

Review

A Review of the Chebyshev Inequality Pertaining to Fractional Integrals

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Abstract: In this article, we give a brief review of a well-known integral inequality that gives information about the integral of the product of two functions using synchronous functions, the Chebyshev inequality. We have compiled the most relevant information about fractional and generalized integrals, which are one of the most dynamic topics in today's mathematical sciences. After presenting the classical formulation of the inequality using Lebesgue integrable functions, the most general results known from the literature are collected in an attempt to present the reader with a current overview of this research topic.

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1. Introduction

We recall the well-known Chebyshev inequality (see [1])

$$\frac{1}{b-a} \int_a^b f(x)g(x) dx \geq \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right), \quad (1)$$

where $a < b$, $a, b \in \mathbb{R}$, and f and g are two Lebesgue integrable functions on $[a, b]$ which are synchronous on $[a, b]$, that is

$$(f(x) - f(y))(g(x) - g(y)) \geq 0 \quad (x, y \in [a, b]).$$

This integral inequality is one of those that provides an upper bound on the product of the mean value of two functions by the mean value of the product of said functions. This has caught the attention of many researchers who have made it an object of study, which has led to an increase in related publications; as we will see below, this development has occurred in the following basic directions: on the one hand, through the approach of synchronous functions (or equivalent arguments); on the other, through new integral operators (fractional or generalized); and finally, through considering the functional

$$T(f, g) = \frac{1}{b-a} \int_a^b f(x)g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right),$$

which is useful to give a lower or upper bound for $T(f, g)$ in the theory of approximations since Chebyshev's inequality follows from $T(f, g) \geq 0$. Each of the above has converted this integral inequality into a current research topic.

Inequality (1) has many applications in diverse research areas such as probability and statistics [2–5], quantum calculus [6,7], time scale calculus [8,9], and economics [10]. Several authors have investigated generalizations of the Chebyshev inequality (1), and these are called Chebyshev-type inequalities. An important way of generalization is via fractional or generalized integral operators. Recently, several kinds of various fractional integrals and derivatives have been investigated by many researchers. Chebyshev-type inequalities were considered via Riemann–Liouville [11–14], Hadamard [15–17], Caputo–Fabrizio [18], Katugampola [19,20], generalized Riemann–Liouville [21–25], Erdélyi–Kober [26,27], Saigo [28], Atangana–Baleanu [29], Raina fractional integrals [30,31], generalized Raina fractional integrals [32,33], and other types of generalized integral operators [34–46].

In the rest of this paper, we present a review of the latest results concerning inequality (1) in the framework of fractional and generalized integral operators.

2. Inequalities

In this section, we recall different definitions of fractional and generalized integrals, as well as different versions of Chebyshev's inequality.

The well-known definition of the Riemann–Liouville fractional integrals reads as follows.

Definition 1 ([47]). Let $a < b$ and $f \in L_{loc}^1([a, b])$. The left and right side Riemann–Liouville fractional integrals of order α , with $\operatorname{Re}(\alpha) > 0$, are defined, respectively, by

$${}^{RL}I_a^\alpha \{f(t)\} = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds,$$

and

$${}^{RL}I_b^\alpha \{f(t)\} = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds,$$

where $t \in (a, b)$ and $\Gamma(\cdot)$ is the Gamma function $\Gamma(\alpha) = \int_0^\infty e^{-s} s^{\alpha-1} ds$.

Belarbi and Dahmani proved the following theorems related to the Chebyshev inequality involving Riemann–Liouville fractional integral operator ${}^{RL}I^\alpha \{f(t)\}$.

Theorem 1 ([11]). Let $f, g : [0, \infty) \rightarrow \mathbb{R}$ be two synchronous functions on $[0, \infty)$. Then, for all $t, \alpha > 0$, we have

$${}^{RL}I_0^\alpha \{fg(t)\} \geq \frac{\Gamma(\alpha+1)}{t^\alpha} {}^{RL}I_0^\alpha \{f(t)\} {}^{RL}I_0^\alpha \{g(t)\}.$$

Theorem 2 ([11]). Let $f, g : [0, \infty) \rightarrow \mathbb{R}$ be two synchronous functions on $[0, \infty)$. Then, for all $t, \alpha, \beta > 0$, we have

$$\begin{aligned} & \frac{t^\alpha}{\Gamma(\alpha+1)} {}^{RL}I_0^\beta \{fg(t)\} + \frac{t^\beta}{\Gamma(\beta+1)} {}^{RL}I_0^\alpha \{fg(t)\} \\ & \geq {}^{RL}I_0^\alpha \{f(t)\} {}^{RL}I_0^\beta \{g(t)\} + {}^{RL}I_0^\beta \{f(t)\} {}^{RL}I_0^\alpha \{g(t)\}. \end{aligned}$$

Dahmani et al. proved the following result for Riemann–Liouville fractional integral operators in the case when the synchronicity of the two functions is replaced by a more general condition. This result generalizes the Chebyshev-type inequality in [48] Theorem 1.

