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Bifurcation Analysis of a Class of Food Chain Model with Two Time Delays

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Abstract: This paper investigates the Hopf bifurcation of a three-dimensional food chain model with two timedelays, focusing on the synergistic effect of time delays in energy transfer between different trophic levels on the stability of the system. By analyzing the distribution of the roots of the characteristic equation, the stability conditions of the internal equilibrium point and the criterion for the existence of the Hopf bifurcation are established. Using the paradigm theory and the central manifold theorem, explicit formulas for determining the bifurcation direction and the stability of the bifurcation periodic solution are obtained. Numerical simulations verify the theoretical results. This study shows that increasing the time delay will lead to the instability of the food chain model through Hopf bifurcation and produce limit cycle oscillations. This work simulates the asymmetric propagation mode of population fluctuations observed in natural ecosystems, providing a theoretical basis for analyzing the coevolution of complex food webs.

Keywords: two time delays; food chain model; Hopf bifurcation

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1. Introduction

Since Lotka [1] and Volterra [2] established the classic predator–prey model, the study of predator–prey systems has become a central research topic in mathematical ecology. Due to the universality and ecological significance of predator–prey interactions in nature, this dynamic system has attracted the continuous attention of mathematical and ecological researchers and has become the main focus of ecological and mathematical ecology research [3–6]. As one of the basic models of predator–prey systems, the food chain model has been widely studied by scholars. The study of the food chain model has advanced the understanding of ecological interactions from linear relationships to complex nonlinear systems.

The three-species food chain model provides a theoretical framework for explaining the complexity of ecosystems by simulating multi-trophic-level interactions. Its dynamic behavior not only reflects the spatiotemporal transmission characteristics of trophic cascades, but also clarifies the key role of nonlinear mechanisms in controlling energy flow and stability regulation between levels [7–13]. In 1991, Hastings and Powell [14] developed a three-species food chain model within the classic Lotka–Volterra framework:

$$\begin{cases} \frac{dx}{dt} = x(1-x) - \frac{a_1 xy}{1+b_1 x}, \\ \frac{dy}{dt} = \frac{a_1 xy}{1+b_1 x} - \frac{a_2 yz}{1+b_2 y} - d_1 y, \\ \frac{dz}{dt} = \frac{a_2 yz}{1+b_2 y} - d_2 z, \end{cases} \tag{1}$$

where population x denotes the prey, population y represents the middle predator, and population z corresponds to the top predator. The parameters a_1 and a_2 represent the predation efficiency of population y on population x and population z on population y, respectively. The coefficients b_1 and b_2 denote the saturation effects of population x density on the predation efficiency of population y and population y density on the predation efficiency of population z, respectively. Additionally, d_1 and d_2 correspond to the natural mortality rates of populations y and z. The study revealed that when mortality rates d_1 and d_2 of intermediate predator y and top predator z reach critical thresholds, the system transitions from a stable equilibrium to chaotic oscillations [14]. This groundbreaking finding not only confirms the universality of deterministic chaos in ecosystems but also advances further exploration into food chain models.

It is well known that both past historical states and current conditions significantly affect population dynamics. In biological systems, time delays are inevitable due to factors such as maturation time, gestation period, and digestion time. Introducing time delays in food chain models can not only enhance the realism of these models, but also increase their complexity. Time delay systems retain all the dynamic characteristics of non-delayed systems while exhibiting richer behavioral patterns. Crucially, the addition of delays can disrupt the originally stable equilibrium state and lead to population fluctuations. Therefore, studying the dynamics of time-delayed food chain models has dual significance—mathematical and biological [15–18].

Traditional food chain models often include maturation delays and gestation delays [19–23]. Tian, B. et al. introduced digestion delay (also referred to as energy transfer delay) into the food chain model [24]. It offers diverse perspectives for advancing the dynamic analysis of food chain models. In natural ecosystems, energy transfer across trophic levels inherently involves hysteresis effects [25]. For example, predators take days to weeks to digest prey (e.g., pythons), while juveniles take months or even years to mature into adults (e.g., salmon). These delays directly regulate population fluctuation patterns. Therefore, introducing energy transfer delays in food chain models can not only reveal the evolutionary dynamics of predator–prey systems more realistically, but also significantly improve simulation accuracy and practical guidance value for ecosystem management.

Food chain models incorporating energy transfer time lags are of considerable research significance. Specifically, the introduction of dual delays in energy transfer—from prey to mesopredator and from mesopredator to apex predator—within tri-trophic systems allows for a more refined characterization of cascading response mechanisms in real-world ecosystems. These dual delays enable a more accurate depiction of the cascading response mechanisms characteristic of real ecosystems. Building upon system (1), this paper extends the framework by introducing a top predator, thereby constructing a three-species food chain model. We incorporate two essential time delays: (i) the delay from prey consumption to its conversion into predator biomass, and (ii) the delay involved in transferring biomass from intermediate predators to top predators. The resulting three-species food chain model is expressed as follows:

$$\begin{cases} \frac{dx}{dt} = x(t)[r_1 - a_{11}x(t)] - a_{12}x(t)y(t), \\ \frac{dy}{dt} = y(t)[S_1(1 - \frac{h_1 y(t)}{x(t-\tau_1)})] - a_{23}y(t)z(t), \\ \frac{dz}{dt} = z(t)[S_2(1 - \frac{h_2 z(t)}{y(t-\tau_2)})]. \end{cases} \tag{2}$$

By analyzing the coupling of energy transfer delays across multiple trophic levels, this work explores how synergistic delays destabilize equilibrium states, leading to Hopf bifurcations and emergent oscillatory regimes. System (2) incorporates both intraspecific competition within populations and the effects of predation behavior and time delays

on population dynamics. In this system, population x denotes the prey, population y represents the middle predator, and population z corresponds to the top predator. The variables x, y and z represent the population densities of the three species, respectively, r_1 is the natural growth rate of population x, a_{11} denotes the intraspecific competition coefficient of population x, and a_{12} and a_{23} are the interspecific interaction coefficients between population x and population y, and between population y and population z, respectively. S_1 and S_2 represent the intrinsic growth rates of predator populations y and z, while h_1 and h_2 correspond to their resource competition coefficients. The time delays τ_1 and τ_2 describe the energy conversion lags from population x to population y and from population y to population z, respectively. All parameters, r_1, a_{ij}, S_1, S_2, h_1 and h_2, are positive constants. Based on different values of the two delays, τ_1 and τ_2, this paper establishes stability conditions for the internal equilibrium of system (2) and derives criteria for the existence of Hopf bifurcations under five distinct scenarios. The introduction of dual delays significantly enhances the modeling depth of ecological dynamics. Compared to single-delay systems [26], this dual-delay framework more realistically reconstructs the cascading response mechanisms across trophic levels by simultaneously capturing the lag effects in energy transfer between different hierarchical layers. This dual-delay coupling significantly alters the population dynamic equilibria. For example, in the Northwest Atlantic cod ecosystem, seasonal fluctuations in plankton abundance propagate through two trophic levels, resulting in a 4–6-month lag in cod population peaks relative to the initial plankton signal. This temporal mismatch critically impacts the timing and sustainability of fishery harvest cycles. Furthermore, by applying normal form theory and the center manifold theorem, we derive explicit formulae to determine the direction of Hopf bifurcations and the stability of periodic solutions in system (2). Finally, numerical simulations are presented to validate the theoretical results.

2. Stability of Equilibrium and Existence of Hopf Bifurcation

Let $t = \frac{T}{r_1}$, $x = \frac{r_1}{a_{11}}x'$, $y = \frac{r_1}{a_{12}}y'$, $z = \frac{r_1}{a_{23}}z'$, the transformed variables, x', y', z', are still denoted as x, y, z and T remains as t; then, system (2) is transformed into

$$\begin{cases} \frac{dx}{dt} = x(t)[1 - x(t)] - x(t)y(t), \\ \frac{dy}{dt} = y(t)[p_1 - p_2\frac{y(t)}{x(t-\tau_1)}] - y(t)z(t), \\ \frac{dz}{dt} = z(t)[p_3 - p_4\frac{z(t)}{y(t-\tau_2)}], \end{cases} \tag{3}$$

where $p_1 = \frac{S_1}{r_1}$, $p_2 = \frac{S_1 h_1 a_{11}}{r_1 a_{12}}$,$p_3 = \frac{S_2}{r_1}$,$p_4 = \frac{S_2 h_2 a_{12}}{r_1 a_{23}}$.

By setting the right-hand-side functions of system (3) to zero, the internal equilibrium, $E(x^*, y^*, z^*)$, can be obtained:

$$x^* = 1 - \frac{q_1 - \sqrt{q_1^2 - 4q_2}}{2p_3},\ y^* = \frac{q_1 - \sqrt{q_1^2 - 4q_2}}{2p_3},\ z^* = \frac{q_1 - \sqrt{q_1^2 - 4q_2}}{2p_4},$$
$$q_1 = p_3 + p_1p_4 + p_2p_4,\ q_2 = p_1p_3p_4.$$

Clearly, when the condition $(H_1) : q_1^2 - 4q_2 \geq 0$, $q_1 - \sqrt{q_1^2 - 4q_2} < 2p_3$ holds, system (3) admits an internal equilibrium.

The Jacobian matrix of system (3) at the equilibrium $E(x^*, y^*, z^*)$ is given by

$$J(E) = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21}e^{-\lambda\tau_1} & a_{22} & a_{23} \\ 0 & a_{32}e^{-\lambda\tau_2} & a_{33} \end{pmatrix},$$

where

$$a_{11} = 1 - 2x^* - y^*,\ a_{12} = -x^*,\ a_{21} = \frac{p_2(y^*)^2}{(x^*)^2},\ a_{22} = p_1 - \frac{2p_2y^*}{x^*} - z^*,$$
$$a_{23} = -y^*,\ a_{32} = \frac{p_4(z^*)^2}{(y^*)^2},\ a_{33} = p_3 - \frac{2p_4z^*}{y^*}.$$

The characteristic equation of system (3) at $E(x^*, y^*, z^*)$ is

$$\lambda^3 + P_{02}\lambda^2 + P_{01}\lambda + P_{00} + (P_{11}\lambda + P_{10})e^{-\lambda\tau_1} + (P_{21}\lambda + P_{20})e^{-\lambda\tau_2} = 0, \tag{4}$$

where

$$P_{01} = a_{33}a_{11} + a_{33}a_{22} + a_{11}a_{22},\ P_{00} = -a_{11}a_{22}a_{33},$$
$$P_{02} = a_{22} - a_{11} - a_{23},\ P_{10} = a_{12}a_{21}a_{33},$$
$$P_{11} = -a_{12}a_{21},\ P_{20} = a_{11}a_{23}a_{32},\ P_{21} = -a_{23}a_{32}.$$

The stability of system (3) at $E(x^*, y^*, z^*)$ and the existence of Hopf bifurcation are analyzed under five distinct cases.

Case 1. $\tau_1 = \tau_2 = 0$.

Equation (6) becomes

$$\lambda^3 + P_{02}\lambda^2 + (P_{01} + P_{11} + P_{21})\lambda + (P_{00} + P_{10} + P_{20}) = 0.$$

For clarity, we introduce the hypothesis

$$(H_2): P_{02} > 0,\ (P_{01} + P_{11} + P_{21})P_{02} - P_{00} + P_{10} + P_{20} > 0,\ (P_{01} + P_{11} + P_{21})(P_{00} + P_{10} + P_{20})(P_{02} - 1) > 0.$$

Based on the Routh–Hurwitz criterion, the following theorem is established.

Theorem 1. *If (H_1) and (H_2) hold, then the internal equilibrium $E(x^*, y^*, z^*)$ of system (3) is asymptotically stable.*

Case 2. $\tau_1 > 0$, $\tau_2 = 0$.

