

Article

Geometric Properties of a General Kohn–Nirenberg Domain in $\mathbb{C}^n$

Kejia Hu ^{1,2} , Hongyi Li ^{2,*} , Di Zhao ² , Yuan Jiang ¹ and Baozhu Li ³¹ School of Electronics and Communication Engineering, Sun Yat-Sen University, Shenzhen 518107, China; hukj6@mail.sysu.edu.cn (K.H.)² School of Mathematical Science, Beihang University, Beijing 100191, China³ Hangzhou Institute of Technology, Xidian University, Hangzhou 311200, China; libaozhu@xidian.edu.cn

* Correspondence: lihongyi@buaa.edu.cn

Abstract: The Kohn–Nirenberg domains are unbounded domains in $\mathbb{C}^n$. In this article, we modify the Kohn–Nirenberg domain $\Omega_{K,L} = \{(z_1, \dots, z_n) \in \mathbb{C}^n : \operatorname{Re} z_n + g |z_n|^2 + \sum_{j=1}^{n-1} (|z_j|^p + K_j |z_j|^{p-q} \operatorname{Re} z_j^q + L_j |z_j|^{p-2q} \operatorname{Im} z_j^{2q}) < 0\}$ and discuss the existence of supporting surface and peak functions at the origin.

Keywords: Kohn–Nirenberg domain; peak function; supporting surface

MSC: 32Q02

1. Introduction

Consider a domain $\Omega \in \mathbb{C}^n$ with a smooth boundary $\partial\Omega$. As the boundary point is a strongly pseudoconvex point $p \in \partial\Omega$, we can find a local system of holomorphic coordinates. Hakim [1] and Pflug [2] show that every strongly pseudoconvex point of $\partial\Omega$ is a peak point. But, this property fails for weakly pseudoconvex boundary points in general. Kohn and Nirenberg have an example that is defined with the boundary point $(0,0) \in \partial\Omega$. The representative example [3] is $\Omega = \{z = (z_1, z_2)^T \in \mathbb{C}^2 : r(z) := \operatorname{Re} z_2 + |z_1|^8 + \frac{15}{7}|z_1|^2 \operatorname{Re} z_1^6 < 0\}$, which is a pseudoconvex domain with a point in the boundary that does not admit any peak function; the supporting surface and the boundary cannot be convexifiable by any local holomorphic coordinates [4–6]. The existence of supporting functions and smooth peak functions and the properties of convexifiability have been established by Pflug [2], Kolař [7], J. Han [8], D. Zhao [9], and J. Byun and H. R. Cho [10]. In [11,12], Taeyong Ahn et al. provide a tool to construct global holomorphic peaks from local holomorphic supporting functions for a class of unbounded domains in $\mathbb{C}^n$. But, it is still an open question whether any Kohn–Nirenberg domain is biholomorphic to a bounded domain. In order to better understand the properties of the Kohn–Nirenberg domain, in [13], Simone Calamai provide some new examples of Kohn–Nirenberg domains that develop some properties and theories about convexifiability in $\mathbb{C}^2$.

Let $\mathcal{O}^\alpha(\Omega)$ ($0 \leq \alpha \leq \infty$) denote the space of functions holomorphic on Ω and of class $\mathcal{C}^\alpha$ on $\overline{\Omega}$. Recall a point $\xi \in \partial\Omega$ is a peak point to Ω if there is a function $f \in \mathcal{O}^\alpha(\Omega)$ satisfying $f(\xi) = 1$ and $|f(z)| < 1$ for all $z \in \overline{\Omega} \setminus \{\xi\}$. We call f a peak function. A holomorphic supporting surface for Ω at ξ is a complex manifold M of co-dimension 1 with the property that there exists a neighborhood $N(\xi)$ of ξ such that $\overline{\Omega} \cap N(\xi) \cap M = \{\xi\}$. In [9], D. Zhao et al. consider a general modification of the Kohn–Nirenberg domain near the origin in $\mathbb{C}^n$, namely the domain $\Omega_K = \{(z_1, \dots, z_n) \in \mathbb{C}^n : \operatorname{Re} z_n + \sum_{j=1}^{n-1} (|z_j|^p +$



Academic Editors: Jay Jahangiri and Alicia Cordero

Received: 21 January 2025

Revised: 6 March 2025

Accepted: 3 April 2025

Published: 5 April 2025

Citation: Hu, K.; Li, H.; Zhao, D.; Jiang, Y.; Li, B. Geometric Properties of a General Kohn–Nirenberg Domain in $\mathbb{C}^n$. *Mathematics* **2025**, *13*, 1200. <https://doi.org/10.3390/math13071200>

Copyright: © 2025 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

$K_j|z_j|^{p-q}\operatorname{Re}(z_j^q)) < 0\}$, where $K = (K_1, \dots, K_{n-1}) \in \mathbb{R}^{n-1}$, $p, q \in \mathbb{Z}^+$ and $p - q - 2 \geq 0$. They proved the following sufficient condition.

