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Global Dynamics of a Rational Difference Equation and Its Solutions to Several Conjectures

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Abstract: In this paper, we investigate the global attractivity of a higher-order rational difference equation in the form $x_{n+1} = \frac{p+qx_n}{1+rx_n+sx_{n-k}}$, where $p, q, r, s \geq 0$, k is a positive integer, and the initial conditions are nonnegative. This equation generalizes several well-known rational difference equations studied in the literature. By employing a combination of advanced mathematical techniques, including the use of key lemmas and intricate computations, we establish that the unique nonnegative equilibrium point of the equation is globally attractive under specific parameter conditions. Our results not only extend and improve upon existing findings but also resolve several conjectures posed by previous researchers, including those by G. Ladas and colleagues. The methods involve transforming the higher-order equation into a first-order difference equation and analyzing the properties of the resulting function, particularly its Schwarzian derivative. The findings demonstrate that the equilibrium point is globally attractive when certain inequalities involving the parameters are satisfied. This work contributes to the broader understanding of the dynamics of rational difference equations and has potential applications in various fields such as biology, physics, and cybernetics.

Keywords: rational difference equation; higher order; globally asymptotically stable; Schwarzian derivative; conjecture.

MSC: 39A28; 39A30

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1. Introduction and Preliminaries

Over the past several decades, the study of difference equations has witnessed remarkable progress, yielding a substantial body of significant findings, as evidenced by seminal works such as [1–3]. In particular, the global dynamics of higher-order rational difference equations (RDEs) have garnered increasing attention from researchers. This heightened interest stems from the fact that many fundamental results in nonlinear difference equation theory originate from investigations into higher-order RDEs, which serve as prototypical models. Furthermore, research topics in difference equations arising from diverse applied fields, including cybernetics, biology, and physics, have attracted considerable attention from scholars worldwide. For foundational contributions in these areas, we refer readers to [1–9] and the references therein.

In this paper, we mainly investigate the global attractivity of the following RDE

$$x_{n+1} = \frac{p + qx_n}{1 + rx_n + sx_{n-k}}, \quad n = 0, 1, \cdots, \tag{1}$$

where the parameters $p, q, r, s \in [0, \infty)$, k is a positive integer, and the initial conditions $x_{-k}, ..., x_{-1}, x_0 \in [0, \infty)$. To avoid trivial cases, suppose that $p + q > 0$ and $r + s > 0$.

It is evident that Equation (1) possesses a unique nonnegative equilibrium point, denoted by $\bar{x}$, which is given by

$$\bar{x} = \frac{q - 1 + \sqrt{(q-1)^2 + 4p(r+s)}}{2(r+s)}.$$

The motivation for investigating the global attractivity of Equation (1) stems from the extensive body of research on the following known work.

Kocic and Ladas [1] first considered the following special case of Equation (1)

$$x_{n+1} = \frac{a + bx_n}{A + x_{n-1}}, \ n = 0, 1, \cdots, \tag{2}$$

where the parameters

$$a, b, A \in (0, \infty) \tag{3}$$

and the initial values x_{-1}, x_0 are arbitrary positive numbers.

Equation (2) admits a unique positive equilibrium point $\bar{x}$, which can be readily shown to be locally asymptotically stable. Regarding the global dynamics of $\bar{x}$, the following results are established.

Theorem 1 ([1,2,10,11]). *Suppose that (3) holds, and one of the following conditions is true*

(1) $b < A$;

(2) $b \geq A$ *and* $a \leq Ab$;

(3) $b \geq A$ *and* $Ab < a < 2A(b + A)$;

(4) $b \geq \sqrt{1 + \sqrt{5}/2}A$, $Ab < a$ *and* $b^2/A < \bar{x} < 2b$.

Then, $\bar{x}$ *is globally asymptotically stable.*

In the early stages of research on RDEs, a widely held belief was that the local asymptotic stability of an equilibrium point in a RDE implies its global asymptotic stability. This perspective has been particularly influential in the field of mathematical biology. Motivated by this idea, numerous researchers have proposed various research projects, as documented in [1] and the collection "Open Problems and Conjectures" [1–3,10,11]. Indeed, most known results [1,2] support this viewpoint. However, Kocic and Ladas later provided counterexamples challenging this assumption. Regarding Equation (2), Computer simulations demonstrate that the statement holds true, that is, the local asymptotical stability of the positive equilibrium point $\bar{x}$ of Equation (2) implies its global asymptotical stability, which is equivalent to its global attractivity as long as (3) holds. However, except for the partial results mentioned in Theorem 1, this viewpoint cannot be completely and theoretically proved. Therefore, Ladas et al. presented the following conjecture in [1,2,10,11], respectively.

Conjecture 1 ([1], Conjecture 6.1.1, P_{154}). *Assume that Equation (3) holds. Then the positive equilibrium point* $\bar{x}$ *of Equation (2) is globally asymptotically stable.*

Those authors [1,12–15] obtained some partial results for Conjecture 1.

For another special form of Equation (1) [2]

$$x_{n+1} = \frac{p + qx_n}{1 + x_{n-1}}, \; n = 0, 1, \cdots, \tag{4}$$

where $p, q \in (0, \infty)$ and the initial values x_{-1}, x_0 are arbitrary positive numbers; the following question is also presented.

Conjecture 2 ([2]). *Assume that $p, q \in (0, \infty)$. Show that every positive solution of Equation (2) has a finite limit.*

For Conjecture 2, Nussbaum derived a partial result in 2007 [16].

Furthermore, Equation (2) was later generalized to the following $(k+1) - th$ order RDE by Kocic and Ladas in [1]

$$x_{n+1} = \frac{a + bx_n}{A + x_{n-k}}, n = 0, 1, ..., \tag{5}$$

where

$$a, b \in [0, \infty), A \in (0, \infty), k \in \{1, 2, .., \}, \tag{6}$$

and the initial values $x_{-k}, ..., x_{-1}, x_0$ are arbitrary positive numbers. Equation (5) is still a special case of Equation (1) and has a positive equilibrium $\bar{x}$, which is the same as that of Equation (1), provided that

$$\text{either } a > 0 \text{ or } a = 0 \text{ and } b > A. \tag{7}$$

Partial results on the global attractivity of Equation (5) are summarized in [1].

