mathematics

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On Rough Parametric Marcinkiewicz Integrals Along Certain Surfaces

Mohammed Ali [1,2,*] and Hussain Al-Qassem [3]

[1] College of Integrative Studies, Abdullah Al-Salem University, Firdous Street, Khaldiya 72303, Kuwait
[2] Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid 22110, Jordan
[3] Department of Mathematics and Statistics, Qatar University, Doha 2713, Qatar; husseink@qu.edu.qa
* Correspondence: myohammed.ali@aasu.edu.kw

Abstract: In this paper, we study rough Marcinkiewicz integrals associated with surfaces defined by $\Psi_{\mathcal{P},\phi} = \{(\mathcal{P}(w), \phi(|w|)) : w \in \mathbb{R}^m\}$. We establish the L^p-boundedness of these integrals when the kernel functions lie in the $L^q(\mathbb{S}^{m-1})$ space. Combining this result with Yano's extrapolation technique, we further obtain the L^p-boundedness under weaker kernel conditions—specifically, when the kernels belong to either the block space $B_q^{(0,-1/2)}(\mathbb{S}^{m-1})$ or $L(\log L)^{1/2}(\mathbb{S}^{m-1})$. Our results extend and refine several previously known results on Marcinkiewicz integrals, offering broader applicability and sharper conclusions.

Keywords: Marcinkiewicz integrals; rough operators; L^p bounds; extrapolation

MSC: 42B20; 42B25; 42B35

1. Introduction

Let $\mathbb{R}^m$ be the m-dimensional Euclidean space with $m \geq 2$, and let $\mathbb{S}^{m-1}$ be the unit sphere in $\mathbb{R}^m$ equipped with the normalized Lebesgue surface measure $d\sigma_m(\cdot)$. Furthermore, let $w' = w/|w|$ for $w \in \mathbb{R}^m \setminus \{0\}$.

Assume that h is a measurable function on $\mathbb{R}^+$ and that Θ is a homogeneous function of degree zero on $\mathbb{R}^m$, integrable over $\mathbb{S}^{m-1}$ and satisfying the condition

$$\int_{\mathbb{S}^{m-1}} \Theta(w')d\sigma_m(w') = 0. \tag{1}$$

For the appropriate mappings $\mathcal{P} : \mathbb{R}^m \to \mathbb{R}^d$ and $\phi : \mathbb{R}^+ \to \mathbb{R}$, we consider the Marcinkiewicz integral $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$, initially defined for $f \in C_0^\infty(\mathbb{R}^{d+1})$ by

$$\mathcal{M}_{\Theta,\mathcal{P},\phi,h}(f)(\tilde{x}) = \left(\int_0^\infty \left| \frac{1}{l^\alpha} \int_{|w|\leq l} f(x - \mathcal{P}(w), x_{d+1} - \phi(|w|)) \frac{\Theta(w)}{|w|^{m-\alpha}} h(|w|)dw \right|^2 \frac{dl}{l} \right)^{1/2}, \tag{2}$$

where $\tilde{x} = (x, x_{d+1}) \in \mathbb{R}^{d+1}$ and $\alpha = \tau + i\kappa$ ($\tau, \kappa \in \mathbb{R}$ with $\tau > 0$).

The study of singular integrals and the corresponding Marcinkiewicz integrals has attracted the attention of many authors in the past two decades. One of the principal motivations for the study of such operators is the requirement of several complex variables and large classes of "subelliptic" equations. Establishing L^p bounds for these integrals is valuable for analyzing the smoothness characteristics of functions and the behavior of integral transformations, including Poisson integrals, singular integrals, and more broadly, singular Radon transforms. For more background information, readers may refer to [1–3].

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Received: 19 March 2025
Revised: 11 April 2025
Accepted: 13 April 2025
Published: 14 April 2025

Citation: Ali, M.; Al-Qassem, H. On Rough Parametric Marcinkiewicz Integrals Along Certain Surfaces. *Mathematics* **2025**, *13*, 1287. https://doi.org/10.3390/math13081287

