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Curves in Multiplicative Equiaffine Space

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Abstract: In this study, the geometric properties of curves in multiplicative equiaffine space are investigated using multiplicative calculus. Fundamental geometric concepts such as multiplicative arc length, multiplicative equiaffine curvature, and torsion are introduced. This study derives the multiplicative Frenet frame and associated Frenet equations, providing a systematic framework for describing the geometric behavior of multiplicative equiaffine curves. Additionally, curves with constant multiplicative curvature and torsion are characterized and supported with illustrative examples.

Keywords: affine space; equiaffine space curves; multiplicative calculus; multiplicative derivative; multiplicative integral

MSC: 53A04; 53A15; 11A05

1. Introduction

The foundational principles of classical analysis, a pivotal element of modern mathematical theory, were laid out by Leibniz and Newton in the late 17th century, primarily focusing on the notation of differential and integral calculus. Rooted in trigonometry, algebra, and analytic geometry, classical analysis incorporates concepts like limits, differentiation, integration, and series expansions. These operations, often regarded as fundamental as those of basic arithmetic, have earned classical analysis the alternative name "summational analysis". Its versatility has led to widespread use across various disciplines that require precise mathematical modeling and optimization.

However, classical analysis faces limitations in certain contexts, prompting the development of alternative methodologies. Among these, Volterra proposed a novel analytical approach in 1887 called Volterra-type or multiplicative analysis, which prioritizes multiplication over addition as its core operation [1]. In this paradigm, multiplication and division replace addition and subtraction, introducing a new structural framework for mathematical modeling. Expanding on Volterra's foundation, Grossman and Katz, between 1972 and 1983, developed what is now recognized as non-Newtonian analysis. This framework introduced novel concepts and definitions, revolutionizing mathematical operations across diverse domains [2,3]. These methods, encompassing geometric, bigeometric, and anageometric analysis, offer innovative tools for problems unsolvable by classical techniques.

Multiplicative analysis, as a counterpart to classical analysis, has gained substantial attention and has encouraged further research. Numerous studies, including [4–8], have significantly contributed to its advancement. Recent works in differential geometry also integrate multiplicative analysis. For instance, Georgiev's contributions [9–11], particularly [10], are considered foundational texts in multiplicative geometry, establishing new definitions and theorems linking geometric structures with multiplicative principles. These

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studies emphasize the interplay between multiplicative geometry and other mathematical disciplines, broadening its scope and applications.

Advances in geometric analysis include Nurkan et al. application of geometric calculus to derive Gram–Schmidt vectors [12]. Similarly, ref. [13] examined spherical indicatrices and helices in non-Newtonian Euclidean spaces, providing fresh characterizations and examples. Aydın et al. classified and visualized rectifying curves within multiplicative Euclidean space using multiplicative spherical curves [14], while [15] extended these concepts to tube surfaces with multiplicative quaternions. Moreover, several intriguing studies in differential geometry have focused on curves. Broscăţeanu et al. [16] analyzed the infinitesimal bending of rectifying curves, Burlacu and Mihai [17] investigated applications of curve theory in road design, and Jianu et al. [18] explored a surface associated with the Catalan triangle.

Further investigations into non-Newtonian geometry include Has, Yılmaz, and Yıldırım's exploration of Lorentz–Minkowski space through non-Newtonian approaches [19], and Özdemir and Ceyhan's study on multiplicative hyperbolic split quaternions and their role in hyperbolic rotation matrices [20]. Has and Yılmaz also analyzed non-Newtonian conics through multiplicative analytic geometry [21]. Additionally, Es's research [22] utilized homothetic multiplicative calculus to derive kinematic expressions for plane motion. Expanding on these, Has and Yılmaz examined magnetic curves within multiplicative Riemannian manifolds using non-Newtonian methods [23]. Moreover, Aydın has studied the multiplicative rectifying submanifolds of multiplicative Euclidean space and made important contributions to this area [24]. Ceyhan has worked on a non-Newtonian approach to the geometric phase through optical fibers via multiplicative quaternions [25].

In the realm of affine geometry, extensive research has illuminated the properties of affine spaces. Works like [26–29] provide comprehensive discussions on both foundational and advanced aspects of affine geometry, bridging classical and contemporary perspectives.

This paper is organized as follows. It first introduces the key concepts of multiplicative calculus. Subsequently, the essential elements of equiaffine geometry relevant to the analysis are discussed. Finally, the curvature of a multiplicative plane curve is computed.

2. Preliminaries

This section delves into two fundamental topics. The initial part discusses the foundational aspects of equiaffine plane curves, emphasizing the determinant in the affine plane and the concept of equiaffine arc-length. The latter part introduces the framework of multiplicative analysis, examining multiplicative operations and structures such as addition, multiplication, and derivatives in the multiplicative context. Furthermore, the multiplicative determinant is presented as a logarithmic counterpart to the classical determinant.