Theorem 1 ([9]). *Given the above domain Ω_K with $|K_j| \leq \frac{p^2}{p^2-q^2}(1 \leq j \leq n-1)$, then*

- (1) Ω_K is a pseudoconvex domain.
- (2) If $q \mid p$ or $q \nmid p$ but $|K_j| < 1(1 \leq j \leq n-1)$, there exists a C^∞ -peak function and a supporting surface at the origin $0 \in \partial\Omega_K$.
- (3) If $q \nmid p$ and $\sum_{j=1}^{n-1}|K_j| > n-1$, there does not exist any C^∞ -peak function and supporting surface at $0 \in \partial\Omega_K$.

In fact, the above general Kohn–Nirenberg domain Ω_K is a special case of a decoupled domain in $\mathbb{C}^n$ [14]. Based on the modification of the domain Ω_K , we define a general Kohn–Nirenberg-type domain as follows:

$$\Omega_{K,L} = \{(z_1, \dots, z_n) \in \mathbb{C}^n : \operatorname{Re} z_n + g|z_n|^2 + \sum_{j=1}^{n-1}(|z_j|^p + K_j|z_j|^{p-q}\operatorname{Re}(z_j^q) + L_j|z_j|^{p-2q}\operatorname{Im}(z_j^{2q})) < 0\}$$

where $K = (K_1, K_2, \dots, K_{n-1}) \in \mathbb{R}^{n-1}$, $L = (L_1, L_2, \dots, L_{n-1}) \in \mathbb{R}^{n-1}$, $g \geq 0$, $p \in \mathbb{Z}^+$, $q \in \mathbb{Z}^+$, and $p - 2q - 1 \geq 0$.

If we do not consider the term $g|z_n|^2$, the general modified domain $\Omega_{K,L}$ is a weighted-bumped domain [12], denoted by Ω_{WB} (use the notation of Definition 2). If we consider this term, Theorem 4.6 [11] has an argument that there exists a global holomorphic supporting function when $g > 0$ and a bound point of Ω_{WB} admits a local holomorphic supporting function. Thus, $\Omega_{K,L}$ keeps the main features. It will be interesting to study whether the existence of supporting surface and peak functions at the origin in [9] can be generalized to the domain $\Omega_{K,L}$.

For the domain $\Omega_{K,L}$, we study the existence of the holomorphic peak function, supporting the surface at the boundary points. The main result of this article is the following theorem.

Theorem 2. *Let $\Omega_{K,L}$ be the above domain with $|K_j| + 2|L_j| \leq \frac{p^2}{p^2-q^2}(1 \leq j \leq n-1)$; we have*

- (a) $\Omega_{K,L}$ is a pseudoconvex domain.
- (b) If $q \mid p$ or $q \nmid p$ and $|K_j| + |L_j| < 1(1 \leq j \leq n-1)$, there exists a holomorphic peak function and supporting surface at the origin $0 \in \partial\Omega_{K,L}$.
- (c) If $q \nmid p$ and $\sum_{j=1}^{n-1}|K_j| > n-1$, there does not exist any supporting surface at $0 \in \partial\Omega_{K,L}$.

Here, $q \mid p$ means that q divides p ; $q \nmid p$ means that q does not divide p . We shall use these notations in this article. The structure of this article is as follows. In Section 2, we provide some basic definitions for the Kohn–Nirenberg domain. In Section 3, we provide the proof of Theorem 2 regarding (1)–(3).

2. Basic Definitions and Lemmas

Let $\Omega = \{(z_1, \dots, z_n) \in \mathbb{C}^n : r(z) < 0\}$ be a domain in $\mathbb{C}^n$ with smooth boundary; its defining function is $r(z)$. Let $\mathcal{O}^\alpha(\Omega)(0 \leq \alpha \leq \infty)$ be the space of holomorphic functions on Ω and C^α -continuous on $\overline{\Omega}$.