Theorem 3 ([13]). Let f and g be two functions of the space $L^\infty([a, b])$ and suppose that for any $\alpha \geq 1$ and for any $a < t \leq x \leq b$, the inequality

$$\left(f(t) - \frac{1}{t-a} \int_a^t f(s) ds\right) \left((x-t)^{\alpha-1} g(t) - \frac{1}{t-a} \int_a^t (x-s)^{\alpha-1} g(s) ds\right) \geq 0$$

is satisfied. Then, we have

$$\frac{1}{t-a} {}^{RL}_a I^\alpha \{fg(t)\} \geq \left(\frac{1}{t-a} \int_a^t f(s) ds\right) \left(\frac{1}{t-a} {}^{RL}_a I^\alpha \{g(t)\}\right).$$

The Hadamard fractional integrals were defined in the following way.

Definition 2 ([15]). The Hadamard fractional integral of order $\alpha \in \mathbb{R}^+$ of a Lebesgue integrable function f , for all $t > 1$, is defined as

$${}_1^H I^{-\alpha} \{f(t)\} = \frac{1}{\Gamma(\alpha)} \int_1^t \ln\left(\frac{t}{s}\right)^{\alpha-1} f(s) \frac{ds}{s},$$

where $\Gamma(\cdot)$ is the Gamma function.

From above definitions, we see the difference between Hadamard and Riemann–Liouville fractional integrals is that the kernel in the Hadamard integral has the form of $\ln\left(\frac{t}{s}\right)$ instead of the form of $(t-s)$, which is involved in the Riemann–Liouville integral. Hadamard fractional calculus is more suitable for describing phenomena unrelated to dilation on the semi-axis, while Riemann–Liouville fractional calculus is better appropriate to describe abnormal convection and diffusion phenomena [49].

Chebyshev's inequality was extended in [15] for the Hadamard fractional integrals.

Theorem 4 ([15]). Let $f, g : [0, \infty) \rightarrow \mathbb{R}$ be two Lebesgue integrable functions which are synchronous on $[0, \infty)$. Then for all $t, \alpha, \beta > 0$, we have

$${}_1^H I^{-\alpha} \{fg(t)\} \geq \frac{\Gamma(\alpha+1)}{(\ln t)^\alpha} {}_1^H I^{-\alpha} \{f(t)\} {}_1^H I^{-\alpha} \{g(t)\}$$

and

$$\begin{aligned} & \frac{(\ln t)^\beta}{\Gamma(\beta+1)} {}_1^H I^{-\alpha} \{fg(t)\} + \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} {}_1^H I^{-\beta} \{fg(t)\} \\ & \geq {}_1^H I^{-\alpha} \{f(t)\} {}_1^H I^{-\beta} \{g(t)\} + {}_1^H I^{-\alpha} \{g(t)\} {}_1^H I^{-\beta} \{f(t)\}. \end{aligned}$$

Recently, Caputo and Fabrizio introduced a new fractional integral operator without a singular kernel in [50]. This fractional integral involves the nonsingular kernel $e^{-(\frac{1-\alpha}{\alpha})(t-s)}$, $0 < \alpha < 1$. The main advantage of the Caputo–Fabrizio integral operator is that the boundary condition of the fractional differential equations with Caputo–Fabrizio derivatives admits the same form as for the integer-order differential equations [18].

Definition 3 ([50]). Let $\alpha \in \mathbb{R}$ such that $0 < \alpha \leq 1$. The Caputo–Fabrizio fractional integral of order α of a Lebesgue integrable function f is defined by

$${}_0^C I^\alpha \{f(t)\} = \frac{1}{\alpha} \int_0^t e^{-(\frac{1-\alpha}{\alpha})(t-s)} f(s) ds$$

for all $t > 0$.

