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An Improved Version of the Parameterized Hardy–Hilbert Inequality Involving Two Partial Sums

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Abstract: In this paper, by employing the Euler–Maclaurin summation formula and real analysis techniques, an improved version of the parameterized Hardy–Hilbert inequality involving two partial sums is established. Based on the obtained inequality, the equivalent conditions of the best possible constant factor related to several parameters are discussed. Our results extend the classical Hardy–Hilbert inequality and improve certain existing results.

Keywords: Hardy–Hilbert inequality; multiple parameters; partial sums; equivalent conditions; best possible constant factor

MSC: 26D15; 26D10; 26A42

1. Introduction

The following double series inequality is known as the Hardy–Hilbert inequality (see [1], Theorem 315)

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}, \tag{1}$$

where $p>1, \frac{1}{p}+\frac{1}{q}=1, a_m \geq 0, b_n \geq 0, 0<\sum_{m=1}^{\infty}a_m^p<\infty, 0<\sum_{n=1}^{\infty}b_n^q<\infty$, the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible.

Since the Hardy–Hilbert inequality was proposed, it has generated a lot of results involving its generalizations, refinements, variants, etc. (see [2–12]).

In 2006, Krnic et al. [13] presented an interesting generalization of inequality (1) by constructing exponentially weight parameters $\lambda_i \in (0,2]\ (i=1,2), \lambda_1+\lambda_2=\lambda \in (0,4]$, as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty}m^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}, \tag{2}$$

where the constant factor $B(\lambda_1,\lambda_2)$, defined by the Beta function below, is the best possible

$$B(u,v):=\int_0^{\infty}\frac{t^{u-1}}{(1+t)^{u+v}}dt\ (u,v>0). \tag{3}$$

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In 2019, Adiyasuren et al. [14] gave a further extension of inequality (2) by imbedding two partial sums $A_m := \sum_{k=1}^{m} a_k$ and $B_n := \sum_{k=1}^{n} b_k$ in the right-hand side of the series, as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < \lambda_1\lambda_2 B(\lambda_1,\lambda_2)\left(\sum_{m=1}^{\infty} m^{-p\lambda_1-1}A_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{-q\lambda_2-1}B_n^q\right)^{\frac{1}{q}}, \tag{4}$$

where $\lambda_i \in (0,1]\cap(0,\lambda)(i=1,2), \lambda \in (0,2], \lambda_1+\lambda_2=\lambda, a_m, b_n \geq 0$; $A_m = o(e^{tm})$, $B_n = o(e^{tn})$ as $t>0\, m,n\to\infty$. The constant factor $\lambda_1\lambda_2 B(\lambda_1,\lambda_2)$ in (4) is the best possible.

Recently, the extension of Hardy–Hilbert's inequality via imbedding partial sums has attracted our interest. In [15], Liao, Wu, and Yang offered a Hardy–Hilbert-type inequality involving the kernel $\frac{1}{(m^{\alpha}+n^{\beta})^{\lambda}}$ and one partial sum B_n. Under the same kernel as used in [15], Yang and Wu [16] generalized the Hardy–Hilbert-type inequality to the form of two partial sums A_m and B_n. In [17], Huang, Wu, and Yang provided a Hardy–Hilbert-type inequality containing the kernel $\frac{1}{(m+n-\eta)^{\lambda}}$ and one partial sum B_n.

What has drawn our special attention is the work carried out by Gu and Yang [18]; there, they established an extension of inequality (4) that contains the same kernel as [16] but differs in the constant factors, written as follows:

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m^{\alpha}+n^{\beta})^{\lambda}} &< \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)}\left(\frac{1}{\beta}k_{\lambda+2}(\lambda_2+1)\right)^{\frac{1}{p}}\left(\frac{1}{\alpha}k\lambda+2(\lambda_1+1)\right)^{\frac{1}{q}}\\ &\times\Big[\sum_{m=1}^{\infty} m^{p[1-\alpha(1+\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}A_m^p\Big]^{\frac{1}{p}}\Big[\sum_{n=1}^{\infty} n^{q[1-\beta(1+\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}B_n^q\Big]^{\frac{1}{q}},\end{aligned} \tag{5}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $p>1, \alpha,\beta\in(0,1], \lambda\in(0,4], \lambda_1\in(0,\frac{2}{\alpha}-1]\cap(0,\lambda+1)$, $\lambda_2\in(0,\frac{2}{\beta}-1]\cap(0,\lambda+1), k_{\lambda}(\lambda_i) := B(\lambda_i,\lambda-\lambda_i)\,(i=1,2)$, and $A_m = o(e^{tm^{\alpha}})$, $B_n = o(e^{tn^{\beta}})$ as $t>0$, $m,n\to\infty$.

It is easy to observe that the constant factor in the inequality (5) is not the best possible except when $\alpha=\beta=1$. This naturally leads us to further explore the issue and ask how to improve the result presented in [18].

Inspired by the above-mentioned papers [14–18], in this paper, by constructing a new kernel shaped like $\frac{1}{[(m-\xi)^{\alpha}+(n-\eta)^{\beta}]^{\lambda}}$, we establish a unified improvement of inequalities (4) and (5); the obtained inequality contains the best possible constant factor. Also, we discuss the equivalent conditions of the best possible constant factor related to several parameters.

2. Preliminaries and Lemmas

For convenience, let us first specify the assumption conditions (H1), which will be employed in the subsequent analysis.

$$\begin{gathered}\text{(H1) } p>1\,(q>1), \tfrac{1}{p}+\tfrac{1}{q}=1, \alpha,\beta\in(0,1], \xi,\eta\in[0,\tfrac{1}{4}], \lambda\in(0,2],\\ \lambda_1\in(0,\tfrac{3}{2\alpha}-1]\cap(0,\lambda), \lambda_2\in(0,\tfrac{3}{2\beta}-1]\cap(0,\lambda), \hat{\lambda}_1 := \tfrac{\lambda-\lambda_2}{p}+\tfrac{\lambda_1}{q}, \hat{\lambda}_2 := \tfrac{\lambda-\lambda_1}{q}+\tfrac{\lambda_2}{p},\\ a_m, b_n\geq 0, A_m := \textstyle\sum_{j=1}^{m} a_j, B_n := \sum_{k=1}^{n} b_k (m,n\in \mathrm{N}=\{1,2,\cdots\}),\\ A_m = o(e^{t(m-\xi)^{\alpha}}), B_n = o(e^{t(n-\eta)^{\beta}})\,(t>0; m,n\to\infty), \text{ and}\\ 0<\sum_{m=1}^{\infty}(m-\xi)^{-p\alpha\hat{\lambda}_1-1}A_m^p<\infty, 0<\sum_{n=1}^{\infty}(n-\eta)^{-q\beta\hat{\lambda}_2-1}B_n^q<\infty.\end{gathered}$$

Lemma 1 ([18,19])**.** (i) Let $(-1)^i\frac{d^i}{dt^i}g(t)>0, t\in[m,\infty)(m<n\in \mathrm{N})$ with $g^{(i)}(\infty)=0\,(i=0,1,2,3)$, and let

$P_i(t), B_i\ (i\in \mathrm{N})$ be the Bernoulli functions and the Bernoulli numbers of

i-order. Then, we obtain the following:

$$\int_m^n P_{2q-1}(t)g(t)dt = \varepsilon_q \frac{B_{2q}}{2q} g(t)|_m^n \, (0 < \varepsilon_q < 1; q = 1, 2, \cdots). \tag{6}$$