Definition 1. For a point $\xi \in \partial\Omega$ and a vector $t = (t_1, \dots, t_n) \in \mathbb{C}^n$, we write $\partial r_\xi(t) = \sum_{j=1}^n \frac{\partial r}{\partial z_j}(\xi) t_j$. The Levi form of $r(z)$ at ξ applied to $t \in \mathbb{C}^n$ is $L(\xi, t) = \sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(\xi) t_j \bar{t}_k$. ξ is called a pseudoconvex point if

$$L(\xi, t) = \sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(\xi) t_j \bar{t}_k \geq 0$$

for all $t \in T_\xi \partial\Omega = \{t \in \mathbb{C}^n : \partial r_\xi(t) = \sum_{j=1}^n \frac{\partial r}{\partial z_j}(\xi) t_j = 0\}$, where $T_\xi \partial\Omega = \{t \in \mathbb{C}^n : \partial r_\xi(t) = 0\}$ is the corresponding complex tangent space.

If the Levi form is positive at boundary point ξ , i.e., $L(\xi, t) > 0$, we call ξ a strong pseudoconvex point.

If all the boundary points are (strong) pseudoconvex points, the domain Ω is called a (strong) pseudoconvex domain.

Definition 2 ([12]). For domain $\Omega_{WB} \in \mathbb{C}^n$, if $\text{Re} z_n + \mathcal{P}(z_1, z_2, \dots, z_{n-1}) < 0$ ($(z_1, z_2, \dots, z_n) \in \mathbb{C}^n$), where

(a) $\mathcal{P}$ is a real-valued weighted polynomial on Ω_{WB} .

(b) All the boundary points of Ω_{WB} are pseudoconvex points and all but the origin are strong pseudoconvex points.

Lemma 1 ([9]). For any real number K with $|K| \leq \frac{m^2}{m^2-1}$ ($1 < m \in \mathbb{Z}^+$), there exist constants $b \in \mathbb{R}$ and $C > 0$ such that

$$1 + K \cos t + b \cos(mt) > C, \quad \text{for all } t \in [0, 2\pi)$$

where $b = b(m)$ and $C = C(m)$ depend only on m .

Lemma 2. If $|K| + 2|L| \leq \frac{m^2}{m^2-1}$ ($1 < m \in \mathbb{Z}^+$), there exist constants $b \in \mathbb{R}$ and $C > 0$ such that

$$1 + K \cos(t) + L \sin(2t) + b \cos(mt) > C, \quad \text{for } \forall t \in [0, 2\pi)$$

Proof. If $|K| + 2|L| \leq \frac{m^2}{m^2-1}$, then $|K + 2L \sin(t)| \leq |K| + 2|L| \leq \frac{m^2}{m^2-1}$. From Lemma 1 and $K \cos(t) + L \sin(2t) = (K + 2L \sin(t)) \cos(t)$, we have constants $b \in \mathbb{R}$ and $C > 0$ such that

$$1 + K \cos(t) + L \sin(2t) + b \cos(mt) > C, \quad \text{for } t \in [0, 2\pi)$$

where $b = b(m)$ and $C = C(m)$ depend only on m .

Set the domain $\Omega_{K,L} = \{(z_1, \dots, z_n) \in \mathbb{C}^n : r(z) < 0\}$; its defining function $r(z)$ on $\mathbb{C}^n$ is as follows

$$r(z) = \text{Re} z_n + g|z_n|^2 + \sum_{j=1}^{n-1} (|z_j|^p + K_j |z_j|^{p-q} \text{Re} z_j^q + L_j |z_j|^{p-2q} \text{Im} z_j^{2q})$$

where $K = (K_1, \dots, K_{n-1}) \in \mathbb{R}^{n-1}$, $L = (L_1, \dots, L_{n-1}) \in \mathbb{R}^{n-1}$, $g \geq 0$, $p \in \mathbb{Z}^+$, $q \in \mathbb{Z}^+$, and $p - 2q - 1 \geq 0$. $\square$

Lemma 3. If $(p^2 - q^2)|K_j| + (p^2 - 4q^2)|L_j| \leq p^2$ ($1 \leq j \leq n-1$) ($g \geq 0$), then $\Omega_{K,L}$ is a pseudoconvex domain.

Proof. For boundary point $\xi = (z_1, \dots, z_n) \in \partial\Omega$ and tangent vector $t = (t_1, \dots, t_n)$, we compute the Levi form and obtain

$$r_{z_j} = \frac{1}{2} p z_j^{\frac{1}{2}p-1} (\bar{z}_j)^{\frac{1}{2}p} + \frac{K_j(p+q)}{4} z_j^{\frac{1}{2}p+\frac{1}{2}q-1} (\bar{z}_j)^{\frac{1}{2}p-\frac{1}{2}q} + \frac{K_j(p-q)}{4} z_j^{\frac{1}{2}p-\frac{1}{2}q-1} (\bar{z}_j)^{\frac{1}{2}p+\frac{1}{2}q} + \frac{L_j(p+2q)}{4i} z_j^{\frac{1}{2}p+q-1} (\bar{z}_j)^{\frac{1}{2}p-q} - \frac{L_j(p-2q)}{4i} z_j^{\frac{1}{2}p-q-1} (\bar{z}_j)^{\frac{1}{2}p+q}$$

where $i = \sqrt{-1}$. $r_{z_j \bar{z}_k} = 0$ ($j \neq k$) and $r_{z_n \bar{z}_n} = g$.