Theorem 5 ([18]). Let $f, g : [0, \infty) \rightarrow \mathbb{R}$ be two Lebesgue integrable functions which are synchronous on $[0, \infty)$. Then, for all $t > 0$ and $0 < \alpha, \beta \leq 1$, we have

$${}_0^C I^\alpha \{fg(t)\} \geq \frac{1-\alpha}{1-e^{-(\frac{1-\alpha}{\alpha})t}} {}_0^C I^\alpha \{f(t)\} {}_0^C I^\alpha \{g(t)\}$$

and

$$\begin{aligned} & \frac{1-e^{-(\frac{1-\beta}{\beta})t}}{1-\beta} {}_0^C I^\alpha \{fg(t)\} + \frac{1-e^{-(\frac{1-\alpha}{\alpha})t}}{1-\alpha} {}_0^C I^\beta \{fg(t)\} \\ & \geq {}_0^C I^\alpha \{f(t)\} {}_0^C I^\beta \{g(t)\} + {}_0^C I^\alpha \{g(t)\} {}_0^C I^\beta \{f(t)\}. \end{aligned}$$

Katugampola defined the following fractional integral operator in [51], which generalizes both the Riemann–Liouville and Hadamard fractional integrals into a single form:

$${}_a^K I^{\alpha, \beta} \{f(t)\} = \frac{\beta^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\beta-1} f(s)}{(t^\beta - s^\beta)^{1-\alpha}} ds,$$

where $f \in X_c^p(a, b)$, $t > a$, $\alpha \in \mathbb{C}$, $\operatorname{Re}(\alpha) > 0$, $\beta > 0$. Here, the space $X_c^p(a, b)$ for $c \in \mathbb{R}$, $1 \leq p \leq \infty$, consists of those complex-valued Lebesgue measurable functions f on $[a, b]$ for which $\|f\|_{X_c^p} < \infty$, where the norm is defined by

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{1/p} < \infty,$$

for $1 \leq p < \infty$, $c \in \mathbb{R}$ and for the case $p = \infty$, $c \in \mathbb{R}$,

$$\|f\|_{X_c^\infty} = \operatorname{ess\,sup}_{a \leq t \leq b} [t^c |f(t)|].$$

Dubey and Goswami and also Set, Mumcu, and Demirbas proved the Chebyshev-type inequality involving the Katugampola fractional integrals.

Theorem 6 ([19,20]). Let $f, g : [0, \infty) \rightarrow \mathbb{R}$ be two Lebesgue integrable functions which are synchronous on $[0, \infty)$. Then, for all $t, \alpha, \beta > 0$, we have

$${}_0^K I^{\alpha, \beta} \{fg(t)\} \geq \frac{{}_0^K I^{\alpha, \beta} \{f(t)\} {}_0^K I^{\alpha, \beta} \{g(t)\}}{{}_0^K I^{\alpha, \beta} \{1\}}.$$

Here, ${}_0^K I^{\alpha, \beta} \{1\} = \frac{t^{\alpha\beta}}{\Gamma(\alpha+1)\beta^\alpha}$, which was correctly indicated in [20,33].

Theorem 7 ([20]). Let $f, g : [0, \infty) \rightarrow \mathbb{R}$ be two Lebesgue integrable functions which are synchronous on $[0, \infty)$. Then, for all $t, \alpha, \beta, \tau > 0$, we have

$$\begin{aligned} & \frac{t^{\tau\beta}}{\Gamma(\tau+1)\beta^\tau} {}_0^K I^{\alpha, \beta} \{fg(t)\} + \frac{t^{\alpha\beta}}{\Gamma(\alpha+1)\beta^\alpha} {}_0^K I^{\tau, \beta} \{fg(t)\} \\ & \geq {}_0^K I^{\alpha, \beta} \{f(t)\} {}_0^K I^{\tau, \beta} \{g(t)\} + {}_0^K I^{\tau, \beta} \{f(t)\} {}_0^K I^{\alpha, \beta} \{g(t)\}. \end{aligned}$$

Sarikaya et al. in [24] defined the (k, s) -Riemann–Liouville fractional integrals of order α , which generalizes all of the fractional integrals above. They provided the following Chebyshev-type inequalities for this type of integrals together with the proof of commutativity and the semigroup properties of the (k, s) -Riemann–Liouville fractional integrals.