2.1. Basics of Equiaffine Space Curves

Equiaffine spaces are a subspace of affine spaces, possessing a fixed volume form, which provides them with additional geometric structure. Equiaffine geometry was considered in the paper due to its intrinsic properties that provide a natural framework for analyzing curves in an affine space without relying on an external Euclidean metric. Unlike classical differential geometry, which depends on metric-based notions of distance and angle, equiaffine geometry allows for the study of curves and surfaces using volume-preserving transformations. This makes it particularly suitable for applications in multiplicative calculus, where traditional metric structures are replaced by multiplicative analogs. By integrating equiaffine concepts with multiplicative calculus, the paper aims to develop a systematic approach to defining arc length, curvature, and torsion in a manner

that is invariant under multiplicative affine transformations. Now, this section provides an overview of the key principles of equiaffine space curves, avoiding in-depth explanations.

Let $\mathbf{R}^3$ be the affine space with the determinant

$$[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1), \quad (1)$$

where $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3) \in \mathbf{R}^3$.

Let $\varphi : I \to \mathbf{R}^3$ represent a regular curve in $\mathbf{R}^3$. Typically, for equiaffine regular curves, the condition of non-degeneracy $[\dot{\varphi}(t)\ \ddot{\varphi}(t)\ \dddot{\varphi}(t)] \neq 0$ for every $t \in I$ is assumed, where

$$\dot{\varphi}(t) = \frac{d\varphi}{dt}(t), \quad \ddot{\varphi}(t) = \frac{d^2\varphi}{dt^2}(t), \quad \dddot{\varphi}(t) = \frac{d^3\varphi}{dt^3}(t). \quad (2)$$

Given this, the equiaffine arc length of φ, measured from $\varphi(t_0)$ ($t_0 \in I$), is given as

$$s(t) = \int_{t_0}^{t} [\dot{\varphi}(t)\ \ddot{\varphi}(t)\ \dddot{\varphi}(t)]^{\frac{1}{6}}\, dt. \quad (3)$$

Let

$$\varphi'(s) = \frac{d\varphi}{ds}(s), \quad \varphi''(s) = \frac{d^2\varphi}{ds^2}(s), \quad \varphi'''(s) = \frac{d^3\varphi}{ds^3}(s). \quad (4)$$

A regular curve φ is parameterized by the equiaffine arc length if it satisfies

$$\left[\frac{d\varphi}{ds}(s)\ \frac{d^2\varphi}{ds^2}(s)\ \frac{d^3\varphi}{ds^3}(s)\right] = 1. \quad (5)$$

for all s.

The set $\left\{\frac{d\varphi}{ds}(s), \frac{d^2\varphi}{ds^2}(s), \frac{d^3\varphi}{ds^3}(s)\right\}$ is referred to as the *equiaffine Frenet frame* of $\varphi(s)$. When we take the derivative of (5) with respect to the parameter s, it is apparent that

$$\left[\frac{d\varphi}{ds}(s)\ \frac{d^2\varphi}{ds^2}(s)\ \frac{d^4\varphi}{ds^4}(s)\right] = 0, \quad (6)$$

where the set is linearly dependent for every $s \in J$, given by

$$\left\{\frac{d\varphi}{ds}(s), \frac{d^2\varphi}{ds^2}(s), \frac{d^4\varphi}{ds^4}(s)\right\}. \quad (7)$$

Therefore, this implies the existence of smooth functions $\kappa(s)$ and $\tau(s)$ on J such that

$$\frac{d^4\varphi}{ds^4}(s) + \kappa(s)\frac{d\varphi}{ds}(s) + \tau(s)\frac{d^2\varphi}{ds^2}(s) = 0, \quad (8)$$

where

$$\kappa(s) = -\left[\frac{d^2\varphi}{ds^2}(s)\ \frac{d^3\varphi}{ds^3}(s)\ \frac{d^4\varphi}{ds^4}(s)\right], \quad (9)$$

and

$$\tau(s) = \left[\frac{d\varphi}{ds}(s)\ \frac{d^3\varphi}{ds^3}(s)\ \frac{d^4\varphi}{ds^4}(s)\right]. \quad (10)$$

The functions $\kappa(s)$ and $\tau(s)$ are called *equiaffine curvature* and *equiaffine torsion* of the curve $\varphi(s)$, respectively. The equiaffine curvature and equiaffine torsion are the equiaffine invariants in $\mathbf{R}^3$.