Equation (4) becomes

$$\lambda^3 + P_{02}\lambda^2 + (P_{01} + P_{21})\lambda + (P_{00} + P_{20}) + (P_{11}\lambda + P_{10})e^{-\lambda\tau_1} = 0. \tag{5}$$

Let $\lambda = i\omega_1 (\omega_1 > 0)$ be a root of Equation (5). Substituting $\lambda = i\omega_1$ into Equation (5) and separating the real and imaginary parts, we obtain

$$\begin{cases} P_{11}\omega_1\cos(\omega_1\tau_1) - P_{10}\sin(\omega_1\tau_1) = \omega_1^3 - P_{01}\omega_1 - P_{21}\omega_1, \\ P_{11}\omega_1\sin(\omega_1\tau_1) + P_{10}\cos(\omega_1\tau_1) = P_{02}\omega_1^2 - P_{00} - P_{20}, \end{cases} \tag{6}$$

Therefore, we have

$$\begin{cases} \sin(\omega_1\tau_1) = \frac{P_{11}\omega_1(P_{02}\omega_1^2 - P_{00} - P_{20}) - P_{10}(\omega_1^3 - P_{10}\omega_1 - P_{21}\omega_1)}{P_{11}^2\omega_1^2 + P_{10}^2}, \\ \cos(\omega_1\tau_1) = \frac{P_{11}\omega_1(\omega_1^3 - P_{10}\omega_1 - P_{21}\omega_1) + P_{10}(P_{02}\omega_1^2 - P_{00} - P_{20})}{P_{11}^2\omega_1^2 + P_{10}^2}. \end{cases}$$

Squaring and adding both equations in (6) yields

$$\omega_1^6 + M_1\omega_1^4 + M_2\omega_1^2 + M_3 = 0, \tag{7}$$

where

$$M_1 = P_{02}^2 - 2P_{01} - 2P_{21},$$
$$M_2 = P_{01}^2 + P_{21}^2 + 2P_{01}P_{21} - 2P_{02}P_{00} - 2P_{02}P_{20} - P_{11}^2,$$
$$M_3 = P_{00}^2 + 2P_{00}P_{20} + P_{20}^2 - P_{10}^2.$$

Lemma 1. *If* $(H_3): M_3 < 0$ *or* $M_3 \geq 0, \Delta > 0, s_{11} > 0, f(s_{11}) \leq 0$ *hold, then Equation (7) admits a positive root, where* Δ, s_{11} *and* $f(s_{11})$ *are demonstrated in the proof of Lemma 1.*

Proof. Let $s_1 = \omega_1^2$. Substituting this into Equation (7), we obtain

$$s_1^3 + M_1 s_1^2 + M_2 s_1 + M_3 = 0. \tag{8}$$

For clarity, let us define

$$f(s_1) = s_1^3 + M_1 s_1^2 + M_2 s_1 + M_3. \tag{9}$$

Obviously, $\lim_{s_1 \to +\infty} f(s_1) = +\infty$.

When $M_3 < 0$, Equation (8) has at least one positive root.

When $M_3 \geq 0$, taking the derivative of Equation (11), we obtain $\frac{df(s_1)}{ds_1} = 3s_1^2 + 2M_1 s_1 + M_2$. When $\Delta = 4M_1^2 - 12M_2 \leq 0$, we can imply that $\frac{df(s_1)}{ds_1} \geq 0$ and $f(s_1)$ is monotonically increasing. Therefore, when $\Delta \leq 0$, Equation (10) has no positive roots; when $\Delta > 0$, $3s_1^2 + 2M_1 s_1 + M_2 = 0$ has two real roots

$$s_{11} = \frac{-2M_1 + \sqrt{\Delta}}{6}, s_{12} = \frac{-2M_1 - \sqrt{\Delta}}{6}.$$

Therefore, when $\Delta > 0$, and $s_{11} > 0$, $f(s_{11}) \leq 0$, Equation (8) admits positive roots. This completes the proof. □

From positive roots s_{11} and s_{12} of Equation (8), the roots of Equation (7) can be derived as $\omega_{1m} = \sqrt{s_{1m}^*}(m = 1, 2)$. For each fixed ω_{1m}, there exists a corresponding sequence $\left\{\tau_{1m}^{(n)} \mid m = 1, 2, n = 0,1,2,\dots\right\}$, where

$$\tau_{1m}^{(n)} = \frac{1}{\omega_{1m}}\left(\arccos\frac{P_{11}\omega_{1m}(\omega_{1m}^3 - P_{10}\omega_{1m} - P_{21}\omega_{1m}) + P_{10}(P_{02}\omega_{1m}^2 - P_{00} - P_{20})}{P_{11}^2\omega_{1m}^2 + P_{10}^2} + 2n\pi\right),$$
$$m = 1, 2, n = 0,1,2\dots$$

When $\tau_1 = \tau_{1m}^{(n)}$, $\pm i\omega_{1m}$ becomes a pair of pure imaginary roots of Equation (5).

Let $\tau_{10} = \min\left\{\tau_{1m}^{(n)} | m = 1, 2, n = 0, 1, 2\dots\right\}$ and ω_{10} be the corresponding ω_{1m}. Suppose $\lambda(\tau_1) = \alpha(\tau_1) + i\omega_1(\tau_1)$ is a root of Equation (5) at $\tau_1 = \tau_{10}$, satisfying $\alpha(\tau_1) = 0$ and $\omega_1(\tau_1) = \omega_{10}$.

Lemma 2. *If* $(H_4): 3s_1^2 + 2M_1 s_1 + M_2 > 0$ *holds, then* $Re\left(\frac{d\lambda}{d\tau_1}\right)^{-1}\Big|_{\lambda = i\omega_{10}} > 0.$

Proof. Differentiating both sides of Equation (5) with respect to τ_1, we obtain

$$\frac{d\lambda}{d\tau_1} = \frac{\lambda(P_{11}\lambda + P_{10})e^{-\lambda\tau_1}}{3\lambda^2 + 2P_{02}\lambda + P_{01} + P_{21} + [P_{11} - \tau_1(P_{11}\lambda + P_{10})]e^{-\lambda\tau_1}}. \tag{10}$$

Substituting $\lambda = i\omega_{10}$ into (10) and extracting the real part yields

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau_1}\right)^{-1}\Big|_{\lambda=i\omega_{10}} = \frac{3\omega_{10}^4 + [2P_{02}^2 - 4(P_{01}+P_{21})]\omega_{10}^2 + [(P_{01}+P_{21})^2 - 2P_{02}(P_{00}+P_{21}) - P_{11}^2]}{[\omega_{10}^3 - (P_{01}+P_{21})\omega_{10}]^2 + [-\omega_{10}^2 P_{02} + P_{00} + P_{20}]^2}.$$

Therefore, when the hypothesis (H_4) : $3s_1^2 + 2M_1 s_1 + M_2 > 0$ holds, we have $\mathrm{Re}\left(\frac{d\lambda}{d\tau_1}\right)^{-1}\Big|_{\lambda=i\omega_{10}} > 0$. This completes the proof. □

From Lemmas 1 and 2, the following theorem is derived.

Theorem 2. *If (H_1), (H_2), (H_3), (H_4) are satisfied, then when $\tau_1 = [0,\tau_{10})$, the internal equilibrium $E(x^*, y^*, z^*)$ of system (3) is asymptotically stable. When $\tau_1 > \tau_{10}$, the internal equilibrium $E(x^*, y^*, z^*)$ of system (3) is unstable. When $\tau_1 = \tau_1^{(n)} (n = 0, 1, 2\ldots)$, system (3) exhibits a Hopf bifurcation at the internal equilibrium $E(x^*, y^*, z^*)$.*

Case 3. $\tau_1 = 0$, $\tau_2 > 0$.

Equation (4) becomes

$$\lambda^3 + P_{02}\lambda^2 + (P_{01}+P_{11})\lambda + (P_{00}+P_{10}) + (P_{21}\lambda + P_{20})e^{-\lambda\tau_2} = 0. \tag{11}$$

Let $\lambda = i\omega_2 (\omega_2 > 0)$ be a root of Equation (11). Substituting $\lambda = i\omega_2$ into Equation (11) and separating the real and imaginary parts, we obtain

$$\begin{cases} P_{21}\omega_2\cos(\omega_2\tau_2) - P_{20}\sin(\omega_2\tau_2) = \omega_2^3 - P_{01}\omega_2 - P_{11}\omega_2, \\ P_{21}\omega_2\sin(\omega_2\tau_2) + P_{20}\cos(\omega_2\tau_2) = P_{02}\omega_2^2 - P_{00} - P_{10}, \end{cases} \tag{12}$$

Therefore, we have

$$\begin{cases} \sin(\omega_2\tau_2) = \frac{(P_{21}P_{02} - P_{20})\omega_2^3 + [P_{20}(P_{01}+P_{11}) - P_{21}(P_{00}+P_{10})]\omega_2}{P_{21}^2\omega_2^2 + P_{20}^2}, \\ \cos(\omega_2\tau_2) = \frac{P_{21}\omega_2^4 + [P_{20}P_{02} - P_{21}(P_{01}+P_{11})]\omega_2^2 - P_{20}(P_{00}+P_{10})}{P_{21}^2\omega_2^2 + P_{20}^2}. \end{cases}$$

Squaring and adding both equations in Equation (12) yields

$$\omega_2^6 + N_1\omega_2^4 + N_2\omega_2^2 + N_3 = 0, \tag{13}$$

where

$$N_1 = P_{02}^2 - 2P_{01} - 2P_{11},$$
$$N_2 = P_{01}^2 + P_{11}^2 + 2P_{01}P_{11} - 2P_{02}P_{00} - 2P_{02}P_{10} - P_{21}^2,$$
$$N_3 = P_{00}^2 + 2P_{00}P_{10} + P_{10}^2 - P_{20}^2.$$

Lemma 3. *If (H_5) : $N_3 < 0$ or $N_3 \geq 0, \Delta_2 > 0$, $s_{21} > 0$, $f(s_{21}) \leq 0$ holds, then Equation (13) admits positive roots, where Δ_2, s_{21} and $f(s_{21})$ are demonstrated in the proof of Lemma 3.*

Proof. Let $s_2 = \omega_2^2$. Substituting this into Equation (13), we obtain

$$s_2^3 + N_1 s_2^2 + N_2 s_2 + N_3 = 0. \tag{14}$$

Define the function

$$g(s_2) = s_2^3 + N_1 s_2^2 + N_2 s_2 + N_3. \tag{15}$$

Obviously, $\lim_{s_2\to+\infty} g(s_2) = +\infty$.

When $N_3 < 0$, Equation (13) has at least one positive root.

When $N_3 \geq 0$, taking the derivative of Equation (15), we obtain $\frac{dg(s_2)}{ds_2} = 3s_2^2 + 2N_1 s_2 + N_2$. When $\Delta_2 = 4N_1^2 - 12N_2 \leq 0$, we can imply that $\frac{dg(s_2)}{ds_2} \geq 0$, and $g(s_2)$ is monotonically increasing. Therefore, when $\Delta_2 \leq 0$, Equation(13) has no positive roots. When $\Delta_2 > 0$, $3s_2^2 + 2N_1 s_2 + N_2 = 0$ has two real roots

$$s_{21} = \frac{-2N_1 + \sqrt{\Delta_2}}{6}, s_{22} = \frac{-2N_1 - \sqrt{\Delta_2}}{6}.$$

Therefore, when $\Delta_2 > 0$, $s_{21} > 0, g(s_{21}) \leq 0$, Equation (14) admits positive roots. This completes the proof. □

From positive roots s_{21} and s_{22} of Equation (14), the roots of Equation (13) can be derived as $\omega_{2m} = \sqrt{s_{2m}^*}(m = 1, 2)$. For each fixed ω_{2m}, there exists a corresponding sequence, $\left\{\tau_{2m}^{(j)} | m = 1, 2, j = 0, 1, 2 \ldots\right\}$, where

$$\tau_{2m}^{(j)} = \frac{1}{\omega_{2m}}(\arccos\frac{P_{21}\omega_{2m}^4 + [P_{20}P_{02} - P_{21}(P_{01} + P_{11})]\omega_{2m}^2 - P_{20}(P_{00} + P_{10})}{P_{21}^2\omega_{2m}^2 + P_{20}^2} + 2n\pi),$$
$$m = 1, 2, j = 0, 1, 2 \ldots$$

When $\tau_2 = \tau_{2m}^{(j)}$, $\pm i\omega_{2m}$ becomes a pair of pure imaginary roots of Equation (11).