Definition 4 ([24]). Let f be a continuous function on the finite real interval $[a, b]$. Then, the (k, s) -Riemann–Liouville fractional integral of f of order $\alpha > 0$ is defined by

$${}^{k,s}_a I^\alpha \{f(t)\} = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} x^s f(x) dx,$$

where $t \in [a, b]$, $k > 0$, $s \in \mathbb{R} \setminus \{-1\}$ and $\Gamma_k(\alpha) = \int_0^\infty e^{-\frac{x^k}{k}} x^{\alpha-1} dx$.

Theorem 8 ([24]). Let $f, g : [0, \infty) \rightarrow \mathbb{R}$ be two synchronous functions on $[0, \infty)$. Then, for all $t > a \geq 0$, $\alpha, \beta > 0$, the following inequalities hold for (k, s) -Riemann–Liouville fractional integrals:

$${}^{k,s}_a I^\alpha \{fg(t)\} \geq \frac{1}{{}^{k,s}_a I^\alpha \{1\}} {}^{k,s}_a I^\alpha \{f(t)\} {}^{k,s}_a I^\alpha \{g(t)\}$$

and

$$\begin{aligned} & {}^{k,s}_a I^\alpha \{fg(t)\} {}^{k,s}_a I^\beta \{1\} + {}^{k,s}_a I^\beta \{fg(t)\} {}^{k,s}_a I^\alpha \{1\} \\ & \geq {}^{k,s}_a I^\alpha \{f(t)\} {}^{k,s}_a I^\beta \{g(t)\} + {}^{k,s}_a I^\alpha \{g(t)\} {}^{k,s}_a I^\beta \{f(t)\}. \end{aligned}$$

Akkurt et al. considered the (k, h) -Riemann–Liouville fractional integral and proved the appropriate extension of Chebyshev inequality via the defined integral.

Definition 5 ([21]). Let (a, b) be a finite interval of the real line $\mathbb{R}$ and $\operatorname{Re}(\alpha) > 0$. Also, let $h(t)$ be an increasing and positive monotone function on (a, b) , having a continuous derivative $h'(t)$ on (a, b) . The left-sided fractional integral of a function f with respect to another function h on $[a, b]$ is defined by

$${}^{k,h}_a I^\alpha \{f(t)\} = \frac{1}{k\Gamma_k(\alpha)} \int_a^t [(h(t) - h(s))^{\frac{\alpha}{k}-1} h'(s) f(s) ds,$$

for $t > a$, $k > 0$.

Theorem 9 ([21]). Let $f, g : [0, \infty) \rightarrow \mathbb{R}$ be two synchronous functions on $[0, \infty)$. Then, for all $t > a \geq 0$, $\alpha, \beta > 0$ and $k > 0$, we have

$${}^{k,h}_a I^\alpha \{fg(t)\} \geq \frac{\Gamma_k(\alpha + k)}{[(h(t) - h(a))^{\frac{\alpha}{k}}]} {}^{k,h}_a I^\alpha \{f(t)\} {}^{k,h}_a I^\alpha \{g(t)\}$$

and

$$\begin{aligned} & \frac{[(h(t) - h(a))^{\frac{\beta}{k}}]}{\Gamma_k(\beta + k)} {}^{k,h}_a I^\alpha \{fg(t)\} + \frac{[(h(t) - h(a))^{\frac{\alpha}{k}}]}{\Gamma_k(\alpha + k)} {}^{k,h}_a I^\beta \{fg(t)\} \\ & \geq {}^{k,h}_a I^\alpha \{f(t)\} {}^{k,h}_a I^\beta \{g(t)\} + {}^{k,h}_a I^\alpha \{g(t)\} {}^{k,h}_a I^\beta \{f(t)\}. \end{aligned}$$

The following generalization of the Riemann–Liouville fractional integral is called the Erdélyi–Kober integral operator [52]:

$${}^{EK}_0 I^{\alpha, \beta, \eta} \{f(t)\} = \frac{\beta t^{-\beta(\eta+\alpha)}}{\Gamma(\alpha)} \int_0^t s^{\beta(\eta+1)-1} (t^\beta - s^\beta)^{\alpha-1} f(s) ds,$$

where $t, \alpha, \beta > 0$, $\eta \in \mathbb{R}$ and $f(t)$ is a real-valued continuous function.

The Erdélyi–Kober integral operator has number of applications in the generalized axially symmetric potential theory and other physical problems in electrostatics, elasticity, etc. (see [53]). Purohit and Kalla proved the following theorems for this integral operator.

