mathematics

MDPI

Article

Exploring q-Fibonacci Numbers in Geometric Function Theory: Univalence and Shell-like Starlike Curves

Abdullah Alsoboh [1,*], Ala Amourah [2,3,*], Omar Alnajar [4], Mamoon Ahmed [5] and Tamer M. Seoudy [6]

[1] Department of Basic and Applied Sciences, College of Applied and Health Sciences, A'Sharqiyah University, Post Box No. 42, Post Code No. 400, Ibra 413, Oman
[2] Mathematics Education Program, Faculty of Education and Arts, Sohar University, Sohar 311, Oman
[3] Applied Science Research Center, Applied Science Private University, Amman 11941, Jordan
[4] Department of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi 43600, Selangor, Malaysia; p117246@siswa.ukm.edu.my
[5] Department of Basic Sciences, Princess Sumaya University for Technology, Amman 11941, Jordan; m.ahmed@psut.edu.jo
[6] Department of Mathematics, Jamoum University College, Umm Al-Qura University, Makkah 21955, Saudi Arabia; tmsaman@uqu.edu.sa
* Correspondence: abdullah.alsoboh@asu.edu.om (A.A.); aamourah@su.edu.om (A.A.)

Abstract: Emphasising their connection with shell-like star-like curves, this work investigates a new subclass of star-like functions defined by q-Fibonacci numbers and q-polynomials. We study the geometric and analytic properties of this subclass, including the computation of intervals of univalence and nonunivalence for some functions. Moreover, we define a sufficient condition for functions in this subclass to satisfy the criteria of the famous class of analytic functions with positive real components. This work improves our understanding of the link between Fibonacci-type sequences and the geometric properties of analytic functions by using subordination ideas and the features of q-Fibonacci sequences. Emphasising the possibility for diverse research in combinatorial and analytical mathematics, the results offer fresh insights and support further study on the applications of calculus in geometric function theory.

Keywords: analytic functions; univalent functions; star-like functions; Fibonacci numbers; Fibonacci polynomials; shell-like curves; quantum calculus

MSC: 30A36; 30C45; 81P68; 11B37

check for updates

Academic Editors: Adolfo Ballester-Bolinches, Jonathan Blackledge and Jay Jahangiri

Received: 27 February 2025
Revised: 18 March 2025
Accepted: 11 April 2025
Published: 15 April 2025

Citation: Alsoboh, A.; Amourah, A.; Alnajar, O.; Ahmed, M.; Seoudy, T.M. Exploring q-Fibonacci Numbers in Geometric Function Theory: Univalence and Shell-like Starlike Curves. *Mathematics* **2025**, *13*, 1294. https://doi.org/10.3390/math13081294

Copyright: © 2025 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction, Preliminaries, and Definitions

The Fibonacci sequence is a widely regarded numerical series in mathematics. The sequence is defined by the recurrence relation $\varphi_{\mathfrak{s}} = \varphi_{\mathfrak{s}-1} + \varphi_{\mathfrak{s}-2}, (\mathfrak{s} \geq 2)$, with initial conditions $\varphi_0 = 0$ and $\varphi_1 = 1$. It is prevalent in nature and art and is utilised in computer science, mathematics, and various other fields. The Fibonacci sequence, intricately linked to generating functions, is essential for comprehending the characteristics of sequences. The generating function is explicitly defined as

$$G(\varkappa) = \sum_{\mathfrak{s}=0}^{\infty} \varphi_{\mathfrak{s}} \varkappa^{\mathfrak{s}} = \frac{\varkappa}{1 - \varkappa - \varkappa^2}.$$

Aside from its mathematical importance, the Fibonacci sequence appears in natural occurrences, affecting development patterns and structural formations in the physical realm. Its recursive structure renders it a potent instrument for analysing sequences, series, and

issues related to iterative processes, highlighting its versatility and significance in both theoretical and practical domains.

Denote the family of all analytic functions defined on the open unit disk as $\mathscr{O}$, where $\mathscr{O}$ is the set of all complex numbers $z = a + ib$ (with $a, b \in \mathbb{R}$) satisfying $|z| < 1$. Geometrically, $\mathscr{O}$ represents the collection of all points in the complex plane that lie strictly inside the unit circle centered at the origin.

The functions $f \in \mathscr{A}$ are normalized to satisfy the following initial conditions:

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

These normalization conditions ensure that the functions are uniquely determined and facilitate the study of their properties within the unit disk. This framework provides a foundation for exploring the analytical and geometric properties of functions in various mathematical contexts, which are central to the broader mathematical investigations discussed in this work.

For every function $f \in \mathscr{A}$, the Taylor–Maclaurin series expansion can be expressed in the following form:

$$f(z) = z + \sum_{s=2}^{\infty} \alpha_s z^s, \quad (z \in \mathscr{O}). \tag{1}$$

This class of functions has numerous significant subclasses, defined as follows:

$$\mathscr{S} = \left\{ f \in \mathscr{A} : \quad f \text{ is univalent (injective) in the open unit disk } \mathscr{O} \right\},$$

$$P = \left\{ p \in \mathscr{A} : \quad p(z) = 1 + \sum_{s=1}^{\infty} p_s z^s \quad \text{with} \quad \Re e\big(p(z)\big) > 0, \ z \in \mathscr{O} \right\},$$

$$P(\beta) = \quad \left\{ p \in \mathscr{A} : p(0) = 1 \quad \text{and} \quad \Re e(p(z)) > \gamma, \ (0 \le \gamma < 1), z \in \mathscr{O} \right\}.$$

Geometric function theory extensively utilises the class P of Caratheodory functions, which pertains to functions possessing positive real components alongside certain analytic characteristics [1]. A function p is classified within $\mathcal{P}$ if and only if it can be represented as

$$p(z) = (1 + \hbar(z))(1 - \hbar(z))^{-1}, \qquad (z \in \mathscr{O}).$$

The function $\hbar$ is categorised as a Schwartz function, fulfilling the criterion $|\hbar(z)| < 1$ for every $z \in \mathscr{O}$. The class $\mathscr{S}$ of univalent functions, which is very pertinent to conformal mappings, along with the generalised class $P(\beta)$, demonstrates substantial applicability in complex analysis, particularly in geometric function theory.

The class of star-like functions, represented as $\mathcal{S}^*$, can be defined by multiple methodologies employing the principle of subordination. In 1992, Ma and Minda [2] established the class $\mathcal{S}^*(\Omega)$ derived from this idea, articulated as follows:

$$s^*(\Omega) = \left\{ f \in \mathscr{A} : \frac{z f'(z)}{f(z)} \prec \Omega(z), \quad \text{where} \quad \Omega \in P \ \text{ and } \ z \in \mathscr{O} \right\}.$$

Contemporary research has produced several subclasses of star-like functions with the application of specific forms of Ω. Scholars demonstrate adaptability in their research by employing several methodologies. Table 1 delineates these subfamilies and underscores the many star-like characteristics arising from distinct Ω choices.

Table 1. Lists several of the star-like classes defined by the subordination principle.

