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Fuzzy Epistemic Logic: Fuzzy Logic of Doxastic Attitudes

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Abstract: In traditional epistemic logic—particularly modal logic—agents are often assumed to have complete and certain knowledge, which is unrealistic in real-world scenarios where uncertainty, imprecision, and the incompleteness of information are common. This study proposes an extension of the logic of doxastic attitudes to a fuzzy setting, representing beliefs or knowledge as continuous values in the interval [0, 1] rather than binary Boolean values. This approach offers a more nuanced and realistic modeling of belief states, capturing the inherent uncertainty and vagueness in human reasoning. We introduce a set of axioms for the fuzzy logic of doxastic attitudes, formalizing how agents reason with regard to uncertain beliefs. The theoretical foundations of this logic are established through proofs of soundness and completeness. To demonstrate practical utility, we present a concrete example, illustrating how the fuzzy logic of doxastic attitudes can model uncertain preferences and beliefs.

Keywords: epistemic logic; logic of doxastic attitudes; fuzzy logic; soundness; completeness

MSC: 68T37; 03B52

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Academic Editor: Hsien-Chung Wu

Received: 19 February 2025
Revised: 24 March 2025
Accepted: 25 March 2025
Published: 27 March 2025

Citation: Zhang, J.; Zhou, X.; Zhang, Y.; Tan, L. Fuzzy Epistemic Logic: Fuzzy Logic of Doxastic Attitudes. *Mathematics* **2025**, *13*, 1105. https://doi.org/10.3390/math13071105

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1. Introduction

Epistemic logic plays a pivotal role in multi-agent systems (MASs) by providing formal frameworks which model agents' knowledge, beliefs, and decision-making processes. For instance, epistemic logic and dynamic epistemic logic, introduced by Hintikka [1] and further developed by researchers such as van Benthem [2] and Eijck and Ditmarsch et al. [3], enable the representation of agents' knowledge and beliefs, as well as the analysis of knowledge propagation and belief revision. These frameworks have been integrated with decision-making models [4], computer science [4], economics [5,6], psychology [7], and philosophy [8]. This interdisciplinary integration highlights the versatility and significance of epistemic logic in addressing complex challenges in both theoretical and practical domains.

Lorini provided the logic of doxastic attitudes [9–11] in order to emphasize the distinction between explicit and implicit belief. In this framework, explicit beliefs are represented as facts in an agent's belief base, while implicit beliefs are those deducible from explicit beliefs and the agent's common ground (i.e., shared information). By leveraging belief bases, the logic of doxastic attitudes enables more efficient belief revision and update mechanisms, allowing agents to adapt their beliefs in response to new information. This approach addresses key limitations associated with traditional epistemic logic, such as the

problem of logical omniscience and exponential growth in model complexity, and offers a more practical and scalable solution for modeling belief dynamics in multi-agent systems.

In the real world, statements such as "It's very hot today" do not accurately convey specific temperature information due to the relative nature of human perception. To characterize the uncertainty in the knowledge, belief, and reasoning of the agent, probabilistic epistemic logic, intuitionistic epistemic logic, and paraconsistent epistemic logic have been explored to offer complementary frameworks to model different dimensions of uncertainty. Probabilistic epistemic logic integrates probability theory with epistemic logic to quantify agents' beliefs and their dynamics under information updates [12,13]. Through assigning numerical probabilities to propositions (e.g., $(P_i(\phi) \geq 0.7)$), risk-based uncertainty is captured, and Bayesian reasoning for belief revision is supported. Intuitionistic epistemic logic, rooted in Brouwer–Heyting–Kolmogorov semantics, rejects the law of excluded middle by modeling constructive uncertainty [14,15]. Here, truth is contingent on an agents' ability to provide evidence or proofs. It is particularly suited to scenarios requiring incremental knowledge acquisition, such as mathematical verification or scientific hypothesis testing. Paraconsistent epistemic logic rejects the principle of explosion to tolerate inconsistencies (e.g., $(\phi \wedge \neg\phi)$) without trivializing inferences. By isolating contradictions via consistency operators (e.g., $(\circ\phi)$) or multi-valued semantics, this logic addresses conflict-driven uncertainty, prevalent in legal debates, fault-tolerant AI, and industrial signal analysis.