For $1 \leq j \leq n-1$, there is

$$\begin{aligned} r_{z_j \bar{z}_j} &= \frac{1}{4} p^2 (z_j)^{\frac{1}{2}p-1} + \frac{K_j(p+q)(p-q)}{8} (z_j)^{\frac{1}{2}p+\frac{1}{2}q-1} (\bar{z}_j)^{\frac{1}{2}p-\frac{1}{2}q-1} + \frac{K_j(p-q)}{8} (p+q) \\ &\quad (z_j)^{\frac{1}{2}p-\frac{1}{2}q-1} (\bar{z}_j)^{\frac{1}{2}p+\frac{1}{2}q-1} + \frac{L_j(p+2q)(p-2q)}{8i} (z_j)^{\frac{1}{2}p+q-1} (\bar{z}_j)^{\frac{1}{2}p-q-1} - \\ &\quad \frac{L_j(p-2q)(p+2q)}{8i} (z_j)^{\frac{1}{2}p-q-1} (\bar{z}_j)^{\frac{1}{2}p+q-1} \\ &= \frac{1}{4} \left\{ p^2 |z_j|^{p-2} + K_j(p^2 - q^2) (z_j)^{\frac{1}{2}p-\frac{1}{2}q-1} (\bar{z}_j)^{\frac{1}{2}p-\frac{1}{2}q-1} \left[\frac{(z_j)^q + (\bar{z}_j)^q}{2} \right] + L_j \right. \\ &\quad \left. (p^2 - 4q^2) (z_j)^{\frac{1}{2}p-q-1} (\bar{z}_j)^{\frac{1}{2}p-q-1} \left[\frac{(z_j)^{2q} - (\bar{z}_j)^{2q}}{2i} \right] \right\} \\ &= \frac{1}{4} \left\{ p^2 |z_j|^{p-2} + (p^2 - q^2) K_j |z_j|^{p-q-2} \operatorname{Re}(z_j)^q + (p^2 - 4q^2) L_j |z_j|^{p-2q-2} \operatorname{Im}(z_j)^{2q} \right\} \end{aligned}$$

If $(p^2 - q^2)|K_j| + (p^2 - 4q^2)|L_j| \leq p^2$ and $g \geq 0$, we have the Levi form

$$\begin{aligned} L(\xi, t) &= \sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k} (\xi) t_j \bar{t}_k \\ &= \sum_{j=1}^{n-1} r_{z_j \bar{z}_j} |t_j|^2 + g |t_n|^2 \\ &\geq \frac{1}{4} \sum_{j=1}^{n-1} (p^2 - (p^2 - q^2)) |K_j| - (p^2 - 4q^2) |L_j| |z_j|^{p-2} |t_j|^2 + g |t_n|^2 \\ &\geq 0. \end{aligned}$$

Thus, the Levi form $L(\xi, t)$ is semi-positive definite, which proves that $\Omega_{K,L}$ is pseudoconvex. $\square$

3. Holomorphic Peak Function and Supporting Surface of the General Modified Domain $\Omega_{K,L}$

Here, we prove the main Theorem 2 regarding (1)–(3).

Proof of Theorem 2 regarding (1). Let $\Omega_{K,L} = \{z \in \mathbb{C}^n : \operatorname{Re} z_n + g|z_n|^2 + \sum_{j=1}^{n-1} (|z_j|^p + K_j |z_j|^{p-q} \operatorname{Re} z_j^q + L_j |z_j|^{p-2q} \operatorname{Im} z_j^{2q}) < 0\}$. Suppose $|K_j| + 2|L_j| \leq \frac{p^2}{p^2 - q^2}$ ($1 \leq j \leq n-1, g \geq 0$), and then

$(p^2 - q^2)|K_j| + (p^2 - 4q^2)|L_j| \leq (p^2 - q^2)|K_j| + 2(p^2 - q^2)|L_j| \leq p^2$ ($1 \leq j \leq n-1, g \geq 0$). So, Lemma 3 implies that $\Omega_{K,L}$ is pseudoconvex. $\square$

Remark 1. Let $\Omega_{K,L}$ be the above domain. If $|K_j| + 2|L_j| \leq \frac{p^2}{p^2 - q^2}$ ($1 \leq j \leq n-1, g \geq 0$), it is easy to see the origin is a weakly pseudoconvex (not strong pseudoconvex) boundary point of $\Omega_{K,L}$.

