

Article

Inequalities with Some Arithmetic Functions

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Abstract: In the paper, some new inequalities are formulated and proved with the classical arithmetic functions φ (of Euler) and ψ (of Dedekind).**Keywords:** arithmetic functions; inequalities; Euler function; Dedekind function**MSC:** 11A25; 26D15

1. Introduction

The Euler function φ and the Dedekind function ψ are two important arithmetic functions in number theory, and certainly two of the most popular examples of multiplicative functions. Inequalities involving these functions have several significant applications in various areas of mathematics like Prime Number Theorem and the distribution of primes, analytic number theory, cryptography and number-theoretic algorithms, among others. Many connections to various problems of number theory and applied mathematics are further discussed in [1]. For example, the famous Riemann hypothesis is equivalent to an inequality for the sum of divisor functions.

For a positive integer $n = \prod_{i=1}^r p_i^{\alpha_i} > 1$, let $\varphi(n)$ and $\psi(n)$, respectively, denote the Euler and Dedekind totient function values, i.e.,

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) = \prod_{i=1}^r p_i^{\alpha_i-1} (p_i - 1), \quad (1)$$

$$\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right) = \prod_{i=1}^r p_i^{\alpha_i-1} (p_i + 1), \quad (2)$$

where p runs through the prime divisors of n , and for any i ($1 \leq i \leq r$), p_i are different primes and α_i are positive integers (see [1]).

Let

$$\varphi(1) = \psi(1) = 1.$$

Another arithmetic function which will be used is the “core” function of n :

$$\gamma(n) = \prod_{p|n} p = \prod_{i=1}^r p_i. \quad (3)$$

Let us also denote

$$\omega(n) = r,$$



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i.e., the number of the distinct prime factors of n , and

$$\Omega(n) = \sum_{i=1}^r \alpha_i,$$

i.e., the total number of prime factors of n (see [1]).

The aim of this paper is to obtain certain new inequalities for these functions.

2. Main Results

Theorem 1. For $n > 1$, one has

$$\psi(n) - \varphi(n) \geq 2^{\omega(n)} \frac{n}{\gamma(n)} > 2^{\omega(n)} \frac{\varphi(n)}{\gamma(n)}. \quad (4)$$

Proof. The first inequality of (4) is proved in [2], where the arithmetic inequality

$$\prod_{i=1}^r (y_i + 1) - \prod_{i=1}^r (y_i - 1) \geq 2^r$$

is used for $y_i \geq 2$. By putting $y_i = p_i$ ($1 \leq i \leq r$) and using (1) and (2), the result follows. The second inequality of (4) follows by the classical inequality $n > \varphi(n)$ for $n \geq 2$. $\square$

Example 1. For example, when $n = 12$, we obtain

$$\begin{aligned} \psi(12) - \varphi(12) &= 24 - 4 = 20 \\ &\geq 8 = 2^2 \frac{12}{6} = 2^{\omega(12)} \frac{12}{\gamma(12)} \\ &> \frac{8}{3} = 2^2 \frac{4}{6} \\ &= 2^{\omega(12)} \frac{\varphi(12)}{\gamma(n)}. \end{aligned}$$

Corollary 1. For $n > 1$,

$$\frac{\psi(n)}{\varphi(n)} - 1 > \frac{2^{\omega(n)}}{\gamma(n)}. \quad (5)$$

This follows by the weaker inequality in (4), by dividing both terms with $\varphi(n)$.

Theorem 2. For $n > 1$, one has

$$\frac{\psi(n)}{n} \geq 1 + \frac{\omega(n)}{\gamma(n)} \geq \frac{\varphi(n)}{n} + \frac{2^{\omega(n)}}{\gamma(n)}. \quad (6)$$

Proof. By (1)–(3), the first inequality of (6) can be rewritten as

$$\prod_{i=1}^r (p_i + 1) \geq \left(\prod_{i=1}^r p_i \right) \left(1 + \frac{r}{\prod_{i=1}^r p_i} \right) = \prod_{i=1}^r p_i + r = \gamma(n) + r. \quad (7)$$