Moreover, Lotfi A. Zadeh et al. introduced fuzzy logic and fuzzy sets as a powerful tool [16] to handle uncertainty and imprecision in various real-world applications. Unlike classical binary logic, fuzzy logic allows for gradual truth values between 0 and 1 and, therefore, it is more suitable for modeling and reasoning about real systems. Fuzzy logic and fuzzy sets have been widely applied in various fields. A series of fuzzy epistemic logic approaches have been proposed to characterize the description and reasoning of knowledge and beliefs with uncertainty. In control systems, fuzzy logic has been extensively adopted in the design of intelligent controllers; for example, in robotics [17] and traffic signal control [18]. In decision-making, fuzzy logic frameworks, such as intuitionistic fuzzy sets and rough fuzzy sets, have been employed to tackle multi-attribute decision problems. This application enhances the ability to handle incomplete or ambiguous data [19,20]. Moreover, fuzzy logic has been applied in artificial intelligence to improve the robustness of learning algorithms, especially in pattern recognition [21,22] and neural networks [23,24].

This study presents the fuzzy logic of doxastic attitudes. In this logic, based on a Gödel algebra, the logic statements in the agents' belief bases are assigned values in the interval [0, 1], rather than just binary values. In this framework, an agent does not either fully believe or completely disbelieve a certain belief. Instead, the agent believes to a certain extent, which is represented by a confidence interval within [0, 1]. We introduce a set of axioms and rules of inference and prove the soundness and completeness of the fuzzy logic of doxastic attitudes. Finally, an example is presented to illustrate the application of this logic framework.

The remainder of this paper is structured as follows. In Section 2, we review some relevant concepts and results of the logic of doxastic attitudes. Section 3 presents the fuzzy logic of doxastic attitudes. Its soundness and completeness are explored in Section 4. An example of a real-life application is given in Section 5. Section 6 provides a comparative analysis with alternative approaches. Finally, Section 7 concludes the paper and outlines future research directions.

2. Preliminaries

This section reviews the logic of doxastic attitudes (LDA), which represents explicit beliefs and implicit beliefs of multiple agents [11]. Let $Atm = \{p, q, \ldots\}$ be a countable

infinite set of atomic propositions and $Agt = \{1, \ldots, n\}$ be a finite set of agents. The logical language of LDA is defined in two steps.

First, the formulas of $LANG_0(Atm, Agt)$ are defined as follows:

$$\alpha ::= p| \perp |\alpha \wedge \alpha|\alpha \vee \alpha|\alpha \rightarrow \alpha|\triangle_i\alpha$$

where $p \in Atm$ and $i \in Agt$. The formula $\triangle_i\alpha$ is read as "agent i explicitly (or actually) believes that α is true".

The language $LANG(Atm, Agt)$ is extended by adding modal operators of implicit belief as follows:

$$\varphi ::= \alpha|\varphi \wedge \varphi|\varphi \vee \varphi|\varphi \rightarrow \varphi|\Box_i\varphi$$

where $\alpha \in LANG_0(Atm, Agt)$ and $i \in Agt$. The formula $\Box_i\varphi$ represents that agent i implicitly (or potentially) believes that φ is true. For notational convenience, we write $LANG_0$ instead of $LANG_0(Atm, Agt)$ and $LANG$ instead of $LANG(Atm, Agt)$ when the context is clear.

A Kripke model is a fundamental concept in modal logic and intuitionistic logic, named after the philosopher and logician Saul Kripke [25]. It provides a formal framework for interpreting and evaluating statements within these logical systems. The semantics of LDA are provided by extending the standard multi-relational Kripke semantics.

Definition 1. *A Kripke model is a 4-tuple $M = (W, \mathcal{D}, \mathcal{N}, V)$ where*

- *W is a set of worlds.*
- *$\mathcal{D} : Agt \times W \longrightarrow 2^{LANG_0}$ is a doxastic function.*
- *$\mathcal{N} : Agt \times W \longrightarrow 2^W$ is a notional function.*
- *$V : Atm \longrightarrow 2^W$ is a valuation function.*

Definition 2. *Let $M = (W, \mathcal{D}, \mathcal{N}, V)$ be a Kripke model. Then,*

$(M, w) \models p \Leftrightarrow w \in V(p)$ *if* $p \in Atm$.
$(M, w) \models \varphi \wedge \psi \Leftrightarrow (M, w) \models \varphi$ *and* $(M, w) \models \psi$.
$(M, w) \models \varphi \vee \psi \Leftrightarrow (M, w) \models \varphi$ *or* $(M, w) \models \psi$.
$(M, w) \models \varphi \rightarrow \psi \Leftrightarrow (M, w) \not\models \varphi$ *or* $(M, w) \models \psi$.
$(M, w) \models \triangle_i\alpha \Leftrightarrow \alpha \in \mathcal{D}(i, w)$.
$(M, w) \models \Box_i\varphi \Leftrightarrow \forall v \in \mathcal{N}(i, w) : (M, v) \models \varphi$.