Let $\tau_{20} = \min\left\{\tau_{2m}^{(j)} | m = 1, 2, n = 0, 1, 2 \ldots\right\}$ and ω_{20} be the corresponding ω_{2m}. Suppose $\lambda(\tau_2) = \alpha(\tau_2) + i\omega_2(\tau_2)$ is a root of Equation (11) at $\tau_2 = \tau_{20}$, satisfying $\alpha(\tau_2) = 0$ and $\omega_2(\tau_2) = \omega_{20}$.

Lemma 4. *If* $(H_6) : 3s_2^2 + 2N_1 s_2 + N_2 > 0$ holds, then $\mathrm{Re}\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\Big|_{\lambda = i\omega_{20}} > 0$.

Proof. Differentiating both sides of Equation (11) with respect to τ_2, we obtain

$$\frac{d\lambda}{d\tau_2} = \frac{\lambda(P_{21}\lambda + P_{20})e^{-\lambda\tau_2}}{3\lambda^2 + 2P_{02}\lambda + P_{01} + P_{11} + [P_{21} - \tau_2(P_{21}\lambda + P_{20})]e^{-\lambda\tau_2}}. \tag{16}$$

Substituting $\lambda = i\omega_{20}$ into Equation (16) and extracting the real part, we derive

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\Big|_{\lambda = i\omega_{20}} = \frac{3\omega_{20}^4 + [2P_{02}^2 - 4(P_{01} + P_{11})]\omega_{20}^2 + [(P_{01} + P_{11})^2 - 2P_{02}(P_{00} + P_{10}) - P_{21}^2]}{[\omega_{20}^3 - (P_{01} + P_{11})\omega_{20}]^2 + [\omega_{20}^2 P_{02} - P_{00} - P_{10}]^2}.$$

Therefore, when the hypothesis $(H_6) : 3s_2^2 + 2N_1 s_2 + N_2 > 0$ holds, we have $\mathrm{Re}\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\Big|_{\lambda = i\omega_{20}} > 0$. This completes the proof. □

From Lemmas 3 and 4, the following theorem is derived.

Theorem 3. *If* $(H_1), (H_2), (H_5), (H_6)$ *are satisfied, then when* $\tau_2 = [0, \tau_{20})$, *the internal equilibrium* $E(x^*, y^*, z^*)$ *of system (3) is asymptotically stable. When* $\tau_2 > \tau_{20}$, *the internal equilibrium* $E(x^*, y^*, z^*)$ *of system (3) is unstable. When* $\tau_2 = \tau_2^{(n)}(n = 0, 1, 2 \ldots)$, *system (3) exhibits a Hopf bifurcation at the internal equilibrium* $E(x^*, y^*, z^*)$.

Case 4. $\tau_1 \in (0, \tau_{10})$, $\tau_2 > 0$, where τ_1 is constrained within its stability interval, while τ_2 is treated as a bifurcation parameter.

The characteristic equation of system (3) at the equilibrium $E(x^*, y^*, z^*)$ is

$$\lambda^3 + P_{02}\lambda^2 + P_{01}\lambda + P_{00} + (P_{11}\lambda + P_{10})e^{-\lambda\tau_1} + (P_{21}\lambda + P_{20})e^{-\lambda\tau_2} = 0. \tag{17}$$

Let $\lambda = i\omega_3(\omega_3 > 0)$ be a root of Equation (17). Substituting $\lambda = i\omega_3$ into Equation (17) and separating the real and imaginary parts, we obtain

$$\begin{cases} P_{11}\omega_3 \sin(\omega_3\tau_1) + P_{10}\cos(\omega_3\tau_1) + P_{21}\omega_3 \sin(\omega_3\tau_2) + P_{20}\cos(\omega_3\tau_2) = P_{02}\omega_3^2 - P_{00}, \\ P_{11}\omega_3 \cos(\omega_3\tau_1) - P_{10}\sin(\omega_3\tau_1) + P_{21}\omega_3 \cos(\omega_3\tau_2) - P_{20}\sin(\omega_3\tau_2) = \omega_3^3 - P_{01}\omega_3, \end{cases} \tag{18}$$

Therefore, we have

$$\begin{cases} \sin(\omega_3\tau_2) = \frac{R_3R_1 - R_2R_4}{R_1^2 + R_2^2}, \\ \cos(\omega_3\tau_2) = \frac{R_2R_3 + R_4R_1}{R_1^2 + R_2^2}, \end{cases}$$

where

$$\begin{gathered} R_1 = P_{21}\omega_3, R_2 = P_{20}, \\ R_3 = P_{02}\omega_3^2 - P_{00} - P_{11}\omega_3 \sin(\omega_3\tau_1) - P_{10}\cos(\omega_3\tau_1), \\ R_4 = \omega_3^3 - P_{01}\omega_3 - P_{11}\omega_3 \cos(\omega_3\tau_1) + P_{10}\sin(\omega_3\tau_1). \end{gathered}$$

Squaring and adding both equations in Equation (18) yields

$$\omega_3^6 + E_1\omega_3^4 + E_2\omega_3^2 + E_3 + (E_4\omega_3^4 + E_5\omega_3^2 + E_6)\cos(\omega_3\tau_1) + (E_7\omega_3^3 + E_8\omega_3)\sin(\omega_3\tau_1) = 0, \tag{19}$$

where

$$\begin{gathered} E_1 = P_{02}^2 - 2P_{01}, E_2 = P_{01}^2 + P_{11}^2 - P_{21}^2 - 2P_{02}P_{00}, \\ E_3 = P_{00}^2 - P_{20}^2 + P_{10}^2, E_4 = -2P_{11}, \\ E_5 = 2P_{01}P_{11} - 2P_{02}P_{10}, E_6 = 2P_{10}P_{00}, \\ E_7 = 2P_{10} - 2P_{02}P_{11}, E_8 = 2P_{11}P_{00} - 2P_{10}P_{01}. \end{gathered}$$

Lemma 5. *If* $(H_7) : E_3 + E_6 < 0$ *holds, then Equation (19) admits at least one positive root.*

Proof. Define the function

$$h(\omega_3) = \omega_3^6 + E_1\omega_3^4 + E_2\omega_3^2 + E_3 + (E_4\omega_3^4 + E_5\omega_3^2 + E_6)\cos(\omega_3\tau_1) + (E_7\omega_3^3 + E_8\omega_3)\sin(\omega_3\tau_1).$$

Since $\lim\limits_{\omega_3 \to +\infty} h(\omega_3) = +\infty$, when $E_3 + E_6 < 0$, Equation (19) possesses at least one positive root. Due to the sixth-degree polynomial nature of ω_3, Equation (19) has finitely many positive real roots, denoted as $\omega_{31}, \omega_{32}, \omega_{33}, \ldots, \omega_{3n}$. For each fixed $\omega_{3i}(i = 1, 2 \ldots, n)$, there exists a corresponding sequence, $\left\{\tau_{3i}^{(j)} | i = 1, 2 \ldots n, j = 1, 2, \ldots\right\}$, where

$$\tau_{3i}^{(j)} = \frac{1}{\omega_{3i}}(\arccos\frac{R_2R_3 + R_4R_1}{R_1^2 + R_2^2} + 2j\pi), i = 1, 2 \ldots, n, j = 0, 1, 2 \ldots$$

When $\tau_3 = \tau_{3i}^{(j)}$, $\pm i\omega_{3i}$ becomes a pair of pure imaginary roots of Equation (17).

Let $\tau_{30} = \min\left\{\tau_{3i}^{(j)} | i = 1, 2 \ldots, n, j = 0, 1, 2 \ldots\right\}$, and ω_{30} be the corresponding ω_{3i}. Suppose $\lambda(\tau_2) = \alpha(\tau_2) + i\omega_3(\tau_2)$ is a root of Equation (17) at the critical delay $\tau_2 = \tau_{30}$, satisfying $\alpha(\tau_2) = 0$ and $\omega_3(\tau_2) = \omega_{30}$. □

Lemma 6. *If* $(H_8) : Q_RP_R + Q_IP_I > 0$ *holds, then* $Re\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\Big|_{\lambda = i\omega_{30}} > 0$, *where* Q_R, P_R, Q_I *and* P_I *are demonstrated in the proof of Lemma 6.*

Proof. Differentiating both sides of Equation (17) with respect to τ_2, we obtain

$$\frac{d\lambda}{d\tau_2} = \frac{\lambda(P_{21}\lambda + P_{20})e^{-\lambda\tau_2}}{3\lambda^2 + 2P_{02}\lambda + P_{01} + P_{11}e^{-\lambda\tau_1} + P_{21}e^{-\lambda\tau_2} - \tau_2(P_{21}\lambda + P_{20})e^{-\lambda\tau_2}}. \tag{20}$$

Substituting $\lambda = i\omega_{30}$ into Equation (20) and extracting the real part, we derive

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\Big|_{\lambda=i\omega_{30}} = \mathrm{Re}\left(\frac{Q_R + Q_I i}{P_R + P_I i}\right) = \frac{Q_R P_R + Q_I P_I}{P_R^2 + P_I^2},$$

where

$$\begin{gathered}
P_R = \omega_{30}^4 - P_{01}\omega_{30}^2 - P_{11}\omega_{30}^2\cos(\omega_{30}\tau_1) + P_{10}\omega_{30}\sin(\omega_{30}\tau_1),\\
P_I = P_{00}\omega_{30} - P_{02}\omega_{30}^3 + P_{11}\omega_{30}^2\sin(\omega_{30}\tau_1) + P_{10}\omega_{30}\cos(\omega_{30}\tau_1),\\
Q_R = 3\omega_{30}^2 - P_{01} - P_{11}\cos(\omega_{30}\tau_1) - P_{21}\cos(\omega_{30}\tau_2) - \tau_1 P_{11}\omega_{30}\sin(\omega_{30}\tau_1) - \tau_1 P_{10}\cos(\omega_{30}\tau_1),\\
Q_I = P_{11}\sin(\omega_{30}\tau_1) - 2P_{02}\omega_{30} + P_{21}\sin(\omega_{30}\tau_2) - \tau_1 P_{11}\omega_{30}\cos(\omega_{30}\tau_1) + \tau_1 P_{10}\sin(\omega_{30}\tau_1).
\end{gathered}$$

Therefore, when the hypothesis (H_8) : $Q_R P_R + Q_I P_I > 0$ holds, we have $\mathrm{Re}\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\Big|_{\lambda=i\omega_{30}} > 0$. This completes the proof. □

From Lemmas 5 and 6, the following theorem is derived.

Theorem 4. *If (H_1), (H_2), (H_7), (H_8) are satisfied, then when $\tau_2 = [0,\tau_{30})$, the internal equilibrium $E(x^*, y^*, z^*)$ of system (3) is asymptotically stable. When $\tau_2 > \tau_{30}$, the internal equilibrium $E(x^*, y^*, z^*)$ of system (3) is unstable. When $\tau_2 = \tau_{30}^{(j)} (n = 0, 1, 2\ldots)$, system (3) exhibits a Hopf bifurcation at the internal equilibrium $E(x^*, y^*, z^*)$.*

Case 5. $\tau_1 > 0$, $\tau_2 \in (0, \tau_{20})$, where τ_2 is constrained within its stability interval, while τ_1 is treated as a bifurcation parameter.