In particular, for $q = 1$, $B_2 = \frac{1}{6}$, we have the following:

$$\frac{1}{12}(g(n) - g(m)) < \int_m^n P_1(t)g(t)dt < 0; \tag{7}$$

For $q = 2$, $B_4 = -\frac{1}{30}$, $n \to \infty$, we have the following:

$$0 < \int_m^\infty P_3(t)g(t)dt < \frac{1}{120}g(m). \tag{8}$$

(ii) ([18,19]) If $f(t)(f(t) > 0) \in C^3[m, \infty)$, $f^{(i)}(\infty) = 0 \, (i = 0, 1, 2, 3)$, then we have the following Euler–Maclaurin summation formula:

$$\sum_{k=m}^n f(k) = \int_m^n f(t)dt + \frac{f(m) + f(n)}{2} + \int_m^n P_1(t)f'(t)dt \, (m < n), \tag{9}$$

$$\int_m^\infty P_1(t)f'(t)dt = -\frac{1}{12}f'(m) + \frac{1}{6}\int_m^\infty P_3(t)f'''(t)dt. \tag{10}$$

Lemma 2. Let $s \in (0, 4]$, $s_2 \in (0, \frac{3}{2\beta}] \cap (0, s)$, $k_s(s_2) := B(s_2, s - s_2)$, and let the weight coefficient $\varpi_s(s_2, m)$ be defined by the following:

$$\varpi_s(s_2, m) := (m - \xi)^{\alpha(s-s_2)} \sum_{n=1}^\infty \frac{\beta(n - \eta)^{\beta s_2 - 1}}{\left[(m - \xi)^\alpha + (n - \eta)^\beta\right]^s} \, (m \in \mathrm{N}) \tag{11}$$

Then, we have the following inequalities:

$$0 < k_s(s_2)(1 - O_1(\frac{1}{(m - \xi)^{\alpha s_2}})) < \varpi_s(s_2, m) < k_s(s_2) \, (m \in \mathrm{N}). \tag{12}$$

where $O_1(\frac{1}{(m-\xi)^{\alpha s_2}}) := \frac{1}{k_s(s_2)} \int_0^{\frac{(1-\eta)^\beta}{(m-\xi)^\alpha}} \frac{u^{s_2-1}}{(1+u)^s} du > 0$.

Proof. For fixed $m \in \mathrm{N}$, a $:= (m - \xi)^\alpha$, we define a positive function $G(t)$ as follows:

$$G(t) := \frac{\beta(t - \eta)^{\beta s_2 - 1}}{\left[a + (t - \eta)^\beta\right]^s} (t > \eta).$$

Below we divide two cases of $s_2 \in (0, \frac{1}{\beta}) \cap (0, s)$ and $s_2 \in [\frac{1}{\beta}, \frac{3}{2\beta}] \cap (0, s)$ to prove the inequalities in (12).

(i) For $s_2 \in (0, \frac{1}{\beta}) \cap (0, s)$, since

$(-1)^i G^{(i)}(t) > 0 \, (t > \eta; i = 0, 1, 2)$,

by using Hermite–Hadamard's inequality [20] and setting $u = \frac{(t-\eta)^\beta}{a}$, we obtain the following:

$$\varpi(s_2, m) = (m - \xi)^{\alpha(s-s_2)} \sum_{n=1}^\infty G(n) < (m - \xi)^{\alpha(s-s_2)} \int_{\frac{1}{2}}^\infty G(t)dt$$

$$= \int_{\frac{(\frac{1}{2}-\eta)^\beta}{a}}^\infty \frac{u^{s_2-1}}{(1+u)^s} du \le \int_0^\infty \frac{u^{s_2-1}}{(1+u)^s} du = B(s_2, s - s_2) = k_\lambda(s_2).$$

On the other hand, in view of the diminishing property of series, setting $u = \frac{(t-\eta)^\beta}{(m-\xi)^\alpha}$, we obtain the following:

$$\varpi(s_2, m) > (m-\xi)^{\alpha(s-s_2)} \int_1^\infty G(t)dt = \int_{\frac{(1-\eta)^\beta}{(m-\xi)^\alpha}}^\infty \frac{u^{s_2-1}}{(1+u)^s} du$$
$$= B(s_2, s-s_2) - \int_0^{\frac{(1-\eta)^\beta}{(m-\xi)^\alpha}} \frac{u^{s_2-1}}{(1+u)^s} du = k_s(s_2)(1 - O_1(\frac{1}{(m-\xi)^{\alpha s_2}})) > 0,$$

where $O_1(\frac{1}{(m-\xi)^{\alpha s_2}}) = \frac{1}{k_s(s_2)} \int_0^{\frac{(1-\eta)^\beta}{(m-\xi)^\alpha}} \frac{u^{s_2-1}}{(1+u)^s} du > 0$, which satisfies the following:

$$0 < \int_0^{\frac{(1-\mu)^\beta}{(m-\xi)^\alpha}} \frac{u^{s_2-1}}{(1+u)^s} du < \int_0^{\frac{(1-\mu)^\beta}{(m-\xi)^\alpha}} u^{s_2-1} du = \frac{1}{s_2} (\frac{(1-\mu)^\beta}{(m-\xi)^\alpha})^{s_2} \ (m \in \mathrm{N}).$$

Hence, we obtain inequality (12).

(ii) For $s_2 \in [\frac{1}{\beta}, \frac{3}{2\beta}] \cap (0, s)$, in view of (9) (for $n \to \infty$), we obtain the following:

$$\sum_{n=1}^\infty G(n) = \int_1^\infty G(t)dt + \frac{1}{2}G(1) + \int_1^\infty P_1(t)G'(t)dt$$
$$= \int_\eta^\infty G(t)dt - h(a),$$
$$h(a) := \int_\eta^1 G(t)dt - \frac{1}{2}G(1) - \int_1^\infty P_1(t)G'(t)dt.$$

It is easy to observe that $-\frac{1}{2}G(1) = \frac{-\beta(1-\eta)^{\beta s_2 - 1}}{2[a+(1-\eta)^\beta]^s}$. Through integration by parts, we obtain the following:

$$\int_\eta^1 G(t)dt = \beta \int_\eta^1 \frac{(t-\eta)^{\beta s_2 - 1}}{[a+(t-\eta)^\beta]^s} dt \overset{u=(t-\eta)^\beta}{=} \int_0^{(1-\eta)^\beta} \frac{u^{s_2-1}}{(a+u)^s} du$$
$$= \frac{1}{s_2} \int_0^{(1-\eta)^\beta} \frac{du^{s_2}}{(a+u)^s} = \frac{1}{s_2} \frac{u^{s_2}}{(a+u)^s} |_0^{(1-\eta)^\beta} + \frac{s}{s_2} \int_0^{(1-\eta)^\beta} \frac{u^{s_2}}{(a+u)^{s+1}} du$$
$$= \frac{1}{s_2} \frac{(1-\eta)^{\beta s_2}}{[(a+(1-\eta)^\beta]^s} + \frac{s}{s_2(s_2+1)} \int_0^{(1-\eta)^\beta} \frac{du^{s_2+1}}{(a+u)^{s+1}}$$
$$= \frac{1}{s_2} \frac{(1-\eta)^{\beta s_2}}{[(a+(1-\eta)^\beta]^s} + \frac{s}{s_2(s_2+1)} \frac{u^{s_2+1}}{(a+u)^{s+1}} |_0^{(1-\eta)^\beta} + \frac{s(s+1)}{s_2(s_2+1)} \int_0^{(1-\eta)^\beta} \frac{u^{s_2+1}}{(a+u)^{s+2}} du$$
$$> \frac{1}{s_2} \frac{(1-\eta)^{\beta s_2}}{[(a+(1-\eta)^\beta]^s} + \frac{s}{s_2(s_2+1)} \frac{u^{s_2+1}}{(a+u)^{s+1}} |_0^{(1-\eta)^\beta} + \frac{s(s+1)}{s_2(s_2+1)[a+(1-\eta)^\beta]^{s+2}} \int_0^{(1-\eta)^\beta} u^{s_2+1} du$$
$$= \frac{1}{s_2} \frac{(1-\eta)^{\beta s_2}}{[(a+(1-\eta)^\beta]^s} + \frac{s}{s_2(s_2+1)} \frac{(1-\eta)^{\beta(s_2+1)}}{[(a+(1-\eta)^\beta]^{s+1}} + \frac{s(s+1)(1-\eta)^{\beta(s_2+2)}}{s_2(s_2+1)(s_2+2)[a+(1-\eta)^\beta]^{s+2}},$$
$$-G'(t) = -\frac{\beta(\beta s_2 - 1)(t-\eta)^{\beta s_2 - 2}}{[a+(t-\eta)^\beta]^s} + \frac{\beta^2 s(t-\eta)^{\beta+\beta s_2-2}}{[a+(t-\eta)^\beta]^{s+1}}$$
$$= \frac{\beta(\beta s - \beta s_2 + 1)(t-\eta)^{\beta s_2 - 2}}{[a+(t-\eta)^\beta]^s} - \frac{\beta^2 sa(t-\eta)^{\beta s_2 - 2}}{[a+(t-\eta)^\beta]^{s+1}},$$