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Our main focus in this paper will be on establishing the L^p boundedness of the operator $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$. When $m = d$, $\mathcal{P}(w) = w$, $\phi \equiv 0$, and $h \equiv 1$, we denote $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$ by $\mathcal{M}_{\Theta,\alpha}$. Furthermore, when $\alpha = 1$, we denote $\mathcal{M}_{\Theta,\alpha}$ by $\mathfrak{M}_\Theta$ which is basically the classical Marcinkiewicz operator introduced by Stein in [1]. The study of the L^p boundedness of $\mathfrak{M}_\Theta$ has received a large amount of attention by many authors for a long time. For instance, it was proved in [1] that $\mathfrak{M}_\Theta$ is bounded on $L^p(\mathbb{R}^m)$ for $p \in (1,2)$ provided that the kernel function Θ belongs to the space $Lip_\mu(\mathbb{S}^{m-1})$ for some $\mu \in (0,1]$. Later on, the authors of [4] proved the L^p boundedness of $\mathfrak{M}_\Theta$ for all $p \in (1,\infty)$ under the condition $\Theta \in C^1(\mathbb{S}^{m-1})$. Thereafter, Walsh [5] confirmed the $L^2(\mathbb{R}^m)$ boundedness of $\mathfrak{M}_\Theta$, whenever $\Theta \in L(logL)^{1/2}(\mathbb{S}^{m-1})$, and he also found that the assumption $\Theta \in L(\log L)^{1/2}(\mathbb{S}^{m-1})$ is optimal in the sense that if $\Theta \in L(logL)^{\epsilon}(\mathbb{S}^{m-1})$ for any $\epsilon \in (0,1/2)$, then the operator $\mathfrak{M}_\Theta$ will not be bounded on $L^2(\mathbb{R}^m)$. The result in Ref. [5] was improved in Ref. [6] for the case where $p \in (1,\infty)$. On the other hand, the authors of [7] obtained the L^p $(1 < p < \infty)$ boundedness of $\mathfrak{M}_\Theta$, if Θ lies in the space $B_q^{(0,-1/2)}(\mathbb{S}^{m-1})$ for $q > 1$. Furthermore, they showed that the assumption $\Theta \in B_q^{(0,-1/2)}(\mathbb{S}^{m-1})$ is optimal in the sense that if $\Theta \in B_q^{(0,\epsilon)}(\mathbb{S}^{m-1})$ for any $\epsilon \in (-1,-1/2)$, then $\mathfrak{M}_\Theta$ may not be bounded on $L^2(\mathbb{R}^m)$. Here $B_q^{(0,\varepsilon)}(\mathbb{S}^{m-1})$ is referring to the block space introduced in Ref. [8].

We point out that the study of the parametric Marcinkiewicz operator $\mathcal{M}_{\Theta,\alpha}$ was initiated in Ref. [9], and it was then continued by many authors. In addition, the study of singular integral operators with rough kernels along surfaces was started in Ref. [10], and it was then continued by many researchers. For instance, the authors in Ref. [11] studied the operator $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$ when $\phi \equiv 0$, $\Theta \in L(\log L)^{1/2}(\mathbb{S}^{m-1}) \cup B_q^{(0,-1/2)}(\mathbb{S}^{m-1})$, $h \in \nabla_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$, and $\mathcal{P}(w) = (P_1(w), P_2(w), \cdots, P_d(w))$ is a polynomial mapping, where each P_j is a real-valued polynomial on $\mathbb{R}^m$. In fact, they established the L^p boundedness of $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$ for all $|1/p - 1/2| < \min\{1/\gamma', 1/2\}$. Here, $\Delta_\gamma(\mathbb{R}_+)$ (with $\gamma > 1$) refers to the collection of all functions h that are defined on $\mathbb{R}^+$ and satisfy

$$\|h\|_{\Delta_\gamma(\mathbb{R}_+)} = \sup_{j\in\mathbb{Z}} \left(\int_{2^j}^{2^{j+1}} |h(l)|^\gamma \frac{dl}{l} \right)^{1/\gamma} < \infty.$$

The authors in Ref. [12] obtained the same results in Ref. [11] for the special cases $\mathcal{P}(w) = w$ and for the case where $\phi \equiv 0$ replaced by the condition $\phi \in C^2(\mathbb{R}^+)$ is a convex and increasing function with $\phi(0) = 0$. The L^p boundedness of $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$ was investigated by many authors under various conditions on Θ, $\mathcal{P}$, ϕ, and h. We refer the readers to consult the following: for background information and a sample of past studies relevant to our current study [13–17], for its extensions and developments [18–23], and for recent advances [24–33].

In light of the results in Ref. [11] concerning operator $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$ in the case $\phi \equiv 0$ and for the results concerning operator $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$ in the case $\mathcal{P}(w) = w$, a question that arises naturally is whether the boundedness of the operator $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$ holds under the same assumptions as in Ref. [11] and for certain classes of functions ϕ?

Our main focus in this paper is to answer the above question in affirmative, as described in the following results.

Theorem 1. *Let $\mathcal{P}$ be a polynomial mapping given by $\mathcal{P}(w) = (P_1(w), P_2(w), \cdots, P_d(w))$, where each P_j is a real-valued polynomial on $\mathbb{R}^m$, and let ϕ be a function satisfying*

$$\phi(l) = \psi(l) + \varphi(l),$$

where ψ is a polynomial, $\varphi^{(k)}(0) = 0$ for all $1 \le k \le M$, $\varphi^{(k)}$ is positive nondecreasing on $\mathbb{R}^+$ for all $1 \le k \le M+1$, and $M = \max\{deg(\psi), deg(\mathcal{P})\}$. Assume that $\Theta \in L^q(\mathbb{S}^{m-1})$ for some $q \in (1,2]$ satisfies the condition (1) and that $h \in \nabla_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$. Then, a positive a constant $C_{p,\Theta,h}$ (independent of ϕ and the coefficients of the polynomials P_j and ψ) exists such that

$$\left\|\mathcal{M}_{\Theta,\mathcal{P},\phi,h}(f)\right\|_{L^p(\mathbb{R}^{d+1})} \le C_{p,\Theta,h}\left(\frac{\gamma}{(q-1)(\gamma-1)}\right)^{1/2}\|f\|_{L^p(\mathbb{R}^{d+1})}$$

for all $|1/p - 1/2| < \min\{1/\gamma', 1/2\}$, where $C_{p,\Theta,h} = C_p\|h\|_{\nabla_\gamma(\mathbb{R}^+)}\|\Theta\|_{L^q(\mathbb{S}^{m-1})}$.