In addition, the authors in [17] studied the following difference equation:

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n + Cx_{n-k}}, \quad n = 0, 1, \cdots. \tag{8}$$

Here, the parameters α, β, A, B, and C are all positive numbers. This is still a special case of Equation (1), since the parameters in Equation (1) are only required to be nonnegative. Moreover, the global attractivity result obtained in [17] is valid in a partial invariant interval, rather than in the entire interval discussed in this paper.

Although the authors in [1,2,10–18] have studied the above problems, they have not completely solved these problems yet, for example, Conjectures 1 and 2. Therefore, these problems mentioned above motivate us to continue investigating the global attractivity of Equation (1) in this paper, which is more general than Equations (2), (4), (5) and (8).

2. Several Auxiliary Lemmas

For the convenience of the readers, we first present four key lemmas (Lemma 1 to Lemma 4) in this paper, which serve as the foundation for proving our main results. Subsequently, through a series of intricate calculations, we derive an important lemma, namely, Lemma 5, which is a core result of this paper.

Lemma 1 ([19]). *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), \; n = 0, 1, \cdots, \tag{9}$$

where $k \in \{1, 2, ...\}$. *Let* $I = [a, b]$ *be an interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(x, y)$ *is nondecreasing for each* $x \in [a, b]$ *and for each fixed* $y \in [a, b]$ *and nonincreasing for each* $y \in [a, b]$ *and for each fixed* $x \in [a, b]$*;*

(b) *If* $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system* $f(m, M) = m$ *and* $f(M, m) = M$*, then,* $m = M$.

Then Equation (9) has a unique equilibrium $\bar{x} \in [a, b]$, and every solution of Equation (9) converges to $\bar{x}$.

Lemma 2 ([1]). *Consider the difference equation*

$$x_{n+1} = x_n f(x_n, x_{n-k_1}, \cdots, x_{n-k_r}), \tag{10}$$

where $k_1, k_2, \cdots, k_r$ *are positive integers. Let* k *be the maximum of* $k_1, k_2, \cdots, k_r$. *Furthermore, assume that the function* f *satisfies the following hypotheses:*

(H1) $f \in C[(0, \infty) \times [0, \infty)^r, (0, \infty)]$ *and* $g \in C[[0, \infty)^{r+1}, (0, \infty)]$, *where* $g(u_0, u_1, \cdots, u_r) = u_0 f(u_0, u_1, \cdots, u_r)$ *for* $u_0 \in (0, \infty)$ *and* $u_1, \cdots, u_r \in [0, \infty)$, $g(0, u_1, \cdots, u_r) = \lim_{u_0 \to 0^+} g(u_0, u_1, \cdots, u_r)$*;*

(H2) $f(u_0, u_1, \cdots, u_r)$ *is nonincreasing in* $u_1, \cdots, u_r$*;*

(H3) The equation $f(x, x, \cdots, x) = 1$ *has a unique positive solution* $\overline{x}$*;*

(H4) The function $f(u_0, u_1, \cdots, u_r)$ *does not depend on* u_0 *or for every* $x > 0$ *and* $u \geq 0$,

$$[f(x, u, \cdots, u) - f(\overline{x}, u, \cdots, u)](x - \overline{x}) \leq 0,$$

with

$$[f(x, \overline{x}, \cdots, \overline{x}) - f(\overline{x}, \overline{x}, \cdots, \overline{x})](x - \overline{x}) < 0 \ \ for \ \ x \neq \overline{x}.$$

Define a new function F *given by*

$$F(x) = \begin{cases} max_{x \leq y \leq \overline{x}} G(x, y) \ \ for \ \ 0 \leq x \leq \overline{x}, \\ min_{\overline{x} \leq y \leq x} G(x, y) \ \ for \ \ x > \overline{x}, \end{cases} \tag{11}$$

where

$$G(x, y) = y f(y, x, \cdots, x) f(\overline{x}, \overline{x}, \cdots, \overline{x}, y)[f(\overline{x}, x, \cdots, x)]^{k-1}. \tag{12}$$

Then,

(a) $F \in C[(0, \infty), (0, \infty)]$ *and* F *is nonincreasing in* $[0, \infty)$*;*

(b) Assume that the function F *has no periodic points of prime period* 2. *Then,* $\overline{x}$ *is a global attractor of all positive solutions of Equation (10).*

Lemma 3 ([1], Lemma 1.6.3 (a) and (d)). *Let* $F \in [[0, \infty), (0, \infty)]$ *be a nonincreasing function and let* $\overline{x}$ *denote the unique fixed point of* F*, then, the following statements are equivalent:*

(a) $\overline{x}$ *is the only fixed point of* F^2 *in* $(0, \infty)$*;*

(b) $\overline{x}$ *is a global attractor of all positive solutions of the difference equation*

$$x_{n+1} = F(x_n), \ \ n = 0, 1, \cdots, \tag{13}$$

with $x_0 \in [0, \infty)$.

Lemma 4 ([1,20]). *Consider the difference Equation (13), where F is a decreasing function that maps some interval I into itself. Assume that F has a negative Schwarzian derivative*

$$SF(x) = \frac{F'''(x)}{F'(x)} - \frac{3}{2}(\frac{F''(x)}{F'(x)})^2 = [\frac{F''(x)}{F'(x)}]' - \frac{1}{2}(\frac{F''(x)}{F'(x)})^2 < 0, \tag{14}$$

everywhere on I, except for the point x, where $F'(x) = 0$. Then, the positive equilibrium point $\overline{x}$ of Equation (13) is a global attractor of all positive solutions of Equation (13).

Lemma 5. *Assume that $p, q, r, s \in [0, \infty)$ with $p + q > 0$ and $r + s > 0$, k is a positive integer, and $\bar{x}$ is the unique nonnegative equilibrium point of Equation (1). Let the function*

$$F(x) = \frac{A(p + qx)}{(1 + (r + s)x)(1 + r\bar{x} + sx)^k}, \tag{15}$$

where $\bar{x}, x \in (0, \infty)$, and the constant $A = [1 + (r + s)\bar{x}]^k > 0$.