The Family of Star-like Functions	Author/s	Ref.
$s^*\left(\frac{1+z}{1-z}\right) = \left\{f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}\right\}$	Janowski	[3]
$s^*(\vartheta) = \left\{f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\vartheta)z}{1-z}, 0 \leq \vartheta < 1\right\}$	Robertson	[4]
$SL(\vartheta) = \left\{f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+\vartheta^2 z^2}{1-\vartheta z-\vartheta^2 z^2}, \vartheta = \frac{1-\sqrt{5}}{2}\right\}$	Sokół	[5]
$\mathcal{SK}(\vartheta) = \left\{f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \frac{3}{3+(\vartheta-3)z-\vartheta^2 z^2}, \vartheta \in (-3,1]\right\}$	Sokół	[6]

Recently, quantum calculus, also known as q-calculus, has garnered significant interest in the fields of physics and mathematics. Its historical origins date back to the 19th century when Jackson [7,8] introduced the q-difference operator and integral. Subsequently, Aral and Gupta [9] expanded upon this foundational research by examining q analogues of many operators, particularly in the realm of geometric function theory. Quantum calculus, grounded in q-differences, extends classical calculus and offers a robust framework for exploring specific subclasses of analytic functions, such as star-like and convex functions. In this context, the parameter q is anticipated to satisfy $0 < q < 1$, ensuring the requisite convergence and essential attributes necessary for these analyses.

Definition 1 ([10]). *The q-bracket $\lceil \kappa \rfloor_q$ is defined as follows:*

$$\lceil \kappa \rfloor_q = \begin{cases} \frac{1-q^\lambda}{1-q}, & 0 < q < 1,\ \lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\} \\ 1, & q \mapsto 0^+,\ \lambda \in \mathbb{C}^* \\ \lambda, & q \mapsto 1^-,\ \lambda \in \mathbb{C}^* \\ q^{\gamma-1} + q^{\gamma-2} + \cdots + q + 1 = \sum_{s=0}^{\gamma-1} q^s, & 0 < q < 1,\ \lambda = \gamma \in \mathbb{N}. \end{cases}$$

Definition 2 ([10]). *The q−derivative, also known as the q−difference operator, of a function f is defined by*

$$\eth_q \langle f(z) \rangle = \begin{cases} (f(z) - f(qz))(z - qz)^{-1}, & \text{if} \quad 0 < q < 1,\ z \neq 0, \\ f'(0), & \text{if} \quad z = 0, \\ f'(z), & \text{if} \quad q \mapsto 1^-,\ z \neq 0. \end{cases}.$$

Additionally, Jackson [8] introduced the concept of the q-integral concerning any function f, characterising it as

$$\int_0^z f(\zeta)\ \eth_q \zeta = (1-q)\, z \sum_{s=1}^{\infty} q^{s-1}\ f(q^{s-1} z). \tag{2}$$

As a specific instance of the q-Jackson integral, we obtain the following equation:

$$\int_0^z \frac{\eth_q \langle f(\zeta) \rangle}{f(\zeta)}\ \eth_q \zeta = \frac{q-1}{\log q} \log_q f(z). \tag{3}$$

By employing the q-Jackson difference operators, Uçar and Özkan [11] introduced a novel class of functions known as q-star-like functions, denoted by s_q^*. This class is formally defined as

$$s_q^* = \left\{ f \in \mathscr{A} : \quad \frac{z\eth_q\langle f(z)\rangle}{f(z)} \prec \frac{1+z}{1-qz} \quad (z \in \mathscr{O}) \right\}.$$

This definition extends the classical concept of star-like functions by incorporating the q-difference operator, thereby providing a more generalized framework for studying geometric properties of analytic functions.

Similarly, Srivastava et al. [12] proposed another class of q star-like functions, denoted by $s^*(\mathcal{L}, q)$, defined as

$$s^*(\mathcal{L}, q) = \left\{ f \in \mathscr{A} : \quad \frac{z\eth_q\langle f(z)\rangle}{f(z)} \prec exp_q(z) = \sum_{n=0}^{\infty} \frac{z}{\lceil s \rfloor_q !} \quad (z \in \mathscr{O}) \right\},$$

where $exp_q(z)$ represents the q-exponential function, which plays a central role in q-calculus. This class generalizes the concept of star-like functions by incorporating the q-exponential function into the subordination condition, providing a broader framework for analysis. The use of q-calculus has facilitated the exploration of new subclasses of analytic functions, leading to significant progress in understanding their geometric and analytic properties. These advancements demonstrate the versatility and depth of q-calculus, showcasing its ability to enrich the theory of analytic functions and reveal new mathematical insights. Such developments highlight the growing importance of q-calculus in both theoretical and applied contexts, opening avenues for further research and innovation in the field, as discussed in [13–19]. The study of univalent functions is essential in geometric function theory, especially for investigating functions that are univalent in both a domain and its inverse. These functions are linked to orthogonal polynomials, such as q-Fibonacci numbers, which facilitate the analysis of coefficient bounds and structural properties [20–31]. Their applications extend across various fields, including low-light imaging for improved contrast, image edge detection for greater precision, and stealth combat aircraft for optimizing radar signatures (see [32–34]).

2. Definition and Example

Motivated by the q-Fibonacci numbers, this section introduces a novel subclass of analytic functions. This subclass not only unifies aspects of the conventional theory of analytic functions but also highlights new geometric and analytic properties that emerge through the use of q-Fibonacci numbers. By incorporating q-Fibonacci numbers, this subclass provides fresh perspectives on the connections between geometric analytic functions, Fibonacci-type sequences, and q-calculus, thereby advancing and enriching the existing theory.

Definition 3. *The function* f *belongs to the class* SL_q *if and only if*

$$\frac{z\eth_q\langle f(z)\rangle}{f(z)} \prec \Upsilon(z; q),$$

where

$$\Upsilon(z; q) = \frac{1 + q\vartheta_q^2 z^2}{1 - \vartheta_q z - q\vartheta_q^2 z^2}, \tag{4}$$

and $\vartheta_q = \frac{1-\sqrt{4q+1}}{2q}$ *is the q-analogue of the Fibonacci numbers, defined recursively by the relation*

$$\begin{aligned} &\varphi_0(q) = \varphi_1(q) - 1 = 0, \\ &\varphi_n(q) = \varphi_{s-1}(q) + q\varphi_{s-2}(q), \qquad \textit{for} \quad (s \geq 2). \end{aligned} \tag{5}$$

The initial terms of the $\mathfrak{q}$-Fibonacci numbers, which serve as a generalization of the classical Fibonacci numbers in the limit as approaches $\mathfrak{q} \mapsto 1^-$, are provided in Table 2.

Table 2. The first initial terms of the $\mathfrak{q}$-Fibonacci sequence.

The $\mathfrak{q}$-Analogue of Fibonacci Numbers	The Classical Fibonacci Numbers
$\varphi_0(\mathfrak{q}) = 0$	$\varphi_0 = 0$
$\varphi_1(\mathfrak{q}) = 1$	$\varphi_1 = 1$
$\varphi_2(\mathfrak{q}) = 1$	$\varphi_2 = 1$
$\varphi_3(\mathfrak{q}) = 1 + \mathfrak{q}$	$\varphi_3 = 2$
$\varphi_4(\mathfrak{q}) = 1 + 2\mathfrak{q}$	$\varphi_4 = 3$
$\varphi_5(\mathfrak{q}) = 1 + 3\mathfrak{q} + \mathfrak{q}^2$	$\varphi_5 = 5$

We aim to establish a novel link between the $\mathfrak{q}$-analogue of Fibonacci numbers and their corresponding Fibonacci polynomials. We define a function $\mathrm{Y}(z;\mathfrak{q})$ in the following approach:

$$\begin{aligned} \mathrm{Y}(z;\mathfrak{q}) &= \frac{1 + \mathfrak{q}\,\vartheta_{\mathfrak{q}}^2\, z^2}{1 - \vartheta_{\mathfrak{q}}\, z - \mathfrak{q}\,\vartheta_{\mathfrak{q}}^2\, z^2} \\ &= 1 + \vartheta_{\mathfrak{q}}\, z + (2\mathfrak{q}+1)\vartheta_{\mathfrak{q}}^2\, z^2 + (3\mathfrak{q}+1)\,\vartheta_{\mathfrak{q}}^3\, z^3 + (2\mathfrak{q}^2 + 4\mathfrak{q} + 1)\,\vartheta_{\mathfrak{q}}^4\, z^4 + \cdots . \end{aligned} \tag{6}$$