Proof of Theorem 2 regarding (2). Let $\Omega_{K,L}$ be the above domain with $|K_j| + 2|L_j| \leq \frac{p^2}{p^2 - q^2}$ ($1 \leq j \leq n-1, g \geq 0$).

For $r(z) = \operatorname{Re} z_n + g|z_n|^2 + \sum_{j=1}^{n-1} (|z_j|^p + K_j|z_j|^{p-q} \operatorname{Re} z_j^q + L_j|z_j|^{p-2q} \operatorname{Im} z_j^{2q})$, we consider two cases.

- (I) The case $q \mid p$ ($p = mq$). Let $z_j = r_j e^{i\theta_j}$ ($1 \leq j \leq n-1$) and $z_n = u + iv$. In polar coordinate system, we have $r(z_1, z_2, \dots, z_n) = u + gu^2 + gv^2 + \sum_{j=1}^{n-1} [1 + K_j \cos(q\theta_j) + L_j \sin(2q\theta_j)] r_j^p$. Then, we consider the coordinate transformation $z_1^* = z_1; z_2^* = z_2; \dots; z_{n-1}^* = z_{n-1}; z_n^* = z_n - \sum_{j=1}^{n-1} b_j z_j^p$, and $b_j \in \mathbb{R}$. In the new coordinate system, after dropping the stars, we have

$$r(z_1, z_2, \dots, z_n) = u + g(u + \sum_{j=1}^{n-1} b_j \cos(p\theta_j) r_j^p)^2 + g(v + \sum_{j=1}^{n-1} b_j \sin(p\theta_j) r_j^p)^2 + \sum_{j=1}^{n-1} [1 + K_j \cos(q\theta_j) + L_j \sin(2q\theta_j) + b_j \cos(p\theta_j)] r_j^p$$

Note that $|K_j| + 2|L_j| \leq \frac{p^2}{p^2 - q^2} = \frac{m^2}{m^2 - 1}$ ($p = mq$). Lemma 2 implies that there exists $C > 0$ such that

$$\sum_{j=1}^{n-1} [1 + K_j \cos(q\theta_j) + L_j \sin(2q\theta_j) + b_j \cos(p\theta_j)] r_j^p > C \sum_{j=1}^{n-1} r_j^p$$

The point 0 belongs to the set $\{(z_1, z_2, \dots, 0) : z_j \in \mathbb{C}, 1 \leq j \leq n-1\} \cap N(0) \cap \overline{\Omega}_{K,L}$, where $N(0)$ is a neighborhood of 0. For all $0 \neq z \in \{(z_1, z_2, \dots, 0) : z_j \in \mathbb{C}, 1 \leq j \leq n-1\} \cap N(0) \cap \overline{\Omega}_{K,L}$, there exists j such that $z_j \neq 0$. We have

$$r(z_1, z_2, \dots, 0) = g \left\{ \left(\sum_{j=1}^{n-1} b_j \cos(p\theta_j) r_j^p \right)^2 + \left(\sum_{j=1}^{n-1} b_j \sin(p\theta_j) r_j^p \right)^2 \right\} + 0 + \sum_{j=1}^{n-1} [1 + K_j \cos(q\theta_j) + L_j \sin(2q\theta_j) + b_j \cos(p\theta_j)] r_j^p > C \sum_{j=1}^{n-1} r_j^p > 0$$

This is a contradiction with the definition $r(z_1, z_2, \dots, z_n) \leq 0$ and implies that

$$\{(z_1, z_2, \dots, 0) : z_j \in \mathbb{C}, 1 \leq j \leq n-1\} \cap N(0) \cap \overline{\Omega}_{K,L} = \{0\}.$$

Thus, in the new coordinates, the complex manifold $\{(z_1, z_2, \dots, 0) : z_j \in \mathbb{C}, 1 \leq j \leq n-1\}$ is a holomorphic supporting surface at the origin $0 \in \partial\Omega_{K,L}$. The holomorphic supporting function is $f(z_1, z_2, \dots, z_n) = z_n$ at 0, and the corresponding holomorphic peak function is $h(z_1, z_2, \dots, z_n) = e^{z_n + z_n^2}$ for the origin 0.