The characteristic equation of system (3) at the equilibrium $E(x^*, y^*, z^*)$ is

$$\lambda^3 + P_{02}\lambda^2 + P_{01}\lambda + P_{00} + (P_{11}\lambda + P_{10})e^{-\lambda\tau_1} + (P_{21}\lambda + P_{20})e^{-\lambda\tau_2} = 0. \tag{21}$$

Let $\lambda = i\omega_4 (\omega_4 > 0)$ be a root of Equation (21). Substituting $\lambda = i\omega_4$ into Equation (21) and separating the real and imaginary parts, we obtain

$$\begin{cases} P_{11}\omega_4\sin(\omega_4\tau_1) + P_{10}\cos(\omega_4\tau_1) + P_{21}\omega_4\sin(\omega_4\tau_2) + P_{20}\cos(\omega_4\tau_2) = P_{02}\omega_4^2 - P_{00}, \\ P_{11}\omega_4\cos(\omega_4\tau_1) - P_{10}\sin(\omega_4\tau_1) + P_{21}\omega_4\cos(\omega_4\tau_2) - P_{20}\sin(\omega_4\tau_2) = \omega_4^3 - P_{01}\omega_4, \end{cases} \tag{22}$$

We have

$$\begin{cases} \sin(\omega_4\tau_1) = \frac{H_1H_3 - H_2H_4}{H_1^2 - H_2^2}, \\ \cos(\omega_4\tau_1) = \frac{H_2H_3 - H_1H_4}{H_1^2 - H_2^2}, \end{cases}$$

where

$$\begin{gathered}
H_1 = P_{11}\omega_4, H_2 = P_{10},\\
H_3 = P_{02}\omega_4^2 - P_{00} - P_{21}\omega_4\sin(\omega_4\tau_2) - P_{20}\cos(\omega_4\tau_2),\\
H_4 = \omega_4^3 - P_{01}\omega_4 - P_{21}\omega_4\cos(\omega_4\tau_2) + P_{20}\sin(\omega_4\tau_2).
\end{gathered}$$

Squaring and adding both equations in Equation (22) yields

$$\omega_4^6 + K_1\omega_4^4 + K_2\omega_4^2 + K_3 + (K_4\omega_4^4 + K_5\omega_4^2 + K_6)\cos(\omega_4\tau_2) + (K_7\omega_4^3 + K_8\omega_4)\sin(\omega_4\tau_2) = 0, \tag{23}$$

where

$$\begin{gathered}
K_1 = P_{02}^2 - 2P_{01}, K_2 = P_{01}^2 - P_{11}^2 + P_{21}^2 - 2P_{02}P_{00},\\
K_3 = P_{00}^2 + P_{20}^2 - P_{10}^2, K_4 = -2P_{11}, K_5 = 2P_{01}P_{21} - P_{02}P_{20},\\
K_6 = P_{20}P_{00}, K_7 = 2P_{20} - 2P_{02}, K_8 = 2P_{21}P_{00} - 2P_{20}P_{01}.
\end{gathered}$$

Lemma 7. *If* $(H_9): K_3 + K_6 < 0$ *holds, then Equation (23) admits at least one positive root.*

Proof. Define the function

$$l(\omega_4) = \omega_4^6 + K_1\omega_4^4 + K_2\omega_4^2 + K_3 + (K_4\omega_4^4 + K_5\omega_4^2 + K_6)\cos(\omega_4\tau_2) + (K_7\omega_4^3 + K_8\omega_4)\sin(\omega_4\tau_2).$$

Since $\lim_{\omega_4 \to +\infty} l(\omega_4) = +\infty$, when $K_3 + K_6 < 0$, Equation (23) possesses at least one positive root. Due to the sixth-degree polynomial nature of ω_4, Equation (23) has finitely many positive real roots, denoted as $\omega_{41}, \omega_{42}, \omega_{43}, \ldots, \omega_{4n}$. For each fixed $\omega_{4i} (i = 1, 2 \ldots, n)$, there exists a corresponding sequence $\left\{\tau_{4i}^{(j)} | i = 1, 2 \ldots, n, j = 1, 2, \ldots\right\}$, where

$$\tau_{4i}^{(j)} = \frac{1}{\omega_{4i}}(\arccos\frac{H_2H_3 - H_1H_4}{H_1^2 - H_2^2} + 2j\pi), i = 1, 2 \ldots, n, j = 0,1,2 \ldots.$$

When $\tau_4 = \tau_{4i}^{(j)}$, $\pm i\omega_{4i}$ becomes a pair of pure imaginary roots of Equation (21).

Let $\tau_{40} = \min\left\{\tau_{4i}^{(j)} | i = 1, 2 \ldots, n, j = 0, 1, 2 \ldots\right\}$, and ω_{40} be the corresponding ω_{4i}. Suppose $\lambda(\tau_1) = \alpha(\tau_1) + i\omega_4(\tau_1)$ is a root of Equation (21) at the critical delay $\tau_1 = \tau_{40}$, satisfying $\alpha(\tau_1) = 0$ and $\omega_4(\tau_1) = \omega_{40}$. □

Lemma 8. *If* (H_{10}): ${Q'}_R{P'}_R + {Q'}_I{P'}_I > 0$ *holds, then* $Re\left(\frac{d\lambda}{d\tau_1}\right)^{-1}\Big|_{\lambda=i\omega_{40}} > 0$, *where* ${Q'}_R$, ${P'}_R$, ${Q'}_I$ *and* ${P'}_I$ *are demonstrated in the proof of Lemma 8.*

Proof. Differentiating both sides of Equation (21) with respect to τ_1, we obtain

$$\frac{d\lambda}{d\tau_1} = \frac{\lambda(P_{11}\lambda + P_{10})e^{-\lambda\tau_1}}{3\lambda^2 + 2P_{02}\lambda + P_{01} + P_{11}e^{-\lambda\tau_1} + P_{21}e^{-\lambda\tau_2} - \tau_1(P_{11}\lambda + P_{10})e^{-\lambda\tau_1} - \tau_2(P_{21}\lambda + P_{00})e^{-\lambda\tau_2}}. \quad (24)$$

Substituting $\lambda = i\omega_{40}$ into Equation (24) and extracting the real part, we derive

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau_1}\right)^{-1}\Big|_{\lambda=i\omega_{40}} = \mathrm{Re}\left(\frac{{Q'}_R + {Q'}_I i}{{P'}_R + {P'}_I i}\right) = \frac{{Q'}_R{P'}_R + {Q'}_I{P'}_I}{{P'}_R^2 + {P'}_I^2},$$

where

$$\begin{gathered}
{P_R}' = -\omega_{40}^4 + P_{01}\omega_{40}^2 + P_{21}\omega_{40}^2\cos(\omega_{40}\tau_2) - P_{20}\omega_{40}\sin(\omega_{40}\tau_2), \\
{P_I}' = P_{02}\omega_{40}^3 - P_{00}\omega_{40} - P_{21}\omega_{40}^2\sin(\omega_{40}\tau_2) - P_{20}\omega_{40}\cos(\omega_{40}\tau_2), \\
{Q_R}' = -3\omega_{40}^2 - P_{01} - P_{21}\cos(\omega_{40}\tau_2) - \tau_2P_{21}\omega_{40}\sin(\omega_{40}\tau_2) - \tau_2P_{00}\cos(\omega_{40}\tau_2), \\
{Q_I}' = 2P_{02}\omega_{40} - P_{21}\sin(\omega_{40}\tau_2) - P_{11}\sin(\omega_{40}\tau_1) + \tau_2P_{21}\omega_{40}\cos(\omega_{40}\tau_2) + \tau_2P_{00}\sin(\omega_{40}\tau_2).
\end{gathered}$$

Therefore, when the hypothesis (H_{10}): ${Q'}_R{P'}_R + {Q'}_I{P'}_I > 0$ holds, we have $\mathrm{Re}\left(\frac{d\lambda}{d\tau_1}\right)^{-1}\Big|_{\lambda=i\omega_{40}} > 0$. This completes the proof. □

From Lemmas 7 and 8, the following theorem is derived.

Theorem 5. *If* (H_1), (H_2), (H_9), (H_{10}) *are satisfied, then when* $\tau_1 = [0, \tau_{40})$, *the internal equilibrium* $E(x^*, y^*, z^*)$ *of system (3) is asymptotically stable. When* $\tau_1 > \tau_{40}$, *the internal equilibrium* $E(x^*, y^*, z^*)$ *of system (3) is unstable. When* $\tau_1 = \tau_{40}^{(j)} (n = 0, 1, 2 \ldots)$, *system (3) exhibits a Hopf bifurcation at the internal equilibrium* $E(x^*, y^*, z^*)$.

3. Computation of Hopf Bifurcation Properties

In this section, taking case 4 as an example, we employ normal form theory and center manifold theory to derive explicit formulas for the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions in system (3) when $\tau_1^* \in (0, \tau_{10})$, $\tau_2 = \tau_2^*$. Other cases can be analyzed analogously and are omitted here for brevity. We translate the internal equilibrium $E(x^*, y^*, z^*)$ to the origin by letting $\overline{x} = x - x^*$, $\overline{y} = y - y^*$, $\overline{z} = z - z^*$; then, system (3) at the internal equilibrium $E(x^*, y^*, z^*)$ can be transformed into

$$\begin{cases} \frac{d\overline{x}}{dt} = (\overline{x} + x^*)[1 - (\overline{x} + x^*)] - (\overline{x} + x^*)(\overline{y} + y^*), \\ \frac{d\overline{y}}{dt} = (\overline{y} + y^*)[p_1 - p_2 \frac{(\overline{y}+y^*)}{\overline{x}(t-\tau_1)+x^*}] - (\overline{y} + y^*)(\overline{z} + z^*), \\ \frac{d\overline{z}}{dt} = (\overline{z} + z^*)[p_3 - p_4 \frac{(\overline{z}+z^*)}{\overline{y}(t-\tau_2)+y^*}]. \end{cases}$$

This yields the linearized equations of system (3) at the equilibrium $E(x^*, y^*, z^*)$. For simplicity, we still denote $x = \overline{x}$, $y = \overline{y}$, $z = \overline{z}$. The system can then be expressed as

$$\begin{cases} \frac{dx}{dt} = (1 - 2x^* - y^*)x - x^* y, \\ \frac{dy}{dt} = p_2 \frac{(y^*)^2}{(x^*)^2} x(t - \tau_1) + (p_1 - z^* - \frac{2p_2 y^*}{x^*})y - y^* z, \\ \frac{dz}{dt} = p_4 \frac{(z^*)^2}{(y^*)^2} y(t - \tau_2) + (p_3 - \frac{p_4 z^*}{y^*})z. \end{cases} \tag{25}$$

Assume $\tau_2^* > \tau_{10}$, $\tau_1^* \in (0, \tau_{10})$. Let $\tau_2 = \tau_2^* + u (u \in R)$, and define the variables

$$h_1 = X(\tau_2, t), h_2 = Y(\tau_2, t), h_3 = Z(\tau_2, t),$$
$$L_u \colon C([-1, 0], \mathbb{R}^3) \to \mathbb{R}^3, F \colon \mathbb{R} \times ([-1, 0], \mathbb{R}^3) \to \mathbb{R}^3.$$

System (25) can then be reformulated as

$$\dot{h}(t) = L_u(h_t) + F(u, h_t), \tag{26}$$

where

$$h_t = (h_1, h_2, h_3)^T \in C\big([-1, 0], \mathbb{R}^3\big),$$
$$L_u(\varphi) = (\tau_2^* + u)\Big[A'\varphi(0) + B'\varphi\Big(-\frac{\tau_1^*}{\tau_2^*}\Big) + C'\varphi(-1)\Big],$$
$$F(u, \varphi) = (\tau_2^* + u)(F_1, F_2, F_3)^T,$$
$$A' = \begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix}, B' = \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, C' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix},$$
$$F(x, y, z) = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \triangleq \begin{pmatrix} f(x, y, z) \\ l(x, y, z) \\ z(x, y, z) \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2} f_{200} x^2 + f_{110} xy + f_{101} xz + f_{011} yz + \frac{1}{2} f_{020} y^2 + \frac{1}{2} f_{002} z^2 + \cdots \\ \frac{1}{2} l_{200} x^2(t - \tau_1) + l_{110} x(t - \tau_1) y(t - \tau_1) + l_{101} x(t - \tau_1) z(t - \tau_1) + l_{011} y(t - \tau_1) z(t - \tau_1) + \cdots \\ \frac{1}{2} k_{200} x^2(t - \tau_2) + k_{110} x(t - \tau_2) y(t - \tau_2) + k_{101} x(t - \tau_2) z(t - \tau_2) + k_{011} y(t - \tau_2) z(t - \tau_2) + \cdots \end{pmatrix},$$
$$l_{ijm} = \frac{\partial_{i+j+m} l(x^*, y^*, z^*)}{\partial^i x(t-\tau_1) \partial^j y(t-\tau_1) \partial^m z(t-\tau_1)},$$
$$f_{ijm} = \frac{\partial_{i+j+m} f(x^*, y^*, z^*)}{\partial^i x \partial^j y \partial^m z},$$
$$k_{ijm} = \frac{\partial_{i+j+m} k(x^*, y^*, z^*)}{\partial^i x(t-\tau_2) \partial^j y(t-\tau_2) \partial^m z(t-\tau_2)}.$$
$$(i, j, k = 1, 2, 3, \ldots).$$