Let φ be a non-degenerate smooth curve in $\mathbf{R}^3$ parameterized by equiaffine arc length s. As a result, the equiaffine equations of Frenet type are presented in matrix form as

$$\begin{bmatrix} \frac{d^2\varphi}{ds^2}(s) \\ \frac{d^3\varphi}{ds^3}(s) \\ \frac{d^4\varphi}{ds^4}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\kappa(s) & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \frac{d\varphi}{ds}(s) \\ \frac{d^2\varphi}{ds^2}(s) \\ \frac{d^3\varphi}{ds^3}(s) \end{bmatrix}. \tag{11}$$

2.2. Basics of Multiplicative Analysis

This section introduces the fundamental definitions and key principles of multiplicative analysis, drawing inspiration from the contributions of Georgiev [9–11]. Subsequently, the discussion transitions to multiplicative equiaffine space curves.

Let $\mathbf{R}_* = (0, \infty)$, where we define the operations of multiplicative addition, multiplication, subtraction, and division as follows:

$$x +_* y = e^{\log x + \log y} = x \cdot y, \tag{12}$$

$$x \cdot_* y = e^{(\log x)(\log y)} = x^{\log y}, \tag{13}$$

$$x -_* y = e^{\log x - \log y} = \frac{x}{y}, \tag{14}$$

$$x /_* y = e^{\frac{\log x}{\log y}} = x^{\frac{1}{\log y}} \text{ for } y \neq 1. \tag{15}$$

These operations define the multiplicative structure $(\mathbf{R}_*, +_*, -_*)$. Each x_* in $\mathbf{R}_*$ is called a multiplicative number, where $x_* = e^x$. The multiplicative zero and unit are $0_* = 1$ and $1_* = e$, respectively.

For $k \in \mathbf{N}$, the power k_* of a multiplicative number x is defined as $x^{k_*} = x \cdot_* x \cdot \cdots \cdot_* x$ (repeated k times), which simplifies to $e^{(\log x) \cdot k}$.

The multiplicative sine and cosine functions are defined as:

$$\sin_* x = e^{\sin(\log x)} \tag{16}$$

and

$$\cos_* x = e^{\cos(\log x)}. \tag{17}$$

Similarly, the multiplicative hyperbolic sine and cosine functions are given by

$$\sinh_* x = e^{\sinh(\log x)} \tag{18}$$

and

$$\cosh_* x = e^{\cosh(\log x)}. \tag{19}$$

Let $A \in \mathbf{R}_*$ and f be a first-order continuously differentiable function. The first-order multiplicative derivative of f, denoted by $f^*(x)$, is defined as

$$f^*(x) = \lim_{h \to 0_*} f(x +_* h) -_* f(x) /_* h = e^{\frac{x f'(x)}{f(x)}}, \tag{20}$$

where f' represents the classical derivative of f.

From now on, the multiplicative derivative given in Equation (20) will also be denoted by $(\ldots)^*$.

For $a, b \in \mathbf{R}_*$ and $f : \mathbf{R}_* \to \mathbf{R}_*$, the multiplicative indefinite integral and the multiplicative Cauchy integral are defined as

$$\int_* f(x) \cdot_* d_*x = e^{\int \frac{1}{x} \log f(x)\, dx} \tag{21}$$

and

$$\int_a^b f(x) \cdot_* d_*x = e^{\int_a^b \frac{1}{x} \log f(x)\, dx}. \tag{22}$$

Finally, for three vectors $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$, and $\mathbf{z} = (z_1, z_2, z_3)$ the multiplicative determinant is defined as

$$\begin{aligned}[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]_* = {} & x_1 \cdot_* y_2 \cdot_* z_3 +_* y_1 \cdot_* z_2 \cdot_* x_3 +_* z_1 \cdot_* x_2 \cdot_* y_3 \\ & -_* z_1 \cdot_* y_2 \cdot_* x_3 -_* x_1 \cdot_* z_2 \cdot_* y_3 -_* y_1 \cdot_* x_2 \cdot_* z_3,\end{aligned} \tag{23}$$

which can be expressed in logarithmic terms as

$$[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]_* = \frac{e^{\left(\log x_1 \cdot \log y_2 \cdot \log z_3 + \log y_1 \cdot \log z_2 \cdot \log x_3 + \log z_1 \cdot \log x_2 \cdot \log y_3\right)}}{e^{\left(\log z_1 \cdot \log y_2 \cdot \log x_3 + \log x_1 \cdot \log z_2 \cdot \log y_3 + \log y_1 \cdot \log x_2 \cdot \log z_3\right)}}. \tag{24}$$

This serves as the multiplicative analogue of the classical determinant in $\mathbf{R}_*^3$.