In fact, it is obvious that $h(0) = 1$. Put $z_n = u + iv$. For $\forall z \in \overline{\Omega}_{K,L} - \{0\}$, we have $r(z_1, z_2, \dots, z_n) \leq 0$; i.e.,

$$u + g(u + \sum_{j=1}^{n-1} b_j \cos(p\theta_j) r_j^p)^2 + g(v + \sum_{j=1}^{n-1} b_j \sin(p\theta_j) r_j^p)^2 + \sum_{j=1}^{n-1} [1 + K_j \cos(q\theta_j) + L_j \sin(2q\theta_j) + b_j \cos(p\theta_j)] r_j^p \leq 0$$

Thus,

$$u \leq -\sum_{j=1}^{n-1} [1 + K_j \cos(q\theta_j) + L_j \sin(2q\theta_j) + b_j \cos(p\theta_j)] r_j^p \leq -C \sum_{j=1}^{n-1} r_j^p \leq 0.$$

If $u = 0$, then $v < 0$ and $|h(z_1, z_2, \dots, z_n)| = e^{u+u^2-v^2} = e^{-v^2} < 1$.

If $u < 0$, then $|h(z_1, z_2, \dots, z_n)| = e^{u+u^2-v^2} = e^{-v^2} < 1$.

So, the function $h(z_1, z_2, \dots, z_n)$ is a local peak function at $0 \in \partial\Omega_{K,L}$. Further, Hakim and Sibony [1,2] show that there is a global peak function with the same regularity as $h(z_1, z_2, \dots, z_n)$.

- (II) The case $q \nmid p$ and $|K_j| + |L_j| < 1$ ($1 \leq j \leq n-1$). There exists holomorphic peak function and supporting surface at the origin. Note that $\sum_{j=1}^{n-1} (|z_j|^p + K_j|z_j|^{p-q} \operatorname{Re}(z_j^q) + L_j|z_j|^{p-2q} \operatorname{Im}(z_j^{2q})) \geq \sum_{j=1}^{n-1} [1 - (|K_j| + |L_j|)] |z_j|^p$; similar to case (I), we have the com-

plex manifold $\{(z_1, z_2, \dots, 0) : z_j \in \mathbb{C}, 1 \leq j \leq n-1\}$, which is also a holomorphic supporting surface at the origin. At the same time, $h(z_1, z_2, \dots, z_n) = e^{z_n + z_n^2}$ is a local peak function at $0 \in \partial\Omega_{K,L}$. In [1,2], Hakim and Sibony show that there is a global peak function with the same regularity as h .

□

Proof of Theorem 2 regarding (3). Assume that there exists supporting surface at the origin $0 \in \partial\Omega_{K,L}$. The support surface M as a complex manifold of co-dimension 1 implies that there are an open neighborhood $N(0) \subset \mathbb{C}^n$ and holomorphic function f on $N(0)$ such that

- (1) $M \cap N(0) = \{z \in N(0) : f(z) = 0\}$;
- (2) $\text{rank}\left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right) = 1$.

We shall study two different cases.

- (I) The case $\frac{\partial f}{\partial z_n} = 0$, there is some j such that $\frac{\partial f}{\partial z_j} \neq 0$. The implicit function theorem implies that

$$M = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_j = \phi(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n)\}$$

Now, let $z_1 = \dots = z_{j-1} = z_{j+1} = \dots = z_n = -\varepsilon$, and then

$$\begin{aligned} r(z) &= -\varepsilon + g\varepsilon^2 + (n-2)\varepsilon^p + (-1)^p \sum_{l \neq j} K_l |\phi(-\varepsilon, \dots, -\varepsilon)|^p + K_j |\phi(-\varepsilon, \dots, -\varepsilon)|^{p-q} \\ &\quad \text{Re}[\phi(-\varepsilon, \dots, -\varepsilon)]^q + L_j |\phi(-\varepsilon, \dots, -\varepsilon)|^{p-2q} \text{Re}[\phi(-\varepsilon, \dots, -\varepsilon)]^{2q} \\ &= -\varepsilon + O(\varepsilon^2) < 0. \end{aligned}$$

If ε is small, then $M \cap \overline{\Omega}_{K,L} \neq \{0\}$ in every small neighborhood of 0. Therefore, we have a contradiction with M as a support surface.

- (II) The case $\frac{\partial f}{\partial z_n} \neq 0$, the implicit function theorem implies that

$$M = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_n = \phi(z_1, \dots, z_{n-1})\}$$

We shall divide this into three different cases.

- (a) When $\phi(z_1, \dots, z_{n-1}) = \sum_{s=t}^{\infty} F_s(z_1, \dots, z_{n-1})$, $t \geq p+1$, F_s is the sum of those terms $C_{\alpha_1, \dots, \alpha_{n-1}} z_1^{\alpha_1} \dots z_{n-1}^{\alpha_{n-1}}$ in the power series for which $\alpha_1 + \dots + \alpha_{n-1} = s$.