By employing the estimate in Theorem 1 along using an extrapolation argument (see Refs. [34–36]), we obtain the following:

Theorem 2. *Assume that $\mathcal{P}$, ϕ, and h are given as in Theorem 1.*
(i) If $\Theta \in L(\log L)^{1/2}(\mathbb{S}^{m-1})$, then

$$\left\|\mathcal{M}_{\Theta,\mathcal{P},\phi,h}(f)\right\|_{L^p(\mathbb{R}^{d+1})} \le C_p\|h\|_{\nabla_\gamma(\mathbb{R}^+)}\left(\|\Theta\|_{L(\log L)^{1/2}(\mathbb{S}^{m-1})} + 1\right)\|f\|_{L^p(\mathbb{R}^{d+1})}$$

for all $|1/p - 1/2| < \min\{1/\gamma', 1/2\}$;

(ii) If $\Theta \in B_q^{(0,-1/2)}(\mathbb{S}^{m-1})$ for some $q > 1$, then

$$\left\|\mathcal{M}_{\Theta,\mathcal{P},\phi,h}(f)\right\|_{L^p(\mathbb{R}^{d+1})} \le C_p\|h\|_{\nabla_\gamma(\mathbb{R}^+)}\left(\|\Theta\|_{q^{(0,-1/2)}(\mathbb{S}^{s-1})} + 1\right)\|f\|_{L^p(\mathbb{R}^{d+1})}$$

for all $|1/p - 1/2| < \min\{1/\gamma', 1/2\}$.

Remark 1.

(i) For any $0 < \mu \le 1$, $\tau > 0$ and $q > 1$, the following inclusions hold and are proper:

$$C^1(\mathbb{S}^{m-1}) \subset Lip_\mu(\mathbb{S}^{m-1}) \subset L^q(\mathbb{S}^{m-1}) \subset L(\log L)^\tau(\mathbb{S}^{m-1}) \subset L^1(\mathbb{S}^{m-1}),$$

$$\bigcup_{r>1} L^r(\mathbb{S}^{m-1}) \subset B_q^{(0,v)}(\mathbb{S}^{m-1}) \subset L^1(\mathbb{S}^{m-1}) \text{ for any } v > -1,$$

$$L(\log L)^{\tau_1}(\mathbb{S}^{m-1}) \subset L(\log L)^{\tau_2}(\mathbb{S}^{m-1}) \text{ for } 0 < \tau_2 < \tau_1,$$

$$B_q^{(0,v_1)}(\mathbb{S}^{m-1}) \subset B_q^{(0,v_2)}(\mathbb{S}^{m-1}) \text{ for } -1 < v_2 < v_1.$$

(ii) For the special case $\phi \equiv 0$, Theorem 2 which gives the main results in Ref. [11] is directly obtained. Thus, our result generalizes the result in Ref. [11].
(iii) Since $Lip_\mu(\mathbb{S}^{m-1}) \subset L(\log L)^{1/2}(\mathbb{S}^{m-1}) \cup B_q^{(0,-1/2)}(\mathbb{S}^{m-1})$, our results extend the results in Refs. [1,4,9].
(iv) Our conditions on Θ in Theorem 2 are known to be the best possible in their respective classes for the special cases $m = d$, $\mathcal{P}(w) = w$, $\phi \equiv 0$, $h \equiv 1$, and $\alpha = 1$ (see Refs. [5,7]).
(v) For the case $\gamma > 2$, our results give the L^p boundedness of $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$ for p in the full range of $(1,\infty)$.
(vi) A model example about the Marcinkiewicz operator $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$ is

$$\left(\int_0^\infty \left|\frac{1}{l^7}\int_{|w|\le l} f\left(x - (w_1^2, w_2^2, \ldots, w_d^2), x_{d+1} - |w|^4\right)\frac{sgn(w_1 \cdot w_2 \cdot \ldots \cdot w_m)}{|w|^{m-7}}dw\right|^2 \frac{dl}{l}\right)^{1/2}.$$

In this example, we choose

$$\begin{aligned}\mathcal{P}(w) &= (w_1^2, w_2^2, \dots, w_d^2),\ \phi(|w|) = |w|^4,\ \Theta(w) = sgn(w_1 \cdot w_2 \cdot \dots \cdot w_m),\\ h &= 1,\ \alpha = 7,\ \textit{and}\ w = (w_1, w_2, \dots, w_m) \in \mathbb{R}^m.\end{aligned}$$

This paper is organized into four sections. Section 1 presents the introduction and main results. Section 2 provides the necessary preliminary lemmas, while Section 3 contains the proof of the main theorem. Finally, Section 4 summarizes the conclusions.