3. Multiplicative Equiaffine Space Curves

In this section, we extend the theory of multiplicative equiaffine space curves by introducing the concept of multiplicative affine space and the multiplicative determinant, which serves as a generalization for volume computation. Additionally, we define key notions such as multiplicative linear independence, the multiplicative equiaffine group, and the multiplicative arc length. The discussion further includes the development of the multiplicative Frenet apparatus and the corresponding Frenet equations, which characterize the curvature and frame transformations of these curves.

The *multiplicative affine space* $\mathbf{R}_*^3$ is defined as a three-dimensional affine space with a multiplicative structure replacing the conventional additive framework. Within this space, a constant volume form is maintained, acting as a generalized determinant to compute the "volume" formed by three vectors. This volume form is expressed multiplicatively as follows:

$$[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]_* = x_1 \cdot_* (y_2 \cdot_* z_3 -_* y_3 \cdot_* z_2) -_* x_2 \cdot_* (y_1 \cdot_* z_3 -_* y_3 \cdot_* z_1) +_* x_3 \cdot_* (y_1 \cdot_* z_2 -_* y_2 \cdot_* z_1), \tag{25}$$

where $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$, and $\mathbf{z} = (z_1, z_2, z_3)$ are vectors in $\mathbf{R}_*^3$.

Proposition 1. *In* $\mathbf{R}_*^3$*, for vectors* $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$, *and* $\mathbf{z} = (z_1, z_2, z_3)$

$$([\mathbf{x}\ \mathbf{y}\ \mathbf{z}]_*)^* = [\mathbf{x}^*\ \mathbf{y}\ \mathbf{z}]_* +_* [\mathbf{x}\ \mathbf{y}^*\ \mathbf{z}]_* +_* [\mathbf{x}\ \mathbf{y}\ \mathbf{z}^*]_*, \tag{26}$$

equality is satisfied.

Proof. Let us assume that

$$[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]_* = x_1 \cdot_* (y_2 \cdot_* z_3 -_* y_3 \cdot_* z_2) -_* x_2 \cdot_* (y_1 \cdot_* z_3 -_* y_3 \cdot_* z_1) +_* x_3 \cdot_* (y_1 \cdot_* z_2 -_* y_2 \cdot_* z_1), \tag{27}$$

Now, consider $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ as matrices with the vectors as columns

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} \tag{28}$$

Then, the determinant of the matrix formed by $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ denoted as $[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]_*$, is given by

$$[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]_* = \det_* \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \tag{29}$$

$$\begin{aligned} &= x_1 \cdot_* (y_2 \cdot_* z_3 -_* y_3 \cdot_* z_2) \\ &-_* x_2 \cdot_* (y_1 \cdot_* z_3 -_* y_3 \cdot_* z_1) \\ &+_* x_3 \cdot_* (y_1 \cdot_* z_2 -_* y_2 \cdot_* z_1) \end{aligned} \tag{30}$$

or

$$\begin{aligned} &= e^{\log x_1 \cdot \log y_2 \cdot \log z_3 + \log x_2 \cdot \log y_3 \cdot \log z_1 + \log x_3 \cdot \log y_1 \cdot \log z_2} \\ &\cdot e^{-(\log x_1 \cdot \log y_3 \cdot \log z_2 + \log x_2 \cdot \log y_1 \cdot \log z_3 + \log x_3 \cdot \log y_2 \cdot \log z_1)} \end{aligned} \tag{31}$$

Now, let us multiplicatively differentiate this last expression, then

$$\begin{aligned} ([\mathbf{x}\ \mathbf{y}\ \mathbf{z}]_*)^* = \Big(&e^{(\log x_1 \cdot \log y_2 \cdot \log z_3 + \log x_2 \cdot \log y_3 \cdot \log z_1 + \log x_3 \cdot \log y_1 \cdot \log z_2)} \\ &\cdot e^{-(\log x_1 \cdot \log y_3 \cdot \log z_2 + \log x_2 \cdot \log y_1 \cdot \log z_3 + \log x_3 \cdot \log y_2 \cdot \log z_1)} \Big)^* \end{aligned} \tag{32}$$

is obtained. Using the multiplicative derivative rule, we have

$$\begin{aligned} &= e^{x\left((\log x_1)^* \cdot (\log y_2 \cdot \log z_3 - \log y_3 \cdot \log z_2) + (\log x_2)^* \cdot (\log y_3 \cdot \log z_1 - \log y_1 \cdot \log z_3) + (\log x_3)^* \cdot (\log y_1 \cdot \log z_2 - \log y_2 \cdot \log z_1)\right)} \\ &+_* e^{x\left((\log y_1)^* \cdot (\log x_3 \cdot \log z_2 - \log x_2 \cdot \log z_3) + (\log y_2)^* \cdot (\log x_1 \cdot \log z_3 - \log x_3 \cdot \log z_1) + (\log y_3)^* \cdot (\log x_2 \cdot \log z_1 - \log x_1 \cdot \log z_2)\right)} \\ &+_* e^{x\left((\log z_1)^* \cdot (\log x_2 \cdot \log y_3 - \log x_3 \cdot \log y_2) + (\log z_2)^* \cdot (\log x_1 \cdot \log y_3 - \log x_3 \cdot \log y_1) + (\log z_3)^* \cdot (\log x_1 \cdot \log y_2 - \log x_2 \cdot \log y_1)\right)}. \end{aligned} \tag{33}$$

