [6,7]. As Frobenius coalgebras in this paper are different from Frobenius coalgebras in [8], so bi-Frobenius algebras in this paper are not a generalization of bi-Frobenius algebras in [8] when n is even.*

Remark 4. *The bi-Frobenius algebra (A, ϕ, y, ψ) is commutative.*

Next, two corollaries will be further described for Equations (4) and (5).

Corollary 1. *If $\pi = Id$, then (A, ϕ, y) is a bi-Frobenius algebra if and only if*

$$\Lambda^2 = \frac{1}{4} I, \tag{7}$$

$$\Lambda = \Lambda^T. \tag{8}$$

Corollary 2. *Let π is a permutation of order 2. Then, (A, ϕ, y) is a bi-Frobenius algebra if and only if*

$$(P_\pi \Lambda)^2 = \frac{1}{4} I, \tag{9}$$

$$\sum_{i=1}^{n} \lambda_{ji} \lambda_{\pi(j)\pi(i)} = \frac{1}{4}. \tag{10}$$

Example 1. *Let $n = 1$, then $A = \mathbb{K}1_A + \mathbb{K}x + \mathbb{K}y$ is a bi-Frobenius algebra with multiplication*

$$x^2 = y,\ x^3 = y^2 = 0,$$

and comultiplication

$$\Delta(x) = 1 \otimes x + x \otimes 1,\ \Delta(y) = 1 \otimes y + y \otimes 1 + x \otimes x,$$

and the antipode $\psi = id$.

Example 2. *Let $n = 2$, then $A = \mathbb{K}1_A + (\mathbb{K}x_1 + \mathbb{K}x_2) + \mathbb{K}y$ is a bi-Frobenius algebra with structure matrix $\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}$. Next, we will discuss the structure matrix Λ when $\pi = Id$ and $\pi = (12)$.*

Case 1: *If $\pi = Id$, we have $\Delta(y) = 1 \otimes y + y \otimes 1 + 2x_1 \otimes x_1 + 2x_2 \otimes x_2$. Then, we have*

$$\begin{cases} \lambda_{11}^2 + \lambda_{12}^2 = \frac{1}{4}, \\ \lambda_{21}^2 + \lambda_{22}^2 = \frac{1}{4}, \\ \lambda_{11}^2 + \lambda_{12}\lambda_{21} = \frac{1}{4}, \\ \lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} = 0, \\ \lambda_{11}\lambda_{12} + \lambda_{12}\lambda_{22} = 0, \\ \lambda_{22}^2 + \lambda_{12}\lambda_{21} = \frac{1}{4}. \end{cases} \tag{11}$$

By Equation (11), we have

(1) $\Lambda = \begin{pmatrix} a & \pm\frac{1}{2}\sqrt{1-4a^2} \\ \pm\frac{1}{2}\sqrt{1-4a^2} & -a \end{pmatrix}$, where $a \in \mathbb{K}$. The multiplication can be found that

$$x_1^2 = ay,\ x_2^2 = -ay,\ x_1x_2 = x_2x_1 = \pm\frac{1}{2}\sqrt{1-4a^2}\, y.$$

(2) $\Lambda = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ or $\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ or $\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ or $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$.

Especially, when $\Lambda = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, the multiplication can be found that

$$x_1^2 = x_2^2 = \frac{1}{2}y,\ x_1x_2 = x_2x_1 = 0.$$

Case 2: *If $\pi = (12)$, we have $\Delta(y) = 1 \otimes y + y \otimes 1 + 2x_1 \otimes x_2 + 2x_2 \otimes x_1$. Then, we have*

$$\begin{cases} \lambda_{21}^2 + \lambda_{11}\lambda_{22} - \frac{1}{4} = 0, \\ \lambda_{11}\lambda_{12} + \lambda_{11}\lambda_{21} = 0, \\ \lambda_{21} - \lambda_{12} = 0, \\ \lambda_{12}\lambda_{22} + \lambda_{21}\lambda_{22} = 0. \end{cases} \tag{12}$$