It is worth noting that the function $\mathrm{Y}(z;\mathfrak{q})$ is not univalent in the domain $\mathscr{O}$. Specifically, it attains the same value at two distinct points:

$$\mathrm{Y}(0;\mathfrak{q}) = 1 \quad \text{and} \quad \mathrm{Y}\left(-\frac{1}{2\mathfrak{q}\vartheta_{\mathfrak{q}}};\mathfrak{q}\right) = 1.$$

Example 1. *If $\mathfrak{q} \to 1^-$ in Definition 3, we obtain the class $\mathsf{SL} = \lim_{\mathfrak{q}\to 1^-} \mathsf{SL}_{\mathfrak{q}}$ defined as follows: A function f belongs to the class SL if and only if*

$$\frac{z\,f'(z)}{f(z)} \prec \mathrm{Y}(z),$$

where

$$\mathrm{Y}(z;1) = \mathrm{Y}(z) = \frac{1 + \vartheta^2 z^2}{1 - \vartheta z - \vartheta^2 z^2}, \tag{7}$$

and $\vartheta = \frac{1-\sqrt{5}}{2}$ corresponds to the classical Fibonacci numbers.

3. Main Results

Theorem 1. *Let*

$$\varphi_{\mathsf{s}}(\mathfrak{q}) = \frac{(1 - \mathfrak{q}\vartheta_{\mathfrak{q}})^{\mathsf{s}} - (\vartheta_{\mathfrak{q}})^{\mathsf{s}}}{\sqrt{4\mathfrak{q}+1}},$$

denotes the sequence of $\mathfrak{q}$-Fibonacci numbers, defined in recurrence relation as

$$\varphi_{\mathsf{s}+2}(\mathfrak{q}) = \varphi_{\mathsf{s}+1}(\mathfrak{q}) + \mathfrak{q}\,\varphi_{\mathsf{s}-1}(\mathfrak{q}), \quad \mathsf{s} = 0, 1, 2, 3, \ldots .$$

If the generating function

$$\mathrm{Y}(z;\mathfrak{q}) = \frac{1 + \mathfrak{q}\vartheta_{\mathfrak{q}}^2 z^2}{1 - \vartheta_{\mathfrak{q}}\, z - \mathfrak{q}\vartheta_{\mathfrak{q}}^2 z^2} = 1 + \sum_{\mathsf{s}=1}^{\infty} \mathrm{p}_{\mathsf{s}}\, z^{\mathsf{s}},$$

then

$$p_s = \left(\varphi_{s+1}(q) + q\varphi_{s-1}(q)\right)\vartheta_q, \qquad s \in \mathbb{N}.$$

Proof. Let us denote $\vartheta_q z = \varpi$ and $|\varpi| < |\vartheta_q|$. Now, we have

$$\begin{aligned}
\frac{1+q\varpi^2}{1-\varpi-q\varpi^2} &= \left(q\varpi+\frac{1}{\varpi}\right)\frac{\varpi}{1-\varpi-q\varpi^2} \\
&= \left(q\varpi+\frac{1}{\varpi}\right)\frac{1}{\sqrt{4q+1}}\left(\frac{\vartheta_q}{\varpi+\vartheta_q}+\frac{q\vartheta_q-1}{q\varpi+(1-q\vartheta_q)}\right) \\
&= \left(q\varpi+\frac{1}{\varpi}\right)\frac{1}{\sqrt{4q+1}}\left(\frac{1}{1+\frac{\varpi}{\vartheta_q}}-\frac{1}{1+\frac{q\varpi}{1-q\vartheta_q}}\right) \\
&= \left(q\varpi+\frac{1}{\varpi}\right)\frac{1}{\sqrt{4q+1}}\sum_{s=1}^{\infty}(-1)^s\left[\left(\frac{\varpi}{\vartheta_q}\right)^s-\left(\frac{q\varpi}{1-q\vartheta_q}\right)^s\right] \\
&= \left(q\varpi+\frac{1}{\varpi}\right)\sum_{s=1}^{\infty}\frac{(1-q\vartheta_q)^s-(\vartheta_q)^s}{\sqrt{4q+1}}t^s \\
&= \left(q\varpi+\frac{1}{\varpi}\right)\sum_{s=1}^{\infty}\varphi_s(q)\varpi^s = 1+\sum_{s=1}^{\infty}(\varphi_{s+1}(q)+q\varphi_{s-1}(q))\vartheta_q^s z^s.
\end{aligned} \tag{8}$$

□

Theorem 2. *A function* f *of the form* (1) *is in the class* SL_q *if and only if there exists a function* $\mu \prec Y$, *where*

$$Y(z;q) = \frac{1+q\vartheta_q^2 z^2}{1-\vartheta_q z - q\vartheta_q^2 z^2}.$$

In this case, f *can be expressed in the integral form*

$$f(z) = z\cdot\exp_q\left(\frac{\log q}{q-1}\int_0^z \frac{\mu(\zeta)-1}{\zeta}\partial_q\zeta\right), \quad z \in \mathscr{O}. \tag{9}$$

Proof. If $f \in SL_q$, then

$$\frac{z\partial_q\langle f(z)\rangle}{f(z)} = Y(\hbar(z);q), \quad |\hbar(z)| < 1,\ z \in \mathscr{O}. \tag{10}$$

Let $\mu(z) = Y(\hbar(z);q)$. By applying the q-Jackson integral, as defined in Equation (3), we establish the equivalence between Equations (9) and (10). Specifically, the q-Jackson integral provides a framework through which the transformation from (9) to (10) can be rigorously derived, ensuring consistency and mathematical validity. □

Theorem 3. *Let* f *be given by* (1) *and belong to the class* SL_q. *Then, the coefficients satisfy the bound*

$$|\alpha_s| \le |\vartheta_q|^{s-1}\varphi_s(q), \tag{11}$$

where $\varphi_s(q)$ *represents the sequence of q-Fibonacci numbers, and* ϑ_q *is given by*

$$\vartheta_q = \frac{1-\sqrt{4q+1}}{2q}.$$

Moreover, this bound is sharp and is attained by the function

$$Y_0(z;q) = \frac{z}{1-\vartheta_q z - q\vartheta_q^2 z^2}.$$

Proof. From the definition of SL_q, there exists a function $\hbar$, with $|\hbar(z)| < 1$ for $z \in \mathscr{O}$, such that for $f \in \mathsf{SL}_q$ given by (1), the following holds:

$$\frac{z\,\eth_q\langle f(z)\rangle}{f(z)} = \frac{1 + q\,\vartheta_q^2\,\hbar^2(z)}{1 - \vartheta_q\,\hbar(z) - q\,\vartheta_q^2\,\hbar^2(z)}.$$

Therefore, we obtain

$$z\eth_q\langle f(z)\rangle - f(z) = \vartheta_q\, z\hbar(z)\,\eth_q\langle f(z)\rangle + \vartheta_q^2\hbar(z)^2\big[z\eth_q\langle f(z)\rangle + f(z)\big]$$

which equivalent to

$$\sum_{s=1}^{\infty}\big(\lceil s\rfloor_q - 1\big)\alpha_s z^s = \vartheta_q\hbar(z)\sum_{s=1}^{\infty}\lceil s\rfloor_q\alpha_s z^s + \vartheta_q^2\hbar^2(z)\sum_{s=1}^{\infty}\big(\lceil s\rfloor_q + 1\big)\alpha_s z^s$$

and so

$$\sum_{s=1}^{\tau}\big(\lceil s\rfloor_q - 1\big)\alpha_s z^s + \sum_{s=\tau+1}^{\infty} c_s z^s = \vartheta_q\hbar(z)\sum_{s=1}^{\tau-1}\lceil s\rfloor_q\alpha_s z^s + \vartheta_q^2\,\hbar^2(z)\sum_{s=1}^{\tau-2}\big(\lceil s\rfloor_q + 1\big)\alpha_s z^s.$$