According to the Riesz representation theorem, there exists a component consisting of a bounded variation function matrix $\eta(\theta, u)$ and a variable function $\theta \in [-1, 0]$, such that

$$L_u(\phi) = \int_{-1}^{0} d\eta(\theta, u)\phi(\theta),\ \theta \in \mathbb{R}.$$

In fact, we can choose

$$\eta(\theta, u) = \begin{cases} (\tau_2^* + u)(A' + B' + C'), & \theta = 0, \\ (\tau_2^* + u)(A' + B'), & \theta \in \left[-\frac{\tau_1^*}{\tau_2^*}, 0\right), \\ (\tau_2^* + u)C', & \theta \in \left(-1, -\frac{\tau_1^*}{\tau_2^*}\right), \\ 0, & \theta = -1 \end{cases}.$$

For $\varphi \in C([-1, 0], \mathbb{R}^3)$, we give the definitions

$$A(u)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^{0} d\eta(\theta, u)\phi(\theta), & \theta = 0. \end{cases}$$

$$R(u)\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ F(u, \phi), & \theta = 0 \end{cases}.$$

Then, Equation (25) can be reformulated as

$$\dot{h}(t) = A(u)h_t + R(u)h_t. \tag{27}$$

Defining A^* as

$$A^*(u)s = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in [-1, 0) \\ \int_{-1}^{0} d\eta^T(\theta, u)\psi(-s), & s = 0 \end{cases}.$$

The bilinear form between A and A^* is given by

$$\langle \psi, \phi \rangle = \overline{\psi}(0)\phi(0) - \int_{-1}^{0} \int_{\xi}^{\theta} \overline{\psi}(\xi - \theta) d\eta(\theta, u)\phi(\xi) d\xi.$$

Clearly, $\pm i\omega_2^*\tau_2^*$ are eigenvalues of $A(0)$, and also of $A^*(0)$. Through straightforward calculation, we obtain the corresponding eigenvectors

$$q(\theta) = (1, q_2, q_3)^T e^{i\omega_2^*\tau_2^*\theta}, q^*(s) = D(1, q_2^*, q_3^*)^T e^{i\omega_2^*\tau_2^* s},$$

where

$$q_2 = \frac{i\omega_2 - a_{11}}{a_{12}}, q_3 = \frac{a_{32}(i\omega_2 - a_{11})e^{-i\omega_2^*\tau_2^*}}{i\omega_2 a_{12}},$$
$$q_2^* = \frac{-i\omega_2 - a_{11}}{a_{21}e^{-i\omega_2^*\tau_1^*}}, q_3^* = \frac{a_{23}(i\omega_2 + a_{11})}{i\omega_2 a_{21}e^{-i\omega_2^*\tau_1^*}},$$
$$\overline{D} = \left[1 + \overline{q}_2^* q_2 + \overline{q}_3^* q_3 + \tau_1^* e^{-i\omega_2^*\tau_1^*} a_{21}\overline{q}_2^* + \tau_2^* e^{-i\omega_2^*\tau_2^*} a_{32}\overline{q}_3^* q_2\right]^{-1}.$$

Let h_t be the solution of Equation (27). Define

$$z(t) = \langle q^*, h_t \rangle,\ W(t, \theta) = h_t(\theta) - 2\mathrm{Re}\{z(t)q(\theta)\} = h_t(\theta) - z(t)q(\theta) - \overline{z}(\theta)\overline{q}(\theta).$$

On the center manifold theorem, we have

$$W(t, \theta) = W(z(t), \overline{z}(t), \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots,$$

$$\dot{z}(t) = i\omega_2^*\tau_2^* z(t) + \overline{q}^*(0)\begin{pmatrix} F_1(u,\phi) \\ F_2(u,\phi) \\ F_3(u,\phi) \end{pmatrix} = i\omega_2^*\tau_2^* z(t) + g(z,\overline{z}), \tag{28}$$

$$g(z,\overline{z}) = g_{20}(\theta)\frac{z^2}{2} + g_{11}(\theta)z\overline{z} + g_{02}(\theta)\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \tag{29}$$

From Equations (28) and (29), we obtain

$$\dot{W} = AW + H(z,\overline{z},\theta), \tag{30}$$

where

$$H(z,\overline{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\overline{z} + H_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots.$$

By comparing the coefficients on both sides of Equation (30), we obtain

$$(2i\omega_2^* - A)W_{20}(\theta) = H_{20}(\theta), -AW_{11}(\theta) = H_{11}(\theta),$$

Then, we have

$$\phi(0) = z\begin{pmatrix} 1 \\ q_2 \\ q_3 \end{pmatrix} + \overline{z}\begin{pmatrix} 1 \\ \overline{q}_2 \\ \overline{q}_3 \end{pmatrix} + W_{20}(0)\tfrac{z^2}{z} + W_{11}(0)z\overline{z} + W_{02}(0)\tfrac{\overline{z}^2}{2} + \cdots,$$

$$\phi(-\tfrac{\tau_1^*}{\tau_2^*}) = z\begin{pmatrix} 1 \\ q_2 \\ q_3 \end{pmatrix} e^{-i\omega_2^*\tau_1^*} + \overline{z}\begin{pmatrix} 1 \\ \overline{q}_2 \\ \overline{q}_3 \end{pmatrix} e^{-i\omega_2^*\tau_1^*} + W_{20}(-\tfrac{\tau_1^*}{\tau_2^*})\tfrac{z^2}{2} + W_{11}(-\tfrac{\tau_1^*}{\tau_2^*})z\overline{z}$$

$$+W_{02}(-\tfrac{\tau_1^*}{\tau_2^*})\tfrac{\overline{z}^2}{2} + \cdots,$$

$$\phi(-1) = z\begin{pmatrix} 1 \\ q_2 \\ q_3 \end{pmatrix} e^{-i\omega_2^*\tau_2^*} + \overline{z}\begin{pmatrix} 1 \\ \overline{q}_2 \\ \overline{q}_3 \end{pmatrix} e^{-i\omega_2^*\tau_2^*} + W_{20}(-1)\tfrac{z^2}{2} + W_{11}(-1)z\overline{z} + W_{02}(-1)\tfrac{\overline{z}^2}{2} + \cdots.$$

By comparing these with the coefficients in Equation (29), we obtain

$$\begin{aligned} g_{20} = {} & \tau_2^*\overline{D}\{f_{200}q_1^2(0) + 2f_{110}q_1(0)q_2(0) + 2f_{101}q_1(0)q_3(0) + 2f_{011}q_2(0)q_3(0) \\ & + f_{020}q_2^2(0) + f_{002}q_3^2(0) + \overline{q}_2^*[l_{200}q_1^2(-\tfrac{\tau_1^*}{\tau_2^*}) + 2l_{110}q_1(-\tfrac{\tau_1^*}{\tau_2^*})q_2(-\tfrac{\tau_1^*}{\tau_2^*}) + \\ & 2l_{101}q_1(-\tfrac{\tau_1^*}{\tau_2^*})q_3(-\tfrac{\tau_1^*}{\tau_2^*}) + 2l_{011}q_2(-\tfrac{\tau_1^*}{\tau_2^*})q_3(-\tfrac{\tau_1^*}{\tau_2^*}) + l_{020}q_2^2(-\tfrac{\tau_1^*}{\tau_2^*}) + l_{002}q_3^2(-\tfrac{\tau_1^*}{\tau_2^*})] + \\ & \overline{q}_3^*[k_{200}q_1^2(-1) + 2k_{110}q_1(-1)q_2(-1) + 2k_{101}q_1(-1)q_3(-1) + 2k_{011}q_2(-1)q_3(-1) \\ & + k_{020}q_2^2(-1) + k_{002}q_3^2(-1)]\}, \end{aligned}$$

$$\begin{aligned} g_{11} = {} & \overline{\tau}_2^*\overline{D}\{f_{200}q_1(0)\overline{q}_1(0) + f_{110}(q_1(0)\overline{q}_2(0) + \overline{q}_1(0)q_2(0)) + f_{101}(q_1(0)\overline{q}_3(0) \\ & + \overline{q}_1(0)q_3(0)) + f_{011}(q_2(0)\overline{q}_3(0) + \overline{q}_2(0)q_3(0)) + f_{020}q_2(0)\overline{q}_2(0) + f_{002}q_3(0)\overline{q}_3(0) \\ & + \overline{q}_2^*[l_{200}q_1(-\tfrac{\tau_1^*}{\tau_2^*})\overline{q}_1(-\tfrac{\tau_1^*}{\tau_2^*}) + l_{110}(q_1(-\tfrac{\tau_1^*}{\tau_2^*})\overline{q}_2(-\tfrac{\tau_1^*}{\tau_2^*}) + \overline{q}_1(-\tfrac{\tau_1^*}{\tau_2^*})q_2(-\tfrac{\tau_1^*}{\tau_2^*})) + \\ & l_{101}(q_1(-\tfrac{\tau_1^*}{\tau_2^*})\overline{q}_3(-\tfrac{\tau_1^*}{\tau_2^*}) + \overline{q}_1(-\tfrac{\tau_1^*}{\tau_2^*})q_3(-\tfrac{\tau_1^*}{\tau_2^*})) + l_{011}(q_2(-\tfrac{\tau_1^*}{\tau_2^*})\overline{q}_3(-\tfrac{\tau_1^*}{\tau_2^*}) \\ & + \overline{q}_2(-\tfrac{\tau_1^*}{\tau_2^*})q_3(-\tfrac{\tau_1^*}{\tau_2^*})) + l_{020}q_2(-\tfrac{\tau_1^*}{\tau_2^*})\overline{q}_2(-\tfrac{\tau_1^*}{\tau_2^*}) + l_{002}q_3(-\tfrac{\tau_1^*}{\tau_2^*})\overline{q}_3(-\tfrac{\tau_1^*}{\tau_2^*})] + \\ & \overline{q}_3^*[k_{200}q_1(-1)\overline{q}_1(-1) + k_{110}(q_1(-1)\overline{q}_2(-1) + \overline{q}_1(-1)q_2(-1)) + k_{101}(q_1(-1)\overline{q}_3(-1) + \\ & \overline{q}_1(-1)q_3(-1)) + k_{011}(q_2(-1)\overline{q}_3(-1) + \overline{q}_2(-1)q_3(-1)) + k_{020}q_2(-1)\overline{q}_2(-1) \\ & + k_{002}q_3(-1)\overline{q}_3(-1)]\}, \end{aligned}$$

$$
\begin{aligned}
g_{02} = {} & \overline{\tau}_2^* \overline{D}\{f_{200}\overline{q}_1^2(0) + 2f_{110}\overline{q}_1(0)\overline{q}_2(0) + 2f_{101}\overline{q}_1(0)\overline{q}_3(0) + 2f_{011}\overline{q}_2(0)\overline{q}_3(0) \\
& + f_{020}\overline{q}_2^2(0) + f_{002}\overline{q}_3^2(0) + \overline{q}_2^*[l_{200}\overline{q}_1^2(-\tfrac{\tau_1^*}{\tau_2^*}) + 2l_{110}\overline{q}_1(-\tfrac{\tau_1^*}{\tau_2^*})\overline{q}_2(-\tfrac{\tau_1^*}{\tau_2^*}) \\
& + 2l_{101}\overline{q}_1(-\tfrac{\tau_1^*}{\tau_2^*})\overline{q}_3(-\tfrac{\tau_1^*}{\tau_2^*}) + 2l_{011}\overline{q}_2(-\tfrac{\tau_1^*}{\tau_2^*})\overline{q}_3(-\tfrac{\tau_1^*}{\tau_2^*}) + l_{020}\overline{q}_2^2(-\tfrac{\tau_1^*}{\tau_2^*}) + \\
& f_{002}\overline{q}_3^2(-\tfrac{\tau_1^*}{\tau_2^*})] + \overline{q}_3^*[k_{200}\overline{q}_1^2(-1) + 2k_{110}\overline{q}_1(-1)\overline{q}_2(-1) + 2k_{101}\overline{q}_1(-1)\overline{q}_3(-1) \\
& + 2k_{011}\overline{q}_2(-1)\overline{q}_3(-1) + k_{020}\overline{q}_2^2(-1) + k_{002}\overline{q}_3^2(-1)]\},
\end{aligned}
$$