By Equation (12), we have

(1) $\Lambda = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{4a} \end{pmatrix}$, *where* $a \in \mathbb{K} \setminus \{0\}$. *The multiplication can be founded that*

$$x_1^2 = ay,\ x_2^2 = \frac{1}{4a}y,\ x_1x_2 = x_2x_1 = 0.$$

(2) $\Lambda = \pm\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. *The multiplication can be founded that*

$$x_1^2 = x_2^2 = 0,\ x_1x_2 = x_2x_1 = \pm\frac{1}{2}y.$$

Example 3. *Let* $n = 3$, *then* $A = \mathbb{K}1_A + (\mathbb{K}x_1 + \mathbb{K}x_2 + \mathbb{K}x_3) + \mathbb{K}y$ *has the structure matrix* $\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{12} & \lambda_{22} & \lambda_{23} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} \end{pmatrix}$. *Let* A *be a bi-Frobenius algebra, then the permutations are given by* $\pi = Id$, $\pi = (12)$, $\pi = (23)$, $\pi = (13)$, $\pi = (123)$, $\pi = (132)$. *We now present the structure matrix* Λ *for the cases* $\pi = Id$, $\pi = (12)$, *and* $\pi = (123)$.

Case 1: *When* $\pi = Id$, *we have* $\Delta(y) = 1 \otimes y + y \otimes 1 + 2x_1 \otimes x_1 + 2x_2 \otimes x_2 + 2x_3 \otimes x_3$. *Then*

$$P = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

such that

$$(PA)^2 = 4A^2 = I_3.$$

It implies that

$$A = \frac{1}{2}Q^T\Omega Q,$$

where Q *is the orthogonal matrix and* Ω *is the diagonal matrix whose diagonal elements are either* 1 *or* -1.

Case 2: *When* $\pi = (12)$, *we have* $\Delta(y) = 1 \otimes y + y \otimes 1 + 2x_1 \otimes x_2 + 2x_2 \otimes x_1 + 2x_3 \otimes x_3$. *Then*

$$P = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

and

$$\begin{cases} -4\lambda_{12}^2 - 4\lambda_{11}\lambda_{22} - 4\lambda_{13}\lambda_{23} + 1 = 0, \\ -4\lambda_{13}^2 - 8\lambda_{11}\lambda_{12} = 0, \\ -4\lambda_{12}\lambda_{13} - 4\lambda_{11}\lambda_{23} - 4\lambda_{13}\lambda_{33} = 0, \\ -4\lambda_{23}^2 - 8\lambda_{12}\lambda_{22} = 0, \\ -4\lambda_{12}^2 - 4\lambda_{11}\lambda_{22} - 4\lambda_{13}\lambda_{23} + 1 = 0, \\ -4\lambda_{12}\lambda_{23} - 4\lambda_{13}\lambda_{22} - 4\lambda_{23}\lambda_{33} = 0, \\ -4\lambda_{12}\lambda_{23} - 4\lambda_{13}\lambda_{22} - 4\lambda_{23}\lambda_{33} = 0, \\ -4\lambda_{12}\lambda_{13} - 4\lambda_{11}\lambda_{23} - 4\lambda_{13}\lambda_{33} = 0, \\ -4\lambda_{33}^2 - 8\lambda_{13}\lambda_{23} + 1 = 0. \end{cases}$$

Let $q = \sqrt{1 - 8\lambda_{13}\lambda_{23}}$, *then there are 16 solutions, as listed in Table 1.*

Table 1. Matrix Coefficients λ_{ij} under Permutation $\pi = (12)$.