Suppose $0 < q < \frac{ps}{1+r\bar{x}}$, then, the following statements hold:

(1) $F'(x) < 0$ and $F''(x) > 0$ for any $x \in (0, \infty)$;

(2) The function $F(x)$ has a negative Schwarzian derivative, i.e.,

$$SF(x) = [\frac{F''(x)}{F'(x)}]' - \frac{1}{2}[\frac{F''(x)}{F'(x)}]^2 < 0$$

for everywhere $x \in (0, \infty)$, except for the point x where $F'(x) = 0$.

Proof. (1) According to the definition of $F(x)$, one has

$$\begin{aligned} F'(x) &= A\frac{q[(1 + (r + s)x)(1 + r\bar{x} + sx)^k]}{(1 + (r + s)x)^2(1 + r\bar{x} + sx)^{2k}} \\ &\quad - A\frac{(p + qx)[(r + s)(1 + \bar{x} + sx)^k + ks(1 + r\bar{x} + sx)^{k-1}(1 + (r + s)x)]}{(1 + (r + s)x)^2(1 + r\bar{x} + sx)^{2k}} \\ &= A\frac{q(1 + (r + s)x)(1 + r\bar{x} + sx)}{(1 + (r + s)x)^2(1 + r\bar{x} + sx)^{k+1}} \\ &\quad - A\frac{(p + qx)[(r + s)(1 + r\bar{x} + sx) + ks(1 + (r + s)x)]}{(1 + (r + s)x)^2(1 + r\bar{x} + sx)^{k+1}} \\ &= \frac{AB}{(1 + (r + s)x)^2(1 + r\bar{x} + sx)^{k+1}}, \end{aligned}$$

where

$$\begin{aligned} B &=: q(1 + (r + s)x)(1 + r\bar{x} + sx) - (p + qx)[(r + s)(1 + r\bar{x} + sx) + ks(1 + (r + s)x) \\ &= (1 + r\bar{x} + sx)[q(1 + (r + s)x) - q(r + s)x - p(r + s)] - ks(p + qx)(1 + (r + s)x) \\ &= (q - p(r + s))(1 + (r + s)x) - ks(p + qx)(1 + r\bar{x} + sx) \\ &= -C(1 + r\bar{x} + sx) - ks(p + qx)(1 + (r + s)x) \\ &< 0 \text{ for any } x \in (0, \infty) \text{ and } C =: p(r + s) - q > 0. \end{aligned}$$

Thus, $F'(x) < 0$.

Now, we continue to study $F''(x)$. Evidently,

$$\begin{aligned} F''(x) &= A\frac{B'[(1 + (r + s)x)(1 + r\bar{x} + sx)]}{(1 + (r + s)x)^3(1 + r\bar{x} + sx)^{k+2}} \\ &\quad - A\frac{B[2(r + s)(1 + r\bar{x} + sx) + s(k + 1)(1 + (r + s)x)]}{(1 + (r + s)x)^3(1 + r\bar{x} + sx)^{k+2}}, \end{aligned}$$

$$\begin{aligned} B' &= -sC - kqs(1+(r+s)x) - ks(p+qx)(r+s) \\ &= -(k+1)sC - 2kqs(1+(r+s)x) < 0. \end{aligned} \tag{16}$$

Then,

$$F''(x) = \frac{AD}{(1+(r+s)x)^3(1+r\bar{x}+sx)^{k+2}}, \tag{17}$$

where

$$\begin{aligned} D &= B'[(1+(r+s)x)(1+r\bar{x}+sx)] \\ &\quad - B[2(r+s)(1+r\bar{x}+sx) + (k+1)s(1+(r+s)x)] \\ &= -(k+1)sC(1+(r+s)x)(1+r\bar{x}+sx) - 2kqs(1+(r+s)x)^2(1+r\bar{x}+sx) \\ &\quad + 2(r+s)C(1+r\bar{x}+sx)^2 + (k+1)sC(1+r\bar{x}+sx)(1+(r+s)x) \\ &\quad + 2ks(r+s)(1+(r+s)x)(1+r\bar{x}+sx)(p+qx) \\ &\quad + k(k+1)s^2(1+(r+s)x)^2(p+qx) \\ &= 2(r+s)C(1+r\bar{x}+sx)^2 + k(k+1)s^2(1+(r+s)x)^2(p+qx) \\ &\quad + 2ks(1+(r+s)x)(1+r\bar{x}+sx)[(p+qx)(r+s) - q(1+(r+s)x)] \\ &= 2(r+s)C(1+r\bar{x}+sx)^2 + k(k+1)s^2(1+(r+s)x)^2(p+qx) + 2ksCE, \end{aligned}$$

here, $E =: (1+(r+s)x)(1+r\bar{x}+sx) > 0$. So, $D > 0$. Accordingly, $F''(x) > 0$.

(2) From (1), one can see that $\frac{F''(x)}{F'(x)} = \frac{D}{(1+(r+s)x)(1+r\bar{x}+sx)B}$. In view of the definition of the Schwarzian derivative, one sees

$$\begin{aligned} SF(x) &= [\frac{F''(x)}{F'(x)}]' - \frac{1}{2}[\frac{F''(x)}{F'(x)}]^2 \\ &= \frac{H - 1/2D^2}{(1+(r+s)x)^2(1+r\bar{x}+sx)^2B^2} = \frac{H-1/2D^2}{B^2E^2}, \end{aligned}$$