Therefore, we have proved that

$$([\mathbf{x}\ \mathbf{y}\ \mathbf{z}]_*)^* = [\mathbf{x}^*\ \mathbf{y}\ \mathbf{z}]_* +_* [\mathbf{x}\ \mathbf{y}^*\ \mathbf{z}]_* +_* [\mathbf{x}\ \mathbf{y}\ \mathbf{z}^*]_*, \tag{34}$$

completing the proof. □

Definition 1. *A set* $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ *of vectors in* $\mathbf{R}_*^3$ *is said to be multiplicatively linearly independent if the equation*

$$\lambda_1 \cdot_* \mathbf{x} +_* \lambda_2 \cdot_* \mathbf{y} +_* \lambda_3 \cdot_* \mathbf{z} = 0_* = 1 \tag{35}$$

holds only when $\lambda_1 = \lambda_2 = \lambda_3 = 1$. *Otherwise, if there exist* $\lambda_1, \lambda_2, \lambda_3 \in \mathbf{R}_*$, *not all equal to* 1, *such that Equation* (35) *holds, then* $\mathbf{x}$, $\mathbf{y}$, *and* $\mathbf{z}$ *are said to be multiplicatively linearly dependent.*

$SL = \left(3, \mathbf{R}_*^{3\times 3}\right) = \left\{A \in \mathbf{R}_*^{3\times 3} : |A|_* = e\right\}$. *The multiplicative equiaffine group* of $\mathbf{R}_*^3$ is produced by the combined action of the multiplicative rotation group and the multiplicative translation group. A representation of a multiplicative equiaffine transformation of $\mathbf{R}_*^3$ can be expressed in the form

$$\bar{\mathbf{C}} = A \cdot_* \mathbf{C} +_* \mathbf{B}, \tag{36}$$

where $\bar{\mathbf{C}}, \mathbf{C}, \mathbf{B} \in \mathbf{R}_*^{3\times 1}$. Note that the transformation given in Equation (36) preserves the volume of a parallelepiped and hence is called a *volume-preserving multiplicative affine transformation.*

Definition 2. *Let* $\varphi = \varphi(t) = (\varphi_1(t), \varphi_2(t), \varphi_3(t))$, *where* $t \in I \subset \mathbf{R}_*$, *is a multiplicative non-degenerate smooth parametric curve in* $\mathbf{R}_*^3$. *This means that*

$$[\varphi^*\ \varphi^{**}\ \varphi^{***}]_* \neq 0_*\ (or\ 1), \tag{37}$$

for any t.

Definition 3. *The multiplicative equiaffine arc length function ρ for the multiplicative equiaffine curve φ in* $\mathbf{R}_*$ *is defined as*

$$\rho(t) = \int_{*t_0}^{t} ([\varphi^* \ \varphi^{**} \ \varphi^{***}]_*)^{\frac{1}{6}*} \cdot_* d_* u = e^{\int_{t_0}^{t} \frac{1}{u}(\log([\varphi^* \ \varphi^{**} \ \varphi^{***}]_*))^{\frac{1}{6}} du}, \tag{38}$$

where $[\varphi^* \ \varphi^{**} \ \varphi^{***}]_*$ *is the multiplicative determinant of the first, second, and third multiplicative derivatives of the curve φ. The parameter ρ is the multiplicative equiaffine arc length parameter.*

Proposition 2. *Let φ be a multiplicative non-degenerate smooth parametric curve in* $\mathbf{R}^3$ *parameterized by the multiplicative equiaffine arc length ρ. Then, we have*

$$[(d_*\varphi /_* d_*\rho \ d_*^2\varphi /_* d_*\rho^{2*} \ d_*^3\varphi /_* d_*\rho^{3*}]_* = 1_*.$$

Proof. Using the chain rule, we write

$$d_*\varphi /_* d_*\rho = (d_*\varphi /_* d_*t) \cdot_* (d_*t /_* d_*\rho)$$

or

$$d_*\varphi /_* d_*\rho = \varphi^* \cdot_* (d_*t /_* d_*\rho). \tag{39}$$

If we take the multiplicative derivative of (39), then

$$d_*^2\varphi /_* d_*\rho^{2*} = \varphi^{**} \cdot_* (d_*t /_* d_*\rho)^{2*} +_* \varphi^* \cdot_* \left(d_*^2 t /_* d_*\rho^{2*}\right). \tag{40}$$