	λ_{11}	λ_{12}	λ_{13}	λ_{22}	λ_{23}	λ_{33}
1	$\frac{2\lambda_{13}^2}{1+q}$	$\frac{1}{4}(-1-q)$	*	$\frac{\lambda_{13}\lambda_{23}-\lambda_{13}\lambda_{23}q}{4\lambda_{13}^2}$	*	$\frac{4\lambda_{13}\lambda_{23}(q^2+3q)}{24\lambda_{13}\lambda_{23}+q-q^3}$
2	$\frac{2\lambda_{13}^2}{-1+q}$	$\frac{1}{4}(1-q)$	*	$\frac{-\lambda_{13}\lambda_{23}-\lambda_{13}\lambda_{23}q}{4\lambda_{13}^2}$	*	$\frac{-4\lambda_{13}\lambda_{23}(q^3-3q)}{24\lambda_{13}\lambda_{23}-q+q^3}$
3	$\frac{2\lambda_{13}^2}{1-q}$	$\frac{1}{4}(-1+q)$	*	$\frac{\lambda_{13}\lambda_{23}+\lambda_{13}\lambda_{23}q}{4\lambda_{13}^2}$	*	$\frac{4\lambda_{13}\lambda_{23}(q^2-3q)}{24\lambda_{13}\lambda_{23}-q+q^3}$
4	$\frac{2\lambda_{13}^2}{-1-q}$	$\frac{1}{4}(1+q)$	*	$\frac{-\lambda_{13}\lambda_{23}+\lambda_{13}\lambda_{23}q}{4\lambda_{13}^2}$	*	$\frac{-4\lambda_{13}\lambda_{23}(q^2+3q)}{24\lambda_{13}\lambda_{23}+q-q^3}$
5	0	$-\frac{1}{2}$	0	λ_{23}^2	*	$\frac{1}{2}$
6	0	$\frac{1}{2}$	0	$-\lambda_{23}^2$	*	$-\frac{1}{2}$
7	*	0	0	$\frac{1}{4\lambda_{11}}$	0	$-\frac{1}{2}$
8	*	0	0	$\frac{1}{4\lambda_{11}}$	0	$\frac{1}{2}$
9	λ_{13}^2	$-\frac{1}{2}$	*	0	0	$\frac{1}{2}$
10	$-\lambda_{13}^2$	$\frac{1}{2}$	*	0	0	$-\frac{1}{2}$
11	λ_{13}^2	$-\frac{1}{2}$	*	$\frac{1}{\lambda_{13}^2}$	$-\frac{1}{\lambda_{13}}$	$\frac{3}{2}$
12	λ_{13}^2	$\frac{1}{2}$	*	$-\frac{1}{\lambda_{13}^2}$	$-\frac{1}{\lambda_{13}}$	$-\frac{3}{2}$
13	0	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$
14	0	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$
15	0	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$
16	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$

Here, the symbol $*$ is used to indicate free variables.

Case 3: $\pi = (123)$, *we have* $\Delta(y) = 1 \otimes y + y \otimes 1 + x_1 \otimes (x_2 + x_3) + x_2 \otimes (x_3 + x_1) + x_3 \otimes (x_1 + x_2)$. *Then,*

$$P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

and

$$\begin{cases} 1 - (\lambda_{11} + \lambda_{13})(\lambda_{22} + \lambda_{23}) - (\lambda_{11} + \lambda_{12})(\lambda_{23} + \lambda_{33}) - (\lambda_{12} + \lambda_{13})^2 = 0, \\ -(\lambda_{11} + \lambda_{13})(\lambda_{12} + \lambda_{13}) - (\lambda_{11} + \lambda_{13})(\lambda_{12} + \lambda_{23}) - (\lambda_{11} + \lambda_{12})(\lambda_{13} + \lambda_{33}) = 0, \\ -(\lambda_{11} + \lambda_{12})(\lambda_{12} + \lambda_{13}) - (\lambda_{11} + \lambda_{13})(\lambda_{12} + \lambda_{22}) - (\lambda_{11} + \lambda_{12})(\lambda_{13} + \lambda_{23}) = 0, \\ -(\lambda_{12} + \lambda_{13})(\lambda_{22} + \lambda_{23}) - (\lambda_{12} + \lambda_{23})(\lambda_{22} + \lambda_{23}) - (\lambda_{12} + \lambda_{22})(\lambda_{23} + \lambda_{33}) = 0, \\ 1 - (\lambda_{11} + \lambda_{13})(\lambda_{22} + \lambda_{23}) - (\lambda_{12} + \lambda_{22})(\lambda_{13} + \lambda_{33}) - (\lambda_{12} + \lambda_{23})^2 = 0, \\ -(\lambda_{11} + \lambda_{12})(\lambda_{22} + \lambda_{23}) - (\lambda_{12} + \lambda_{22})(\lambda_{12} + \lambda_{23}) - (\lambda_{12} + \lambda_{22})(\lambda_{13} + \lambda_{23}) = 0, \\ -(\lambda_{12} + \lambda_{13})(\lambda_{23} + \lambda_{33}) - (\lambda_{22} + \lambda_{23})(\lambda_{13} + \lambda_{33}) - (\lambda_{13} + \lambda_{23})(\lambda_{23} + \lambda_{33}) = 0, \\ -(\lambda_{11} + \lambda_{13})(\lambda_{23} + \lambda_{33}) - (\lambda_{12} + \lambda_{23})(\lambda_{13} + \lambda_{33}) - (\lambda_{13} + \lambda_{23})(\lambda_{13} + \lambda_{33}) = 0, \\ 1 - (\lambda_{11} + \lambda_{12})(\lambda_{23} + \lambda_{33}) - (\lambda_{12} + \lambda_{22})(\lambda_{13} + \lambda_{33}) - (\lambda_{13} + \lambda_{23})^2 = 0. \end{cases}$$

Let $p = \lambda_{12} + \lambda_{13}$, then there are 20 solutions, as listed in Table 2.