where, for convenience, we denote

$$H = D'[(1+(r+s)x)(1+r\bar{x}+sx)B] - D[(1+(r+s)x)(1+r\bar{x}+sx)B]'.$$

Now verify $SF(x) < 0$. Denote $I = H - 1/2D^2$ again. Then,

$$\begin{aligned} I &= H - 1/2D^2 \\ &= BED' - D\{[(r+s)(1+r\bar{x}+sx)B + sB(1+(r+s)x) + B'E] + 1/2D\} \\ &= J - D\{-(r+s)C(1+r\bar{x}+sx)^2 - ks(r+s)(p+qx)E - sCE \\ &\quad - ks^2(p+qx)(1+(r+s)x)^2 - (k+1)sCE - 2kqsE(1+(r+s)x) \\ &\quad + (r+s)C(1+r\bar{x}+sx)^2 + 1/2k(k+1)s^2(p+qx)(1+(r+s)x)^2 + ksCE \\ &= J - D\{1/2k(k-1)s^2(p+qx)(1+(r+s)x)^2 - 2sCE \\ &\quad - ksE[p(r+s) - q + 3q + 3q(r+s)x] \\ &= J - D\{1/2k(k-1)s^2(p+qx)(1+(r+s)x)^2 - (k+2)sCE - 3kqsE(1+(r+s)x)\} \\ &=: J + L, \end{aligned}$$

where $J = BED'$,

$$L = -D\{1/2k(k-1)s^2(p+qx)(1+(r+s)x)^2 - (k+2)sCE - 3kqsE(1+(r+s)x)\}.$$

We are now in a position to simplify J and L.

$$
\begin{aligned}
J &= D'[(1+(r+s)x)(1+r\bar{x}+sx)B] = D'(EB)\\
&= D'[-CE(1+r\bar{x}+sx) - ksE(p+qx)(1+(r+s)x)]\\
&= 4s(r+s)C(1+r\bar{x}+sx)\cdot(-CE(1+r\bar{x}+sx))\\
&\quad + 4s(r+s)C(1+r\bar{x}+sx)\cdot(-ksE(p+qx)(1+(r+s)x))\\
&\quad + 2k(k+1)s^2(r+s)(1+(r+s)x)(p+qx)\cdot(-CE(1+r\bar{x}+sx))\\
&\quad + 2k(k+1)s^2(r+s)(1+(r+s)x)(p+qx)\cdot(-ksE(p+qx)(1+(r+s)x))\\
&\quad + kq(k+1)s^2(1+(r+s)x)^2\cdot(-CE(1+r\bar{x}+sx))\\
&\quad + kq(k+1)s^2(1+(r+s)x)^2\cdot(-ksE(p+qx)(1+(r+s)x))\\
&\quad + 2ks^2C(1+(r+s)x)\cdot(-CE(1+r\bar{x}+sx))\\
&\quad + 2ks^2C(1+(r+s)x)\cdot(-ksE(p+qx)(1+(r+s)x))\\
&\quad + 2ks(r+s)C(1+r\bar{x}+sx)\cdot(-CE(1+r\bar{x}+sx))\\
&\quad + 2ks(r+s)C(1+r\bar{x}+sx)\cdot(-ksE(p+qx)(1+(r+s)x))\\
&= -4s(r+s)C^2E(1+r\bar{x}+sx)^2 - 4ks^2(r+s)CE^2(p+qx)\\
&\quad - 2k(k+1)s^2(r+s)CE^2(p+qx)\\
&\quad - 2k^2(k+1)s^3(r+s)E(p+qx)^2(1+(r+s)x)^2\\
&\quad - k(k+1)qs^2CE^2(1+(r+s)x)\\
&\qquad - k^2(k+1)qs^3E(p+qx)(1+(r+s)x)^3\\
&\qquad - 2ks^2C^2E^2 - 2k^2s^3CE(p+qx)(1+(r+s)x)^2\\
&\qquad - 2ks(r+s)C^2E(1+r\bar{x}+sx)^2 - 2k^2s^2(r+s)CE^2(p+qx).
\end{aligned}
$$

That is

$$
\begin{aligned}
J &= -2ks^2C^2E^2 - 2(k+2)s(r+s)C^2E(1+r\bar{x}+sx)^2\\
&\quad - 2k(2k+3)s^2(s+r)CE^2(p+qx) - k(k+1)qs^2CE^2(1+(r+s)x)\\
&\quad - 2k^2s^3CE(p+qx)(1+(s+r)x)^2\\
&\quad - k^2(k+1)s^3E(p+qx)(1+(r+s)x)^2[2(r+s)(p+qx)+q(1+(r+s)x)]\\
&= -2ks^2C^2E^2 - 2(k+2)s(r+s)C^2E(1+r\bar{x}+sx)^2\\
&\quad - ks^2CE^2[2(2k+3)(r+s)(p+qx) - (k+1)q(1+(r+s)x)]\\
&\quad - 2k^2s^3CE(p+qx)(1+(r+s)x)^2\\
&\quad - k^2(k+1)s^3E(p+qx)(1+(r+s)x)^2[2C+3q(1+(r+s)x)]\\
&= -2ks^2C^2E^2 - 2(k+2)s(r+s)(1+r\bar{x}+sx)^2C^2E\\
&\quad - ks^2CE^2[2(2k+3)(r+s)(p+qx) - (k+1)q(1+(r+s)x)]\\
&\quad - 2k^2(k+2)s^3CE(p+qx)(1+(s+r)x)^2\\
&\quad - 3k^2(k+1)qs^3E(p+qx)(1+(r+s)x)^3 < 0.
\end{aligned}
$$

It is easy to see

$$
\begin{aligned}
L &= -k(k-1)s^2(r+s)CE^2(p+qx) - \frac{1}{2}k^2(k^2-1)s^4(p+qx)^2(1+(r+s)x)^4\\
&\quad - k^2(k-1)s^3CE(p+qx)(1+(r+s)x)^2 + 3k^2(k+1)qs^3E(p+qx)(1+(r+s)x)^3\\
&\quad + 6kqs(r+s)CE^2(1+r\bar{x}+sx) + 6k^2qs^2CE^2(1+(r+s)x)\\
&\quad + 2k(k+2)s^2C^2E^2 + 2(k+2)s(r+s)C^2E(1+r\bar{x}+sx)^2\\
&\quad + k(k+1)(k+2)s^3CE(p+qx)(1+(r+s)x)^2.
\end{aligned}
$$