Hence, for $m \geq 2$, we have

$$\begin{aligned}
&\left|\sum_{s=1}^{\tau}\big(\lceil s\rfloor_q - 1\big)\alpha_s z^s + \sum_{s=\tau+1}^{\infty} c_s z^s\right|^2\\
&= \left|\vartheta_q\hbar(z)\sum_{s=1}^{\tau-1}\lceil s\rfloor_q\alpha_s z^s + \vartheta_q^2\,\hbar^2(z)\sum_{s=1}^{\tau-2}\big(\lceil s\rfloor_q + 1\big)\alpha_s z^s\right|^2\\
&\leq \left|\vartheta_q\sum_{s=1}^{\tau-1}\lceil s\rfloor_q\alpha_s z^s + \vartheta_q^2\hbar(z)\sum_{s=1}^{\tau-1}\lceil s\rfloor_q\alpha_{s-1} z^{s-1}\right|^2\\
&\leq \sum_{s=1}^{\tau-1}\left|\vartheta_q\lceil s\rfloor_q\alpha_s z^s + \vartheta_q^2\lceil s\rfloor_q\alpha_{s-1}\,\hbar(z)\,z^{s-1}\right|^2\\
&\leq \sum_{s=1}^{\tau-1}\left|\vartheta_q\lceil s\rfloor_q\alpha_s z^s\right|^2 + \left|\vartheta_q^2\lceil s\rfloor_q\alpha_{s-1}\,z^{s-1}\right|^2 + 2\left|\vartheta_q^3\lceil s\rfloor_q^2\alpha_s\alpha_{s-1}\,z^{2s-1}\right|^2
\end{aligned}$$

By integrating this inequality over the contour defined by $z = rexp(i\phi)$ and taking the limit as $r \mapsto 1^-$, we derive the following:

$$\sum_{s=1}^{\tau}([n-1]_q)^2|\alpha_s|^2 + \sum_{s=\tau+1}^{\infty}|c_s|^2 \quad \leq \vartheta_q^2\sum_{s=1}^{\tau-1}\lceil s\rfloor_q^2|\alpha_s|^2 + \vartheta_q^4\sum_{s=1}^{\tau-2}\lceil s+1\rfloor_q^2|\alpha_s|^2 + 2\vartheta_q^3\sum_{s=2}^{\tau-1}\lceil s\rfloor_q^2|\alpha_s||\alpha_{s-1}|.$$

Thus, we have

$$|\alpha_s| \leq \frac{1}{(\lceil s\rfloor_q - 1)^2}\left(\sum_{\kappa=1}^{s-1}\Big(\vartheta_q^2\lceil\kappa\rfloor_q^2 - (\lceil\kappa\rfloor_q - 1)^2\Big)|\alpha_\kappa|^2 + \vartheta_q^4\sum_{\kappa=1}^{s-2}(\lceil\kappa\rfloor_q + 1)^2|\alpha_\kappa|^2 + 2\vartheta_q^3\sum_{\kappa=1}^{s-1}\lceil\kappa\rfloor_q^2|\alpha_\kappa||\alpha_{\kappa-1}|\right). \tag{12}$$

For $s = 1$, the estimation (11) holds true. Let the inequality in (11) hold for $s,\ 1 < s \leq \tau$. Then, from (12), we have

$$|\alpha_{\tau+1}|^2 \leq \frac{1}{\lceil\tau\rfloor_q^2}\left\{\sum_{\kappa=1}^{\tau}\Big(\vartheta_q^2\lceil\kappa\rfloor_q^2 - (\lceil\kappa\rfloor_q - 1)^2\Big)|\alpha_\kappa|^2 + \vartheta_q^4\sum_{\kappa=1}^{\tau-1}(\lceil\kappa\rfloor_q + 1)^2|\alpha_\kappa|^2 + 2\sum_{\kappa=1}^{\tau}\Big|\vartheta_q^3\Big|\lceil\kappa\rfloor_q^2|\alpha_\kappa||\alpha_{\kappa-1}|\right\}.$$

Therefore, we have $|\alpha_{\tau+1}|^2 \leq$

$$\left(\frac{\lceil\tau\rfloor_{\mathfrak{q}}-1}{\lceil\tau\rfloor_{\mathfrak{q}}}\right)^2 \frac{1}{(\lceil\tau\rfloor_{\mathfrak{q}}-1)^2}\left\{\begin{array}{c}\sum_{\kappa=1}^{\tau-1}\left(\vartheta_{\mathfrak{q}}^2\lceil\kappa\rfloor_{\mathfrak{q}}^2-(\lceil\kappa\rfloor_{\mathfrak{q}}-1)^2\right)|\alpha_\kappa|^2+\vartheta_{\mathfrak{q}}^4\sum_{\kappa=1}^{\tau-2}(\lceil\kappa\rfloor_{\mathfrak{q}}+1)^2|\alpha_\kappa|^2\\ +2\left|\vartheta_{\mathfrak{q}}^3\right|\sum_{\kappa=1}^{\tau-1}\lceil\kappa\rfloor_{\mathfrak{q}}^2|\alpha_\kappa||\alpha_{\kappa-1}|+\left(\vartheta_{\mathfrak{q}}^2\lceil\tau\rfloor_{\mathfrak{q}}^2-(\lceil\tau\rfloor_{\mathfrak{q}}-1)^2\right)|\alpha_\tau|^2\\ +\vartheta_{\mathfrak{q}}^4(\lceil\tau\rfloor_{\mathfrak{q}})^3|\alpha_{\tau-1}|^2+2\left|\vartheta_{\mathfrak{q}}^3\right|\lceil\tau\rfloor_{\mathfrak{q}}^2|\alpha_\tau||\alpha_{\tau-1}|\end{array}\right\}$$

$$=\left(\frac{\lceil\tau\rfloor_{\mathfrak{q}}-1}{\lceil\tau\rfloor_{\mathfrak{q}}}\right)^2\left\{\begin{array}{c}|\alpha_\tau|^2+\frac{1}{(\lceil\tau\rfloor_{\mathfrak{q}}-1)^2}\left(\vartheta_{\mathfrak{q}}^2[\tau]_{\mathfrak{q}}^2+\vartheta_{\mathfrak{q}}^4\lceil\tau\rfloor_{\mathfrak{q}}^2|\alpha_{\tau-1}|^2\right.\\ \left.+2\vartheta_{\mathfrak{q}}^3\lceil\tau\rfloor_{\mathfrak{q}}^2|\alpha_\tau||\alpha_{\tau-1}|-(\lceil\tau\rfloor_{\mathfrak{q}}-1)^2|\alpha_\tau|^2\right)\end{array}\right\}$$

$$=\frac{1}{\lceil\tau\rfloor_{\mathfrak{q}}^2}\left(\vartheta_{\mathfrak{q}}\lceil\tau\rfloor_{\mathfrak{q}}|\alpha_\tau|+\vartheta_{\mathfrak{q}}^2\lceil\tau\rfloor_{\mathfrak{q}}|\alpha_{\tau-1}|\right)^2\leq\left(\vartheta_{\mathfrak{q}}|\alpha_\tau|+\mathfrak{q}\vartheta_{\mathfrak{q}}^2|\alpha_{\tau-1}|\right)^2$$

$$=\left(\vartheta_{\mathfrak{q}}^\tau\varphi_\tau(\mathfrak{q})+\mathfrak{q}\vartheta_{\mathfrak{q}}^\tau\varphi_{\tau-1}(\mathfrak{q})\right)^2=\vartheta_{\mathfrak{q}}^\tau\left(\varphi_\tau(\mathfrak{q})+\mathfrak{q}\vartheta_{\mathfrak{q}}^\tau\varphi_{\tau-1}(\mathfrak{q})\right)^2=\left(\vartheta_{\mathfrak{q}}^\tau\varphi_{\tau+1}(\mathfrak{q})\right)^2.$$