2. Some Lemmas

In this section, we give the auxiliary lemmas which will play major roles in proving the main results of this work. Let $\mu \geq 2$. For suitable mappings $\mathcal{P} : \mathbb{R}^m \to \mathbb{R}^d$, $\phi : \mathbb{R}^+ \to \mathbb{R}$, and $h : \mathbb{R}^+ \to \mathbb{C}$, we consider the family of measures $\{\mho_{\Theta,\mathcal{P},\phi,h,l} := \mho_l : l \in \mathbb{R}^+\}$ and its related maximal operators $\mho_h^*$ and $M_{h,\mu}$ on $\mathbb{R}^{d+1}$, given by

$$\int_{\mathbb{R}^{d+1}} f d\mho_l = \frac{1}{l^\alpha} \int_{l/2 \leq |w| \leq l} f(\mathcal{P}(w), \phi(|w|)) \frac{\Theta(y) h(|w|)}{|w|^{m-\alpha}} dw,$$

$$\mho_h^* f(\tilde{x}) = \sup_{l \in \mathbb{R}^+} ||\mho_l| * f(\tilde{x})|,$$

and

$$M_{h,\mu} f(\tilde{x}) = \sup_{j \in \mathbb{Z}} \int_{\mu^j}^{\mu^{j+1}} ||\mho_l| * f(\tilde{x})| \frac{dl}{l},$$

where $|\mho_l|$ is defined similarly to the definition of $\mho_l$, replacing Θ by $|\Theta|$ and h by $|h|$.

The following lemma comes from the the results in Ref. [37].

Lemma 1. *Let $\mathcal{P}$, ϕ, h, and Θ be given as in Theorem 1. Then, there exists a constant $C_p > 0$ such that for $f \in L^p(\mathbb{R}^{d+1})$ with $p > \gamma'$, we have*

$$\|\mho_h^*(f)\|_{L^p(\mathbb{R}^{d+1})} \leq C_{p,\Theta,h} \|f\|_{L^p(\mathbb{R}^{d+1})} \tag{3}$$

and

$$\|M_{h,\mu}(f)\|_{L^p(\mathbb{R}^{d+1})} \leq C_{p,\Theta,h} (\ln \mu) \|f\|_{L^p(\mathbb{R}^{d+1})}. \tag{4}$$

Proof. It is easy to check that Hölder's inequality gives that

$$\begin{aligned}||\mho_l| * f(\tilde{x})| &\leq C \|\Theta\|_{L^1(\mathbb{S}^{m-1})}^{1/\gamma'} \|h\|_{\nabla_\gamma(\mathbb{R}_+)}\\ &\times \left(\frac{1}{l} \int_{l/2}^{l} \int_{\mathbb{S}^{m-1}} |\Theta(w)| |f(x - \mathcal{P}(tw), x_{d+1} - \phi(t))|^{\gamma'} d\sigma_m(w) dt \right)^{1/\gamma'}.\end{aligned}$$

By Minkowski's inequality we have

$$\|\mho_h^*(f)\|_{L^p(\mathbb{R}^{d+1})} \leq C \|\Theta\|_{L^1(\mathbb{S}^{m-1})}^{1/\gamma'} \|h\|_{\nabla_\gamma(\mathbb{R}^+)} \left(\|\Upsilon_l^*(|f|^{\gamma'})\|_{L^{(p/\gamma')}(\mathbb{R}^{d+1})} \right)^{1/\gamma'}, \tag{5}$$

where

$$\int_{\mathbb{R}^{d+1}} f d\Upsilon_l = \frac{1}{l^\alpha} \int_{1/2l \leq |w| \leq l} f(\mathcal{P}(w), \phi(|w|)) \frac{\Theta(w)}{|w|^{m-\alpha}} dw$$

and

$$\mathrm{Y}^{*}_{\Theta}(f) \quad = \quad \sup_{l\in\mathbb{R}_{+}} ||\mathrm{Y}_{l}| * f|.$$

It is clear that

$$\mathrm{Y}^{*}_{\Theta}(f)(\tilde{x}) \quad \leq \quad 2W^{*}(f)(\tilde{x}), \tag{6}$$

where

$$W^{*}(f)(\tilde{x}) \quad = \quad \left(\sup_{j\in\mathbb{Z}} \int_{2^{j}\leq|w|\leq 2^{j+1}} |f(x-\mathcal{P}(w), x_{d+1}-\phi(|w|))| \frac{|\Theta(w)|}{|w|^{m}} dw\right).$$

By Theorem 1.1 in [37], we deduce that

$$\|W^{*}(f)\|_{L^{p}(\mathbb{R}^{d+1})} \quad \leq \quad C_{p}\|\Theta\|_{L^{1}(\mathbb{S}^{m-1})}\|f\|_{L^{p}(\mathbb{R}^{d+1})} \tag{7}$$

for all $1 < p < \infty$. Thus, by (5)–(7), we prove the inequality (3) which directly gives (4). The proof of this lemma is complete. □