Relation (7) follows from the fact that

$$\prod_{i=1}^r (p_i + 1) \geq \prod_{i=1}^r p_i + \sum_{i=1}^r p_i \geq \prod_{i=1}^r p_i + r$$

with an equality only for $r = 1$. The second inequality of (6) can be rewritten as

$$\prod_{i=1}^r p_i - \prod_{i=1}^r (p_i - 1) + r - 2^r \geq 0. \quad (8)$$

For $r = 1$, the statement (8) is obvious; for $r = 2$, we obtain

$$p_1 p_2 - (p_1 - 1)(p_2 - 1) + 2 - 4 = p_1 + p_2 - 3 > 0.$$

For $r \geq 3$, we have that $p_1 \geq 2, p_2 \geq 3, p_3, \dots, p_r \geq 5$, and

$$\begin{aligned} \prod_{i=1}^r p_i - \prod_{i=1}^r (p_i - 1) + r - 2^r &> \prod_{i=1}^r (p_i - 1) \left(\prod_{i=1}^r \left(1 + \frac{1}{p_i - 1} \right) - \frac{2^r}{\prod_{i=1}^r (p_i - 1)} \right) \\ &\geq 8 \prod_{i=4}^r (p_i - 1) \left(1 - \frac{2^r}{8 \prod_{i=4}^r (p_i - 1)} \right) \geq 0, \end{aligned}$$

which proves the theorem. $\square$

We can note that relation (6) offers an improvement of the first inequality of (4).

Example 2. For example, when $n = 12$, we obtain

$$\begin{aligned} \frac{\psi(12)}{12} &= \frac{24}{12} = 2 \\ &> 1 + \frac{2}{6} = 1 + \frac{\omega(12)}{\gamma(12)} \\ &> 1 = \frac{4}{12} + \frac{4}{6} \\ &= \frac{\varphi(12)}{12} + \frac{2^{\omega(12)}}{\gamma(12)}. \end{aligned}$$

Corollary 2. For $n > 1$

$$\varphi(n) \leq n - \frac{n\omega(n)}{\gamma(n)}. \quad (9)$$

Indeed, the second inequality of (6) can be written as

$$\varphi(n) \leq n - \frac{(2^{\omega(n)} - \omega(n))n}{\gamma(n)}. \quad (10)$$

Obviously, relation (10) is stronger than (9), as $2^r - r \geq r$, i.e., $2^r \geq 2r$ for each $r \geq 1$. Relation (9) can be rewritten also as

$$\frac{n}{\varphi(n)} \geq \frac{\gamma(n)}{\gamma(n) - \omega(n)}.$$

The following refinement of this inequality holds true.

Theorem 3. For $n > 1$ such that $\omega(n) \geq 2$,

$$\frac{n}{\varphi(n)} \geq 1 + \frac{2\omega(n)}{\gamma(n)} \geq \frac{\gamma(n)}{\gamma(n) - \omega(n)}. \quad (11)$$

Proof. The first inequality of (11) can be written as

$$\prod_{i=1}^r p_i \left(\prod_{i=1}^r p_i - \prod_{i=1}^r (p_i - 1) \right) \geq 2r \prod_{i=1}^r (p_i - 1). \quad (12)$$

Obviously, for $r = 1$, (12) is valid only for $p_1 = 2$.

Let $r = \omega(n) \geq 2$. As

$$\prod_{i=1}^r p_i > \prod_{i=1}^r (p_i - 1),$$

it will be sufficient to prove that

$$\prod_{i=1}^r p_i - \prod_{i=1}^r (p_i - 1) > 2r.$$

Since $p_1 \geq 2, p_2 \geq 3$, we obtain that

$$\begin{aligned} \prod_{i=1}^r p_i - \prod_{i=1}^r (p_i - 1) - 2r &\geq 6 \prod_{i=3}^r p_i - 2 \prod_{i=3}^r (p_i - 1) - 2r \\ &> 4 \prod_{i=3}^r p_i - 2r \begin{cases} = 0, & \text{if } r = 2 \\ > 0, & \text{if } r \geq 3 \end{cases}. \end{aligned}$$

The second inequality of (11) can be written as

$$\gamma(n)^2 \leq \gamma(n) + 2\omega(n)\gamma(n) - \omega(n)\gamma(n) - 2\omega(n)^2$$

or

$$\gamma(n) \geq 2\omega(n),$$

which is true because

$$\gamma(n) = \prod_{i=1}^r p_i \geq 2^r \geq 2r. \quad \square$$

Example 3. For example, when $n = 12$, we obtain

$$\begin{aligned} \frac{12}{\varphi(12)} &= \frac{12}{4} = 3 \\ &> 1 + \frac{4}{6} = 1 + \frac{2^{\omega(12)}}{\gamma(12)} \\ &> \frac{6}{4} \\ &= \frac{\gamma(12)}{\gamma(12) - \omega(12)}. \end{aligned}$$