Theorem 10 ([27]). Let $f, g : [0, \infty) \rightarrow \mathbb{R}$ be two synchronous functions on $[0, \infty)$. Then,

$${}^{EK}_0 I^{\alpha, \beta, \eta} \{fg(t)\} \geq \frac{\Gamma(1 + \alpha + \eta)}{\Gamma(1 + \eta)} {}^{EK}_0 I^{\alpha, \beta, \eta} \{f(t)\} {}^{EK}_0 I^{\alpha, \beta, \eta} \{g(t)\},$$

for all $t, \alpha, \beta > 0$ and $\eta > -1$.

Theorem 11 ([27]). Let $f, g : [0, \infty) \rightarrow \mathbb{R}$ be two synchronous functions on $[0, \infty)$. Then,

$$\begin{aligned} & \frac{\Gamma(1 + \eta)}{\Gamma(1 + \alpha + \eta)} {}^{EK}_0 I^{\gamma, \delta, \zeta} \{fg(t)\} + \frac{\Gamma(1 + \zeta)}{\Gamma(1 + \gamma + \zeta)} {}^{EK}_0 I^{\alpha, \beta, \eta} \{fg(t)\} \\ & \geq {}^{EK}_0 I^{\alpha, \beta, \eta} \{f(t)\} {}^{EK}_0 I^{\gamma, \delta, \zeta} \{g(t)\} + {}^{EK}_0 I^{\gamma, \delta, \zeta} \{f(t)\} {}^{EK}_0 I^{\alpha, \beta, \eta} \{g(t)\}, \end{aligned}$$

for all $t, \alpha, \beta, \gamma, \delta > 0$ and $\eta, \zeta > -1$.

In [54], Saigo defined an integral operator which includes both the Riemann–Liouville and the Erdélyi–Kober fractional integral operators.

Definition 6 ([54]). Let $\alpha > 0, \beta, \eta \in \mathbb{R}$. Then, the Saigo fractional integral of order α for a real-valued continuous function f is defined by

$${}_0^S I^{\alpha, \beta, \eta} \{f(t)\} = \frac{t^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} {}_2F_1\left(\alpha + \beta, -\eta; \alpha; 1 - \frac{s}{t}\right) f(s) ds$$

where $\Gamma(\cdot)$ is the Gamma function, ${}_2F_1$ denotes the Gaussian hypergeometric function

$${}_2F_1(a, b; c; t) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} t^n, \quad |t| < 1,$$

and $(a)_n = \Gamma(a + n)/\Gamma(a)$.

Purohit and Raina proved new generalizations of the Chebyshev inequality for the Saigo integral operator in [28].

Theorem 12 ([28]). Let $f, g : [0, \infty) \rightarrow \mathbb{R}$ be two synchronous functions on $[0, \infty)$. Then,

$${}_0^S I^{\alpha, \beta, \eta} \{fg(t)\} \geq \frac{\Gamma(1 - \beta)\Gamma(1 + \alpha + \eta)t^\beta}{\Gamma(1 - \beta + \eta)} {}_0^S I^{\alpha, \beta, \eta} \{f(t)\} {}_0^S I^{\alpha, \beta, \eta} \{g(t)\},$$

for all $t > 0, \alpha > \max\{0, -\beta\}, \beta < 1, \beta - 1 < \eta < 0$.

Theorem 13 ([28]). Let $f, g : [0, \infty) \rightarrow \mathbb{R}$ be two synchronous functions on $[0, \infty)$. Then,

$$\begin{aligned} & \frac{\Gamma(1 - \beta + \eta)}{\Gamma(1 - \beta)\Gamma(1 + \alpha + \eta)t^\beta} {}_0^S I^{\gamma, \delta, \zeta} \{fg(t)\} + \frac{\Gamma(1 - \delta + \zeta)}{\Gamma(1 - \delta)\Gamma(1 + \gamma + \zeta)t^\delta} {}_0^S I^{\alpha, \beta, \eta} \{fg(t)\} \\ & \geq {}_0^S I^{\alpha, \beta, \eta} \{f(t)\} {}_0^S I^{\gamma, \delta, \zeta} \{g(t)\} + {}_0^S I^{\gamma, \delta, \zeta} \{f(t)\} {}_0^S I^{\alpha, \beta, \eta} \{g(t)\}, \end{aligned}$$

for all $t > 0, \alpha > \max\{0, -\beta\}, \gamma > \max\{0, -\delta\}, \beta < 1, \delta < 1, \beta - 1 < \eta < 0, \delta - 1 < \zeta < 0$.