This is because $\varphi_\tau(\mathfrak{q})$ represents the sequence of the $\mathfrak{q}$-analog of Fibonacci numbers. We have proven the inequality in (11) by induction, which establishes a key property of this sequence. In addition, we have

$$\frac{z}{1-\vartheta_{\mathfrak{q}}\,z-\mathfrak{q}\vartheta_{\mathfrak{q}}^2z^2}=z+\vartheta_{\mathfrak{q}}\,z^2+(\mathfrak{q}+1)\vartheta_{\mathfrak{q}}^2z^3+(2\mathfrak{q}+1)\vartheta_{\mathfrak{q}}^3\,z^4+\quad(\mathfrak{q}^2+3\mathfrak{q}+1)\vartheta_{\mathfrak{q}}^4z^5+\cdots.$$

□

Corollary 1 ([5]). *If the function* f *belongs to the class* SL*, then*

$$\lim_{n\to\infty}|\alpha_{\mathsf{s}}|\leq|\vartheta|^{\mathsf{s}-1}\varphi_{\mathsf{s}}.$$

This result is sharp for the function

$$\mathrm{Y}_0(z;1)=\frac{z}{1-\vartheta\,z-\vartheta^2\,z^2},\qquad\vartheta=\frac{1-\sqrt{5}}{2}.$$

Theorem 4. *Consider the function* $\mathrm{Y}_0(z;\mathfrak{q})$ *given by the form in* (4)*. The image of the unit circle under this function traces out the curve* $\Omega_{\mathfrak{q}}$*, which satisfies the following parametric equations:*

$$\varkappa=\frac{\sqrt{4\mathfrak{q}+1}}{2(1+2\mathfrak{q}-2\mathfrak{q}\cos\phi)}\ \text{ and }\ \mathsf{y}=\frac{\frac{\sin\phi}{2(1+\cos\phi)}(4\mathfrak{q}\cos\phi-1)}{(1+2\mathfrak{q}-2\mathfrak{q}\cos\phi)},\quad(\phi\in[0,2\pi)\setminus\{\pi\}).\tag{13}$$

Proof. The proof is established through a series of straightforward calculations, which rely on fundamental properties and techniques from the underlying mathematical framework. Recall that the curve known as the conchoid of Sluze is described by the following equation:

$$\varrho(\varkappa-\varrho)\left(\varkappa^2+\mathsf{y}^2\right)+\eta^2\varkappa^2=0,\quad(\varrho,\eta>0).\tag{14}$$

When $\eta=2\varrho$, the equation for the conchoid of Sluze (14) simplifies to the following form:

$$\varkappa^3+(\varkappa-\varrho)\mathsf{y}^2+3\varrho\varkappa^2=0,\tag{15}$$

which represents the well-known trisectrix of Maclaurin, as shown in Figure 1.

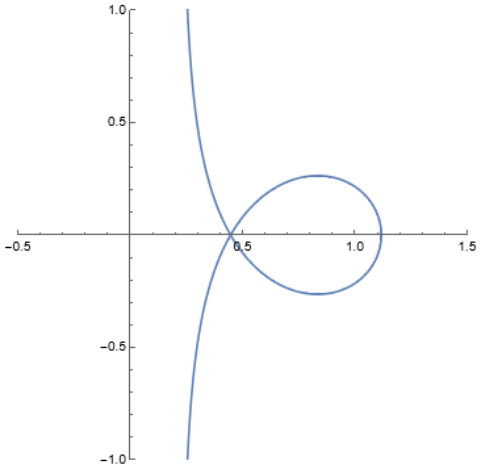


Figure 1. Curve running towards trisectrix of Maclaurin reaching convergence.

Derived from Equation (13), the Cartesian equation for the curve is

$$\Omega_{\mathfrak{q}}: \quad \left[2(1+4\mathfrak{q})\varkappa - \sqrt{1+4\mathfrak{q}}\right]y^2 = \left(\sqrt{4\mathfrak{q}+1}-2\varkappa\right)\left(\sqrt{4\mathfrak{q}+1},\varkappa-1\right)^2. \tag{16}$$

If we express Equation (16) in the form

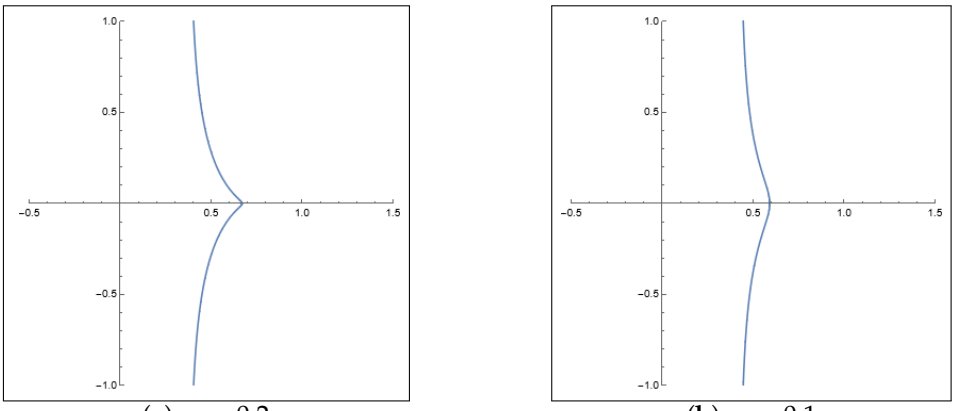
$$\left(\frac{\sqrt{4\mathfrak{q}+1}}{4\mathfrak{q}+1}-\varkappa\right)^3 + \frac{(4\mathfrak{q}-1)\sqrt{4\mathfrak{q}+1}}{2(4\mathfrak{q}+1)}\left(\frac{\sqrt{4\mathfrak{q}+1}}{4\mathfrak{q}+1}-\varkappa\right)^2 + \left[\left(\frac{\sqrt{4\mathfrak{q}+1}}{4\mathfrak{q}+1}-\varkappa\right) - \frac{\sqrt{4\mathfrak{q}+1}}{2(4\mathfrak{q}+1)}\right]y^2 = 0, \tag{17}$$

then we can observe that the image of a unit circle under the function $Y(z;\mathfrak{q})$ is translated into a curve with Equation (15), where

$$a = \frac{1-2\mathfrak{q}\vartheta_q}{2(4\mathfrak{q}+1)} = \frac{\sqrt{4\mathfrak{q}+1}}{2(4\mathfrak{q}+1)}.$$

As a result, the curve $\Omega_{\mathfrak{q}}$ displays a shell-like configuration and is symmetric about the real axis, as illustrated in Figure 1. The symmetry and geometric structure originate from the analytical properties of the function $Y(z;\mathfrak{q})$ and its dependence on the parameter $\mathfrak{q}$.

1. In the range of $\mathfrak{q}$ where $\mathfrak{q} \in \left(0, \frac{1}{4}\right)$, the curve takes a similar appearance of a conchoid without any loops. See Figure 2 for visual representations of $\Omega_{\mathfrak{q}}$ corresponding to various $\mathfrak{q}$ values.