Table 2. Matrix Coefficients λ_{ij} under Permutation $\pi = (\mathbf{123})$.

	λ_{11}	λ_{12}	λ_{13}	λ_{22}	λ_{23}	λ_{33}
1	$\frac{-1-4\lambda_{12}\lambda_{13}}{4p}$	*	*	$\frac{1-2\lambda_{12}(1+2p)}{2(1+2\lambda_{13})}$	$\frac{-1-2p}{2}$	$\frac{1-2\lambda_{13}-4\lambda_{12}\lambda_{13}-4\lambda_{13}^2}{2(1+2\lambda_{12})}$
2	$\frac{-1-4\lambda_{12}\lambda_{13}}{4p}$	*	*	$\frac{1+2\lambda_{12}(1-2p)}{2(-1+2\lambda_{13})}$	$\frac{1-2p}{2}$	$\frac{1+2\lambda_{13}-4\lambda_{12}\lambda_{13}-4\lambda_{13}^2}{2(-1+2\lambda_{12})}$
3	*	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3+2\lambda_{11}}{2(-1+2\lambda_{11})}$
4	*	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3-2\lambda_{11}}{2(1+2\lambda_{11})}$	$\frac{1}{2}$	$-\frac{1}{2}$
5	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	*	$\frac{1}{2}$	$\frac{3+2\lambda_{22}}{-2+4\lambda_{22}}$
6	$\frac{1}{2}$	$-\frac{1}{2}$	*	$\frac{1}{2}$	$1-\lambda_{13}$	$\frac{-1-4\lambda_{13}+4\lambda_{13}^2}{4}$
7	*	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3+2\lambda_{11}}{2(-1+2\lambda_{11})}$	$-\frac{1}{2}$	$\frac{1}{2}$
8	*	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3-2\lambda_{11}}{2(1+2\lambda_{11})}$
9	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	*	$-\frac{1}{2}$	$\frac{3-2\lambda_{22}}{2(1+2\lambda_{22})}$
10	$-\frac{1}{2}$	$\frac{1}{2}$	*	$-\frac{1}{2}$	$-1-\lambda_{13}$	$\frac{1-4\lambda_{13}-4\lambda_{13}^2}{4}$
11	$\frac{1}{2}$	*	$-\frac{1}{2}$	$\frac{-1-4\lambda_{12}+4\lambda_{12}^2}{4}$	$1-\lambda_{12}$	$\frac{1}{2}$
12	$-\frac{1}{2}$	*	$\frac{1}{2}$	$\frac{1-4\lambda_{12}-4\lambda_{12}^2}{4}$	$-1-\lambda_{12}$	$-\frac{1}{2}$
13	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
14	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
15	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
16	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
17	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
18	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
19	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$
20	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Here, the symbol $*$ is used to indicate free variables.

Based on the commutative and cocommutative of A, we can naturally obtain the properties of its Nakayama automorphism, Nakayama co-automorphism, and the formula of ψ^2.

Proposition 2. *If (A, ϕ, y, ψ) is bi-Frobenius algebra, $\mathcal{N}$ is the Nakayama automorphism of (A, ϕ), and ${}^c\mathcal{N}$ is the coNakayama automorphism of (A, y), then $\mathcal{N} = id$, ${}^c\mathcal{N} = id$ and $\psi^2 = id$.*