Remark 2. *Let $0 < m_1 < m_2 < \cdots < m_M$ be non-negative integers. Then, for any $w \in \mathbb{R}^m$, we can write $\mathcal{P}(w) = \sum_{r=1}^{M} \mathcal{P}^{(r)}(w) + \mathcal{R}^{(r)}(|w|)$, where $\mathcal{P}^{(r)}(w) = (P_1^r(w), P_2^r(w), \ldots, P_d^r(w))$, $\{P_\nu^r(w) : 1 \leq \nu \leq d, 1 \leq r \leq M\}$ are real-valued homogeneous polynomials of degree m_r with $|w|^{m_r} \notin span\{P_1^r, \ldots, P_d^r\}$, $\mathcal{R}^{(r)}(l) = (\mathcal{R}_1^{(r)}(l), \mathcal{R}_2^{(r)}(l), \ldots, \mathcal{R}_d^{(r)}(l))$, and $\{\mathcal{R}_\nu^{(r)}(l) : 1 \leq \nu \leq d, 1 \leq r \leq M\}$ are polynomials on $\mathbb{R}$ of a degree less than m_r. Let τ_r denote the number of elements of $\{\beta = (\beta_1, \beta_2 \ldots, \beta_m) \in (\mathbb{N} \cup \{0\})^m : |\beta| = m_r\} = \{\beta(1), \beta(2), \ldots, \beta(\tau_r)\}$. Write $P_k^r(w) = \sum_{s=1}^{\tau_r} a_{s,k} w^{\beta(s)}$, and define the linear mapping $L_r : \mathbb{R}^d \to \mathbb{R}^{\tau_r}$ by $L_r(\zeta) = \left(\sum_{k=1}^{d} a_{1,k}^r \zeta_k, \ldots, \sum_{k=1}^{d} a_{\tau_r,k}^r \zeta_k\right)$. For $1 \leq r \leq M$, set $\boldsymbol{P}_r(w) = \sum_{k=1}^{r} \mathcal{P}^{(k)}(w) + \mathcal{W}(|w|)$ and $\boldsymbol{P}_0(w) = \mathcal{W}(|w|)$. Hence, we have $\mathcal{P}(w) = \boldsymbol{P}_M(w)$. For $1 \leq r \leq M$, we let $\mho_l^{(r)} = \mho_{\Theta,\boldsymbol{P}_r,\phi,h,l}$ and $\mho_h^{(r)*} f(\tilde{x}) = \sup_{l\in\mathbb{R}^+} ||\mho_l^{(r)}| * f(\tilde{x})|$.*

Now, we have the following result concerning the measures $\mho_l^{(r)}$:

Lemma 2. *Let $\mathcal{P}$, ϕ, Θ, and h be given as in Theorem 1. Let $\{\mho_l^{(r)} : l \in \mathbb{R}^+, 1 \leq r \leq M\}$ be a family of Borel measures defined as in Remark 2. Then, for $\mu \geq 2$, there exist positive constants δ_r and C such that*

$$\left(\int_{\mu^j}^{\mu^{j+1}} \left|\hat{\mho}_l^{(r)}(\zeta, \zeta_{m+1})\right|^2 \frac{dl}{l}\right)^{1/2} \leq C_{\Theta,h}(\ln\mu)^{1/2}, \tag{8}$$

$$\left(\int_{\mu^j}^{\mu^{j+1}} \left|\hat{\mho}_l^{(r)}(\zeta, \zeta_{m+1})\right|^2 \frac{dl}{l}\right)^{1/2} \leq C_{\Theta,h}(\ln\mu)^{1/2}\left(\mu^{jm_r}|L_r(\zeta)|\right)^{-\frac{1}{4m_r\ln\mu}}, \tag{9}$$

$$\left(\int_{\mu^j}^{\mu^{j+1}} \left|\hat{\mho}_l^{(r)}(\zeta, \zeta_{m+1}) - \hat{\mho}_l^{(r-1)}(\zeta, \zeta_{m+1})\right|^2 \frac{dl}{l}\right)^{1/2} \leq C_{\Theta,h}(\ln\mu)^{1/2}\left(\mu^{jm_r}|L_r(\zeta)|\right)^{\frac{1}{4m_r\ln\mu}}, \tag{10}$$

where $C_{\Theta,h} = C\|\Theta\|_{L^q(\mathbb{S}^{m-1})}\|h\|_{\nabla_\gamma(\mathbb{R}^+)}$.

Proof. By the definition of $\mho_l^{(r)}$, it is easy to obtain (8). In addition, the same arguments as in Proposition 5.1 in [38] lead to (9). By a simple change of variables, we obtain

$$\left(\int_{\mu^j}^{\mu^{j+1}}\left|\hat{\mho}_l^{(r)}(\zeta,\zeta_{m+1})-\hat{\mho}_l^{(r-1)}(\zeta,\zeta_{m+1})\right|^2\frac{dl}{l}\right)^{1/2}\leq C_{\Theta,h}(\ln\mu)^{1/2}\Big(\mu^{jm_r}|L_r(\zeta)|\Big) \tag{11}$$

which when combined with the trivial estimate (8), we conclude that

$$\left(\int_{\mu^j}^{\mu^{j+1}}\left|\hat{\mho}_l^{(r)}(\zeta,\zeta_{m+1})-\hat{\mho}_l^{(r-1)}(\zeta,\zeta_{m+1})\right|^2\frac{dl}{l}\right)^{1/2}\leq C_{\Theta,h}(\ln\mu)^{1/2}\Big(\mu^{jm_r}|L_r(\zeta)|\Big)^{\frac{1}{4m_r\ln\mu}}. \tag{12}$$

This concludes the proof of the lemma. □

By employing similar arguments as that employed in [38], we obtain the following:

Lemma 3. *Let $\mathcal{P}$ and ϕ be given as in Theorem 1, and let $\mu \geq 2$, $h \in \nabla_\gamma(\mathbb{R}^+)$ with $\gamma > 1$ and $\Theta \in L^q\big(\mathbb{S}^{\alpha-1}\big)$ with $1 < q \leq 2$. Then, there exists $C_{p,\Theta,h} > 0$ such that*

$$\left\|\left(\sum_{j\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}|\mho_l * \mathcal{T}_j|^2\frac{dl}{l}\right)^{1/2}\right\|_{L^p(\mathbb{R}^{d+1})} \leq C_{p,\Theta,h}(\ln\mu)^{1/2}\left\|\left(\sum_{j\in\mathbb{Z}}|\mathcal{T}_j|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^{d+1})} \tag{13}$$

for all $|1/p-1/2| < \min\{1/\gamma', 1/2\}$, where $\{\mathcal{T}_j(\cdot,\cdot),\ j\in\mathbb{Z}\}$ is any set of functions on $\mathbb{R}^{d+1}$.

Proof. Since $\nabla_\gamma(\mathbb{R}^+) \subseteq \nabla_2(\mathbb{R}^+)$ for any $\gamma \geq 2$, it suffices to prove this lemma only for the case $\gamma \in (1,2]$. In this case, we have $|1/p-1/2| < 1/\gamma'$, which means that $p \in (\frac{2\gamma}{3\gamma-2}, \frac{2\gamma}{2-\gamma})$. First, if $p \in [2, \frac{2\gamma}{2-\gamma})$, then by duality there exists a function $\mathcal{G} \in L^{(p/2)'}(\mathbb{R}^{d+1})$ such that $\|\mathcal{G}\|_{L^{(p/2)'}(\mathbb{R}^{d+1})} \leq 1$ and

$$\left\|\left(\sum_{j\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}|\mho_l * \mathcal{T}_j|^2\frac{dl}{l}\right)^{1/2}\right\|^2_{L^p(\mathbb{R}^{d+1})} = \int_{\mathbb{R}^{d+1}}\sum_{j\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}|\mho_l * \mathcal{T}_j(\tilde{x})|^2\frac{dl}{l}|\mathcal{G}(\tilde{x})|d\tilde{x}.$$

Employing Schwartz's inequality, we deduce that

$$\begin{aligned}|\mho_l * \mathcal{T}_j(\tilde{w})|^2 &\leq C\|\Theta\|_{L^q(\mathbb{S}^{m-1})}\|h\|^\gamma_{\nabla_\gamma(\mathbb{R}^+)}\int_{\frac{1}{2}l}^{l}\int_{\mathbb{S}^{m-1}}|\Theta(w)|\\ &\times |\mathcal{T}_j(x-\mathcal{P}(tw), x_{d+1}-\phi(t))|^2|h(t)|^{2-\gamma}d\sigma_m(w)\frac{dt}{t}.\end{aligned}$$

Thanks to Hölder's inequality and Lemma 1, we obtain

$$\left\|\left(\sum_{j\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}|\mho_l * \mathcal{T}_j|^2\frac{dl}{l}\right)^{1/2}\right\|^2_{L^p(\mathbb{R}^{d+1})}$$

$$\leq C\|\Theta\|_{L^q(\mathbb{S}^{m-1})}\|h\|^\gamma_{\nabla_\gamma(\mathbb{R}^+)}\left\|\sum_{j\in\mathbb{Z}}|\mathcal{T}_j|^2\right\|_{L^{(p/2)}(\mathbb{R}^{d+1})}\left\|M_{|h|^{2-\gamma},\mu}(\overline{\mathcal{G}})\right\|_{L^{(p/2)'}(\mathbb{R}^{d+1})}$$

$$\begin{aligned}
&\leq C(\ln\mu)\|\Theta\|_{L^q(\mathbb{S}^{m-1})}\|h\|^{\gamma}_{\nabla_\gamma(\mathbb{R}_+)}\left\|\left(\sum_{j\in\mathbb{Z}}|\mathcal{T}_j|^2\right)^{1/2}\right\|^2_{L^p(\mathbb{R}^{d+1})}\left\|\mathcal{U}^*_{|h|^{2-\gamma}}(\overline{\mathcal{G}})\right\|_{L^{(p/2)'}(\mathbb{R}^{d+1})}\\
&\leq C^2_{p,\Theta,h}(\ln\mu)\left\|\left(\sum_{j\in\mathbb{Z}}|\mathcal{T}_j|^2\right)^{1/2}\right\|^2_{L^p(\mathbb{R}^{d+1})}, \qquad (14)
\end{aligned}$$

where $\overline{\mathcal{G}}(\tilde{x}) = \mathcal{G}(-\tilde{x})$.