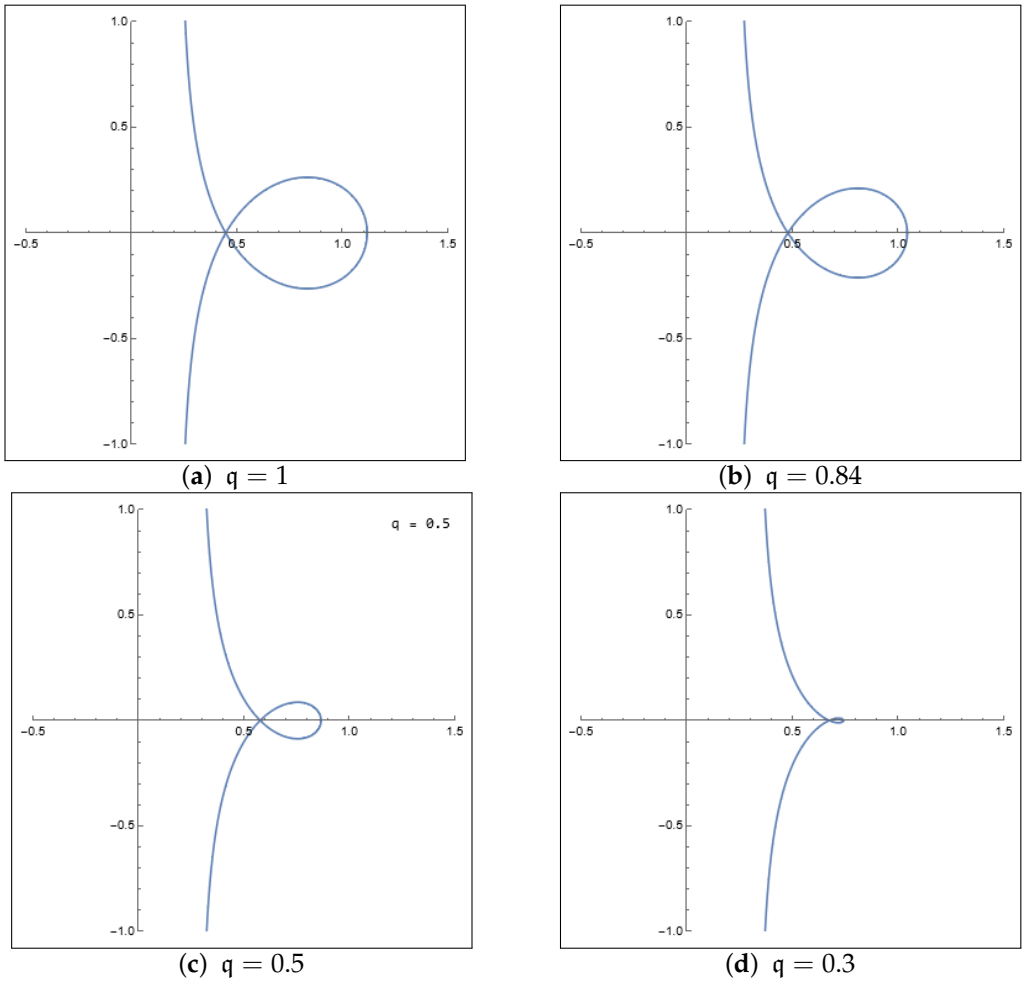
(a) $\mathfrak{q} = 0.2$ **(b)** $\mathfrak{q} = 0.1$

Figure 2. The curve $\Omega_{\mathfrak{q}}$ for various values of $\mathfrak{q}$ within the range where $\mathfrak{q} \in \left(0, \frac{1}{4}\right)$.

2. In the range of $\mathfrak{q}$ where $\mathfrak{q} \in \left(\frac{1}{4}, 1\right)$, we observe that

$$\mathrm{Y}\left(\exp\left(\pm i \arccos\left(4\mathfrak{q}\right)^{-1}\right)\right) = \frac{\sqrt{4\mathfrak{q}+1}}{4\mathfrak{q}+1},$$

implies that the curve $\Omega_{\mathfrak{q}}$ intersects itself on the real axis at the point $\frac{\sqrt{4\mathfrak{q}+1}}{4\mathfrak{q}+1}$. Consequently, $\Omega_{\mathfrak{q}}$ forms a loop that intersects the real axis at two points, $e_1 = \frac{\sqrt{4\mathfrak{q}+1}}{2}$ and $e_2 = \frac{\sqrt{4\mathfrak{q}+1}}{4\mathfrak{q}+1}$. See Figure 3 for visual representations of $\Omega_{\mathfrak{q}}$ corresponding to various $\mathfrak{q}$ values.



(**a**) $\mathfrak{q} = 1$ (**b**) $\mathfrak{q} = 0.84$

(**c**) $\mathfrak{q} = 0.5$ (**d**) $\mathfrak{q} = 0.3$

Figure 3. The curve $\Omega_{\mathfrak{q}}$ for various values of $\mathfrak{q}$ where $\mathfrak{q} \in \left(\frac{1}{4}, 1\right)$.

The curve $\mathrm{Y}(e^{i\phi}; \mathfrak{q})$, where $\phi \in [0, 2\pi) \setminus \{\pi\}$, exhibits a vertical asymptote and a loop, arising from the characteristics of Y. Furthermore, the curve intersects itself at the coordinates

$$\mathrm{Y}\left(e^{\pm i \arccos(1/4\mathfrak{q})}\right) = \frac{\sqrt{4\mathfrak{q}+1}}{4\mathfrak{q}+1}.$$

These intersections, along with the presence of the asymptote and loop, highlight the intricate geometric structure of the curve, which is influenced by the parameter $\mathfrak{q}$ and the functional properties of Y. Furthermore, we have the following limits:

(a) $\lim\limits_{\xi \to \pi} \Re e\left(\mathrm{Y}(e^{i\xi}); \mathfrak{q}\right) = \epsilon.$

(b) $\lim\limits_{\xi \to \pi^-} \Im m\left(\mathrm{Y}(e^{i\xi}); \mathfrak{q}\right) = -\infty$ and $\lim\limits_{\xi \to \pi^+} \Im m\left(\mathrm{Y}(e^{i\xi}); \mathfrak{q}\right) = \infty.$

□

In the following theorem, we establish a sufficient condition for the function $\mathrm{Y}_0(\mathsf{z};\mathfrak{q})$ to belong to the class $\mathrm{Y}(\beta)$.

Theorem 5. *The function* $\mathrm{Y}(\mathsf{z};\mathfrak{q})$ *belongs to the class* $\mathsf{P}(\beta)$*, with the parameter*

$$\beta = \frac{\sqrt{4\mathfrak{q}+1}}{2(4\mathfrak{q}+1)}.$$

Proof. The proof is direct. Let us examine the image of the unit circle under the mapping Y_0. From Theorem 4 (after performing the necessary calculations), we derive the following relationship:

$$\Re e\Big\{\mathrm{Y}\big(e^{it};\mathfrak{q}\big)\Big\} = \frac{\sqrt{4\mathfrak{q}+1}}{2(1+2\mathfrak{q}-2\mathfrak{q}\cos\phi)} \geq \frac{\sqrt{4\mathfrak{q}+1}}{2(4\mathfrak{q}+1)} = \beta, \tag{18}$$

where $\phi \in [0, 2\pi)$, since $\mathrm{Y}(0;\mathfrak{q}) = 1$—it follows that $\mathrm{Y} \in \mathsf{P}\Big(\frac{\sqrt{4\mathfrak{q}+1}}{2(4\mathfrak{q}+1)}\Big)$. □

We construct a sufficient condition for the univalency of the function $\mathrm{Y}(\mathsf{z};\mathfrak{q})$ in the following theorem within a well-defined disk.