and for $\frac{1}{\beta} \le s_2 \le \frac{3}{2\beta}, 0 < \beta \le 1, s_2 < s \le 4$, the following is obtained:

$$(-1)^i \frac{d^i}{dt^i} [\frac{(t-\eta)^{\beta s_2 - 2}}{[a+(t-\eta)^\beta]^s}] > 0, (-1)^i \frac{d^i}{dt^i} [\frac{(t-\eta)^{\beta s_2 - 2}}{[a+(t-\eta)^\beta]^{s+1}}] > 0 \ (i = 0, 1, 2, 3).$$

By (7) and (9), for $n \to \infty$, we obtain the following:

$$\frac{-1}{12} g(m) < \int_m^\infty P_1(t)g(t)dt < 0, \sum_{k=m}^\infty f(k) = \int_m^\infty f(t)dt + \frac{f(m)}{2} + \int_m^\infty P_1(t)f'(t)dt.$$

By using the above inequalities, in view of (8) and (10), we deduce the following:

$$\beta(\beta s-\beta s_2+1)\int_1^\infty P_1(t)\frac{(t-\eta)^{\beta s_2-2}}{[a+(t-\eta)^\beta]^s}dt > -\frac{\beta(\beta s-\beta s_2+1)(1-\eta)^{\beta s_2-2}}{12[a+(1-\eta)^\beta]^s},$$

$$\begin{aligned}
&-\beta^2 as\int_1^\infty P_1(t)\frac{(t-\eta)^{\beta s_2-2}}{[a+(t-\eta)^\beta]^{s+1}}dt\\
&= \frac{\beta^2 as(1-\eta)^{\beta s_2-2}}{12[a+(1-\eta)^\beta]^{s+1}} - \frac{\beta^2 as}{6}\int_1^\infty P_3(t)\,[\frac{(t-\eta)^{\beta s_2-2}}{[a+(t-\eta)^\beta]^{s+1}}]''\,dt\\
&> \frac{\beta^2 as(1-\eta)^{\beta s_2-2}}{12[a+(1-\eta)^\beta]^{s+1}} - \frac{\beta^2 as}{720}\Big[\frac{(t-\eta)^{\beta s_2-2}}{[a+(t-\eta)^\beta]^{s+1}}\Big]''_{t=1}\\
&= \frac{\beta^2 as(1-\eta)^{\beta s_2-2}}{12[a+(1-\eta)^\beta]^{s+1}}\\
&-\frac{\beta^2 as}{720}\{\frac{(\beta s_2-2)(\beta s_2-3)(1-\eta)^{\beta s_2-4}}{[a+(1-\eta)^\beta]^{s+1}} + \frac{\beta(s+1)(5-\beta-2\beta s_2)(1-\eta)^{\beta s_2+\beta-4}}{[a+(1-\eta)^\beta]^{s+2}} + \frac{\beta^2(s+1)(s+2)(1-\eta)^{\beta s_2+2\beta-4}}{[a+(1-\eta)^\beta]^{s+3}}\}\\
&> \frac{\beta^2[a+(1-\eta)^\beta-(1-\eta)^\beta]s}{12[a+(1-\eta)^\beta]^{s+1}}(1-\eta)^{\beta s_2-2} - \frac{\beta^2[a+(1-\eta)^\beta]s}{720}\\
&\quad\times\{\frac{(s+1)(s+2)\beta^2}{[a+(1-\eta)^\beta]^{s+3}}(1-\eta)^{2\beta} + \frac{\beta(s+1)(5-\beta-2\beta s_2)}{[a+(1-\eta)^\beta]^{s+2}}(1-\eta)^\beta + \frac{(2-\beta s_2)(3-\beta s_2)}{[a+(1-\eta)^\beta]^{s+1}}\}(1-\eta)^{\beta s_2-4}\\
&= \frac{\beta^2 s(1-\eta)^{\beta s_2-2}}{12[a+(1-\eta)^\beta]^s} - \frac{\beta^2 s(1-\eta)^{\beta s_2+\beta-2}}{12[a+(1-\eta)^\beta]^{s+1}} - \frac{\beta^2 s}{720}\\
&\quad\times\{\frac{(s+1)(s+2)\beta^2}{[a+(1-\eta)^\beta]^{s+2}}(1-\eta)^{2\beta} + \frac{\beta(s+1)(5-\beta-2\beta s_2)}{[a+(1-\eta)^\beta]^{s+1}}(1-\eta)^\beta + \frac{(2-\beta s_2)(3-\beta s_2)}{[a+(1-\eta)^\beta]^s}\}(1-\eta)^{\beta s_2-4}.
\end{aligned}$$

Then, we obtain the following:

$$h(a) > \frac{(1-\eta)^{\beta s_2-4}}{[a+(1-\eta)^\beta]^s}h_1 + \frac{s(1-\eta)^{\beta s_2+\beta-4}}{[a+(1-\eta)^\beta]^{s+1}}h_2 + \frac{s(s+1)(1-\eta)^{\beta s_2+2\beta-4}}{[a+(1-\eta)^\beta]^{s+2}}h_3,$$

where

$$\begin{aligned}
h_1 &:= \tfrac{1}{s_2}(1-\eta)^4 - \tfrac{\beta}{2}(1-\eta)^3 - \tfrac{\beta-\beta^2 s_2}{12}(1-\eta)^2 - \tfrac{\beta^2 s(2-\beta s_2)(3-\beta s_2)}{720},\\
h_2 &:= \tfrac{(1-\eta)^4}{s_2(s_2+1)} - \tfrac{\beta^2}{12}(1-\eta)^2 - \tfrac{\beta^3(s+1)(5-\beta-2\beta s_2)}{720},
\end{aligned}$$

$h_3 := \frac{(1-\eta)^4}{s_2(s_2+1)(s_2+2)} - \frac{\beta^4(s+2)}{720}$. We find $h_1 = \frac{g(s_2)}{720s_2}$, with the following:

$$\begin{aligned}
g(s_2) &:= 720(1-\eta)^4 - [360\beta(1-\eta)^3 + 60\beta(1-\eta)^2 + 6s\beta^2]s_2\\
&\quad +[60\beta^2(1-\eta)^2 + 5s\beta^3]s_2^2 - s\beta^4 s_2^3\ (s_2 \in [\tfrac{1}{\beta}, \tfrac{3}{2\beta}]).
\end{aligned}$$