Again, take the multiplicative derivative of (40), then

$$\begin{aligned} d_*^3\varphi /_* d_*\rho^{3*} = {} & \varphi^{***} \cdot_* (d_*t /_* d_*\rho)^{3*} \\ & +_* \varphi^{**} \cdot_* 2 \cdot_* (d_*t /_* d_*\rho) \cdot_* \left(d_*^2 t /_* d_*\rho^{2*}\right) \\ & +_* \varphi^{**} \cdot_* (d_*t /_* d_*\rho) \cdot_* \left(d_*^2 t /_* d_*\rho^{2*}\right) \\ & +_* \varphi^* \cdot_* \left(d_*^3 t /_* d_*\rho^{3*}\right) \end{aligned}$$

Thus, we have

$$[(d_*\varphi /_* d_*\rho \ d_*^2\varphi /_* d_*\rho^{2*} \ d_*^3\varphi /_* d_*\rho^{3*}]_* = (d_*t /_* d_*\rho)^{6*} \cdot_* [\varphi^* \ \varphi^{**} \ \varphi^{***}]_*.$$

According to the Mean Value Theorem in analysis, since

$$[\varphi^* \ \varphi^{**} \ \varphi^{***}]_* = (d_*\rho /_* d_*t)^{6*}$$

from (38), we get

$$[(d_*\varphi /_* d_*\rho \ d_*^2\varphi /_* d_*\rho^{2*} \ d_*^3\varphi /_* d_*\rho^{3*}]_* = 1_*.$$

This completes the proof. □

Definition 4. *Let φ be a multiplicative non-degenerate smooth curve in $\mathbf{R}_*^3$ parameterized by the multiplicative equiaffine arc length s. The multiplicative equiaffine Frenet equations are given in matrix form as*

$$\begin{bmatrix} d_*^2\varphi/_*d_*s^{2_*} \\ d_*^3\varphi/_*d_*s^{3_*} \\ d_*^4\varphi/_*d_*s^{4_*} \end{bmatrix} = \begin{bmatrix} 0_* & 1_* & 0_* \\ 0_* & 0_* & 1_* \\ -_*\kappa_*(s) & -_*\tau_*(s) & 0_* \end{bmatrix} \begin{bmatrix} d_*\varphi/_*d_*s \\ d_*^2\varphi/_*d_*s^{2_*} \\ d_*^3\varphi/_*d_*s^{3_*} \end{bmatrix}, \tag{41}$$

where the functions $\kappa_*(s)$ and $\tau_*(s)$ are called the multiplicative equiaffine curvature and multiplicative equiaffine torsion of the curve $\varphi(s)$, respectively.

Therefore, this implies the existence of smooth functions $\kappa_*(s)$ and $\tau_*(s)$ such that

$$d_*^4\varphi/_*d_*s^{4_*} +_* \kappa_*(s) \cdot_* d_*\varphi/_*d_*s +_* \tau_*(s) \cdot_* d_*^2\varphi/_*d_*s^{2_*} = 0_*, \tag{42}$$

where

$$\kappa_*(s) = -_*[d_*^2\varphi/_*d_*s^{2_*} \quad d_*^3\varphi/_*d_*s^{3_*} \quad d_*^4\varphi/_*d_*s^{4_*}]_* \tag{43}$$

and

$$\tau_*(s) = \left[d_*\varphi/_*d_*s \quad d_*^3\varphi/_*d_*s^{3_*} \quad d_*^4\varphi/_*d_*s^{4_*}\right]_*. \tag{44}$$

The multiplicative equiaffine curvature $\kappa_*(s)$ and torsion $\tau_*(s)$ are multiplicative equiaffine invariants in $\mathbf{R}_*^3$.

Theorem 1. *Let φ be a curve in $\mathbf{R}_*^3$ parameterized by multiplicative equiaffine arc length. Then the curve φ and $A \cdot_* \varphi +_* B$ have the same multiplicative equiaffine curvature and torsion, where $A \in SL$ and $B \in \mathbf{R}_*^{3\times 1}$.*