$$
\begin{aligned}
g_{21} = {} & \overline{\tau}_2^* \overline{D}\Big\{f_{200}(\tfrac{1}{2}W_{20}^{(1)}(0)\overline{q}_1(0) + W_{11}^{(1)}(0)q_1(0)) + 2f_{110}(\tfrac{1}{2}W_{20}^{(1)}(0)\overline{q}_2(0) \\
& + W_{20}^{(2)}(0)\overline{q}_1(0) + W_{11}^{(1)}(0)q_2(0) + W_{11}^{(2)}(0)q_1(0)) + 2f_{101}(\tfrac{1}{2}W_{20}^{(1)}(0)\overline{q}_3(0) + \\
& W_{20}^{(3)}(0)\overline{q}_1(0) + W_{11}^{(1)}(0)q_3(0) + W_{11}^{(3)}(0)q_1(0)) + 2f_{011}(\tfrac{1}{2}W_{20}^{(2)}(0)\overline{q}_3(0) + \\
& W_{20}^{(3)}(0)\overline{q}_2(0) + W_{11}^{(2)}(0)q_3(0) + W_{11}^{(3)}(0)q_2(0)) + f_{020}(\tfrac{1}{2}W_{20}^{(2)}(0)\overline{q}_2(0) + \\
& W_{11}^{(2)}(0)q_2(0)) + f_{002}(\tfrac{1}{2}W_{20}^{(3)}(0)\overline{q}_3(0) + W_{11}^{(3)}(0)q_3(0)) + \\
& \overline{q}_2^*[l_{200}(\tfrac{1}{2}W_{20}^{(1)}(-\tfrac{\tau_1^*}{\tau_2^*})\overline{q}_1(-\tfrac{\tau_1^*}{\tau_2^*}) + W_{11}^{(1)}(-\tfrac{\tau_1^*}{\tau_2^*})q_1(-\tfrac{\tau_1^*}{\tau_2^*})) + 2l_{110}(\tfrac{1}{2}W_{20}^{(1)}(-\tfrac{\tau_1^*}{\tau_2^*})\overline{q}_2(-\tfrac{\tau_1^*}{\tau_2^*}) \\
& + W_{20}^{(2)}(-\tfrac{\tau_1^*}{\tau_2^*})\overline{q}_1(-\tfrac{\tau_1^*}{\tau_2^*}) + W_{11}^{(1)}(-\tfrac{\tau_1^*}{\tau_2^*})q_2(-\tfrac{\tau_1^*}{\tau_2^*}) + W_{11}^{(2)}(-\tfrac{\tau_1^*}{\tau_2^*})q_1(-\tfrac{\tau_1^*}{\tau_2^*})) + \\
& 2l_{101}(\tfrac{1}{2}W_{20}^{(1)}(-\tfrac{\tau_1^*}{\tau_2^*})\overline{q}_3(-\tfrac{\tau_1^*}{\tau_2^*}) + W_{20}^{(3)}(-\tfrac{\tau_1^*}{\tau_2^*})\overline{q}_1(-\tfrac{\tau_1^*}{\tau_2^*}) + W_{11}^{(1)}(-\tfrac{\tau_1^*}{\tau_2^*})q_3(-\tfrac{\tau_1^*}{\tau_2^*}) + \\
& W_{11}^{(3)}(-\tfrac{\tau_1^*}{\tau_2^*})q_1(-\tfrac{\tau_1^*}{\tau_2^*})) + 2l_{011}(\tfrac{1}{2}W_{20}^{(2)}(-\tfrac{\tau_1^*}{\tau_2^*})\overline{q}_3(-\tfrac{\tau_1^*}{\tau_2^*}) + W_{20}^{(3)}(-\tfrac{\tau_1^*}{\tau_2^*})\overline{q}_2(-\tfrac{\tau_1^*}{\tau_2^*}) + \\
& W_{11}^{(2)}(-\tfrac{\tau_1^*}{\tau_2^*})q_3(-\tfrac{\tau_1^*}{\tau_2^*}) + W_{11}^{(3)}(-\tfrac{\tau_1^*}{\tau_2^*})q_2(-\tfrac{\tau_1^*}{\tau_2^*})) + l_{020}(\tfrac{1}{2}W_{20}^{(2)}(-\tfrac{\tau_1^*}{\tau_2^*})\overline{q}_2(-\tfrac{\tau_1^*}{\tau_2^*}) \\
& + W_{11}^{(2)}(-\tfrac{\tau_1^*}{\tau_2^*})q_2(-\tfrac{\tau_1^*}{\tau_2^*})) + l_{002}(\tfrac{1}{2}W_{20}^{(3)}(-\tfrac{\tau_1^*}{\tau_2^*})\overline{q}_3(-\tfrac{\tau_1^*}{\tau_2^*}) + W_{11}^{(3)}(-\tfrac{\tau_1^*}{\tau_2^*})q_3(-\tfrac{\tau_1^*}{\tau_2^*}))] \\
& + \overline{q}_3^*[k_{200}(\tfrac{1}{2}W_{20}^{(1)}(-1)\overline{q}_1(-1) + W_{11}^{(1)}(-1)q_1(-1)) + 2k_{110}(\tfrac{1}{2}W_{20}^{(1)}(-1)\overline{q}_2(-1) \\
& + W_{20}^{(2)}(-1)\overline{q}_1(-1) + W_{11}^{(1)}(-1)q_2(-1) + W_{11}^{(2)}(-1)q_1(-1)) + \\
& 2k_{101}(\tfrac{1}{2}W_{20}^{(1)}(-1)\overline{q}_3(-1) + W_{20}^{(3)}(-1)\overline{q}_1(-1) + W_{11}^{(1)}(-1)q_3(-1) + W_{11}^{(3)}(-1)q_1(-1)) \\
& + 2k_{011}(\tfrac{1}{2}W_{20}^{(2)}(-1)\overline{q}_3(-1) + W_{20}^{(3)}(-1)\overline{q}_2(-1) + W_{11}^{(2)}(-1)q_3(-1) + \\
& W_{11}^{(3)}(-1)q_2(-1)) + k_{020}(\tfrac{1}{2}W_{20}^{(2)}(-1)\overline{q}_2(-1) + W_{11}^{(2)}(-1)q_2(-1)) + \\
& k_{002}(\tfrac{1}{2}W_{20}^{(3)}(-1)\overline{q}_3(-1) + W_{11}^{(3)}(-1)q_3(-1))]\Big\},
\end{aligned}
$$

where

$$
\begin{aligned}
W_{20}(\theta) &= \frac{ig_{20}q(0)}{\omega_2\tau_2^*}e^{i\omega_2^*\tau_2^*\theta} + \frac{i\overline{g}_{02}\overline{q}(0)}{3\omega_2\tau_2^*}e^{-i\omega_2^*\tau_2^*\theta} + E_1e^{2i\omega_2^*\tau_2^*\theta}, \\
W_{11}(\theta) &= \frac{-ig_{11}q(0)}{\omega_2\tau_2^*}e^{i\omega_2^*\tau_2^*} + \frac{i\overline{g}_{11}\overline{q}(0)}{\omega_2\tau_2^*}e^{-i\omega_2^*\tau_2^*\theta} + E_2, \\
E_1 &= [2i\omega_2^*\tau_2^* - \textstyle\int_{-\tau_2}^0 e^{2i\omega_2^*\tau_2^*\theta}d\eta(\theta)]^{-1}F_{z^2} \\
&= \begin{pmatrix} 2i\omega_2^*\tau_2^* - a_{11} & -a_{12} & 0 \\ -a_{21}e^{-i\omega_2^*\tau_1^*} & 2i\omega_2^*\tau_2^* - a_{22} & -a_{23} \\ 0 & -a_{32}e^{-i\omega_2^*\tau_2^*} & 2i\omega_2^*\tau_2^* \end{pmatrix}^{-1} \times \tau_2^* \begin{pmatrix} f_{200}q_1^2(0) + 2f_{110}q_1(0)q_2(0) + \\ +2f_{101}q_1(0)q_3(0) + 2f_{011}q_2(0)q_3(0) \\ +f_{020}q_2^2(0) + f_{002}q_3^2(0) + \overline{q}_2^* \\ \\ l_{200}q_1^2(-\frac{\tau_1^*}{\tau_2^*}) + 2l_{110}q_1(-\frac{\tau_1^*}{\tau_2^*})q_2(-\frac{\tau_1^*}{\tau_2^*}) + \\ 2l_{011}q_2(-\frac{\tau_1^*}{\tau_2^*})q_3(-\frac{\tau_1^*}{\tau_2^*}) + \\ 2l_{101}q_1(-\frac{\tau_1^*}{\tau_2^*})q_3(-\frac{\tau_1^*}{\tau_2^*}) + l_{020}q_2^2(-\frac{\tau_1^*}{\tau_2^*}) \\ +l_{002}q_3^2(-\frac{\tau_1^*}{\tau_2^*}) \\ \\ k_{200}q_1^2(-1) + 2k_{110}q_1(-1)q_2(-1) + \\ 2k_{101}q_1(-1)q_3(-1) + \\ 2k_{011}q_2(-1)q_3(-1) + \\ k_{020}q_2^2(-1) + k_{002}q_3^2(-1) \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
E_2 &= -\left[\int_{-\tau_2}^{0} d\eta(\theta)\right]^{-1} F_{z\bar{z}} \\
&= -\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix}^{-1} \times \tau_2^* \begin{pmatrix}
f_{200}q_1(0)\bar{q}_1(0) + f_{110}(q_1(0)\bar{q}_2(0) + \bar{q}_1(0)q_2(0)) + \\
f_{101}(q_1(0)\bar{q}_3(0) + \bar{q}_1(0)q_3(0)) + f_{011}(q_2(0)\bar{q}_3(0) + \\
\bar{q}_2(0)q_3(0)) + f_{020}q_2(0)\bar{q}_2(0) + f_{002}q_3(0)\bar{q}_3(0) \\
\\
l_{200}q_1(-\frac{\tau_1^*}{\tau_2^*})\bar{q}_1(-\frac{\tau_1^*}{\tau_2^*}) + l_{110}(q_1(-\frac{\tau_1^*}{\tau_2^*})\bar{q}_2(-\frac{\tau_1^*}{\tau_2^*}) + \bar{q}_1(-\frac{\tau_1^*}{\tau_2^*})q_2(-\frac{\tau_1^*}{\tau_2^*})) \\
+l_{101}(q_1(-\frac{\tau_1^*}{\tau_2^*})\bar{q}_3(-\frac{\tau_1^*}{\tau_2^*}) + \bar{q}_1(-\frac{\tau_1^*}{\tau_2^*})q_3(-\frac{\tau_1^*}{\tau_2^*})) + l_{011}(q_2(-\frac{\tau_1^*}{\tau_2^*})\bar{q}_3(-\frac{\tau_1^*}{\tau_2^*}) \\
+\bar{q}_2(-\frac{\tau_1^*}{\tau_2^*})q_3(-\frac{\tau_1^*}{\tau_2^*})) + l_{020}q_2(-\frac{\tau_1^*}{\tau_2^*})\bar{q}_2(-\frac{\tau_1^*}{\tau_2^*}) + l_{002}q_3(-\frac{\tau_1^*}{\tau_2^*})\bar{q}_3(-\frac{\tau_1^*}{\tau_2^*}) \\
\\
k_{200}q_1(-1)\bar{q}_1(-1) + k_{110}(q_1(-1)\bar{q}_2(-1) + \bar{q}_1(-1)q_2(-1)) \\
+k_{101}(q_1(-1)\bar{q}_3(-1) + \bar{q}_1(-1)q_3(-1)) + k_{011}(q_2(-1)\bar{q}_3(-1) + \\
\bar{q}_2(-1)q_3(-1)) + k_{020}q_2(-1)\bar{q}_2(-1) + k_{002}q_3(-1)\bar{q}_3(-1)]
\end{pmatrix},
\end{aligned}
$$

Thus, through calculation, we obtain

$$
\begin{aligned}
c_1(0) &= \frac{i}{2\omega_2^*\tau_2^*}\left[g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right] + \frac{g_{21}}{2}, \\
\mu_2 &= \frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_2^*)\}}, \\
\beta_2 &= 2\mathrm{Re}\{c_1(0)\}, \\
T_2 &= -\frac{\mathrm{Im}\{c_1(0)\} + \mu_2 \mathrm{Im}\{\lambda'(\tau_2^*)\}}{\omega_2}.
\end{aligned}
$$

Among them, μ_2 determines the direction of the Hopf bifurcation; when $\mu_2 > 0$, $\tau_2 > \tau_{30}$ ($\mu_2 < 0, \tau_2 < \tau_{30}$), the system undergoes a supercritical (subcritical) Hopf bifurcation near the equilibrium point. β_2 determines the stability of the bifurcated periodic solution, and $\beta_2 > 0$ ($\beta_2 < 0$) indicates that the bifurcated periodic solution restricted to the center manifold is unstable (asymptotically stable). T_2 determines the increase or decrease in the period, and in $T_2 > 0$ ($T_2 < 0$) the part outside of the brackets indicates that the period increases, while that contained within the brackets indicates that the period decreases [27].