For $\beta \in (0,1], s \in (0,4], \eta \in [0, \frac{1}{4}]$, we obtain the following:

$$\begin{aligned}
g'(s_2) &= -[360\beta(1-\eta)^3 + 60\beta(1-\eta)^2 + 6s\beta^2]\\
&\quad +2[60\beta^2(1-\eta)^2 + 5s\beta^3]s_2 - 3s\beta^4 s_2^2\\
&\le -360\beta(1-\eta)^3 - 60\beta(1-\eta)^2 - 6s\beta^2\\
&\quad +2[60\beta^2(1-\eta)^2 + 5s\beta^3]\tfrac{3}{2\beta} - 3s\beta^2\\
&= 120\beta(1-\eta)^2(3\eta-2) + 6s\beta^2\\
&\le 120\beta[(1-\eta)^2[3\eta - 2 + \tfrac{1}{5(1-\eta)^2}]\\
&\le 120\beta[(1-\eta)^2[\tfrac{3}{4} - 2 + \tfrac{1}{5(1-\frac{1}{4})^2}] = -2\beta[(1-\eta)^2\tfrac{161}{3} < 0.
\end{aligned}$$

Hence, we deduce the following:

$$\begin{aligned}
h_1 &\ge \tfrac{g(\frac{3}{2\beta})}{720s_2} = \tfrac{1}{720s_2}\{720(1-\eta)^4 - [360\beta(1-\eta)^3 + 60\beta(1-\eta)^2 + 6s\beta^2]\tfrac{3}{2\beta}\\
&\quad +[60\beta^2(1-\eta)^2 + 5s\beta^3]\tfrac{9}{4\beta^2} - s\beta^4\tfrac{27}{8\beta^3}\}\\
&= \tfrac{1}{720s_2}[720(1-\eta)^4 - 540(1-\eta)^3 + 45(1-\eta)^2 - \tfrac{9}{8}s\beta]\\
&= \tfrac{(1-\eta)^2}{16s_2}[16(1-\eta)^2 - 12(1-\eta) + 1 - \tfrac{1}{40(1-\eta)^2}s\beta]\\
&\ge \tfrac{(1-\eta)^2}{16s_2}[16\eta^2 - 20\eta + 5 - \tfrac{1}{10(1-\eta)^2}]\ (s\beta \le 4)\\
&\ge \tfrac{(1-\eta)^2}{16s_2}[16(\tfrac{1}{4})^2 - 20(\tfrac{1}{4}) + 5 - \tfrac{8}{45}] = \tfrac{37(1-\eta)^2}{720s_2} > 0\ (\eta \in [0, \tfrac{1}{4}]).
\end{aligned}$$

For $s_2 \in [\frac{1}{\beta}, \frac{3}{2\beta}], s \in (0,4]$, we still can obtain the following:

$$\begin{aligned} h_2 &\geq \tfrac{4\beta^2(1-\eta)^4}{15} - \tfrac{\beta^2}{12}(1-\eta)^2 - \tfrac{\beta^2(s+1)}{144} \\ &\geq \beta^2[\tfrac{4(1-\frac{1}{4})^4}{15} - \tfrac{1}{12}(1-\tfrac{1}{4})^2 - \tfrac{5}{144}] = \tfrac{\beta^2}{360} > . \end{aligned}$$

$$h_3 \geq \beta^3[\frac{8(3/4)^4}{105} - \frac{1}{120}] = \frac{53}{3360}\beta^3 > 0.$$

Hence, we have $h(a) > 0$, and then by setting $u = \frac{(t-\eta)^\beta}{(m-\xi)^\alpha}$, the following is established:

$$\begin{aligned} \varpi_s(s_2, m) &= (m-\xi)^{\alpha(s-s_2)} \sum_{n=1}^{\infty} G(n) < (m-\xi)^{\alpha(s-s_2)} \int_\eta^\infty G(t)dt \\ &= \int_0^\infty \tfrac{u^{s_2-1}du}{(1+u)^s} = B(s_2, s-s_2) = k_s(s_2). \end{aligned}$$

On the other hand, by (9), for $n \to \infty$, we obtain the following:

$$\begin{aligned} \sum_{n=1}^{\infty} G(n) &= \int_1^\infty G(t)dt + \tfrac{1}{2}G(1) + \int_1^\infty P_1(t)G'(t)dt \\ &= \int_1^\infty G(t)dt + H(a), \\ H(a) &:= \tfrac{1}{2}G(1) + \int_1^\infty P_1(t)G'(t)dt. \end{aligned}$$

We obtain $\frac{1}{2}G(1) = \frac{\beta(1-\eta)^{\beta s_2-1}}{2[a+(1-\eta)^\beta]^s}$, and the following:

$$G'(t) = -\frac{\beta(\beta s - \beta s_2 + 1)(t-\eta)^{\beta s_2-2}}{[a+(t-\eta)^\beta]^s} + \frac{\beta^2 sa(t-\eta)^{\beta s_2-2}}{[a+(t-\eta)^\beta]^{s+1}},$$

For $s_2 \in [\frac{1}{\beta}, \frac{3}{2\beta}] \cap (0, s), 0 < s \leq 4$, by (7), we find the following:

$$\begin{aligned} -\beta(\beta s - \beta s_2 + 1)\textstyle\int_1^\infty P_1(t)\frac{(t-\eta)^{\beta s_2-2}}{[a+(t-\eta)^\beta]^s}dt > 0, \text{ and} \\ \beta^2 as\textstyle\int_1^\infty P_1(t)\frac{(t-\eta)^{\beta s_2-2}}{[a+(t-\eta)^\beta]^{s+1}}dt > -\frac{\beta^2 as(1-\eta)^{\beta s_2-2}}{12[a+(1-\eta)^\beta]^{s+1}} > -\frac{\beta(1-\eta)^{\beta s_2-2}}{4[a+(1-\eta)^\beta]^s}. \end{aligned}$$

Hence, we obtain the following:

$$H(a) > \frac{\beta(1-\eta)^{\beta s_2-1}}{2[a+(1-\eta)^\beta]^s} - \frac{\beta(1-\eta)^{\beta s_2-2}}{4[a+(1-\eta)^\beta]^s} = \frac{\beta(1-\eta)^{\beta s_2-2}(1-2\eta)}{4[a+(1-\eta)^\beta]^s} > 0,$$

and then we obtain the following:

$$\begin{aligned} \varpi_s(s_2, m) &= (m-\xi)^{\alpha(s-s_2)} \sum_{n=1}^{\infty} G(n) > (m-\xi)^{\alpha(s-s_2)} \int_1^\infty G(t)dt \\ &= (m-\xi)^{\alpha(s-s_2)}(\int_\eta^\infty G(t)dt - \int_\eta^1 G(t)dt) \\ &= k_s(s_2)\left[1 - \tfrac{1}{k_s(s_2)}\int_0^{\frac{(1-\eta)^\beta}{(m-\xi)^\alpha}} \tfrac{u^{s_2-1}}{(1+u)^s}du\right] > 0, \end{aligned}$$

where, we set $O_1(\frac{1}{(m-\xi)^{\alpha s_2}}) = \frac{1}{k_s(s_2)}\int_0^{\frac{(1-\eta)^\beta}{(m-\xi)^\alpha}} \frac{u^{s_2-1}}{(1+u)^s}du > 0$. Therefore, inequality (12) is derived. This completes the proof of Lemma 2. □

Lemma 3. Let $s \in (0,4]$, $s_1 \in (0, \frac{3}{2\alpha}] \cap (0,s), s_2 \in (0, \frac{3}{2\beta}] \cap (0,s)$, $k_s(s_1) = B(s_1, s-s_1)$. Then, we have the following inequality:

$$\begin{aligned} I &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{[(m-\xi)^{\alpha}+(n-\eta)^{\beta}]^{s}} \le \left(\frac{1}{\beta}ks\left(s_2\right)\right)^{\frac{1}{p}}\left(\frac{1}{\alpha}k_s\left(s_1\right)\right)^{\frac{1}{q}} \\ &\times \left\{\sum_{m=1}^{\infty}(m-\xi)^{p[1-\alpha(\frac{s-s_2}{p}+\frac{s_1}{q})]-1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}(n-\eta)^{q[1-\beta(\frac{s-s_1}{q}+\frac{s_2}{p})]-1}b_n^q\right\}^{\frac{1}{q}} \end{aligned} \tag{13}$$

Proof. In the same way as the proof of Lemma 2, we can obtain the following inequalities for the next weight coefficient:

$$\begin{aligned} &0 < k_s(s_1)(1-O_2(\frac{1}{(n-\eta)^{\beta s_1}})) < \omega_s(s_1,n) \\ &:= (n-\eta)^{\beta(s-s_1)}\sum_{m=1}^{\infty}\frac{\alpha(m-\xi)^{\alpha s_1-1}}{[(m-\xi)^{\alpha}+(n-\eta)^{\beta}]^{s}} < k_s(s_1) = B\left(s_1, s-s_1\right)(n\in \mathrm{N}), \end{aligned} \tag{14}$$

where $O_2(\frac{1}{(n-\eta)^{\beta s_1}}) := \frac{1}{k_s(s_1)}\int_0^{\frac{(1-\xi)^{\varepsilon}}{(n-\eta)^{\beta}}}\frac{u^{s_1-1}}{(1+u)^{s}}du > 0$.