Now, if $p \in (\frac{2\gamma}{3\gamma-2}, 2)$, by duality, there exists a class of functions $\{g_j(\tilde{x}, l)\}$ on $\mathbb{R}^{d+1} \times \mathbb{R}^+$ such that

$$\left\|\left\|\|g_j\|_{L^2([\mu^j,\mu^{j+1}],\frac{dl}{l})}\right\|_{l^2}\right\|_{L^{p'}(\mathbb{R}^{d+1})} \leq 1$$

and

$$\begin{aligned}
&\left\|\left(\sum_{j\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}|\mathcal{U}_l * \mathcal{T}_j|^2\frac{dl}{l}\right)^{1/2}\right\|_{L^p(\mathbb{R}^{d+1})} = \int_{\mathbb{R}^{d+1}}\sum_{j\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}(\mathcal{U}_l * \mathcal{T}_j(\tilde{x}))g_j(\tilde{x}, l)\frac{dl}{l}d\tilde{x}\\
&\leq C_p(\ln\mu)^{1/2}\left\|\mathcal{H}(g_j)\right\|^{1/2}_{L^{(p'/2)}(\mathbb{R}^{d+1})}\left\|\left(\sum_{j\in\mathbb{Z}}|\mathcal{T}_j|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^{d+1})}, \qquad (15)
\end{aligned}$$

where

$$\mathcal{H}(g_j)(\tilde{x}) = \sum_{j\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}|\mathcal{U}_l * g_j(\tilde{x}, l)|^2\frac{dl}{l}.$$

Since $p < 2$ we have $p' > 2$. Hence, by duality, there exists a function $\mathcal{V} \in L^{(p'/2)'}(\mathbb{R}^{d+1})$ satisfying $\|\mathcal{V}\|_{L^{(p'/2)'}(\mathbb{R}^{d+1})} \leq 1$ and

$$\begin{aligned}
&\left\|\mathcal{H}(g_j)\right\|_{L^{(p'/2)}(\mathbb{R}^{d+1})} = \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^{d+1}}\int_{\mu^j}^{\mu^{j+1}}|\mathcal{U}_l * g_j(\tilde{x}, l)|^2\frac{dl}{l}\mathcal{V}(\tilde{x})d\tilde{x}\\
&\leq C\|\Theta\|_{L^q(\mathbb{S}^{m-1})}\left\|\left(\sum_{j\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}|g_j(\tilde{x}, l)|^2\frac{dl}{l}\right)\right\|_{L^{(p'/2)}(\mathbb{R}^{d+1})}\\
&\times \|h\|^{\gamma}_{\nabla_\gamma(\mathbb{R}^+)}\left\|\mathcal{U}^*_{|g|^{2-\gamma}}(\mathcal{V})\right\|_{L^{(p'/2)'}(\mathbb{R}^{d+1})} \leq C^2_{p,\Theta,h}. \qquad (16)
\end{aligned}$$

By the inequalities (15) and (16), we obtain (13) if $p \in (\frac{2\gamma}{3\gamma-2}, 2)$, which, in turn, along with (14) concludes the proof of the lemma. □

3. Proof of Theorem 1

Let $\Theta \in L^q(\mathbb{S}^{m-1})$ for some $1 < q \leq 2$ and $h \in \nabla_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$. Set $\mu = 2^{\gamma' q'}$. By Minkowski's inequality, we obtain

$$\begin{aligned}
&\mathcal{M}_{\Theta,\mathcal{P},\phi,h}(f)(\tilde{x})\\
&\leq \sum_{j=0}^{\infty}\left(\int_{\mathbb{R}^+}\left|\frac{1}{l^\alpha}\int_{2^{-j-1}l<|w|\leq 2^{-j}l} f(x-\mathcal{P}(w), x_{d+1}-\phi(|w|))\frac{\Theta(w)}{|w|^{m-\alpha}}h(|w|)dw\right|^2\frac{dt}{t}\right)^{1/2}\\
&\leq \frac{2^\tau}{(2^\tau-1)}\left(\int_{\mathbb{R}^+}|\mathcal{U}_l * f(\tilde{x})|^2\frac{dt}{t}\right)^{1/2} = C\left(\int_{\mathbb{R}^+}\left|\mathcal{U}^{(M)}_l * f(\tilde{x})\right|^2\frac{dt}{t}\right)^{1/2}. \qquad (17)
\end{aligned}$$

For $j \in \mathbb{Z}$, let $\{\mathcal{A}_j\}$ be a collection of $C^\infty((0,\infty))$ functions satisfying the following:

$$\begin{aligned} 0 &\leq \mathcal{A}_j \leq 1\,, \ \sum_{j\in\mathbb{Z}} \mathcal{A}_j(l) = 1, \\ \text{supp}\,(\mathcal{A}_j) &\subseteq [\mu^{-j-1}, \mu^{-j+1}], \ \text{and} \ \left|\frac{d^n \mathcal{A}_j(l)}{dl^n}\right| \leq \frac{C_n}{l^n}, \end{aligned}$$

where C_n is independent of $\{\mu^j; j \in \mathbb{Z}\}$. Define the operator $\widehat{T_j(f)}(\zeta, \zeta_{d+1}) = \mathcal{A}_j(|L_M(\zeta)|)$ $\hat{f}(\zeta, \zeta_{d+1})$. Thus, for any $f \in C_0^\infty(\mathbb{R}^{d+1})$, Minkowski's inequality yields

$$\left(\int_{\mathbb{R}^+} \left|\mho_l^{(M)} * f(\tilde{x})\right|^2 \frac{dt}{t}\right)^{1/2} \leq C \sum_{s\in\mathbb{Z}} \mathcal{F}_s(f)(\tilde{x}), \tag{18}$$

where

$$\mathcal{F}_s(f)(\tilde{x}) = \left(\int_{\mathbb{R}^+} |\mathcal{J}_s(f)(\tilde{x}, t)|^2 \frac{dt}{t}\right)^{1/2},$$

$$\mathcal{J}_s(f)(\tilde{x}, t) = \sum_{j\in\mathbb{Z}} \mho_l^{(M)} * T_{j+s} * f(\tilde{x}) \chi_{_{[\mu^j, \mu^{j+1})}}(t).$$

Thus, to prove Theorem 1, it suffices to show that

$$\|\mathcal{F}_s(f)\|_{L^p(\mathbb{R}^{d+1})} \leq C_{p,\Theta,h} (\ln \mu)^{1/2}\, 2^{-\frac{\varepsilon|s|}{2}} \|f\|_{L^p(\mathbb{R}^{d+1})}. \tag{19}$$

for all $|1/p - 1/2| < \min\{1/\gamma', 1/2\}$ and for some $\varepsilon > 0$.