The Atangana–Baleanu fractional integral operator was defined as follows.

Definition 7 ([55]). The fractional integral associate to the Atangana–Baleanu fractional derivative with non-local kernel of a function $f \in H^1(a, b)$ is defined by

$${}^{AB}I_a^\alpha \{f(t)\} = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_a^t f(s)(t-s)^{\alpha-1} ds$$

where $a < b$ and $\alpha \in (0, 1]$, $B(\alpha) > 0$ is normalization function, $\Gamma(\cdot)$ is the Gamma function, and $H^1(a, b)$ is the Sobolev space of order one defined as

$$H^1(a, b) = \{g \in L^2(a, b) : g' \in L^2(a, b)\},$$

where

$$L^2(a, b) = \{g(z) : \left(\int_a^b g^2(z) dz\right)^{\frac{1}{2}} < \infty\}.$$

In [56], Abdeljawad and Baleanu introduced the right fractional integral with Mittag-Leffler kernel of order $\alpha \in (0, 1]$.

Definition 8 ([56]). The right Atangana–Baleanu fractional integral of a function $f \in H^1(a, b)$ is defined by

$${}^{AB}I_b^\alpha \{f(t)\} = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_t^b f(s)(s-t)^{\alpha-1} ds.$$

Applications of the the Atangana–Baleanu operators have been explored in fields as diverse as chaos theory, heat transfer, and variational problems [57].

Theorem 14 ([29]). Let $f, g : [0, \infty) \rightarrow \mathbb{R}$ be two Lebesgue integrable functions which are synchronous on $[0, \infty)$, $a, b \in [0, \infty)$ and $a < b$. Then, we have the following inequality for Atangana–Baleanu fractional integral operators

$$\begin{aligned} & \frac{(b-a)^\alpha}{B(\alpha)\Gamma(\alpha)} \left[{}^{AB}I_a^\alpha \{fg(b)\} - \frac{1-\alpha}{B(\alpha)}fg(b) \right] \\ & \geq \left[{}^{AB}I_a^\alpha \{f(b)\} - \frac{1-\alpha}{B(\alpha)}f(b) \right] \left[{}^{AB}I_a^\alpha \{g(b)\} - \frac{1-\alpha}{B(\alpha)}g(b) \right], \end{aligned}$$

and

$$\begin{aligned} & \frac{(b-a)^\alpha}{B(\alpha)\Gamma(\alpha)} \left[{}^{AB}I_b^\alpha \{fg(a)\} - \frac{1-\alpha}{B(\alpha)}fg(a) \right] \\ & \geq \left[{}^{AB}I_b^\alpha \{f(a)\} - \frac{1-\alpha}{B(\alpha)}f(a) \right] \left[{}^{AB}I_b^\alpha \{g(a)\} - \frac{1-\alpha}{B(\alpha)}g(a) \right], \end{aligned}$$

where $\alpha \in (0, 1]$, $B(\alpha) > 0$ is normalization function and $\Gamma(\cdot)$ is the Gamma function.

Raina introduced fractional integral in [58].

Definition 9 ([58]). The Raina fractional integral operator of f is defined by

$${}^RI_{a,\beta,\omega,\sigma}^\alpha \{f(t)\} = \int_a^t (t-s)^{\beta-1} \mathcal{F}_{\alpha,\beta}^\sigma[\omega(t-s)^\alpha] f(s) ds,$$

where

$$\mathcal{F}_{\alpha,\beta}^\sigma(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\alpha k + \beta)} x^k$$

and $t > a > 0$, $\alpha, \beta, \omega \in \mathbb{C}$, $\operatorname{Re}(\alpha), \operatorname{Re}(\beta) > 0$, $f(t)$ is such that the integral on the right side exists.

Raina's fractional operators are important due to their level of generality. More precisely, by specifying the coefficient $\omega(k)$, we can obtain many fractional integral operators, e.g., Riemann–Liouville, Prabhakar [59], and Salim [60]. Usta et al. proved the following theorems for the Raina fractional integral operator.