By applying Hölder's inequality [20], we obtain the following:

$$\begin{aligned} I_s &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{[(m-\xi)^{\alpha}+(n-\eta)^{\beta}]^{s}}\left\{\frac{(m-\xi)^{\alpha(1-s_1)/q}[\beta(n-\eta)^{\beta-1}]^{1/p}}{(n-\eta)^{\beta(1-s_2)/p}[\alpha(m-\xi)^{\alpha-1}]^{1/q}}a_m\right\} \\ &\times\left\{\frac{(n-\eta)^{\beta(1-s_2)/p}[\alpha(m-\xi)^{\alpha-1}]^{1/q}}{(m-\xi)^{\alpha(1-s_1)/q}[\beta(n-\eta)^{\beta-1}]^{1/p}}b_n\right\} \\ &\le \left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\beta}{[(m-\xi)^{\alpha}+(n-\eta)^{\beta}]^{s}}\frac{(m-\xi)^{\alpha(1-s_1)(p-1)}(n-\eta)^{\beta-1}}{(n-\eta)^{\beta(1-s_2)}[\alpha(m-\xi)^{\alpha-1}]^{p-1}}a_m^p\right\}^{\frac{1}{p}} \\ &\times\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\alpha}{[(m-\xi)^{\alpha}+(n-\eta)^{\beta}]^{s}}\frac{(n-\eta)^{\beta(1-s_2)(q-1)}(m-\xi)^{\alpha-1}}{(m-\xi)^{\alpha(1-s_1)}[\beta(n-\eta)^{\beta-1}]^{q-1}}b_n^q\right\}^{\frac{1}{q}} \\ &= \frac{1}{\alpha^{1/q}\beta^{1/p}}\left\{\sum_{m=1}^{\infty}\omega_s(s_2,m)(m-\xi)^{p[1-\alpha(\frac{s-s_2}{p}+\frac{s_1}{q})]-1}a_m^p\right\}^{\frac{1}{p}} \\ &\times\left\{\sum_{n=1}^{\infty}\omega_s(s_1,n)(n-\eta)^{q[1-\beta(\frac{s-s_1}{q}+\frac{s_2}{p})]-1}b_n^q\right\}^{\frac{1}{q}}. \end{aligned}$$

Then, by combining (12) and (14), we derive inequality (13). The Lemma 3 is proved. □

Remark 1. In particular, for $s = \lambda+2 \in (2,4], \lambda \in (0,2]$, we obtain the following:

$$\begin{aligned} s_1 &= \lambda_1+1 \in (1, \tfrac{3}{2\alpha}]\cap(1,\lambda+1), \lambda_1 \in (0, \tfrac{3}{2\alpha}-1]\cap(0,\lambda), \\ s_2 &= \lambda_2+1 \in (1, \tfrac{3}{2\beta}]\cap(1,\lambda+1), \lambda_2 \in (0, \tfrac{3}{2\beta}-1]\cap(0,\lambda). \end{aligned}$$

In (13), replacing a_m and b_n by $(m-\xi)^{\alpha-1}A_m$ and $(n-\eta)^{\beta-1}B_n$, respectively, in view of the assumption (H1) and (5), we obtain the following:

$$\begin{aligned} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(m-\xi)^{\alpha-1}(n-\eta)^{\beta-1}A_m B_n}{[(m-\xi)^{\alpha}+(n-\eta)^{\beta}]^{\lambda+1}} &< \left(\frac{1}{\beta}k_{\lambda+2}\left(\lambda_2+1\right)\right)^{\frac{1}{p}}\left(\frac{1}{\alpha}k_{\lambda+2}\left(\lambda_1+1\right)\right)^{\frac{1}{q}} \\ &\times\left[\sum_{m=1}^{\infty}(m-\xi)^{-p\alpha\hat{\lambda}_1-1}A_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\eta)^{-q\beta\hat{\lambda}_2-1}B_n^q\right]^{\frac{1}{q}}. \end{aligned} \tag{15}$$

Lemma 4. For $t > 0$, we have the following inequalities:

$$\sum_{m=1}^{\infty}e^{-t(m-\xi)^{\alpha}}a_m \le t\alpha\sum_{m=1}^{\infty}e^{-t(m-\xi)^{\alpha}}(m-\xi)^{\alpha-1}A_m \tag{16}$$

$$\sum_{n=1}^{\infty}e^{-t(n-\eta)^{\beta}}b_n \le t\beta\sum_{n=1}^{\infty}e^{-t(n-\eta)^{\beta}}(n-\eta)^{\beta-1}B_n. \tag{17}$$

Proof. In view of $A_m e^{-t(m-\xi)^{\alpha}} = o\left(1\right)(m\to\infty)$, by Abel summation by parts formula, we obtain the following:

$$\sum_{m=1}^{\infty} e^{-t(m-\xi)^{\alpha}} a_m = \lim_{m\to\infty} A_m e^{-t(m-\xi)^{\alpha}} + \sum_{m=1}^{\infty} A_m \left[e^{-t(m-\xi)^{\alpha}} - e^{-t(m-\xi+1)^{\alpha}} \right]$$
$$= \sum_{m=1}^{\infty} A_m \left[e^{-t(m-\xi)^{\alpha}} - e^{-t(m-\xi+1)^{\alpha}} \right].$$

We set $g(x) = e^{-t(x-\xi)^{\alpha}}, x \in (\xi, \infty)$. Then, we find the following:

$$g'(x) = -t\alpha(x-\xi)^{\alpha-1} e^{-t(x-\xi)^{\alpha}} = -t\alpha h(x),$$

where, for $\alpha \in (0,1], h(x) := (x-\xi)^{\alpha-1} e^{-t(x-\xi)^{\alpha}}$ is decreasing in (ξ, ∞). By the differentiation mid-value theorem, we obtain the following:

$$\sum_{m=1}^{\infty} e^{-t(m-\xi)^{\alpha}} a_m = -\sum_{m=1}^{\infty} A_m (g(m+1) - g(m))$$
$$= -\sum_{m=1}^{\infty} A_m g'(m+\theta_m) = t\alpha \sum_{m=1}^{\infty} h(m+\theta_m) A_m \, (\theta_m \in (0,1))$$
$$\leq t\alpha \sum_{m=1}^{\infty} h(m) A_m = t\alpha \sum_{m=1}^{\infty} (m-\xi)^{\alpha-1} e^{-t(m-\xi)^{\alpha}} A_m,$$

Thus, inequality (16) is proved. In the same way as above, we can derive inequality (17). The proof of Lemma 4 is complete. □