Proposition 3. *Let (A, ϕ, y, ψ) be a bi-Frobenius algebra. Define the linear map $f : A \longrightarrow A$ by*

$$f\begin{pmatrix}1\\x_1\\x_2\\\vdots\\x_n\\y\end{pmatrix}=\begin{pmatrix}1&0&0&\cdots&0&0\\k_1&d_{11}&d_{12}&\cdots&d_{1n}&d_{1,n+1}\\k_2&d_{21}&d_{22}&\cdots&d_{2n}&d_{2,n+1}\\\vdots&\vdots&&\vdots&\vdots&\\k_n&d_{n1}&d_{n2}&\cdots&d_{nn}&d_{n,n+1}\\k_{n+1}&d_{n+1,1}&d_{n+1,2}&\cdots&d_{n+1,n}&d_{n+1,n+1}\end{pmatrix}\begin{pmatrix}1\\x_1\\x_2\\\vdots\\x_n\\y\end{pmatrix},$$

where $k_i, d_{ij} \in \mathbb{K}$, *for* $i, j \in \{1, 2, \cdots, n+1\}$. *Denote the coefficient matrix as* $\tilde{D}$ *and the block matrix* $D = (d_{ij})_{n\times n}$. *If* f *is an automorphism of* A, *then the following conditions hold:*

(i) $|D| \neq 0$.
(ii) $k_i = 0$, *for all* $i \in \{1, 2, \cdots, n+1\}$.
(iii) $d_{i,n+1} = d_{n+1,i} = 0$, $i \in \{1, 2, \cdots, n\}$.
(iv) $d_{n+1,n+1}P = D^T PD$.
(v) $d_{n+1,n+1}\Lambda = D\Lambda D^T$.
(vi) $\Lambda PD = D\Lambda P$.

Proof. It is evident that (i) holds. Suppose f is a morphism of algebra, then for all $i \in \{1, 2, \cdots, n\}$,

$$\begin{aligned}&f(x_i)f(y)\\=\;&(k_i + d_{i1}x_1 + \cdots + d_{in}x_n + d_{i,n+1}y)(k_{n+1} + d_{n+1,1}x_1 + \cdots + d_{n+1,n+1}y)\\=\;&k_ik_{n+1} + (k_id_{n+1,1} + d_{i1}k_{n+1})x_1 + \cdots + (k_id_{n+1,n} + k_{n+1}d_{i,n})x_n\\&+(k_id_{n+1,n+1} + k_{n+1}d_{i,n+1})y\\=\;&0.\end{aligned}$$

Thus, it follows that $k_i = 0$ and $k_{n+1} = 0$.

Furthermore,

$$\begin{aligned}&f(x_i)f(x_j)\\=\;&(d_{i1}x_1 + \cdots + d_{in}x_n + d_{i,n+1}y)(d_{j1}x_1 + \cdots + d_{jn+1}y)\\=\;&\sum_{s,t=1}^{n}\lambda_{st}d_{is}d_{jt}y,\end{aligned}\tag{13}$$

$$\begin{aligned}&f(x_ix_j)\\=\;&\lambda_{ij}f(y)\\=\;&\lambda_{ij}(d_{n+1,1}x_1 + \cdots + d_{n+1,n}x_n + d_{n+1,n+1}y),\end{aligned}\tag{14}$$

As Equations (13) and (14) are equal, then it follows that

$$d_{n+1,i} = 0,$$

and

$$\lambda_{ij}d_{n+1,n+1} = \sum_{s,t=1}^{n}\lambda_{st}d_{is}d_{jt},$$

which means

$$d_{n+1,n+1}\Lambda = D\Lambda D^T.$$

Next, suppose f is a morphism of coalgebra, then

$$\begin{aligned}&(f\otimes f)\Delta(x_i)\\=\;&(f\otimes f)(1\otimes x_i + x_i\otimes 1)\\=\;&2k_i(1\otimes 1) + d_{i1}(1\otimes x_1 + x_1\otimes 1) + \cdots + d_{i,n+1}(1\otimes y + y\otimes 1)\\=\;&2k_i\Delta(1) + d_{i1}\Delta(x_1) + \cdots + d_{in}\Delta(x_n) + d_{i,n+1}(1\otimes y + y\otimes 1).\end{aligned}\tag{15}$$

$$\begin{aligned} & \Delta f(x_i) \\ = \; & k_i\Delta(1) + d_{i1}\Delta(x_1) + \cdots + d_{in}\Delta(x_n) + d_{i,n+1}\Delta(y) \\ = \; & k_i\Delta(1) + d_{i1}\Delta(x_1) + \cdots + d_{in}\Delta(x_n) + d_{i,n+1}(1\otimes y + y\otimes 1 \\ & + \sum_{i=1}^{n} x_i \otimes (x_{\pi(i)} + x_{\pi^{-1}(i)})). \end{aligned} \tag{16}$$