Theorem 4. For $n > 1$,

$$\psi(n) \geq \frac{n}{2} \left(1 + \frac{n}{\varphi(n)} \right) \geq n \left(1 + \frac{\omega(n)}{\gamma(n)} \right). \quad (13)$$

Proof. From the first inequality of (11), one has

$$1 + \frac{n}{\varphi(n)} \geq 2 \left(1 + \frac{\omega(n)}{\gamma(n)} \right),$$

so the second inequality of (13) follows.

The first inequality of (13) is due to Ch. R. Wall but without a proof; it has been proved in [3] in the form

$$\frac{2\psi(n)}{n} \geq 1 + \frac{n}{\varphi(n)}. \quad \square$$

Example 4. For example, when $n = 12$, we obtain

$$\begin{aligned} \psi(12) &= 24 \\ &= 6 \left(1 + \frac{12}{4} \right) = \frac{n}{2} \left(1 + \frac{12}{\varphi(12)} \right) \\ &> 16 = 12 \left(1 + \frac{2}{6} \right) \\ &= 12 \left(1 + \frac{\omega(12)}{\gamma(12)} \right). \end{aligned}$$

Another, independent, result is contained in the following Theorem 5.

Theorem 5. For $n > 1$

$$\frac{n\omega(n)}{\gamma(n)} \leq n - \varphi(n) \leq \psi(n) - n. \quad (14)$$

Proof. The first inequality of (14) is exactly Corollary 2 (see (9)). The second inequality, written in the form

$$\varphi(n) + \psi(n) \geq 2n$$

is due to Ch. R. Wall [4]. $\square$

Example 5. For example, when $n = 12$, we obtain

$$\begin{aligned} \frac{12\omega(12)}{\gamma(12)} &= \frac{24}{6} = 4 \\ &< 8 = 12 - \varphi(12) \\ &< 12 = 24 - 12 \\ &= \psi(12) - 12. \end{aligned}$$

Remark 1. The last inequality of Theorem 5 gives $\psi(n) \geq 2n - \varphi(n)$. We will show that this cannot be compared to the first inequality of Theorem 4, i.e., $\psi(n) \geq \frac{n}{2} \left(1 + \frac{n}{\varphi(n)} \right)$. Suppose that one has the inequality $\psi(n) \geq 2n - \varphi(n) \geq \frac{n}{2} \left(1 + \frac{n}{\varphi(n)} \right)$. Then, the last inequality gives $\frac{3n}{2} \geq \frac{\varphi(n)}{n} + \frac{n}{2\varphi(n)}$. Let $\frac{\varphi(n)}{n} = x$. Then, we obtain $2x^2 + 1 \leq 3x$, or $(x-1)(2x-1) \leq 0$. This inequality is not true, as $x-1 \leq 0$, but $2x-1$ can have positive and negative signs infinitely often. Also, clearly $(x-1)(2x-1) \geq 0$ is also not true. Thus, while the weaker inequalities in both theorems are the same, the results of Theorems 4 and 5 are independent of each other.

Remark 2. We must mention that the relation (14) refines the first inequality of (6).

Theorem 6. For $n > 1$, one has

$$\psi(n) \geq n + 2^{\omega(n)+\Omega(n)-2} \geq n + 1. \quad (15)$$

Proof. By using (2) and the definitions of $\omega(n)$ and $\Omega(n)$, one has that

$$\psi(n) - n = \prod_{i=1}^r p_i^{\alpha_i-1} \left(\prod_{i=1}^r (p_i + 1) - \prod_{i=1}^r p_i \right).$$

Since

$$\prod_{i=1}^r p_i^{\alpha_i-1} \geq 2^{\sum_{i=1}^r \alpha_i - r} = 2^{\Omega(n) - \omega(n)},$$

then in order to prove (15), we have to prove that

$$\prod_{i=1}^r (p_i + 1) - \prod_{i=1}^r p_i \geq 2^{2r-2}. \quad (16)$$

For $r = 1$ in (16), there is an equality, while for $r = 2$, one has

$$\begin{aligned} (p_1 + 1)(p_2 + 1) - p_1 p_2 &= p_1 + p_2 + 1 \\ &\geq 2 + 3 + 1 \\ &= 6 > 4 \\ &= 2^2. \end{aligned}$$