Proof. Let β be a curve other than φ and let it be represented as follows:

$$\beta = A \cdot_* \varphi +_* B. \tag{45}$$

If the derivatives are taken in the multiplicative sense, the following equalities are obtained:

$$d_*\beta/_*d_*s = A \cdot_* d_*\varphi/_*d_*s, \tag{46}$$

$$d_*^2\beta/_*d_*s^{2_*} = A \cdot_* d_*^2\varphi/_*d_*s^{2_*}, \tag{47}$$

$$d_*^3\beta/_*d_*s^{3_*} = A \cdot_* d_*^3\varphi/_*d_*s^{3_*}, \tag{48}$$

$$d_*^4\beta/_*d_*s^{4_*} = A \cdot_* d_*^4\varphi/_*d_*s^{4_*}. \tag{49}$$

Moreover, from the properties of the multiplicative determinant, we obtain the following expressions:

$$\begin{aligned} &\left[d_*\beta/_*d_*s \quad d_*^2\beta/_*d_*s^{2_*} \quad d_*^3\beta/_*d_*s^{3_*}\right]_* \\ &= \left[A \cdot_* d_*\varphi/_*d_*s \quad A \cdot_* d_*^2\varphi/_*d_*s^{2_*} \quad A \cdot_* d_*^3\varphi/_*d_*s^{3_*}\right]_* \\ &= \left[d_*\varphi/_*d_*s \quad d_*^2\varphi/_*d_*s^{2_*} \quad d_*^3\varphi/_*d_*s^{3_*}\right]_* \\ &= 1_*. \text{ (or } e \text{)} \end{aligned} \tag{50}$$

This last equation implies that β is parameterized by multiplicative equiaffine arc length.

If Definition 4 is considered, the following equalities are found between the curvatures and torsions of these two curves:

$$\begin{aligned}\kappa_*^{\beta}(s) &= -_*\left[d_*^2\beta/_*d_*s^{2_*} \quad d_*^3\beta/_*d_*s^{3_*} \quad d_*^4\beta/_*d_*s^{4_*}\right]_* \\ &= -_*\left[d_*^2\varphi/_*d_*s^{2_*} \quad d_*^3\varphi/_*d_*s^{3_*} \quad d_*^4\varphi/_*d_*s^{4_*}\right]_* \\ &= \kappa_*^{\varphi}(s),\end{aligned} \tag{51}$$

and

$$\begin{aligned}\tau_*^{\beta}(s) &= \left[d_*\beta/_*d_*s \quad d_*^3\beta/_*d_*s^{3_*} \quad d_*^4\beta/_*d_*s^{4_*}\right]_* \\ &= \left[d_*\varphi/_*d_*s \quad d_*^3\varphi/_*d_*s^{3_*} \quad d_*^4\varphi/_*d_*s^{4_*}\right]_* \\ &= \tau_*^{\varphi}(s).\end{aligned} \tag{52}$$

The proof is thus completed. □

Now, based on the above explanations, we can provide the following three examples in the multiplicative equiaffine space.

First, consider the curve $\psi(s)$ in the multiplicative equiaffine space, given by multiplicative operations as

$$\psi(s) = \left(s, e^{1/2} \cdot_* e^{(\log s)^2}, e^{1/6} \cdot_* e^{(\log s)^3}\right), \tag{53}$$

or by classical operations as

$$\psi(s) = \left(s, e^{1/2(\log s)^2}, e^{1/6(\log s)^3}\right). \tag{54}$$

The curvatures $\kappa_*(s)$ and $\tau_*(s)$ of this curve are identically equal to 0_* or 1. The graph of this curve can be seen in Figure 1.

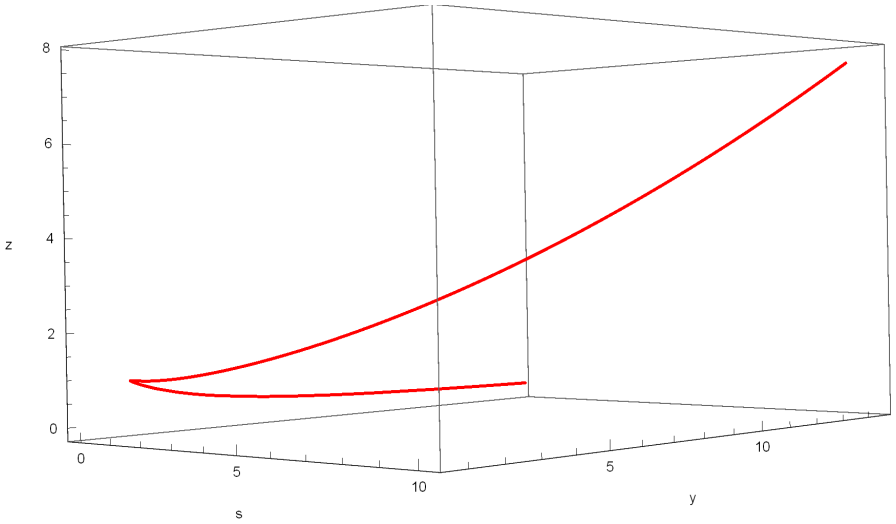


Figure 1. Graph of Equation (54).