As Equations (15) and (16) are euqal, then it follows that

$$k_i = 0,\ d_{i,n+1} = 0.$$

Similarly, using $(f\otimes f)\Delta(y) = \Delta f(y)$, we can obtain

$$k_{n+1} = 0,$$

and

$$d_{n+1,n+1}P = D^T PD.$$

Suppose f is a morphism of a bi-Frobenius algebra, then the matrix $\tilde{D}$ can be denoted as

$$\tilde{D} = \begin{pmatrix} 1 & & \\ & D & \\ & & d_{n+1,n+1} \end{pmatrix}.$$

Moreover, it satisfies $\psi\circ f = f\circ\psi$. As

$$\psi\circ f(1) = f\circ\psi(1) = 1,\ \psi\circ f(y) = f\circ\psi(y) = d_{n+1,n+1}y,$$

it suffices to consider $\psi\circ f(x_i)$ and $f\circ\psi(x_i)$.

$$\begin{aligned} \psi\circ f(x_i) = \; & \psi(d_{i1}x_1 + d_{i2}x_2 + \cdots + d_{in}x_n) \\ = \; & \sum_{j=1}^{n} d_{ij}\psi(x_j) \\ = \; & \sum_{j,k=1}^{n} d_{ij}(\lambda_{\pi(k)j} + \lambda_{\pi^{-1}(k)j})x_k, \end{aligned} \tag{17}$$

$$\begin{aligned} f\circ\psi(x_i) = \; & f(\sum_{k=1}^{n}(\lambda_{\pi(k)i} + \lambda_{\pi^{-1}(k)i})x_k) \\ = \; & \sum_{k=1}^{n}(\lambda_{\pi(k)i} + \lambda_{\pi^{-1}(k)i})(d_{k1}x_1 + d_{k2}x_2 + \cdots + d_{kn}x_n) \\ = \; & \sum_{k,j=1}^{n}(\lambda_{\pi(k)i} + \lambda_{\pi^{-1}(k)i})d_{kj}x_j. \end{aligned} \tag{18}$$

As Equations (17) and (18) are euqal, then

$$\Lambda PD = D\Lambda P. \tag{19}$$

□

Example 4. *Recall Example 2 Case 2. Let $n = 2$ and $\pi = (12)$, then $A = \mathbb{K}1_A + (\mathbb{K}x_1 + \mathbb{K}x_2) + \mathbb{K}y$ is a bi-Frobenius algebra with structure matrix $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$. The multiplication is*

$$x_1^2 = y,\ x_2^2 = \frac{1}{4}y,\ x_1x_2 = x_2x_1 = 0,$$

and the comultiplication is

$$\Delta(y) = 1 \otimes y + y \otimes 1 + 2x_1 \otimes x_2 + 2x_2 \otimes x_1.$$

Define $\tilde{D} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & d_{11} & d_{12} & 0 \\ 0 & d_{21} & d_{22} & 0 \\ 0 & 0 & 0 & d_{33} \end{pmatrix}$, *where* $d_{ij} \in \mathbb{K}$. *By Proposition 3, there exists*

$$\begin{cases} d_{11}d_{21} = 0, \\ d_{33} - d_{11}d_{22} - d_{12}d_{21} = 0, \\ 4d_{11}^2 + d_{12}^2 - 4d_{33} = 0, \\ 4d_{11}d_{21} + d_{12}d_{22} = 0, \\ 4d_{21}^2 + d_{22}^2 - d_{33} = 0, \\ d_{11}d_{22} - d_{21}d_{12} \neq 0. \end{cases} \tag{20}$$

By Equation (20), we have $\tilde{D} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k^2 \end{pmatrix}$ *or* $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 4k & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & 0 & 4k^2 \end{pmatrix}$, *where* $k \in \mathbb{K} \setminus \{0\}$.

Then, the automorphism f *of* A *is*

$$f\begin{pmatrix} 1 \\ x_1 \\ x_2 \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k^2 \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ y \end{pmatrix},$$

or

$$f\begin{pmatrix} 1 \\ x_1 \\ x_2 \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 4k & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & 0 & 4k^2 \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ y \end{pmatrix}.$$

Author Contributions: Writing—original draft, Y.W.; Writing—review & editing, Z.F. All authors have read and agreed to the published version of the manuscript.

Funding: Supported by the Fundamental Research Funds for the Central Universities.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: The authors thank Xiaowu Chen for making valuable comments.

Conflicts of Interest: The authors declare no conflict of interest.

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