4. Numerical Simulations

In this section, numerical simulations are conducted under five scenarios with selected parameters.

Case 1. $\tau_1 = \tau_2 = 0$.

Selecting parameters $p_1 = 0.7635, p_2 = 0.6279, p_3 = 0.7720, p_4 = 0.9329$, system (3) becomes

$$
\begin{cases}
\frac{dx}{dt} = x(t)[1 - x(t)] - x(t)y(t), \\
\frac{dy}{dt} = y(t)[0.7635 - 0.6279\frac{y(t)}{x(t)}] - y(t)z(t), \\
\frac{dz}{dt} = z(t)[0.7720 - 0.9329\frac{z(t)}{y(t)}].
\end{cases}
$$

The internal equilibrium is calculated as $E(0.7584, 0.2416, 0.2919)$. Under the conditions (H_1) and (H_2), Theorem 1 confirms that the internal equilibrium $E(0.7584, 0.2416, 0.2919)$ is asymptotically stable, as illustrated in Figure 1.

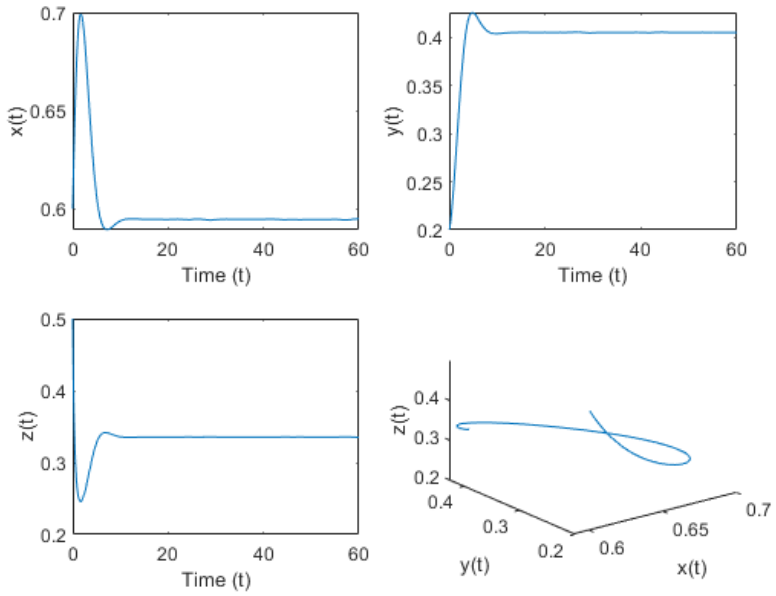

Figure 1. Time series curves and phase portraits of system (3) with $\tau_1 = \tau_2 = 0$.

Case 2. $\tau_1 > 0, \tau_2 = 0$.

Selecting parameters $p_1 = 0.6232, p_2 = 0.8923, p_3 = 0.9809, p_4 = 0.6470$, system (3) becomes

$$\begin{cases} \frac{dx}{dt} = x(t)[1 - x(t)] - x(t)y(t), \\ \frac{dy}{dt} = y(t)[0.6236 - 0.8923\frac{y(t)}{x(t-\tau_1)}] - y(t)z(t), \\ \frac{dz}{dt} = z(t)[0.9809 - 0.6470\frac{z(t)}{y(t)}]. \end{cases}$$

The internal equilibrium is calculated as $E(0.7760, 0.2240, 0.1477)$. Under the conditions $(H_1), (H_2), (H_3)$ and (H_4), the critical delay is $\tau_{10} = 5.4532$. According to Theorem 2, when $\tau_1 = 3 < \tau_{10}$, the internal equilibrium $E(0.7760, 0.2240, 0.1477)$ is asymptotically stable, as shown in Figure 2. when $\tau_1 = 9 > \tau_{10}$, the internal equilibrium $E(0.7760, 0.2240, 0.1477)$ becomes unstable, and a Hopf bifurcation occurs, as illustrated in Figure 3.

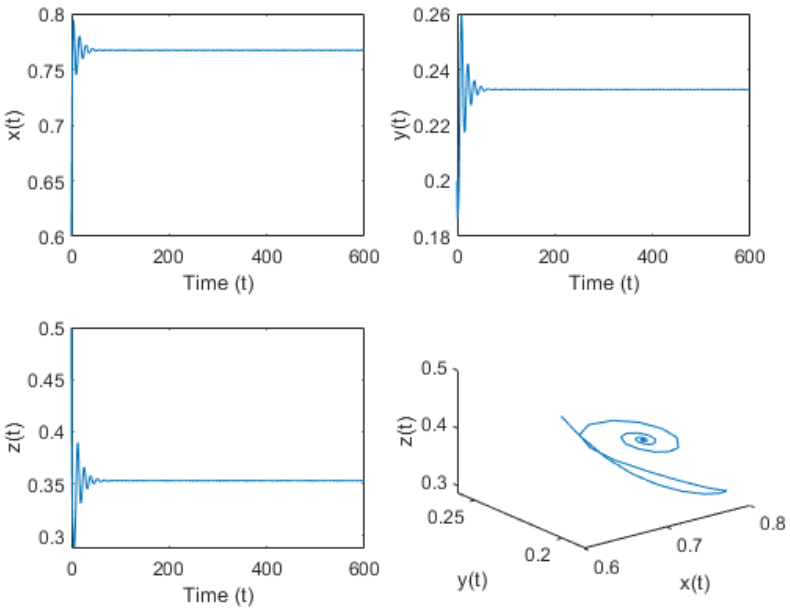

Figure 2. Time series curves and phase portraits of system (3) with $\tau_1 = 3, \tau_2 = 0$.

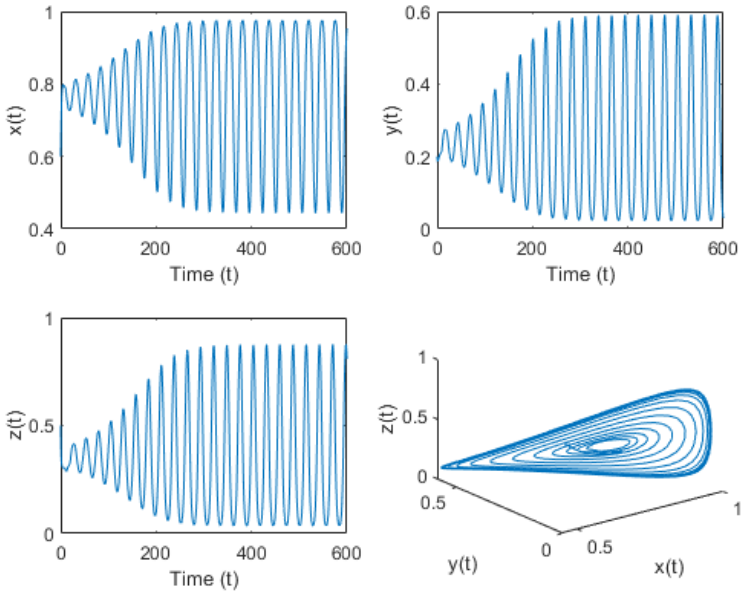

Figure 3. Time series curves and phase portraits of system (3) with $\tau_1 = 9, \tau_2 = 0$.

Case 3. $\tau_1 = 0, \tau_2 > 0$.

Selecting parameters $p_1 = 0.8250, p_2 = 0.2249, p_3 = 0.9595, p_4 = 0.9715$, system (3) becomes

$$\begin{cases} \frac{dx}{dt} = x(t)[1 - x(t)] - x(t)y(t), \\ \frac{dy}{dt} = y(t)[0.8250 - 0.2249\frac{y(t)}{x(t)}] - y(t)z(t), \\ \frac{dz}{dt} = z(t)[0.9595 - 0.8715\frac{z(t)}{y(t-\tau_2}]. \end{cases}$$

The internal equilibrium is calculated as $E(0.5174, 0.4826, 0.4384)$. Under the conditions $(H_1), (H_2), (H_5)$ and (H_6),the critical delay is $\tau_{20} = 4.5291$. According to Theorem 3, when $\tau_2 = 2 < \tau_{20}$, the internal equilibrium $E(0.5174, 0.4826, 0.4384)$ is asymptotically stable, as shown in Figure 4. When $\tau_2 = 5 > \tau_{20}$, the internal equilibrium $E(0.5174, 0.4826, 0.4384)$ becomes unstable, and a Hopf bifurcation occurs, as illustrated in Figure 5.

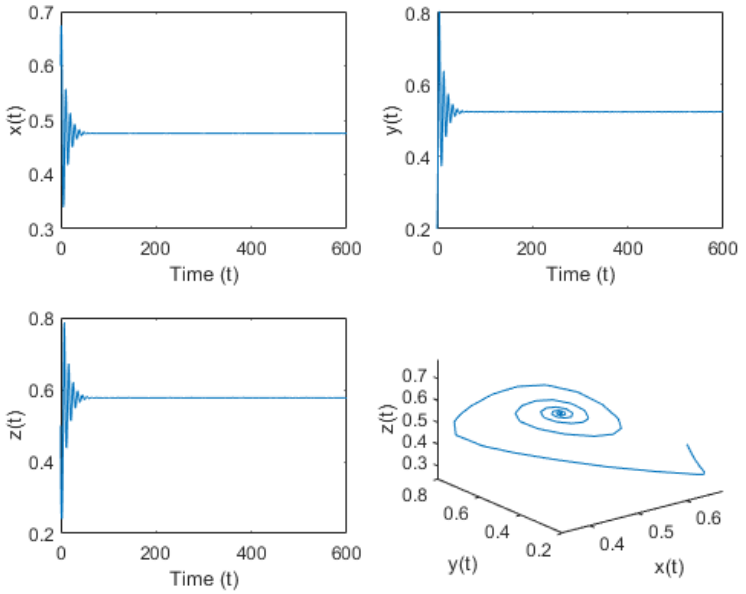

Figure 4. Time series curves and phase portraits of system (3) with $\tau_1 = 0, \tau_2 = 2$.

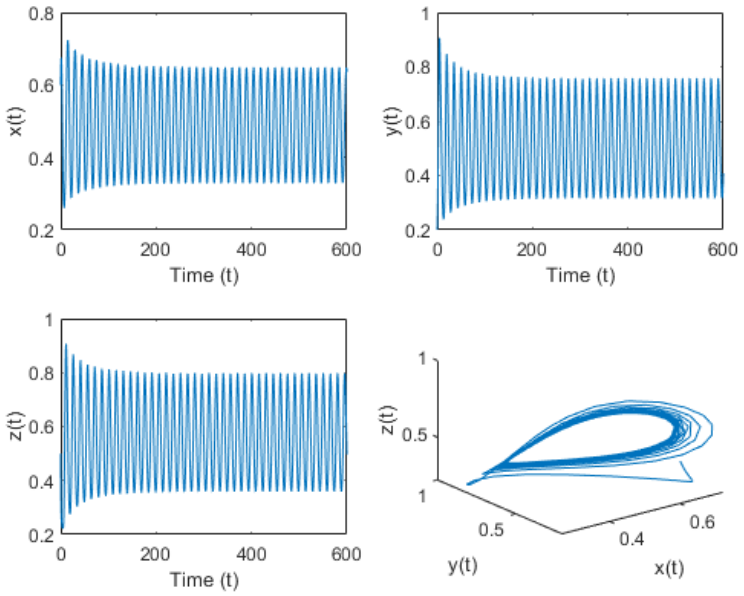

Figure 5. Time series curves and phase portraits of system (3) with $\tau_1 = 0, \tau_2 = 5$.