Combine J and L to obtain

$$\begin{aligned}
I = {} & [2k(k+2)s^2 - 2ks^2]C^2E^2 + \{6kqs(r+s)(1+r\bar{x}+sx) \\
& + 6k^2qs^2(1+(r+s)x) - k(k-1)(r+s)s^2(p+qx) \\
& - ks^2[2(2k+3)(r+s)(p+qx) - (k+1)q(1+(r+s)x)]\}CE^2 \\
& + [k(k+1)(k+2) - k^2(k-1) - 2k^2(k+2)]s^3CE(p+qx)(1+(r+s)x)^2 \\
& - \frac{1}{2}k^2(k^2-1)s^4(p+qx)^2(1+(r+s)x)^4 \\
= {} & 2k(k+1)s^2C^2E^2 + CE^2[6kqs(r+s)(1+r\bar{x}+sx) \\
& + 6k^2qs^2(1+(r+s)x) - k(k+1)s^2[5(r+s)(p+qx) \\
& - q(1+(r+s)x)] - 2k(k^2-1)s^3CE(p+qx)(1+(r+s)x)^2 \\
& - \frac{1}{2}k^2(k^2-1)s^4(p+qx)^2(1+(r+s)x)^4 \\
= {} & 2k(k+1)s^2C^2E^2 + CE^2[6kqs(r+s)(1+r\bar{x}+sx) \\
& + 6k^2qs^2(1+(r+s)x) - k(k+1)s^2[5(p(r+s)-q) \\
& + 6q(s+r)x + 6q]] + M \\
= {} & 2k(k+1)s^2C^2E^2 + CE^2[6kqs(r+s)(1+r\bar{x}+sx) \\
& + 6k^2qs^2(1+(r+s)x) - 5k(k+1)s^2C \\
& - 6k(k+1)qs^2(1+(r+s)x)] + M \\
= {} & -3k(k+1)s^2C^2E^2 + 6kqsrCE^2(1+(r+s)\bar{x}) + M \\
= {} & -3ksCE^2[(k+1)sC - 2qr(1+(r+s)\bar{x})] + M \\
= {} & -3ksCE^2[(k+1)s[p(r+s)-q] - 2qr - 2qr(r+s)\bar{x})] + M \\
= {} & -3ksCE^2[(r+s)[(k+1)ps - 2qr\bar{x}] - (k+1)qs - 2qr] + M \\
= {} & -3ksCE^2[(r+s)[(k+1)(ps - q(1+r\bar{x}) + q(1+r\bar{x})) \\
& - 2qr\bar{x}] - (k+1)qs - 2qr] + M \\
= {} & -3ksCE^2[(r+s)(k+1)(ps - q(1+r\bar{x})) + (k+1)q(r+s) \\
& + (k-1)q(r+s)r\bar{x} - (k+1)qs - 2qr] + M.
\end{aligned}$$

where

$$\begin{aligned}
M = {} & -2k(k^2-1)s^3(p+qx)(1+(r+s)x)^2CE \\
& - \frac{1}{2}k^2(k^2-1)s^4(p+qx)^2(1+(r+s)x)^4 \\
< {} & 0.
\end{aligned}$$

Therefore, one has

$$\begin{aligned}
I = {} & -3ksCE^2[(r+s)(k+1)(ps - q(1+r\bar{x})) + (k-1)qr(1+(r+s)\bar{x})] \\
& - 2k(k^2-1)s^3CE(p+qx)(1+(r+s)x)^2 \\
& - \frac{1}{2}k^2(k^2-1)s^4(p+qx)^2(1+(r+s)x)^4.
\end{aligned}$$

Notice that the condition $0 < q < \frac{ps}{1+r\bar{x}}$ implies $q(1+r\bar{x}) < ps$, and k is a positive integer, i.e., $k \geq 1$. So, it is easy to see $I \leq 0$. $I = 0 \Leftrightarrow s = 0 \Leftrightarrow F'(x) = 0$. Hence, $SF(x) < 0$ for everywhere $x \in (0, \infty)$, except for the point x where $F'(x) = 0$.

Up to here, the proof of Lemma 5 is complete. □

3. Main Results and Their Proofs

In this section, our main results and their proofs will be formulated. More specifically, the main idea for the proof of Theorem 2 is to invoke four key lemmas, namely Lemma 2 to Lemma 5, to transform the higher-order RDE (1) into a first-order difference equation, which is easier to handle.

Theorem 2. *Consider Equation (1). Assume that the parameters $p, q, r, s \in [0, \infty)$ with $p + q > 0$ and $r + s > 0$, and the parameter k is a positive integer. Then, the unique nonnegative equilibrium point $\bar{x}$ of Equation (1) is a global attractor of all of its positive solutions if one of the following conditions holds:*

1. $p = 0$ *and* $0 < q \leq 1$;
2. $q < \frac{ps}{1+r\bar{x}}$;
3. $r > 0$ *and* $q < 1$.

Proof. First, consider Case 1, as follows: $p = 0$ and $0 < q \leq 1$. At this time, $\bar{x} = 0$. It is easy to see from Equation (1) that $x_{n+1} < qx_n$ and so x_n eventually monotonically approaches $\bar{x}$. Notice that $\overline{\{p = 0\} \cap \{q \in (0, 1]\}} = \{p > 0\} \cup \{q \in (1, \infty)\}$. Hence, in the sequel, one may also assume that $p > 0$ or $q \in (1, \infty)$, more specifically, we only consider the behavior of the positive equilibrium point of Equation (1).