We let $z_j = \varepsilon e^{i \frac{(\chi(K_j)+1)}{q} \pi}$, $1 \leq j \leq n-1$, where $\chi(K_j)$ is defined by

$$\chi(K_j) = \begin{cases} 0 & K_j > 0 \\ 1 & K_j < 0 \end{cases} (1 \leq j \leq n-1).$$

Moreover, $z_n = \phi(z_1, \dots, z_{n-1})$. If ε is sufficiently small,

$$\begin{aligned} r(z) &= \text{Re}[\phi(\varepsilon e^{i \frac{\chi(K_1)+1}{q} \pi}, \dots, \varepsilon e^{i \frac{\chi(K_{n-1})+1}{q} \pi})] + g|\phi(\cdot)|^2 + (n-1)\varepsilon^p - \varepsilon^p \sum_{j=1}^{n-1} |K_j| \\ &= [(n-1) - \sum_{j=1}^{n-1} |K_j|] \varepsilon^p + O(\varepsilon^{p+1}) < 0. \end{aligned}$$

Hence, M is not a supporting surface. It is a contradiction.

- (b) When $\phi(z_1, \dots, z_{n-1}) = \sum_{s=t}^{\infty} F_s(z_1, \dots, z_{n-1})$, $1 \leq t \leq p-1$, $F_t(z_1, \dots, z_{n-1}) \neq 0$. We can suppose $F_t(t_1, \dots, t_{n-1}) = \lambda_1 \neq 0$ and choose θ such that $\lambda_1 e^{i\theta t} < 0$. Let $z_1 = t_1 \varepsilon e^{i\theta}$, $z_2 = t_2 \varepsilon e^{i\theta}$, $\dots$, $z_{n-1} = t_{n-1} \varepsilon e^{i\theta}$; it is easy to see that

$$F_t(z_1, \dots, z_{n-1}) = F_t(t_1, \dots, t_{n-1}) \varepsilon^t e^{i\theta t} = \lambda_1 e^{i\theta t} \varepsilon^t = -|\lambda_1| \varepsilon^t.$$

Then,

$$\begin{aligned} r(z) &= \operatorname{Re}[\phi(z_1, \dots, z_{n-1})] + g|\phi(z_1, \dots, z_{n-1})|^2 + \varepsilon^p \sum_{j=1}^{n-1} |t_j|^p + \varepsilon^p \cos q\theta \\ &\quad \left(\sum_{j=1}^{n-1} K_j |t_j|^p \right) + \varepsilon^p \sin 2q\theta \left(\sum_{j=1}^{n-1} L_j |t_j|^p \right) \\ &= -|\lambda_1| \varepsilon^t + O(\varepsilon^{t+1}) < 0 \end{aligned}$$

if ε is sufficiently small. Hence, M is not a supporting surface. It is a contradiction.

(c) Then, the only remaining case is when

$$\phi(z_1, \dots, z_{n-1}) = \sum_{s=p}^{\infty} F_s(z_1, \dots, z_{n-1}).$$

Let $z_j^l = \varepsilon e^{i \frac{\chi(K_j)+l}{q} \pi}$ ($1 \leq j \leq n-1$), where $\chi(K_j)$ is defined by

$$\chi(K_j) = \begin{cases} 0 & K_j > 0 \\ 1 & K_j < 0 \end{cases}$$

and $z_n^l = \phi(z_1^l, \dots, z_{n-1}^l)$. Then,

$$\begin{aligned} r(z^l) &= \operatorname{Re}[\phi(z_1^l, \dots, z_{n-1}^l)] + g|\phi(z_1^l, \dots, z_{n-1}^l)|^2 + (n-1)\varepsilon^p + \varepsilon^p \cos l\pi \left(\sum_{j=1}^{n-1} |K_j| \right) \\ &= \varepsilon^p [\operatorname{Re}(\lambda e^{i \frac{p\pi}{q} l}) + O(\varepsilon) + (n-1) + \cos l\pi \left(\sum_{j=1}^{n-1} |K_j| \right)], \end{aligned}$$

where $\lambda = F_p(e^{i \frac{\chi(K_1)}{q} \pi}, \dots, e^{i \frac{\chi(K_{n-1})}{q} \pi})$. We take $l = 1, 3, \dots, 2q-1$, and then

$$\sum_{l=1,3,\dots,2q-1} r(z^l) = \varepsilon^p \left\{ \operatorname{Re} \lambda \left(\sum_{l=1,3,\dots,2q-1} e^{i \frac{p\pi}{q} l} \right) + O(\varepsilon) + q(n-1) + \sum_{l=1,3,\dots,2q-1} \cos l\pi \left(\sum_{j=1}^{n-1} |K_j| \right) \right\}$$

Since l takes odd integers, we obtain

$$\sum_{l=1,3,\dots,2q-1} \cos l\pi = -q.$$

If $q \nmid p$, we have $\sum_{l=1,3,\dots,2q-1} e^{i \frac{p\pi}{q} l} = 0$. Therefore, when ε is small,

$$\sum_{l=1,3,\dots,2q-1} r(z^l) = \varepsilon^p \left\{ O(\varepsilon) + q[(n-1) - \sum_{j=1}^{n-1} |K_j|] \right\} < 0.$$

Since $M \cap \overline{\Omega}_{K,L} = \{0\}$ and $z^l \in M$, it follows that $r(z^l) > 0$ for all l .