3. Main Results

Theorem 1. Under the assumption (H1), we have the following improved version of Hardy–Hilbert's inequality:

$$I := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{\left[(m-\xi)^{\alpha} + (n-\eta)^{\beta}\right]^{\lambda}} < \lambda(\lambda+1)(\alpha k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}} (\beta k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}$$
$$\times \left[\sum_{m=1}^{\infty} (m-\xi)^{-p\alpha\hat{\lambda}_1 - 1} A_m^p \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} (n-\eta)^{-q\beta\hat{\lambda}_2 - 1} B_n^q \right]^{\frac{1}{q}}. \tag{18}$$

In particular, for $\lambda_1 + \lambda_2 = \lambda$, we have, with regards to the conditions, the following:

$$0 < \sum_{m=1}^{\infty} (m-\xi)^{-p\alpha\lambda_1 - 1} A_m^p < \infty, \; 0 < \sum_{n=1}^{\infty} (n-\eta)^{-q\beta\lambda_2 - 1} B_n^q < \infty,$$

with the following inequality:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{\left[(m-\xi)^{\alpha} + (n-\eta)^{\beta}\right]^{\lambda}} < \alpha^{\frac{1}{p}} \beta^{\frac{1}{q}} \lambda(\lambda+1) B(\lambda_1+1, \lambda_2+1)$$
$$\times \left[\sum_{m=1}^{\infty} (m-\xi)^{-p\alpha\lambda_1 - 1} A_m^p \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} (n-\eta)^{-q\beta\lambda_2 - 1} B_n^q \right]^{\frac{1}{q}}. \tag{19}$$

Proof. In view of the following expression:

$$\frac{1}{\left[(m-\xi)^{\alpha} + (n-\eta)^{\beta}\right]^{\lambda}} = \frac{1}{\Gamma(\lambda)} \int_0^{\infty} t^{\lambda-1} e^{-[(m-\xi)^{\alpha} + (n-\eta)^{\beta}]t} dt$$

by (16) and (17), the following can be assumed:

$$
\begin{aligned}
I &= \tfrac{1}{\Gamma(\lambda)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n \int_0^{\infty} t^{\lambda-1} e^{-[(m-\xi)^{\alpha}+(n-\eta)^{\beta}]t} dt \\
&= \tfrac{1}{\Gamma(\lambda)} \int_0^{\infty} t^{\lambda-1} \left[\sum_{m=1}^{\infty} e^{-(m-\xi)^{\alpha} t} a_m \right] \left[\sum_{n=1}^{\infty} e^{-(n-\eta)^{\beta} t} b_n \right] dt \\
&\leq \tfrac{\alpha\beta}{\Gamma(\lambda)} \int_0^{\infty} t^{\lambda+1} \left[\sum_{m=1}^{\infty} e^{-(m-\xi)^{\alpha} t} (m-\xi)^{\alpha-1} A_m \right] \left[\sum_{n=1}^{\infty} e^{-(n-\eta)^{\beta} t} (n-\eta)^{\beta-1} B_n \right] dt \\
&= \tfrac{\alpha\beta}{\Gamma(\lambda)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (m-\xi)^{\alpha-1} (n-\eta)^{\beta-1} A_m B_n \int_0^{\infty} t^{\lambda+1} e^{-[(m-\xi)^{\alpha}+(n-\eta)^{\beta}]t} dt \\
&= \alpha\beta \tfrac{\Gamma(\lambda+2)}{\Gamma(\lambda)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \tfrac{(m-\xi)^{\alpha-1}(n-\eta)^{\beta-1}}{[(m-\xi)^{\alpha}+(n-\eta)^{\beta}]^{\lambda+2}} A_m B_n.
\end{aligned}
$$

Then, by aid of (15), we obtain inequality (18). This completes the proof of Theorem 1. □

Theorem 2. Under the assumption (H1), if $\lambda_1 + \lambda_2 = \lambda$, then the following constant:

$$\lambda(\lambda+1)(\alpha k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}} (\beta k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}$$

in (18) is the best value.

Proof. It suffices to prove that the constant factor, written as follows:

$$\alpha^{\frac{1}{p}} \beta^{\frac{1}{q}} \lambda(\lambda+1) B(\lambda_1+1, \lambda_2+1) = \alpha^{\frac{1}{p}} \beta^{\frac{1}{q}} \lambda_1 \lambda_2 B(\lambda_1, \lambda_2)$$

in (19) is the best value. In order to prove this assertion, for any $0 < \varepsilon < \min\{p\lambda_1, q\lambda_2\}$, we construct a pair of terms $\widetilde{a}_m, \widetilde{b}_n$ by $\widetilde{a}_m := m^{\alpha(\lambda_1 - \frac{\varepsilon}{p})-1}$, $\widetilde{b}_n := n^{\beta(\lambda_2 - \frac{\varepsilon}{q})-1}$, accordingly, and the partial sums $\widetilde{A}_m := \sum_{i=1}^{m} \widetilde{a}_i$, $\widetilde{B}_n := \sum_{k=1}^{n} \widetilde{b}_k$ $(m, n \in \mathrm{N})$, which satisfy the condition in (H1):

$$0 < \sum_{m=1}^{\infty} m^{-p\alpha\lambda_1 - 1} \widetilde{A}_m^p < \infty, \; and \; 0 < \sum_{n=1}^{\infty} n^{-q\beta\lambda_2 - 1} \widetilde{B}_n^q < \infty.$$

Since $0 < \lambda_1 - \frac{\varepsilon}{p} < \frac{3}{2\alpha} - 1, 0 < \alpha(\lambda_1 - \frac{\varepsilon}{p}) < \frac{3}{2} - \alpha < 2$, by using (9) and (7), we obtain the following:

$$
\begin{aligned}
\widetilde{A}_m &:= \sum_{i=1}^{m} \widetilde{a}_i = \sum_{i=1}^{m} i^{\alpha(\lambda_1 - \frac{\varepsilon}{p})-1} = \int_1^m t^{\alpha(\lambda_1 - \frac{\varepsilon}{p})-1} dt \\
&\quad + \tfrac{1}{2}[m^{\alpha(\lambda_1 - \frac{\varepsilon}{p})-1} + 1] + \tfrac{\varepsilon_0}{12}[\alpha(\lambda_1 - \tfrac{\varepsilon}{p}) - 1][m^{\alpha(\lambda_1 - \frac{\varepsilon}{p})-2} - 1] \\
&= \tfrac{1}{\alpha(\lambda_1 - \frac{\varepsilon}{p})} (m^{\alpha(\lambda_1 - \frac{\varepsilon}{p})} + c_1 + O_1(m^{\alpha(\lambda_1 - \frac{\varepsilon}{p})-1})) \\
&\leq \tfrac{m^{\alpha(\lambda_1 - \frac{\varepsilon}{p})}}{\alpha(\lambda_1 - \frac{\varepsilon}{p})} (1 + |c_1| m^{-\alpha(\lambda_1 - \frac{\varepsilon}{p})} + |O_1(m^{-1})|) \; (\varepsilon_0 \in (0,1); m \in \mathrm{N}, \mathrm{m} \to \infty).
\end{aligned}
$$

In the same way as above, for $0 < \beta(\lambda_2 - \frac{\varepsilon}{q}) < 2$, we obtain the following:

$$\widetilde{B}_n := \sum_{k=1}^{n} \widetilde{b}_k \leq \frac{n^{\beta(\lambda_2 - \frac{\varepsilon}{q})}}{\beta(\lambda_2 - \frac{\varepsilon}{q})} (1 + |c_2| n^{-\beta(\lambda_2 - \frac{\varepsilon}{q})} + |O_2(n^{-1})|) \; (n \in \mathrm{N}; n \to \infty)$$

(c_i $(i = 1, 2)$ are the constants). We observe that
$\widetilde{A}_m = o(e^{tm^{\alpha}}), \widetilde{B}_n = o(e^{tn^{\beta}})$ $(t > 0; m, n \to \infty)$.