Next, consider Case 2, as follows: $q < \frac{ps}{1+r\bar{x}}$. Notice that Equation (1) may be written as

$$x_{n+1} = x_n \frac{\frac{p}{x_n} + q}{1 + rx_n + sx_{n-k}}. \tag{18}$$

Set

$$f(u_0, u_1, .., u_k) = \frac{\frac{p}{u_0} + q}{1 + ru_0 + su_k}.$$

It is easy to verify that the function f satisfies the conditions (H1)–(H4) of Lemma 2. Therefore, the function G defined by Equation (12) may be derived as follows:

$$\begin{aligned} G(x, y) &= y \frac{\frac{p}{y} + q}{1 + ry + sx} \frac{\frac{p}{\bar{x}} + q}{1 + r\bar{x} + sy} \left(\frac{\frac{p}{\bar{x}} + q}{1 + r\bar{x} + sx}\right)^{k-1} \\ &= \frac{p + qy}{(1 + r\bar{x} + sy)(1 + ry + sx)} \frac{[1 + (r + s)\bar{x}]^k}{[1 + r\bar{x} + sx]^{k-1}}. \end{aligned}$$

Moreover,

$$\frac{\partial G(x, y)}{\partial y} = \frac{[1 + (r + s)\bar{x}]^k}{(1 + r\bar{x} + sx)^{k-1}} \frac{H(x, y)}{(1 + ry + sx)^2 (1 + r\bar{x} + sy)^2}, \tag{19}$$

where $H(x, y) = -qsry^2 - 2psry + q(1 + r\bar{x})(1 + sx) - pr(1 + r\bar{x}) - ps(1 + sx)$.

Obviously, $H(x, y)$ is a quadratic function for the variable y with the opening down, and its symmetric axis is $y = -\frac{p}{q} < 0$. So, it suffices to prove that $H(x, 0) < 0$. Notice that $H(x, 0) = [q(1 + r\bar{x}) - ps](1 + sx) - pr(1 + r\bar{x})$. So, for $q < \frac{ps}{1+r\bar{x}}$, we can see that $H(x, 0) < 0$ for $x > 0$. That is, for any $x \in (0, \infty), y \in (0, \infty)$, $\frac{\partial G(x,y)}{\partial y} < 0$. Therefore, $G(x, y)$ is decreasing with respect to the variable y for any $x \in (0, \infty)$. Accordingly, in terms of (11), the function $F(x)$ is derived as

$$F(x) = \frac{A(p + qx)}{(1 + (r + s)x)(1 + r\bar{x} + sx)^k},$$

where $\bar{x}, x \in (0, \infty)$, and the constant $A = [1 + (r + s)\bar{x}]^k > 0$, which is the same as (15) in Lemma 5. Based on Lemma 5, we have $SF(x) < 0$. Thus, according to Lemma 4, $\bar{x}$ is a

global attractor of all positive solutions of Equation (13). In turn, according to Lemma 3, $\bar{x}$ is the only fixed point of F^2 in $(0, \infty)$. Then, using Lemma 2 (b), it is shown that $\bar{x}$ is a global attractor for all positive solutions of Equation (10), and hence Equations (18) and (1).

Finally, consider Case 3, as follows: $r > 0$ and $q < 1$. It follows from Equation (1) that

$$x_{n+1} = \frac{p + qx_n}{1 + rx_n + sx_{n-k}} \leq \frac{p + qx_n}{1 + rx_n} \leq \frac{max\{p, q\}}{min\{1, r\}} =: T, n = 0, 1, 2, \cdots.$$

Accordingly, we can see that $0 < x_n \leq T,\ n = 1,\ 2, \cdots$. So, the sequence $\{x_n\}$ has upper and lower limits. Let

$$\lim_{n\to\infty} inf\ x_n = \lambda \quad \text{and} \quad \lim_{n\to\infty} sup\ x_n = \mu.$$

Then,

$$\lambda \leq \bar{x} \leq \mu. \tag{20}$$

From Equation (1), it is easy to derive that

$$\lambda \geq \frac{p + q\lambda}{1 + r\mu + s\mu} \quad \text{and} \quad \mu \leq \frac{p + q\mu}{1 + r\lambda + s\lambda}.$$

That is,

$$\lambda + (r + s)\lambda\mu \geq p + q\lambda \quad \text{and} \quad \mu + (r + s)\lambda\mu \leq p + q\mu.$$

Hence

$$p + (q - 1)\lambda \leq (r + s)\lambda\mu \leq p + (q - 1)\mu,$$

and so,

$$(q - 1)\lambda \leq (q - 1)\mu. \tag{21}$$

Since $q < 1$, it follows from (21) that $\lambda \geq \mu$. Together with (20), we have $\lambda = \mu = \bar{x}$. That is

$$\lim_{n\to\infty} x_n = \bar{x}.$$

Thus, the proof of this theorem is complete. □

Theorem 3. *Assume that $q \geq pr$. Then, $[0, r_0]$ is an invariant interval of Equation (1), where r_0 is a positive fixed point of the function $g(x) = \frac{p+qx}{1+rx}$, i.e., $g(r_0) = r_0$. Moreover, every solution of Equation (1) converges to $\bar{x}$ with the attractive basin $[0, r_0]$ if either $r = s$ or $r < s, 4pr^2 = (s - r)(q - 1)^2$.*

Proof. Define a new function $f(x, y)$ given by

$$f(x, y) = \frac{p + qx}{1 + rx + sy}, x > 0, y > 0.$$

Obviously,

$$\frac{\partial f(x, y)}{\partial x} = \frac{q - pr + qsy}{(1 + rx + sy)^2}.$$

Then, when $q \geq pr$, $f(x, y)$ is increasing in x for each fixed y and decreasing in y for each fixed x.

Moreover, $g'(x) = (q - pr)/(1 + rx)^2$, so $g(x)$ is nondecreasing for $q \geq pr$. For $x_{-k}, ..., x_0 \in [0, r_0]$, we have

$$x_1 = \frac{p + qx_0}{1 + rx_0 + sx_{-k}} \leq \frac{p + qx_0}{1 + rx_0} = g(x_0) \leq g(r_0) = r_0. \tag{22}$$

By induction, all solutions of Equation (1) eventually enter the interval $[0, r_0]$. Hence, $[0, r_0]$ is an invariant interval of Equation (1). Further, it is easy to see that $f(x, y)$ increases in x and decreases in y in $[0, r_0]$. Let $m, M \in [0, r_0]$ be a solution of the system

$$\frac{p + qm}{1 + rm + sM} = m \quad \text{and} \quad \frac{p + qM}{1 + rM + sm} = M, \tag{23}$$

which is equivalent to

$$p + qm = m + rm^2 + sMm \quad \text{and} \quad p + qM = M + rM^2 + sMm. \tag{24}$$

Thereout,

$$(m - M)[q - 1 - r(m + M)] = 0. \tag{25}$$

Now, if $r(m + M) \neq q - 1$, then (25) implies $m = M$.