We can prove that there is a strict inequality in (16) for $r \geq 2$. Having in mind that $p_1 \geq 2, p_2 \geq 3$ for each i ($3 \leq i \leq r$): $p_i \geq 5$, we obtain

$$\begin{aligned} \prod_{i=1}^r (p_i + 1) - \prod_{i=1}^r p_i - 2^{2r-2} &\geq 12 \prod_{i=3}^r (p_i + 1) - 6 \prod_{i=3}^r p_i - 2^{2r-2} \\ &> 6 \prod_{i=3}^r (p_i + 1) - 2^{2r-2} \\ &\geq 6 \times 6^{r-2} - 2^{2r-2} \\ &= 6^{r-1} - 4^{r-1} \\ &> 0, \end{aligned}$$

which proves (16) and therefore (15).

From the above proof, it follows that there is equality in (15) only for $n = 2^\alpha$ for $\alpha = 1, 2, \dots$ or for n being a prime number.

The second inequality of (15) follows from $\Omega(n) + \omega(n) \geq 2$. $\square$

Example 6. For example, when $n = 12$, we obtain

$$\begin{aligned} \psi(12) &= 24 \\ &> 20 = 12 + 8 \\ &= 12 + 2^{m(12) + \Omega(12) - 2} \\ &> 13 \\ &= 12 + 1. \end{aligned}$$

Theorem 7. Let

$$\lambda = \frac{\ln 2}{\ln 3}.$$

Then

$$\varphi(n) \geq n \times \begin{cases} \frac{1}{(\gamma(n))^{1-\lambda}}, & \text{if } n > 1 \text{ is odd} \\ \frac{1}{2^\lambda (\gamma(n))^{1-\lambda}}, & \text{if } n > 1 \text{ is even} \end{cases}. \quad (17)$$

Proof. Let us consider the function

$$f_1(x) = \frac{x-1}{x^\lambda}$$

for $x > 1$. Then, an easy computation gives

$$f_1'(x) = \frac{1}{x^{\lambda+1}}((1-\lambda)x + \lambda) > 0$$

for $0 < \lambda < 1$. Thus, the function f_1 is strictly increasing. Particularly, for $p \geq 3$,

$$f_1(p) \geq f_1(3) = \frac{2}{3^\lambda} = 1$$

for $3^\lambda = 2$, i.e., it is valid only for

$$\lambda = \frac{\ln 2}{\ln 3} = 0.63092\dots$$

Thus, we have the inequality

$$p-1 \geq p^\lambda \quad (18)$$

for $p \geq 3$ with equality only for $p = 3$.

Now, it is well known that

$$\frac{\varphi(n)}{n} = \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

where p runs through the prime divisors of n . Now, for $n > 1$ odd, by (18) we obtain

$$\prod_{p|n} \left(1 - \frac{1}{p}\right) = \frac{\prod_{p|n} (p-1)}{\prod_{p|n} p} \geq \frac{\prod_{p|n} p^\lambda}{\prod_{p|n} p}.$$

Thus, the first inequality of (17) follows. When n is even, then let $p_1 = 2$ be the least prime divisor of n . Then, by (18)

$$\frac{\varphi(n)}{n} = \frac{1}{2} \prod_{p|n, p \geq 3} \left(1 - \frac{1}{p}\right) \geq \frac{1}{\prod_{p|n} p} \left(\frac{\prod_{p|n} p}{2}\right)^\lambda.$$

Thus, the second inequality of (17) follows as well. $\square$

Remark 3. As $\lambda > \frac{1}{2}$ and $\gamma(n) \geq 2$, by (17) we obtain the weaker inequality

$$\varphi(n) \geq n \times \begin{cases} \frac{1}{\sqrt{\gamma(n)}}, & \text{if } n > 1 \text{ is odd,} \\ \frac{1}{\sqrt{2\gamma(n)}}, & \text{if } n > 1 \text{ is even,} \end{cases}$$

proved in [5].