Case 4. $\tau_1 \in (0, \tau_{10}), \tau_2 > 0$.

Selecting parameters $p_1 = 0.9622, p_2 = 0.5952, p_3 = 0.9616, p_4 = 0.5689$, system (3) becomes

$$\begin{cases} \frac{dx}{dt} = x(t)[1 - x(t)] - x(t)y(t), \\ \frac{dy}{dt} = y(t)[0.9622 - 0.5952\frac{y(t)}{x(t-\tau_1)}] - y(t)z(t), \\ \frac{dz}{dt} = z(t)[0.9616 - 0.5689\frac{z(t)}{y(t-\tau_2)}]. \end{cases}$$

The internal equilibrium is calculated as $E(0.7448, 0.2552, 0.2240)$. Under the conditions (H_7) and (H_8), the critical delay is $\tau_{30} = 0.5712$. According to Theorem 4, when $\tau_1 = 3.5 \in (0, \tau_{10}) = (0, 5.4532), \tau_2 = 0.1 < \tau_{30}$, the internal equilibrium $E(0.7448, 0.2552, 0.2240)$ is asymptotically stable, as shown in Figure 6. When $\tau_1 = 3.5$, $\tau_2 = 1 > \tau_{30}$, the internal equilibrium $E(0.7448, 0.2552, 0.2240)$ becomes unstable, and a Hopf bifurcation occurs, as illustrated in Figure 7.

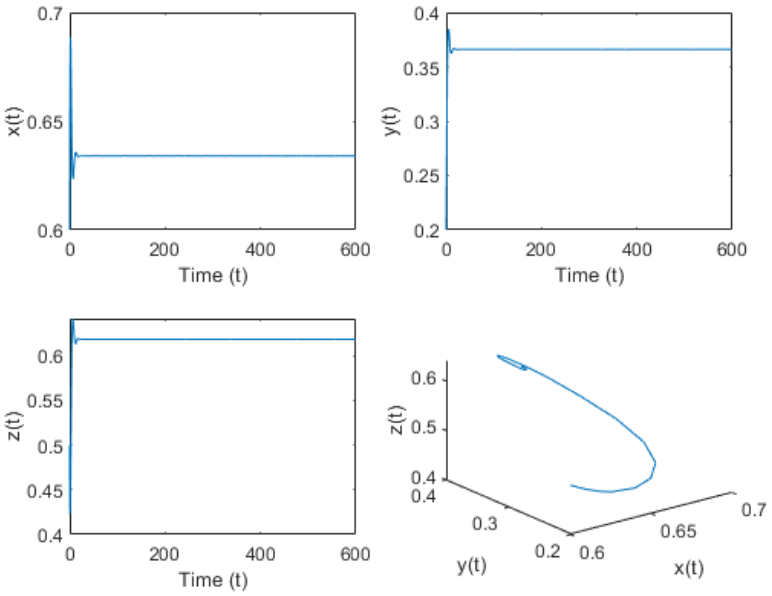

Figure 6. Time series curves and phase portraits of system (3) with $\tau_1 = 3.5, \tau_2 = 0.1$.

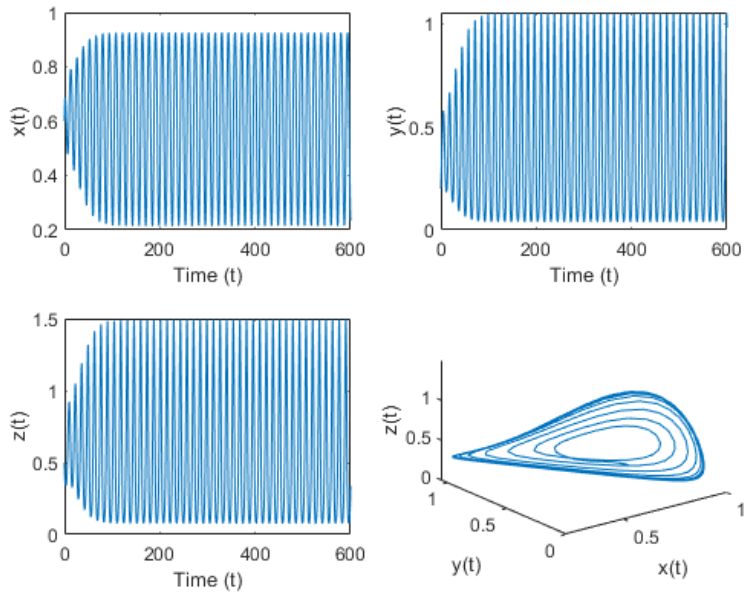

Figure 7. Time series curves and phase portraits of system (3) with $\tau_1 = 3.5, \tau_2 = 1$.

Case 5. $\tau_1 > 0, \tau_2 \in (0, \tau_{20})$.

Selecting parameters $p_1 = 0.6188, p_2 = 0.1308, p_3 = 0.7958, p_4 = 0.9659$, system (3) becomes

$$\begin{cases} \frac{dx}{dt} = x(t)[1 - x(t)] - x(t)y(t), \\ \frac{dy}{dt} = y(t)[0.9024 - 0.9299\frac{y(t)}{x(t-\tau_1)}] - y(t)z(t), \\ \frac{dz}{dt} = z(t)[0.9898 - 0.5335\frac{z(t)}{y(t-\tau_2)}]. \end{cases}$$

The internal equilibrium is calculated as $E(0.7200, 0.2800, 0.1509)$. Under the conditions (H_9) and (H_{10}), the critical delay is $\tau_{40} = 3.7957$. According to Theorem 5, when $\tau_1 = 3 < \tau_{40}$, $\tau_2 = 2 \in (0, \tau_{20}) = (0, 4.5291)$, the internal equilibrium $E(0.7200, 0.2800, 0.1509)$ is asymptotically stable, as shown in Figure 8. When $\tau_1 = 5 > \tau_{40}, \tau_2 = 2$, the internal equilibrium $E(0.7200, 0.2800, 0.1509)$ becomes unstable, and a Hopf bifurcation occurs, as illustrated in Figure 9.

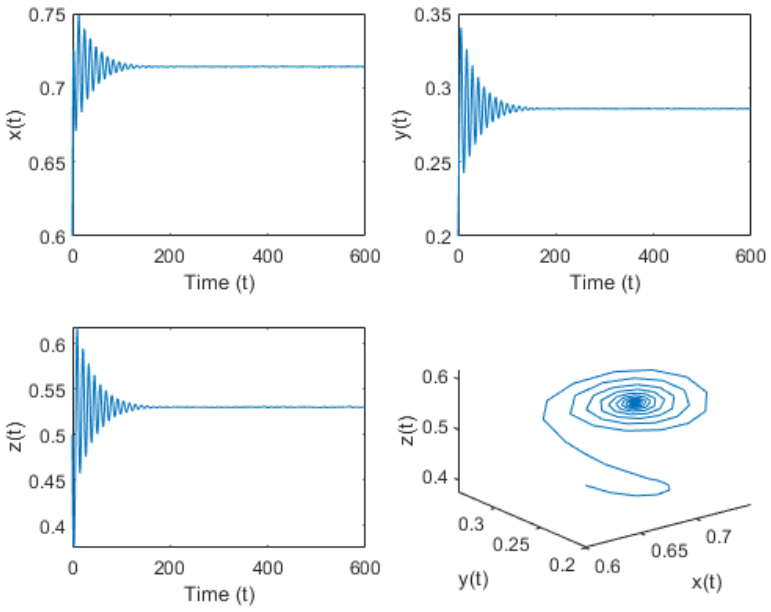

Figure 8. Time series curves and phase portraits of system (3) with $\tau_1 = 3, \tau_2 = 2$.

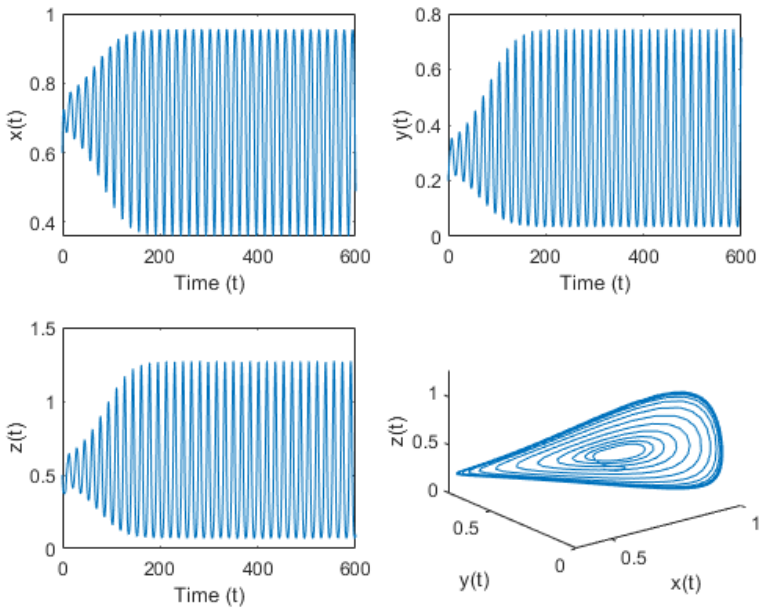

Figure 9. Time series curves and phase portraits of system (3) with $\tau_1 = 5, \tau_2 = 2$.

Collectively, when either time delay takes a value of zero, the system transitions into a single-delay configuration governed by one energy transfer mechanism. Through analytical investigation, we derive critical conditions for Hopf bifurcations induced by individual time delays, as demonstrated in Case 2 and Case 3. When one time delay operates within its stability interval while the other exceeds zero, the system experiences dual energy transfer delays. Subsequent analysis reveals critical Hopf bifurcation conditions arising from the cooperative effects of dual time delays, as presented in Case 4 and Case 5. This comprehensive examination delineates both isolated and synergistic temporal influences on system stability thresholds.

5. Conclusions

This paper investigates the dynamical behaviors of a three-species food chain model (prey–middle predator–top predator) with dual time delays. In system (2), the two delays, τ_1 and τ_2, represent the time lags in energy conversion from prey to middle predator biomass and from middle predator to top predator biomass, respectively. First, the stability and Hopf bifurcation conditions of the system are analyzed under five distinct delay scenarios: (1) $\tau_1 = \tau_2 = 0$; (2) $\tau_1 > 0, \tau_2 = 0$; (3) $\tau_1 = 0, \tau_2 > 0$; (4) τ_2 varies while τ_1 remains its stability interval; (5) τ_1 varies while τ_2 remains within its stability interval. Second, the Hopf bifurcation properties for scenario (5) are examined in detail using normal form theory and the center manifold theorem. Finally, numerical simulations are performed to further explore the effects of time delays on the dynamics of the food chain model.

From a theoretical perspective, this study analyzes the Hopf bifurcation behavior of the dual-delay three-dimensional food chain model and confirms the analytical results through numerical simulations. The findings demonstrate that energy conversion delays significantly influence and may destabilize the food chain model. The main contributions are novel, revealing how cross-trophic energy transfer delays can induce periodic population oscillations. These results provide a theoretical foundation for understanding the synchronized population fluctuations frequently observed in natural ecosystems. By characterizing the system's critical transitions from stable states to periodic solutions under the coupled influence of dual delays, the research enables the development of targeted intervention strategies (such as adjusting the population sizes of specific species) to maintain ecological balance. Overall, the study offers valuable theoretical insights for predicting dynamical

behaviors, formulating ecosystem management strategies, and sustaining equilibrium in real-world food chain models.

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