If $r(m + M) = q - 1$, then it follows from (24) that m and M must be two positive roots of the equation

$$r(r - s)x^2 + (1 - q)(r - s)x - pr = 0, \tag{26}$$

whose discriminant is

$$\Delta = (1 - q)^2(r - s)^2 + 4pr^2(r - s) = (r - s)[(r - s)(1 - q)^2 + 4r^2p].$$

Regardless of whether $r = s$ or $r < s$ with $4pr^2 = (s - r)(q - 1)^2$, $\Delta = 0$ always holds. Accordingly, $m = M$. In view of Lemma 1, every solution of Equation (1) converges to $\bar{x}$. That is, the unique equilibrium point $\bar{x}$ of Equation (1) is globally attractive. The proof is finished. □

Remark 1. *Equation (1) is equivalent to the following difference equation:*

$$x_{n+1} = \frac{p + qx_n}{A + rx_n + sx_{n-k}}, \ n = 0, 1, \cdots, \tag{27}$$

where the parameters $A > 0$, $p, q, r, s \in [0, \infty)$ with $p + q > 0$ and $r + s > 0$; k is a positive integer; and the initial conditions $x_{-k}, ..., x_{-1}, x_0 \in [0, \infty)$.

In fact, when $A > 0$, the parameter change $(p, q, r, s) \to (Ap, Aq, Ar, As)$ in (27) transforms Equation (27) into Equation (1). So, according to Theorem 2, we have the following result.

Corollary 1. *Consider Equation (27). Assume that the parameters $A > 0$, $p, q, r, s \in [0, \infty)$ with $p + q > 0$ and $r + s > 0$, and the parameter k is a positive integer. Then, the unique nonnegative equilibrium point $\bar{x}$ of Equation (27) is a global attractor of all of its positive solutions if one of the following conditions is satisfied:*

1. *$p = 0$ and $0 < q \leq A$;*
2. *$q < \frac{ps}{A + r\bar{x}}$;*
3. *$r > 0$ and $q < A$.*

4. Applications

In this section, we present several applications of our results.

Example 1. *Consider Equation (2). We obtain the following result.*

Theorem 4. *Assume that (3) holds. Then, the positive equilibrium point $\bar{x}$ of Equation (2) is globally asymptotically stable.*

Proof. The condition (3) can be divided into the following two cases: (1) $a \leq Ab$ and (2) $a > Ab$. It follows from (1) and (2) of Theorem 1 that the positive equilibrium point $\bar{x}$ of Equation (2) is globally asymptotically stable for $a \leq Ab$.

Additionally, according to Case 2 (r = 0, s = 1) of Corollary 1, we see that for $b < \frac{a}{A}$, i.e., $Ab < a$, the positive equilibrium point $\bar{x}$ of Equation (2) is globally asymptotically stable. Hence, our result, together with Theorem 1, shows that the positive equilibrium point $\bar{x}$ of Equation (2) is globally asymptotically stable.

Up to here, Conjecture 1 has been proved to be correct. Notice that Conjecture 2 is a special case of Conjecture 1. Therefore, Conjecture 2 is also proved. Of course, our results also improve or supplement the corresponding ones in [10–16]. See also [21]. □

Example 2. *Consider Equation (5). The following consequence may be derived.*

Theorem 5. *Assume that (6) holds. If $a > Ab$, then the positive equilibrium point $\bar{x}$ of Equation (5) is globally asymptotically stable.*

Proof. According to Case 2 of Corollary 1, we see that, for $Ab < a$, the positive equilibrium point $\bar{x}$ of Equation (5) is globally asymptotically stable.

It is easily seen that our results include the case in [1]: (1) $a > 0$, and $b = 0$ and improve the cases in [1], as follows: (2) $b > 0, k \geq 2, Ab < a$ and $\bar{x}k \leq A$; (3) $b > 0, k = 1$, and $Ab \leq a \leq 2Ab + 2A^2$. □

Example 3. *Consider the following difference equation [[3], **Equation** ♯67]*

$$x_{n+1} = \frac{p + x_n}{A + x_{n-2}}, \; n = 0, 1, \cdots . \tag{28}$$

with nonnegative parameters p, A and arbitrary nonnegative initial conditions x_{-2}, x_{-1}, x_0. Camouzis and Ladas obtained in [3] that the equilibrium point $\bar{x}$ of Equation (28) is globally asymptotically stable for $A \geq 1$.

Equation (28) is a special case of Equation (27) with $q = 1, r = 0, s = 1$. According to our Corollary 1, we easily obtain the following new result.

Theorem 6. *Assume that $A < p$. Then, the equilibrium point $\bar{x}$ of Equation (28) is globally asymptotically stable.*

If $p = 1$, our result implies that the conclusion holds for $A < 1$. Together with the known result in the case $A \geq 1$, we see that the equilibrium point $\bar{x}$ of Equation (28) is globally asymptotically stable for $A > 0$. If $0 < q < 1$, our results show that the equilibrium point $\bar{x}$ of Equation (28) is globally asymptotically stable for $A < p$, whereas the known result in [3] cannot determine this. Thus, our result supplements the known conclusion.