Remark 4. The number $\lambda = \frac{\ln 2}{\ln 3}$ is an irrational number. Indeed, as $0 < \lambda < 1$, it cannot be an integer. If it would be rational, i.e.,

$$\frac{\ln 2}{\ln 3} = \frac{a}{b}$$

for some integers $a, b > 1$, then we would obtain $2^b = 3^a$, which is impossible, as the left side is even and the right side is odd. But λ is even a transcendental number, according to the famous theorem of Gelfond–Schneider [6]. If a and b are algebraic numbers with $a \notin \{0, 1\}$ and b is not rational, then a^b is transcendental. In our case, $3^\lambda = 2$, and since λ is irrational, by the above theorem, if λ were algebraic, we would obtain a contradiction.

Example 7. For example, when $n = 12$, we obtain

$$\varphi(12) = 4 > 3.4642 \dots > \frac{12}{\sqrt{2\gamma(12)}},$$

and when $n = 15$, we obtain

$$\varphi(15) = 8 > 3.8730 \dots > \frac{15}{\sqrt{\gamma(15)}}.$$

Theorem 8. Let

$$\mu = \frac{\ln 4}{\ln 3}.$$

Then

$$\psi(n) \leq n \times \begin{cases} (\gamma(n))^{\mu-1}, & \text{if } n > 1 \text{ is odd} \\ \frac{3}{2^\mu} (\gamma(n))^{\mu-1}, & \text{if } n > 1 \text{ is even} \end{cases}. \quad (19)$$

Proof. Let us define

$$f_2(x) = \frac{x^\mu}{x+1}$$

for $x > 1$. For the derivative of this function, one has

$$f_2'(x) = \frac{x^{\mu-1}}{(x+1)^2} (x(\mu-1) + \mu) > 0,$$

as $\mu > 1$. This f_2 is strictly increasing and implying

$$f_2(p) \geq f_2(3) = \frac{3^\mu}{4} = 1$$

for $p \geq 3$. This implies the inequality

$$p+1 \leq p^\mu \quad (20)$$

for $p \geq 3$, with μ satisfying $\frac{3^\mu}{4}$, i.e., $\mu = \frac{\ln 4}{\ln 3} = 1.26185 \dots$.

Now, the proof of (19) follows by applying (20) in the same manner as in the proof of Theorem 7. $\square$

Remark 5. As $\mu - 1 = 0.26185 \dots < \frac{1}{2}$, it is easy to see from (19) that we obtain the weaker relation

$$\psi(n) \leq n \times \begin{cases} \sqrt{\gamma(n)}, & \text{if } n > 1 \text{ is odd,} \\ \sqrt{2\gamma(n)}, & \text{if } n > 1 \text{ is even.} \end{cases}$$

Example 8. For example, when $n = 12$, we obtain

$$\psi(12) = 24 < 41.0 < 12 \times 3.464 < 12\sqrt{2\gamma(12)},$$

and when $n = 15$, we obtain

$$\psi(15) = 24 < 58.0 < 15 \times 3.872 < 15\sqrt{\gamma(15)}.$$

Remark 6. From the proof of Theorem 7, it follows that there is an equality in the first part of (17) only for $n = 3^k$, where $k \geq 1$ is an integer, and for $n = 2^k$ in the second part. This similarly applies for relation (19).

Remark 7. As $\mu = \frac{2\ln 2}{\ln 3}$, from Remark 4, we obtain that μ is also a transcendental number.

Theorem 9. For each $n > 1$,

$$\varphi(n)\psi(n) \leq n^2 - \omega(n) \left(\frac{n}{\gamma(n)} \right)^2. \quad (21)$$

Proof. When n is prime, (21) is obviously true. Let us assume that (21) is valid for some $n > 1$ with $\omega(n) \geq 2$, and let $p \geq 2$ be not a divisor of n . Then,