Theorem 15 ([30]). Let $f, g : [0, \infty) \rightarrow \mathbb{R}$ be two synchronous functions on $[0, \infty)$. Then, for all $t, \alpha, \beta > 0$ and $\omega \in \mathbb{R}$, we have

$$\begin{aligned} & {}_0^R I^{\alpha, \beta, \omega, \sigma} \{1\} + {}_0^R I^{\alpha, \beta, \omega, \sigma} \{fg(t)\} \\ & \geq {}_0^R I^{\alpha, \beta, \omega, \sigma} \{f(t)\} + {}_0^R I^{\alpha, \beta, \omega, \sigma} \{g(t)\}, \end{aligned}$$

where the coefficients $\sigma(k)$ ($k \in \mathbb{N}_0$) is a bounded sequence of positive real numbers.

Theorem 16 ([30]). Let $f, g : [0, \infty) \rightarrow \mathbb{R}$ be two synchronous functions on $[0, \infty)$. Then, for all $t, \alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ and $\omega_1, \omega_2 \in \mathbb{R}$, we have

$$\begin{aligned} & t^{\beta_2} \mathcal{F}_{\alpha_2, \beta_2+1}^{\sigma_2} [\omega_2 t^{\alpha_2}] {}_0^R I^{\alpha_1, \beta_1, \omega_1, \sigma_1} \{fg(t)\} + t^{\beta_1} \mathcal{F}_{\alpha_1, \beta_1+1}^{\sigma_1} [\omega_1 t^{\alpha_1}] {}_0^R I^{\alpha_2, \beta_2, \omega_2, \sigma_2} \{fg(t)\} \\ & \geq {}_0^R I^{\alpha_1, \beta_1, \omega_1, \sigma_1} \{f(t)\} {}_0^R I^{\alpha_2, \beta_2, \omega_2, \sigma_2} \{g(t)\} + {}_0^R I^{\alpha_2, \beta_2, \omega_2, \sigma_2} \{f(t)\} {}_0^R I^{\alpha_1, \beta_1, \omega_1, \sigma_1} \{g(t)\}, \end{aligned}$$

where the coefficients $\sigma_1(k), \sigma_2(k)$ ($k \in \mathbb{N}_0$) are bounded sequences of positive real numbers.

Vivas-Cortez et al. introduced and investigated the generalized Raina fractional integral in [33].

Definition 10 ([33]). Let $0 \leq a < b$ and $f \in L^1([a, b])$. The generalized (left) Raina fractional integral operator of f associated with the parameters $\alpha, \beta > 0$, $\omega \in \mathbb{R}$, $\theta \in (0, 1]$, $\eta \geq 0$, and σ any bounded arbitrary sequence of real (or complex) numbers, are defined by the following integral transform for any $t \in (a, b)$:

$${}_a^R I^{\alpha, \beta, \omega, \sigma, \theta, \eta} \{f(t)\} = \int_a^t \left(\frac{t^{\theta+\eta} - s^{\theta+\eta}}{\theta + \eta} \right)^{\beta-1} s^{\theta+\eta-1} \mathcal{F}_{\alpha, \beta}^{\sigma} \left[\omega(t^{\theta+\eta} - s^{\theta+\eta})^{\alpha} \right] f(s) ds.$$

Theorem 17 ([33]). Let $\alpha, \beta > 0$, $\theta \in (0, 1]$ and $\eta \geq 0$, and suppose that f and g are two synchronous functions defined on $[0, \infty)$. Then, for all $t > 0$, we have

$${}_0^R I^{\alpha, \beta, \omega, \sigma, \theta, \eta} \{fg(t)\} \geq \frac{(\theta + \eta)^{\beta}}{\mathcal{F}_{\alpha, \beta+1}^{\sigma} [\omega(t^{\theta+\eta})^{\alpha}](t^{\theta+\eta})^{\beta}} {}_0^R I^{\alpha, \beta, \omega, \sigma, \theta, \eta} \{f(t)\} {}_0^R I^{\alpha, \beta, \omega, \sigma, \theta, \eta} \{g(t)\}.$$

Nápoles Valdés and Rabossi investigated the generalized k -proportional fractional integral as follows.