Theorem 6. *The function* $\mathrm{Y}(\mathsf{z};\mathfrak{q})$ *is univalent within the disk*

$$\mathbb{D}_{\mathfrak{q}} = \left\{\mathsf{z} : |\mathsf{z}| < \mathsf{r}_{\mathfrak{q}} = \frac{1+\mathfrak{q}-\sqrt{(1+\mathfrak{q})^2+1}}{\mathfrak{q}\vartheta_{\mathfrak{q}}}\right\}, \tag{19}$$

and is not univalent in any disk with a radius equal to or greater than $\mathsf{r}_{\mathfrak{q}}$.

Proof. Let $\mathrm{Y}(\mathsf{z};\mathfrak{q}) = \mathrm{Y}(w;\mathfrak{q})$ for some $\mathsf{z}, \omega \in \mathscr{O}$. By performing the necessary calculations, we arrive at the following equation:

$$\vartheta_{\mathfrak{q}}(\mathsf{z}-\omega)\left[\omega - \frac{2\mathfrak{q}\vartheta_{\mathfrak{q}}\mathsf{z}+1}{\mathfrak{q}\vartheta_{\mathfrak{q}}(\vartheta_{\mathfrak{q}}\mathsf{z}-2)}\right] = 0. \tag{20}$$

The function $g_{\mathfrak{q}}(\mathsf{z}) = \frac{2\mathfrak{q}\vartheta_{\mathfrak{q}}\mathsf{z}+1}{\mathfrak{q}\vartheta_{\mathfrak{q}}(\vartheta_{\mathfrak{q}}\mathsf{z}-2)}$, transforms the circle $|\mathsf{z}| = r_{\mathfrak{q}}$ (where $r_{\mathfrak{q}} < 2/\vartheta_{\mathfrak{q}}$) into another a circle, with its diameter extending between the points $g_{\mathfrak{q}}(-r_{\mathfrak{q}})$ and $g_{\mathfrak{q}}(r_{\mathfrak{q}})$. The center of the result circle is positioned at

$$m = \frac{-r_{\mathfrak{q}}}{\mathfrak{q}}\left(\frac{4\mathfrak{q}+1}{4-\vartheta_{\mathfrak{q}}^2 r_{\mathfrak{q}}^2}\right),$$

and its radius is given by

$$\rho = \frac{-2\big(\mathfrak{q}\vartheta_{\mathfrak{q}}^2 r_{\mathfrak{q}}^2+1\big)}{\mathfrak{q}\,\vartheta_{\mathfrak{q}}\big(4-\vartheta_{\mathfrak{q}}^2 \mathsf{r}_{\mathfrak{q}}^2\big)}.$$

Consequently, the function $g_{\mathfrak{q}}$ transforms the circle $|\mathsf{z}| = \mathsf{r}_{\mathfrak{q}}$ into another circle with a diameter stretching from the point $g_{\mathfrak{q}}(\mathsf{r}_{\mathfrak{q}}) = \mathsf{r}_{\mathfrak{q}}$ to the point $g_{\mathfrak{q}}(-\mathsf{r}_{\mathfrak{q}})$. For all $\mathfrak{q} \in (0,1)$, we have $g_{\mathfrak{q}}(-\mathsf{r}_{\mathfrak{q}}) > g_{\mathfrak{q}}(\mathsf{r}_{\mathfrak{q}}) = \mathsf{r}_{\mathfrak{q}}$, which follows from the inequality

$$\eth_{\mathfrak{q}}\langle f(g_{\mathfrak{q}}(\varkappa))\rangle > 0, \quad \forall \varkappa \in \mathbb{R},$$

where

$$f(g_{\mathfrak{q}}(\mathsf{z})) = \frac{-(4\mathfrak{q}+1)}{\mathfrak{q}(2-q\vartheta_{\mathfrak{q}}\mathsf{z})(2-\vartheta_{\mathfrak{q}}\mathsf{z})}.$$

Thus, if $|w| \leq r_q$ and $|z| \leq r_q$, the third factor in Equation (20) vanishes only when $\omega = z - r_q$. As a result, Equation (20) cannot hold for $|w| < r_q$ and $|z| < r_q$. This confirms that the function $Y(z;q)$ is univalent in the disk $\mathbb{D}_q$, as defined by Equation (19).

On the other hand, the derivative of the function $Y(z;q)$ is given by

$$\begin{aligned} \eth_q \langle f(Y(z;q)) \rangle &= \frac{\vartheta_q \left(1 + 2q[2]_q \vartheta_q z - q^2 \vartheta_q^2 z^2\right)}{\left(1 - \vartheta_q z - q\vartheta_q^2 z^2\right)\left(1 - q\vartheta_q z - q^3 \vartheta_q^2 z^2\right)} \\ &= \frac{\vartheta_q (z - r_q)\left(z - \frac{1+q+\sqrt{(1+q)^2+1}}{q\vartheta_q}\right)}{\left(1 - \vartheta_q z - q\vartheta_q^2 z^2\right)\left(1 - q\vartheta_q z - q^3 \vartheta_q^2 z^2\right)}. \end{aligned} \tag{21}$$

The function $Y(z;q)$ vanishes at the point $z = r_q$, and therefore it follows that $Y(z;q)$ is not univalent to $|z| \geq r_q$. □

4. Conclusions

Emphasising their connection with shell-like star-like curves, this work presents a fresh method for exploring q-Fibonacci numbers and their corresponding polynomials. We investigated the geometric and analytic characteristics of a newly defined subclass of star-like functions by using the interval of univalence and nonunivalence for some functions. We also developed a sufficient condition for functions in this subclass such that they may be labelled as analytic functions with positive real components. Using subordination ideas and the natural properties of Fibonacci sequences helped us to obtain the results, therefore providing a strong analytical foundation for our findings.

This work highlights the critical connection between Geometric Function Theory and q-calculus, offering fresh insights into the structural properties of functions defined by q-Fibonacci sequences. By bridging sequential and analytical domains, it enhances the theoretical understanding of Fibonacci numbers and paves the way for multidisciplinary research opportunities. The study establishes a novel framework that links Fibonacci-type sequences with the geometric properties of analytic functions, uncovering new geometric and analytical features that enrich the field. This approach not only advances the theoretical foundation of Geometric Function Theory but also opens new avenues for exploration, such as extending the results to higher-dimensional spaces, applying them in mathematical physics, or developing computational techniques inspired by this theory. Additionally, the work lays the groundwork for future investigations and encourages researchers to further explore the applications of q-calculus in geometric function theory and related fields, ensuring its continued relevance and fostering innovation within the mathematical community.

Author Contributions: Conceptualization, A.A. (Abdullah Alsoboh) and A.A. (Ala Amourah); Methodology, A.A. (Abdullah Alsoboh) and O.A.; Software, O.A. and T.M.S.; Validation, M.A.; Formal analysis, A.A. (Abdullah Alsoboh) and A.A. (Ala Amourah); Investigation, A.A. (Ala Amourah), M.A. and T.M.S.; Data curation, A.A. (Abdullah Alsoboh) and T.M.S.; Writing—original draft, A.A. (Abdullah Alsoboh) and A.A. (Ala Amourah). All authors have read and agreed to the published version of the manuscript.

Funding: This research work was funded by Umm Al-Qura University, Saudi Arabia, under grant number: 25UQU4350561GSSR05.