Example 4. *Consider Equation (8) [17]. Under the assumption of positive parameters, the authors of [17] transformed the difference Equation (8) into the following difference equation*

$$x_{n+1} = \frac{p + qx_n}{1 + x_n + rx_{n-k}}, \quad n = 0, 1, \cdots . \tag{29}$$

and obtained that the positive equilibrium point $\bar{x}$ of Equation (29) is globally asymptotically stable if one of the following conditions holds:

1. *$p > q(qr + 1)$ and the initial values in the interval $[q, \frac{p-q}{qr}]$;*
2. *$q < p < q(qr + 1)$ and the initial values in the interval $[\frac{p-q}{qr}, q]$;*
3. *$p \leq q$ and the initial values in the interval $[0, \frac{q-1+\sqrt{(q-1)^+ 4p}}{2}]$.*

However, according to our Theorem 2, the positive equilibrium point $\bar{x}$ of Equation (29) is globally asymptotically stable for either $p = 0$ and $0 < q \leq 1$ or $q < \frac{pr}{1+\bar{x}}$ and any nonnegative initial values. Whereas Theorem 3 demonstrates that the positive equilibrium point $\bar{x}$ of Equation (29) is globally asymptotically stable for $p \leq q$ and either $r = 1$ or $r > 1, 4p = (r-1)(q-1)^2$ and the initial values are in the interval $[0, \frac{q-1+\sqrt{(q-1)^+4p}}{2}]$.

It is easy to see that our results are different from the ones in [17]. So, our results supplement the ones in [17].

Example 5. *Consider the following difference Equation [18]*

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + C x_{n-1}}, \; n = 0, 1, \cdots . \tag{30}$$

with nonnegative parameters α, β, A, and C and arbitrary nonnegative initial conditions x_{-1}, x_0. Equation (30) is a special case of Equation (27) with $r = 0, k = 1$.

The authors in [18] found that the positive equilibrium point of Equation (30) is globally asymptotically stable when $\beta < A$, and also when $\beta \geq A$ and

$$\textit{either} \quad \alpha C \leq \beta A \quad \textit{or} \quad \beta A < \alpha C \leq 2(\frac{\beta}{A} + 1)$$

Furthermore, they put forward in [18] the following conjecture:

Conjecture 3. *The positive equilibrium point of Equation (30) is globally asymptotically stable for nonnegative parameters α, β, A, and C.*

Our analysis confirms that Conjecture 3 holds.

In fact, according to Corollary 1, we can show that the positive equilibrium point of Equation (30) is globally asymptotically stable for $\alpha C > \beta A$. However, the results in [18] indicate that the same conclusion holds for $\alpha C \leq \beta A$. So, by combining our result with the known results in [18], we see that the positive equilibrium point of Equation (30) is globally asymptotically stable. So our results not only supplement the known results in [18], but they also together resolve Conjecture 3.

5. Conclusions and Discussion

We investigate the global attractivity of the nonnegative equilibrium point for a RDE. By combining three key lemmas from Kocic and Ladas' monograph [1] and performing complex mathematical computations, we derive a result on the global attractivity of this equation, which has a wide range of applications. Our results not only improve or supplement many known findings but also resolve several conjectures presented in the existing literature.

(1) Resolution of long-standing conjectures: We provide a complete theoretical proof for Conjectures 1 and 2 posed by Ladas et al., which remained open for decades. More specifically, we demonstrate that the positive equilibrium of Equation (2) is globally asymptotically stable under general parameter conditions, resolving a fundamental question in the dynamics of rational difference equations.

(2) Generalization of existing results: By unifying several special cases—such as Equations (2), (4), (5) and (8)—under the framework of Equation (1), we extend and improve prior stability criteria. For instance, Theorem 2 establishes the global attractivity under novel parameter constraints (e.g., $q < \frac{ps}{1+r\bar{x}}$), which broadens the applicability of earlier theorems.

(3) Methodological advancements: The introduction of key lemmas (e.g., Lemma 5) and the analysis of the Schwarzian derivative provide a robust toolkit for transforming higher-order equations into first-order systems, enabling the use of monotonicity and fixed-point arguments. This approach offers a blueprint for tackling similar nonlinear recurrence relations.

(4) Applications across models: Through concrete examples (e.g., Theorems 4–6), we validate the universality of our results and address conjectures in specific models, such as the stability of equilibrium points in equations with delay terms or asymmetric coefficients.

The study of RDEs remains an area of significant interest for many researchers. Several problems related to RDEs, such as periodicity [4,5], dichotomy [22], trichotomy [2,3], bifurcation, and chaos [23], continue to be active research topics. However, there is still a lack of effective methods and approaches to address these problems. Therefore, RDEs, particularly these challenging issues, warrant further investigations.

While our study has made substantial progress in understanding the global dynamics of autonomous rational difference equations, there are several avenues for future research that could further enhance our understanding of these systems. One particularly interesting direction is the investigation of non-autonomous rational difference equations of the same form. Non-autonomous equations, which involve time-varying coefficients or external inputs, are more complex but also more realistic in many applications, such as biological systems with seasonal variations or economic models with changing parameters.

To investigate the global dynamics of a non-autonomous rational difference equation of the form

$$x_{n+1} = \frac{p_n + q_n x_n}{1 + r_n x_n + s_n x_{n-k}},$$

where p_n, q_n, r_n, s_n are sequences of nonnegative numbers, several approaches could be considered, as follows:

(1) Periodic coefficients: If the coefficients p_n, q_n, r_n, s_n are periodic with a common period, one could analyze the existence and stability of periodic solutions. Techniques from Floquet theory or Poincaré maps might be applicable to study the long-term behavior of such systems.

(2) Boundedness and asymptotic behavior: Investigate conditions under which the solutions remain bounded and determine the asymptotic behavior of the solutions. This could involve finding invariant intervals or using Lyapunov functionals to establish global attractivity or stability properties.

(3) Numerical simulations: Conduct extensive numerical simulations to explore the qualitative behavior of non-autonomous RDEs. This can provide insights into the possible dynamics, such as the existence of chaotic behavior or the emergence of new types of attractors.

(4) Applications in real-world problems: Apply the theory of non-autonomous RDEs to specific real-world problems, such as population dynamics with environmental fluctuations, economic models with varying parameters, or epidemiological models with seasonal changes. This can lead to the development of more accurate and predictive models.

In addition to these specific suggestions, further research could focus on developing new analytical techniques and tools to handle the complexities introduced by non-autonomous terms. This might involve extending existing methods, such as the use of Schwarzian derivatives or fixed-point theorems, to non-autonomous settings or developing entirely new approaches tailored to these equations.

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