First, we estimate $\|\mathcal{F}_s(f)\|_{L^2(\mathbb{R}^{d+1})}$ as follows: By Plancherel's Theorem, Fubini's Theorem, and Lemma 2, we obtain

$$\begin{aligned} \|\mathcal{F}_s(f)\|^2_{L^2(\mathbb{R}^{d+1})} &\leq \sum_{j\in\mathbb{Z}} \int_{\mathcal{O}_{j+s}} \left(\int_{\mu^j}^{\mu^{j+1}} \left|\hat{\mho}_l^{(M)}(\zeta, \zeta_{d+1})\right|^2 \frac{dt}{t}\right) \left|\hat{f}(\zeta, \zeta_{d+1})\right|^2 d\zeta d\zeta_{d+1} \\ &\leq C^2_{p,\Theta,h} (\ln \mu) \sum_{j\in\mathbb{Z}} \int_{\mathcal{O}_{j+s}} \left|\mu^{j m_M} L_M(\zeta)\right|^{\pm \frac{\varepsilon}{q'\gamma'}} \left|\hat{f}(\zeta, \zeta_{d+1})\right|^2 d\zeta d\zeta_{d+1} \\ &\leq C^2_{p,\Theta,h} (\ln \mu)\, 2^{-\varepsilon|s|} \sum_{j\in\mathbb{Z}} \int_{\mathcal{O}_{j+s}} \left|\hat{f}(\zeta, \zeta_{d+1})\right|^2 d\zeta d\zeta_{d+1} \\ &\leq C^2_{p,\Theta,h} (\ln \mu)\, 2^{-\varepsilon|s|} \|f\|^2_{L^2(\mathbb{R}^{d+1})}, \end{aligned} \tag{20}$$

where $\mathcal{O}_j = \left\{(\zeta, \zeta_{d+1}) \in \mathbb{R}^d \times \mathbb{R} : |L_M(\zeta)| \in [\mu^{-j-1}, \mu^{-j+1}]\right\}$ and $\varepsilon \in (0, 1)$.

Now, let us estimate $\|\mathcal{F}_s(f)\|_{L^p(\mathbb{R}^{d+1})}$. By utilizing Lemma 3 and the Littlewood–Paley theory, we deduce

$$\begin{aligned} \|\mathcal{F}_s(f)\|_{L^p(\mathbb{R}^{d+1})} &\leq C \left\| \left(\sum_{j\in\mathbb{Z}} \int_{\mu^j}^{\mu^{j+1}} \left(\left|\mho_l^{(M)} * T_{j+s} * f\right|\right)^2 \frac{dl}{l}\right)^{1/2} \right\|_{L^p(\mathbb{R}^{d+1})} \\ &\leq C_{p,\Theta,h} (\ln \mu)^{1/2} \left\| \left(\sum_{j\in\mathbb{Z}} \left|T_{j+s} * f\right|^2\right)^{1/2} \right\|_{L^p(\mathbb{R}^{d+1})} \\ &\leq C_{p,\Theta,h} \left(\frac{\gamma}{(\gamma-1)(q-1)}\right)^{1/2} \|f\|_{L^p(\mathbb{R}^{d+1})}. \end{aligned} \tag{21}$$

Consequently, by interpolating between (20) with (21), we obtain (19), which, in turn, along with (17) and (18) completes the proof of Theorem 1.

4. Conclusions

In this paper, we obtained sharp L^p bounds for parametric Marcinkiewicz integrals $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$, whenever their kernel functions belong to the space $L^q(\mathbb{S}^{m-1})$ for some $q > 1$. These estimates allow us to employ Yano's extrapolation argument to prove the L^p boundedness of $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$ whenever the singular kernels functions Θ are either in the space $L(\log L)^{1/2}(\mathbb{S}^{m-1})$ or in the space $B_q^{(0,-1/2)}(\mathbb{S}^{m-1})$ for some $q > 1$. Our results improve or extend several known results as those in [1,4–7,9–12,15]. In future work, we aim to prove the boundedness of $\mathcal{M}_{\Theta,\mathcal{P},\phi,h}$ on the weighted spaces $L^p(\omega, \mathbb{R}^{d+1})$ for a wider range of p, provided that $\Theta \in L(\log L)^{1/2}(\mathbb{S}^{m-1}) \cup B_q^{(0,-1/2)}(\mathbb{S}^{m-1})$ and $h \in \nabla_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$.

Author Contributions: Formal analysis and writing—original draft preparation: M.A. and H.A.-Q. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflict of interest.

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