Next, consider the curve $\Phi(s)$ given by

$$\Phi(s) = (-_*s, \sin_* s, \cos_* s), \tag{55}$$

or

$$\Phi(s) = \left(1/s, e^{\log(\sin s)}, e^{\log(\cos s)}\right), \tag{56}$$

where $\kappa_*(s) = 0_* = 1$ and $\tau_*(s) = 1_* = e$. The graph of this curve can be seen in Figure 2.

Lately, consider the curve $\Psi(s)$ given by

$$\Psi(s) = \left(e^{5/216} \cdot_* e^{1/s}, e^{\sqrt{s}} \cdot_* e^{\log(\sin\sqrt{s})}, e^{\sqrt{s}} \cdot_* e^{\log(\cos\sqrt{s})}\right), \tag{57}$$

or

$$\Psi(s) = \left(e^{\frac{5}{216s}}, e^{\sqrt{s}\log(\sin(\sqrt{s}))}, e^{\sqrt{s}\log(\cos(\sqrt{s}))}\right) \tag{58}$$

where $\kappa_*(s) = -_* e^{2/27} = e^{-2/27}$ and $\tau_*(s) = -_* e^{1/3} = e^{-1/3}$. The graph of this curve can be seen in Figure 3.

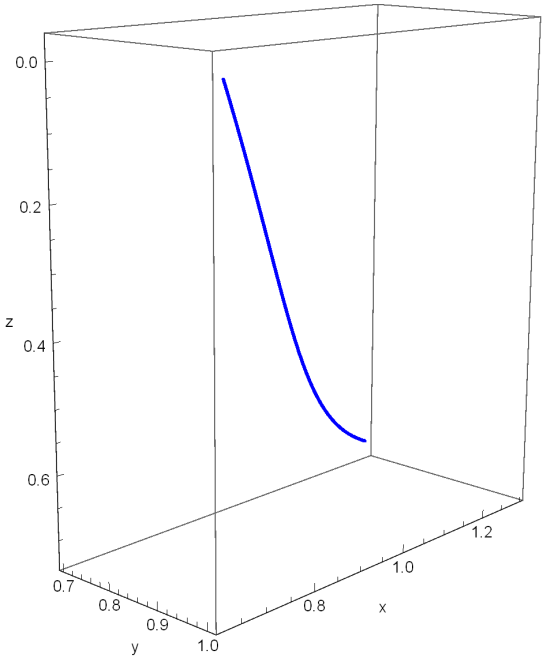


Figure 2. Graph of Equation (56).

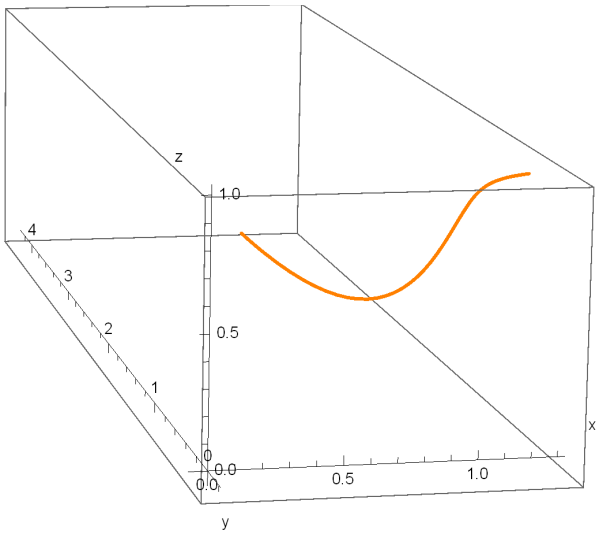


Figure 3. Graph of Equation (58).

4. Conclusions

This study introduces a comprehensive framework for analyzing curves in multiplicative equiaffine space by integrating concepts from multiplicative calculus and equiaffine differential geometry. Key contributions include the definitions of multiplicative equiaffine

arc length, curvature, and torsion, alongside the development of the multiplicative Frenet apparatus and corresponding Frenet equations. Through these tools, the study systematically classifies space curves with constant multiplicative curvature and torsion, providing new insights into the geometric behavior of these curves.

The presented examples demonstrate how multiplicative calculus extends classical geometric concepts, showcasing its potential to generalize and enhance traditional methodologies. These advancements contribute not only to the theoretical development of affine geometry but also highlight potential applications in fields such as physics, engineering, and mathematical modeling, particularly in scenarios involving exponential growth, scaling phenomena, or non-Newtonian frameworks.

Future research could explore the broader applicability of this framework, examining its integration with other geometric and physical systems and its potential to address complex real-world problems.

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