Informed Consent Statement: Not applicable.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: The authors extend their appreciation to Umm Al-Qura University, Saudi Arabia, for funding this research work through grant number: 25UQU4350561GSSR05.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Duren, P.L. Univalent functions. In *Grundlehren der Mathematischen Wissenschaften Series*; Springer: New York, NY, USA, 1983.
2. Ma, W.; Minda, D. A unified treatment of some special classes of univalent functions. In *Proceedings of the Conference on Complex Analysis*; International Press Inc.: Tianjin, China, 1992; pp. 157–169.
3. Janowski, W. Extremal problems for a family of functions with positive real part and for some related families. *Ann. Pol. Math.* **1970**, *23*, 159–177. [CrossRef]
4. Robertson, M.S. Certain classes of star-like functions. *Mich. Math. J.* **1985**, *32*, 135–140. [CrossRef]
5. Sokół, J. On star-like functions connected with Fibonacci numbers. *Zesz. Nauk. Politech. Rzesz. Mat.* **1999**, *23*, 111–116.
6. Sokół, J. A certain class of star-like functions. *Comput. Math. Appl.* **2011**, *62*, 611–619. [CrossRef]
7. Jackson, F.H. On *q*-functions and a certain difference operator. *Trans. R. Soc. Edinb.* **1909**, *46*, 253–281. [CrossRef]
8. Jackson, F.H. On *q*-definite integrals. *Q. J. Pure Appl. Math.* **1910**, *41*, 193–203.
9. Aral, A.; Gupta, V. Generalized *q*-Baskakov operators. *Math. Slovaca* **2011**, *61*, 619–634. [CrossRef]
10. Alsoboh; Oros, G. A Class of Bi-univalent Functions in a Leaf-Like Domain Defined through Subordination via q-calculus. *Mathematics* **2004**, *12*, 1594. [CrossRef]
11. Uçar, H.; Özkan, E. Coefficient inequality for q-star-like functions. *Appl. Math. Comput.* **2016**, *276*, 122–126.
12. Srivastava, H.M.; Khan, B.; Khan, N.; Tahir, M.; Ahmad, S.; Khan, N. Upper bound of the third Hankel determinant for a subclass of q-star-like functions associated with the q-exponential function. *Bull. Des Sci. Math.* **2021**, *167*, 102942. [CrossRef]
13. Khan, B.; Srivastava, H.M.; Khan, N.; Darus, M.; Tahir, M.; Ahmad, Q.Z. Coefficient Estimates for a Subclass of Analytic Functions Associated with a Certain Leaf-Like Domain. *Mathematics* **2020**, *8*, 1334. [CrossRef]
14. Arif, M.; Barkub, O.; Srivastava, H.M.; Abdullah, S.; Khan, S.A. Some Janowski Type Harmonic q-Star-like Functions Associated with Symmetrical Points. *Mathematics* **2020**, *8*, 629. [CrossRef]
15. Srivastava, H.M.; Aouf, M.K.; Mostafa, A.O. Some properties of analytic functions associated with fractional q-calculus operators. *Miskolc Math. Notes* **2019**, *20*, 1245–1260. [CrossRef]
16. Srivastava, H.M.; El-Deeb, S.M. A certain class of analytic functions of complex order connected with a q-analogue of integral operators. *Miskolc Math. Notes* **2020**, *21*, 417–433. [CrossRef]
17. Shafiq, M.; Srivastava, H.M.; Khan, N.; Ahmad, Q.Z.; Darus, M.; Kiran, S. An Upper Bound of the Third Hankel Determinant for a Subclass of q-Star-like Functions Associated with k-Fibonacci Numbers. *Symmetry* **2020**, *12*, 1043. [CrossRef]
18. Mahmood, S.; Ahmad, Q.Z.; Srivastava, H.M.; Khan, N.; Khan, B.; Tahir, M. A certain subclass of meromorphically q-star-like functions associated with the Janowski functions. *J. Inequal. Appl.* **2019**, *2019*, 88. [CrossRef]
19. Mahmood, S.; Srivastava, H.M.; Khan, N.; Ahmad, Q.Z.; Khan, B.; Ali, I. Upper bound of the third Hankel determinant for a subclass of q-star-like functions. *Symmetry* **2019**, *11*, 347. [CrossRef]
20. Alsoboh, A.; Amourah, A.; Sakar, F.M.; Ogilat, O.; Gharib, G.M.; Zomot, N. Coefficient Estimation Utilizing the Faber Polynomial for a Subfamily of Bi-Univalent Functions. *Axioms* **2023**, *12*, 512. [CrossRef]
21. Al-Hawary, T.; Amourah, A.; Alsoboh, A.; Ogilat, O.; Harny, I.; Darus, M. Applications of q-Ultraspherical polynomials to bi-univalent functions defined by q-Saigo's fractional integral operators. *AIMS Math.* **2024**, *9*, 17063–17075. [CrossRef]
22. Amourah, A.; Frasin, B.; Salah, J.; Yousef, F. Subfamilies of Bi-Univalent Functions Associated with the Imaginary Error Function and Subordinate to Jacobi Polynomials. *Symmetry* **2025**, *17*, 157. [CrossRef]
23. Alsoboh, A.; Darus, M. On subclasses of harmonic univalent functions defined by Jackson (p,q)-derivative. *J. Anal.* **2019**, *10*, 123–130.
24. Illafe, M.; Mohd, M.H.; Yousef, F.; Supramaniam, S. Investigating inclusion, neighborhood, and partial sums properties for a general subclass of analytic functions. *Int. J. Neutrosophic Sci.* **2025**, *25*, 501–510.
25. Amourah, A.; Alsoboh, A.; Breaz, D.; El-Deeb, S.M. A Bi-Star-like Class in a Leaf-like Domain Defined through Subordination via q-Calculus. *Mathematics* **2024**, *12*, 1735. [CrossRef]
26. Alsoboh, A.M.; Caglar, M.; Buyankara, M. Fekete-Szeg Inequality for a Subclass of Bi-Univalent Functions Linked to q-Ultraspherical Polynomials. *Contemp. Math.* **2024**, *5*, 2531–2545.
27. Elhaddad, S.; Aldweby, H.; Darus, M. Some properties on a class of harmonic univalent functions defined by *q*-analogue of Ruscheweyh operator. *J. Math. Anal.* **2018**, *9*, 28–35.
28. Illafe, M.; Mohd, M.H.; Yousef, F.; Supramaniam, S. A Subclass of bi-univalent functions defined by aSymmetric q-derivative operator and Gegenbauer polynomials. *Eur. J. Pure Appl. Math.* **2024**, *17*, 2467–2480. [CrossRef]
29. Amourah, A.; Frasin, B.A.; Swamy, S.R.; Sailaja, Y. Coefficient bounds for Al-Oboudi type bi-univalent functions connected with a modified sigmoid activation function and *k*-Fibonacci numbers. *J. Math. Computer Sci.* **2022**, *27*, 105–117. [CrossRef]
30. Dziok, J.; Raina, R.K.; Sokół, J. Certain results for a class of convex functions related to a shell-like curve connected with Fibonacci numbers. *Comput. Math. Appl.* **2011**, *61*, 2605–2613. [CrossRef]

31. Dziok, J.; Raina, R.K.; Sokół, J. On ϑ_q-convex functions related to a shell-like curve connected with Fibonacci numbers. *Appl. Math. Comput.* **2011**, *218*, 996–1002.
32. Sundari, K.S.; Keerthi, B.S. Enhancing low-light images using Sakaguchi type function and Gegenbauer polynomial. *Sci. Rep.* **2024**, *14*, 29679. [CrossRef]
33. Nithiyanandham, E.K.; Keerthi, B.S. Image edge detection enhancement using coefficients of Sakaguchi type functions mapped onto petal shaped domain. *Heliyon* **2024**, *10*, e31430. [CrossRef]
34. Al-Ameedee, S.A.; Al-Hakeem, M.B.H.; Alghafil, A.K.H. Fekete-Szegö inequalities for higher-order derivatives of multivalent analytic function with application to stealth combat aircraft. *J. Int. Math.* **2024**, *27*, 721–727. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.