If there exists a constant $M (\leq \alpha^{\frac{1}{p}} \beta^{\frac{1}{q}} \lambda(\lambda+1) B(\lambda_1+1, \lambda_2+1))$, such that (19) is valid when we replace $\alpha^{\frac{1}{p}} \beta^{\frac{1}{q}} \lambda(\lambda+1) B(\lambda_1+1, \lambda_2+1)$ by M, then, in particular, for $\xi = \eta = 0$, using a substitution of $a_m = \widetilde{a}_m$, $b_n = \widetilde{b}_n$, $A_m = \widetilde{A}_m$ and $B_n = \widetilde{B}_n$ in (19), we acquire the following:

$$\widetilde{I} := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\widetilde{a}_m \widetilde{b}_n}{(m^{\alpha} + n^{\beta})^{\lambda}} < M \left(\sum_{m=1}^{\infty} m^{-p\alpha\lambda_1 - 1} \widetilde{A}_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} n^{-q\beta\lambda_2 - 1} \widetilde{B}_n^q \right)^{\frac{1}{q}}.$$

Note that $a(x) \to 0$ as $x \to \infty$, we obtain the following:

$$\lim_{x\to\infty} \frac{(1+a(x))^p - 1}{a(x)} = \lim_{x\to\infty} \frac{p(1+a(x))^{p-1}a'(x)}{a'(x)} = \lim_{x\to\infty} p(1+a(x))^{p-1} = p,$$

This yields $(1+a(x))^p = 1 + O(a(x))\,(x \to \infty)$.

Thereby, we deduce the following:

$$\begin{aligned}&(1+|c_1|m^{-\alpha(\lambda_1-\frac{\varepsilon}{p})} + |O_1(m^{-1})|)p\\ &= 1 + O(|c_1|m^{-\alpha(\lambda_1-\frac{\varepsilon}{p})} + |O_1(m^{-1})|)\,(m \in \text{ N}; m \to \infty).\end{aligned}$$

Hence, the following is validated:

$$\begin{aligned}\sum_{m=1}^{\infty} m^{-p\alpha\lambda_1-1}\widetilde{A}_m^p &\le \left[\frac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\right]^p \sum_{m=1}^{\infty} m^{-\alpha\varepsilon-1}(1+|c_1|m^{-\alpha(\lambda_1-\frac{\varepsilon}{p})} + |O_1(m^{-1})|)^p\\ &= \left[\frac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\right]^p \sum_{m=1}^{\infty} m^{-\alpha\varepsilon-1}[1+O(|c_1|m^{-\alpha(\lambda_1-\frac{\varepsilon}{p})} + |O_1(m^{-1})|)]\\ &= \left[\frac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\right]^p \left[\sum_{m=2}^{\infty} m^{-\alpha\varepsilon-1} + 1 + \sum_{m=1}^{\infty} O(|c_1|m^{-\alpha(\lambda_1+\frac{\varepsilon}{q})-1} + |O_1(m^{-\alpha\varepsilon-2})|)\right]\\ &= \left[\frac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\right]^p \left(\sum_{m=2}^{\infty} m^{-\alpha\varepsilon-1} + O_1(1)\right)\\ &< \left[\frac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\right]^p (\int_1^{\infty} x^{-\alpha\varepsilon-1}dx + O_1(1)) = \left[\frac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\right]^p (\frac{1}{\alpha\varepsilon} + O_1(1)).\end{aligned}$$

In the same way as above, we obtain the following:

$$\sum_{n=1}^{\infty} n^{-q\beta\lambda_2-1}\widetilde{B}_n^q < \left[\frac{1}{\beta(\lambda_2-\frac{\varepsilon}{q})}\right]^q (\frac{1}{\beta\varepsilon} + O_2(1)).$$

Then, we obtain the following:

$$\widetilde{I} < \frac{M}{\varepsilon}\frac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\frac{1}{\beta(\lambda_2-\frac{\varepsilon}{q})}\left(\frac{1}{\alpha} + \varepsilon O_1(1)\right)^{\frac{1}{p}}\left(\frac{1}{\beta} + \varepsilon O_2(1)\right)^{\frac{1}{q}}.$$

In view of (14), for $\xi = \eta = 0, s = \lambda \in (0,1], s_1 = \lambda_1 - \frac{\varepsilon}{p} \in (0, \frac{3}{2\alpha} - 1] \cap (0, \lambda)$, we obtain the following:

$$\begin{aligned}\widetilde{I} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{m^{\alpha(\lambda_1-\frac{\varepsilon}{p})-1}}{(m^\alpha+n^\beta)^\lambda} n^{\beta(\lambda_2-\frac{\varepsilon}{q})-1} = \frac{1}{\alpha}\sum_{n=1}^{\infty} n^{-\beta\varepsilon-1}\left[n^{\beta(\lambda_2+\frac{\varepsilon}{p})}\sum_{m=1}^{\infty}\frac{\alpha m^{\alpha(\lambda_1-\frac{\varepsilon}{p})-1}}{(m^\alpha+n^\beta)^\lambda}\right]\\ &\ge \frac{1}{\alpha}k_\lambda(\lambda_1-\frac{\varepsilon}{p})\sum_{n=1}^{\infty} n^{-\beta\varepsilon-1}\left(1 - O_2(\frac{1}{n^{\beta(\lambda_1-\frac{\varepsilon}{p})}})\right)\\ &= \frac{1}{\alpha}k_\lambda(\lambda_1-\frac{\varepsilon}{p})\left(\sum_{n=1}^{\infty} n^{-\beta\varepsilon-1} - \sum_{n=1}^{\infty} O_2(\frac{1}{n^{\beta(\lambda_1+\frac{\varepsilon}{q})+1}})\right)\\ &> \frac{1}{\alpha}k_\lambda(\lambda_1-\frac{\varepsilon}{p})(\int_1^{\infty} y^{-\beta\varepsilon-1}dy - O_3(1))\\ &= \frac{1}{\alpha}B(\lambda_1-\frac{\varepsilon}{p}, \lambda_2+\frac{\varepsilon}{p})\left(\frac{1}{\beta\varepsilon} - O_3(1)\right).\end{aligned}$$

Based on the above results, we obtain the following:

$$\begin{aligned}&\frac{1}{\alpha}B(\lambda_1-\frac{\varepsilon}{p}, \lambda_2+\frac{\varepsilon}{p})\left(\frac{1}{\beta} - \varepsilon O_3(1)\right)\\ &< \varepsilon\widetilde{I} < M\frac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\frac{1}{\beta(\lambda_2-\frac{\varepsilon}{q})}\left(\frac{1}{\alpha} + \varepsilon O_1(1)\right)^{\frac{1}{p}}\left(\frac{1}{\beta} + \varepsilon O_2(1)\right)^{\frac{1}{q}}.\end{aligned}$$

Thereby, by setting $\varepsilon \to 0^+$, with the help of the continuity of the Beta function, we obtain the following:

$$\frac{1}{\alpha\beta}B(\lambda_1,\lambda_2) \leq M\frac{1}{\alpha\beta\lambda_1\lambda_2}\left(\frac{1}{\alpha}\right)^{\frac{1}{p}}\left(\frac{1}{\beta}\right)^{\frac{1}{q}},$$

that is written as follows:

$$\alpha^{\frac{1}{p}}\beta^{\frac{1}{q}}\lambda(\lambda+1)B(\lambda_1+1,\lambda_2+1) = \alpha^{\frac{1}{p}}\beta^{\frac{1}{q}}\lambda_1\lambda_2 B(\lambda_1,\lambda_2) \leq M.$$

Hence, $M = \alpha^{\frac{1}{p}}\beta^{\frac{1}{q}}\lambda(\lambda+1)B(\lambda_1+1,\lambda_2+1)$ is the best possible constant factor in (19). The proof of Theorem 2 is complete. □

Theorem 3. Under the assumption (H1), if the constant

$$\lambda(\lambda+1)(\alpha k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}}(\beta k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}$$

in (18) is the best value, then for the following:

$$\lambda - \lambda_1 - \lambda_2 \leq \min\{p(\frac{2}{\alpha} - 1 - \lambda_1), q(\frac{2}{\beta} - 1 - \lambda_2)\},$$

we have $\lambda_1 + \lambda_2 = \lambda$.