Definition 11 ([41]). Let $f \in X_F^q(0, \infty)$ and F be a continuous and positive function on $[0, \infty)$ with $F(0) = 0$. The (left side) generalized k -proportional fractional integral operator with general kernel of order γ of f is defined by

$${}_a^{NR} I^{\gamma, \frac{\gamma}{k}, \alpha} \{f(t)\} = \frac{1}{\alpha^{\gamma} k \Gamma_k(\gamma)} \int_a^t \frac{G(\mathbb{F}(t, s), \alpha) F(s) f(s)}{\mathbb{F}(t, s)^{1-\frac{\gamma}{k}}} ds,$$

where the proportionality index $\alpha \in (0, 1)$, $\gamma \in \mathbb{C}$, $\operatorname{Re}(\gamma) > 0$, $t > a$, $\mathbb{F}(t, s) = \int_s^t F(r) dr$, $G(\mathbb{F}(t, s), 1) = 1$, and the space $X_F^q(0, \infty)$ for $1 \leq q \leq \infty$, consists of those real-valued Lebesgue measurable functions f on $[0, \infty)$ for which $\|f\|_{X_F^q} < \infty$, where the norm is defined by

$$\|f\|_{X_F^q} = \left(\int_a^b |f(t)|^q F(t) dt \right)^{1/q} < \infty,$$

for $1 \leq q < \infty$, and for the case $q = \infty$,

$$\|f\|_{X_F^\infty} = \operatorname{ess\,sup}_{0 \leq t < \infty} [F(t)|f(t)|].$$

It is shown in [41] that many integral operators are particular cases of the generalized k -proportional fractional integral, e.g., Hadamard, Riemann–Liouville, Katugampola, and the generalized proportional fractional integral operator [61].

Theorem 18 ([41]). Let $f, g : [0, \infty) \rightarrow \mathbb{R}$ be two synchronous functions on $[0, \infty)$. Then, for all $t > a \geq 0$, $\alpha \in (0, 1)$, $\gamma, \delta \in \mathbb{C}$, $\operatorname{Re}(\gamma), \operatorname{Re}(\delta) > 0$,

$${}^{NR}_a I^{F, \frac{\gamma}{k}, \alpha} \{fg(t)\} \geq \frac{{}^{NR}_a I^{F, \frac{\gamma}{k}, \alpha} \{f(t)\} {}^{NR}_a I^{F, \frac{\gamma}{k}, \alpha} \{g(t)\}}{{}^{NR}_a I^{F, \frac{\gamma}{k}, \alpha} \{1\}}$$

and

$$\begin{aligned} & \frac{{}^{NR}_a I^{F, \frac{\gamma}{k}, \alpha} \{fg(t)\}}{{}^{NR}_a I^{F, \frac{\gamma}{k}, \alpha} \{1\}} + \frac{{}^{NR}_a I^{F, \frac{\delta}{k}, \alpha} \{fg(t)\}}{{}^{NR}_a I^{F, \frac{\delta}{k}, \alpha} \{1\}} \\ & \geq \frac{{}^{NR}_a I^{F, \frac{\gamma}{k}, \alpha} \{f(t)\} {}^{NR}_a I^{F, \frac{\delta}{k}, \alpha} \{g(t)\} + {}^{NR}_a I^{F, \frac{\delta}{k}, \alpha} \{f(t)\} {}^{NR}_a I^{F, \frac{\gamma}{k}, \alpha} \{g(t)\}}{{}^{NR}_a I^{F, \frac{\gamma}{k}, \alpha} \{1\} {}^{NR}_a I^{F, \frac{\delta}{k}, \alpha} \{1\}} \end{aligned}$$

hold.

3. Conclusions

In this work, we presented several Chebyshev-type inequalities known from the literature within the framework of fractional integral operators and different variants of them.

The results presented in the thesis do not exhaust what is known in the literature. For example, in [23,35,39], generalized integral operators are defined and in that framework a number of Chebyshev-type inequalities are presented, which contain as particular cases results reported in the literature. In some papers [13,44], the synchronicity of the two functions is relaxed via more general conditions.

We note that by choosing measures appropriately for the fractional integrals of Section 2, one could easily see that Theorems 1, 4, 5, 6, 8–10, 12, 14, 15, 17 and 18 are consequences of the measurable version of the Chebyshev inequality

$$\int_a^b d\mu \int_a^b fg d\mu \geq \int_a^b f d\mu \int_a^b g d\mu,$$

observed by Andréief in [62] (see [63], where $a < b$, $a, b \in \mathbb{R}$, μ is a non-negative measure, and f and g are two measurable functions synchronous on $[a, b]$).

Finally, we note that possible future generalizations of Chebyshev's inequality can be considered through newer definitions of general fractional integral operators or through appropriate relaxation of the synchronicity condition.

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