Thus,

$$\sum_{l=1,3,\dots,2q-1} r(z^l) > 0.$$

Hence, we obtain a contradiction. This completes the proof.

□

There are still some problems yet to be answered for the domain $\Omega_{K,L}$.

Problem 1. When $q \nmid p$ and $\sum_{j=1}^{n-1} |K_j| > n-1$, the peak functions of the $\Omega_{K,L}$ at the origin are still not clear.

Moreover, the behavior of invariant metrics (e.g., Kobayashi and Carathéodory) on the Kohn–Nirenberg domain could be studied using the techniques in [11]. Such metrics are central to understanding hyperbolic geometry in complex domains, and their properties here might reveal new phenomena in extremal map constructions or boundary asymptotics. Thus, another nature problem is as follows.

Problem 2. *The Bergman metric and Carathéodory metrics of $\Omega_{K,L}$ in the statement of Theorem 2 are positive and complete.*

Author Contributions: Conceptualization, K.H. and D.Z.; Methodology, D.Z. and K.H.; Formal analysis, D.Z. and K.H.; Writing—original draft, K.H.; Writing—review and editing, H.L. and D.Z.; Supervision, H.L., Y.J. and B.L.; Funding acquisition, K.H. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the China Postdoctoral Science Foundation, certificate number 2023M744095, and National Natural Science Foundation of China, grant number 61771001.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Sibony, M.N. Quelques conditions pour l’existence de fonctions pics dans des domaines pseudoconvexes. *Duke Math. J.* **1997**, *42*, 399–406.
2. Pflug, P. Über polynomiabile Funktionen und Holomorphiegebiete. *Math. Z.* **1974**, *139*, 133–139.
3. Kohn, J.J.; Nirenberg, L. A pseudoconvex domain not admitting a holomorphic support function. *Math. Ann.* **1973**, *201*, 265–268. [[CrossRef](#)]
4. Fornæss, J.E.; Stensfnes, B. *Lectures on Counterexamples in Several Complex Variables*; Princeton University Press: Princeton, NJ, USA, 1987; Volume 33.
5. Han, J.; Zhao, D.; Gao, Z. Peak function and support surface of a Kohn–Nirenberg domain. *J. Math. Anal. Appl.* **2010**, *365*, 410–414. [[CrossRef](#)]
6. Byun, J.; Cho, H.R. The explicit automorphism group of the Kohn–Nirenberg domain. *Math. Z.* **2009**, *263*, 295–305.
7. Kolář, M. Convexifiability and supporting functions in $\mathbb{C}^2$. *Math. Res. Lett.* **1995**, *2*, 503–513. [[CrossRef](#)]
8. Han, J.; Zhao, D. Explicit description for the automorphism group of the general Kohn–Nirenberg domain. *J. Math. Anal. Appl.* **2011**, *5*, 569–574.
9. Zhao, D.; Han, J.; Li, H. Peak function and support surface of a general Kohn–Nirenberg domain in $\mathbb{C}^n$. *Complex Var. Elliptic Equ.* **2013**, *58*, 635–646. [[CrossRef](#)]
10. Byun, J.; Krantz, S.G.; On special forms of holomorphic automorphisms of domains in complex two-space. *Complex Var. Elliptic Equ.* **2010**, *55*, 395–405. [[CrossRef](#)]
11. Ahn, T.; Gaussier, H.; Kim, K.T. Bergman and Carathéodory metrics of the Kohn–Nirenberg domains. *arXiv* **2014**, arXiv:1406.3406.
12. Ahn, T.; Gaussier, H.; Kim, K.T. Positivity and Completeness of Invariant Metrics. *J. Geom. Anal.* **2016**, *26*, 1173–1185. [[CrossRef](#)]
13. Simone, C. A bounded Kohn–Nirenberg domain. *Bull. Korean Math. Soc.* **2014**, *51*, 1339–1345.
14. Fornæss, J.E.; McNeal, J. A construction of peak functions on some finite type domains. *Am. J. Math.* **1994**, *116*, 737–755.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.