Proof. For $\hat{\lambda}_1 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} = \frac{\lambda-\lambda_1-\lambda_2}{p} + \lambda_1, \hat{\lambda}_2 = \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p} = \frac{\lambda-\lambda_1-\lambda_2}{q} + \lambda_2$, we find

$$\hat{\lambda}_1 + \hat{\lambda}_2 = \lambda, 0 < \hat{\lambda}_1, \hat{\lambda}_2 < \frac{\lambda}{p} + \frac{\lambda}{q} = \lambda, \text{and } B(\hat{\lambda}_1+1, \hat{\lambda}_2+1) \in \mathrm{R}_+.$$

For $\lambda - \lambda_1 - \lambda_2 \leq p(\frac{3}{2\alpha} - 1 - \lambda_1)$, we have $\hat{\lambda}_1 \leq \frac{3}{2\alpha} - 1$;
for $\lambda - \lambda_1 - \lambda_2 \leq q(\frac{3}{2\beta} - 1 - \lambda_2)$, we have $\hat{\lambda}_2 \leq \frac{3}{2\beta} - 1$. Utilizing a substitution of $\lambda_i = \hat{\lambda}_i\,(i = 1, 2)$ in (19), we still obtain the following:

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{[(m-\xi)^{\alpha}+(n-\eta)^{\beta}]^{\lambda}} &< \alpha^{\frac{1}{p}}\beta^{\frac{1}{q}}\lambda(\lambda+1)B(\hat{\lambda}_1+1,\hat{\lambda}_2+1)\\ &\times\left[\sum_{m=1}^{\infty}(m-\xi)^{-p\alpha\hat{\lambda}_1-1}A_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\eta)^{-q\beta\hat{\lambda}_2-1}B_n^q\right]^{\frac{1}{q}}.\end{aligned} \tag{20}$$

By using Hölder's inequality [20], we obtain the following:

$$\begin{aligned}&B(\hat{\lambda}_1+1,\hat{\lambda}_2+1) = k_{\lambda+2}(\tfrac{\lambda-\lambda_2}{p} + \tfrac{\lambda_1}{q} + 1)\\ &= \int_0^{\infty}\frac{1}{(1+u)^{\lambda+2}}u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}}du = \int_0^{\infty}\frac{1}{(1+u)^{\lambda+2}}\left(u^{\frac{\lambda-\lambda_2}{p}}\right)\left(u^{\frac{\lambda_1}{q}}\right)du\\ &\geq \left[\int_0^{\infty}\frac{1}{(1+u)^{\lambda+2}}u^{\lambda-\lambda_2}du\right]^{\frac{1}{p}}\left[\int_0^{\infty}\frac{1}{(1+u)^{\lambda+2}}u^{\lambda_1}du\right]^{\frac{1}{q}}\\ &= \left[\int_0^{\infty}\frac{1}{(1+v)^{\lambda+2}}v^{\lambda_2}dv\right]^{\frac{1}{p}}\left[\int_0^{\infty}\frac{1}{(1+u)^{\lambda+2}}u^{\lambda_1}du\right]^{\frac{1}{q}}\\ &= (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}.\end{aligned} \tag{21}$$

If the constant $\lambda(\lambda+1)(\alpha k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}}(\beta k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}$ in (18) is the best value, then, by comparing with the constant factors in (18) and (20), we have the following inequality:

$$\lambda(\lambda+1)(\alpha k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}}(\beta k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}} \leq \alpha^{\frac{1}{p}}\beta^{\frac{1}{q}}\lambda(\lambda+1)B(\hat{\lambda}_1+1,\hat{\lambda}_2+1),$$

which yields the following:

$$B(\hat{\lambda}_1+1,\hat{\lambda}_2+1) \geq (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}},$$

namely, (21) keeps the form of equality. We observe that (21) keeps the form of equality if, and only if, there exist constants A and B such that they are not both zero satisfying $Au^{\lambda-\lambda_2} = Bu^{\lambda_1}$ *a.e.* in R_+ (see [20]). Assuming that $A \neq 0$, we have $u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A}$ *a.e.* in R_+, and then $\lambda-\lambda_2-\lambda_1 = 0$. Hence, we have $\lambda_1+\lambda_2 = \lambda$. This completes the proof of Theorem 3. □

4. Special Cases for Improved Inequality

Below, we show that several new inequalities of Hardy–Hilbert type can be derived from the current result via the special values of parameters. From the perspective of applications, the parameterized Hardy–Hilbert inequality includes lots of new inequalities of the Hardy–Hilbert type; these new inequalities provide accurate upper bounds for certain double series, which can be applied to the estimations of double series sums.

(1) Putting $\lambda_1+\lambda_2 = \lambda = 1$ in (19), then for $\lambda_1 \in (0, \frac{3}{2\alpha}-1] \cap (0,1), \lambda_2 \in (0, \frac{3}{2\beta}-1] \cap (0,1)$, we have the following inequality:

$$\begin{aligned} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m-\xi)^{\alpha}+(n-\eta)^{\beta}} &< \alpha^{\frac{1}{p}}\beta^{\frac{1}{q}}\lambda_1\lambda_2 B(\lambda_1,\lambda_2) \\ \times\left[\sum_{m=1}^{\infty}(m-\xi)^{-p\alpha\lambda_1-1}A_m^p\right]^{\frac{1}{p}}&\left[\sum_{n=1}^{\infty}(n-\eta)^{-q\beta\lambda_2-1}B_n^q\right]^{\frac{1}{q}}, \end{aligned} \tag{22}$$

where the constant factor $\alpha^{\frac{1}{p}}\beta^{\frac{1}{q}}\lambda_1\lambda_2 B(\lambda_1,\lambda_2)$ is the best possible.

(2) Choosing $\alpha = \beta = 1, \varsigma = \xi+\eta \in (0, \frac{1}{2}], \lambda_1, \lambda_2 \in (0, \min\{\frac{1}{2}, \lambda\})$ in (19), we obtain the following:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n-\varsigma)^{\lambda}} < \lambda_1\lambda_2 B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty}(m-\xi)^{-p\lambda_1-1}A_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\eta)^{-q\lambda_2-1}B_n^q\right]^{\frac{1}{q}}. \tag{23}$$

where the constant factor $\lambda_1\lambda_2 B(\lambda_1,\lambda_2)$ is the best possible.

Remark 2. Taking $\xi = \eta = 0$ in inequality (23), we obtain inequality (4). Hence, inequality (19) is a generalization of inequality (4) presented in [14]. Moreover, since the constant factor in inequality (19) is the best possible, the inequality (19) is an improvement of the existing inequality (5) reported in [18].

5. Conclusions

This paper focuses on dealing with the refinement and generalization of the discrete Hardy–Hilbert inequality. The main methodological approach of this paper lies in constructing weight coefficients, introducing partial sums, and applying the special functions to the estimation of weight functions. Some analytical tools, such as the Euler–Maclaurin summation formula, Abel summation by parts formula, and the differentiation mid-value theorem, are employed in the process of deriving the main results. This shows the usefulness and applicability of these technologies in classical analysis. As a consequence, an improvement of the Hardy–Hilbert inequality involving two partial sums is established in Theorem 1, which is a unified improvement of the existing results in [14,18]. In addition, the characterizations of equivalent conditions of the best value related to several parameters are provided by Theorem 2 and Theorem 3, respectively. At the end of the paper, two new inequalities of Hardy–Hilbert type are established to illustrate the application of the current result.

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