$$\begin{aligned} & (np)^2 - \omega(np) \left(\frac{np}{\gamma(np)} \right)^2 - \varphi(np)\psi(np) \\ &= n^2 p^2 - (\omega(n) + 1) \left(\frac{n}{\gamma(n)} \right)^2 - \varphi(n)\psi(n)(p^2 - 1) \\ &\geq n^2 p^2 - (\omega(n) + 1) \left(\frac{n}{\gamma(n)} \right)^2 - \left(n^2 - \omega(n) \left(\frac{n}{\gamma(n)} \right)^2 \right) (p^2 - 1) \\ &= n^2 p^2 - \omega(n) \left(\frac{n}{\gamma(n)} \right)^2 - \left(\frac{n}{\gamma(n)} \right)^2 \\ &\quad - n^2 p^2 + \omega(n) \left(\frac{n}{\gamma(n)} \right)^2 p^2 + n^2 - \omega(n) \left(\frac{n}{\gamma(n)} \right)^2 \\ &= \omega(n) \left(\frac{n}{\gamma(n)} \right)^2 (p^2 - 2) + n^2 - \left(\frac{n}{\gamma(n)} \right)^2 \\ &> 0. \end{aligned}$$

Let $p \geq 2$ be a divisor of n . Then,

$$\begin{aligned} & (np)^2 - \omega(np) \left(\frac{np}{\gamma(np)} \right)^2 - \varphi(np)\psi(np) \\ &= n^2 p^2 - \omega(n) \left(\frac{n}{\gamma(n)} \right)^2 p^2 - \varphi(n)\psi(n)p^2 \\ &\geq n^2 p^2 - \omega(n) \left(\frac{n}{\gamma(n)} \right)^2 p^2 - \left(n^2 - \omega(n) \left(\frac{n}{\gamma(n)} \right)^2 \right) p^2 \\ &= 0, \end{aligned}$$

which proves the theorem.

An alternative proof of the theorem is the following:

$$\frac{\varphi(n)\psi(n)}{n^2} = \left(1 - \left(\frac{1}{p_1} \right)^2 \right) \cdots \left(1 - \left(\frac{1}{p_r} \right)^2 \right) \leq 1 - r \left(\frac{1}{p_1} \cdots \frac{1}{p_r} \right)^2,$$

on the basis of inequality $(x_1 + 1) \cdots (x_r + 1) \geq x_1 \cdots x_r + r$, applied to $x_i = (p_i)^2 - 1$. $\square$

Example 9. For example, when $n = 12$, we obtain

$$\varphi(12)\psi(12) = 4 \times 24 = 96 < 136 = 144 - 2\left(\frac{12}{\gamma(6)}\right)^2 = 12^2 - \omega(12)\left(\frac{12}{\gamma(12)}\right)^2.$$

Remark 8. Let $\sigma(n)$ denote the sum of the divisors of n . By the known inequality for $n > 1$

$$\varphi(n)\sigma(n) < n^2,$$

we obtain from (17) the following relation for $\sigma(n)$:

$$\sigma(n) < n \times \begin{cases} (\gamma(n))^{1-\lambda}, & \text{if } n > 1 \text{ is odd,} \\ \frac{1}{2^\lambda}(\gamma(n))^{1-\lambda}, & \text{if } n > 1 \text{ is even.} \end{cases}$$

As $\gamma(n) \leq n$ and $1 - \lambda < \frac{1}{2}$, this improves the inequality

$$\sigma(n) < n\sqrt{n}$$

by C. C. Lindner (see [1]).

Remark 9. By using the known inequality for $n > 1$ (see [7])

$$\sigma(n) < \frac{\pi^2}{6}\psi(n) \quad (22)$$

and combining with relation (19), one can obtain another upper bound for $\sigma(n)$. For example, when $n > 1$ is odd, we obtain from (22)

$$\sigma(n) < \frac{\pi^2}{6}n(\gamma(n))^{\mu-1}. \quad (23)$$

Since $\gamma(n) \leq n$, a simple computation shows that the right side of (23) is less than $\frac{6}{\pi^2}n\sqrt{n}$ for $n \geq 77$, so we attain an improvement of the inequality for $n \geq 9$

$$\sigma(n) < \frac{6}{\pi^2}n\sqrt{n}$$

due to U. Annapurna (see [8]).

3. Conclusions

Euler's arithmetical function φ and Dedekind's arithmetical function ψ have many theoretical and practical applications in many fields of pure and applied mathematics.

In the authors' book [9], a lot of inequalities related to the arithmetic functions φ and ψ were given. For a brief survey of some inequalities for arithmetic functions, see paper [10].

In the present paper, some new inequalities with these functions were formulated, and their validity was proven. The obtained results improve or complement various results in the literature. They are sharp, as there is equality in the relations for $r = 1$ (the number of prime factors).

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