Definition 3. *A Kripke model $M = (W, \mathcal{D}, \mathcal{N}, V)$ is said to be a* ***quasi-notional doxastic model (QNDM)*** *if and only if it satisfies*

$$(C1^*) \quad \mathcal{N}(i, w) \subseteq \bigcap_{\alpha \in \mathcal{D}(i,w)} ||\alpha||_M,$$

with $||\alpha||_M = \{v \in W : (M, v) \models \alpha\}$.

Global consistency and belief correctness are fundamental concepts in logic. Global consistency ensures that the QNDM is serial, while belief correctness means that an agent's set of notional worlds must include the actual world.

Definition 4. *The QNDM $M = (W, \mathcal{D}, \mathcal{N}, V)$ satisfies global consistency (GC) if and only if, for each $i \in Agt$ and $w \in W$, $\mathcal{N}(i, w) \neq \emptyset$.*

Definition 5. *The QNDM $M = (W, \mathcal{D}, \mathcal{N}, V)$ satisfies belief correctness (BC) if and only if, for each $i \in Agt$ and $w \in W$, $w \in \mathcal{N}(i, w)$.*

For ease of searching, the following table lists some symbols and meanings commonly used in this article (Table 1).

Table 1. Symbol table.

Symbols	Meanings
$p, q, p_1 \cdots$	atomic propositions
i, j	agents
α, γ	the formulas in $LANG_0$
φ, ψ	the formulas in $LANG$
w, v, x, y	the worlds in Kripke model

3. Fuzzy Logic of Doxastic Attitudes

In many real-world situations, information or belief is often uncertain, imprecise, and incomplete. Traditional binary logic is insufficient for modeling such complex scenarios Fuzzy logic has emerged as a powerful tool to address these challenges, as it allows for the representation of beliefs and knowledge as continuous values within the interval [0, 1] [16]. This section will extend the logic of doxastic attitudes to a fuzzy framework using a Gödel algebra. The formulas of the fuzzy logic of doxastic attitudes (FLDA) are defined as follows.

Definition 6. *The formulas of $FLANG_0(Atm, Agt)$ are defined as follows:*

$$\gamma ::= \ p \mid t \mid \bot \mid \gamma \wedge \gamma \mid \gamma \vee \gamma \mid \gamma \to \gamma \mid \triangle_i \gamma,$$

where $p \in Atm$, $t \in [0,1]$ and $i \in Agt$.

The formulas of $FLANG(Atm, Agt)$ are defined as follows:

$$\psi \quad ::= \quad \gamma \mid \psi \wedge \psi \mid \psi \vee \psi \mid \psi \to \psi \mid \Box_i \psi,$$

where $\gamma \in FLANG_0$ and $i \in Agt$.

A substitution of $FLANG(Atm, Agt)$ is a pair $\theta = ((p_1, \cdots, p_n), (\varphi_1, \cdots, \varphi_n))$, where $n \geq 0$ and for any $i \leq n$, $p_i \in Atm$ and $\varphi_i \in FLANG(Atm, Agt)$. The substitution $\theta : FLANG(Atm, Agt) \to FLANG(Atm, Agt)$ is a partial function defined as follows: for any $\varphi \in FLANG(Atm, Agt)$, if it is satisfied that for each $j \leq n$, the fact that p_j occurs in a subformula $\triangle_i \gamma$ of φ implies $\varphi_j \in FLANG_0(Atm, Agt)$, then $\varphi\theta$ is obtained from φ by replacing each p_i with φ_i $(i \leq n)$; otherwise, $\varphi\theta$ is not well-defined.

As usual, $\neg\gamma \stackrel{\triangle}{=} \gamma \to \bot$, $\neg\psi \stackrel{\triangle}{=} \psi \to \bot$, and $\leftrightarrow$ can be defined using $\to$. For notational convenience, we write $FLANG_0$ instead of $FLANG_0(Atm, Agt)$ and $FLANG$ instead of $FLANG(Atm, Agt)$ when the context is clear. Moreover, the above definition ensures that the results of substitutions are also well-formed formulas.

Definition 7. *A fuzzy Kripke model is a 4-tuple $M = (W, \mathcal{D}, \mathcal{N}, V)$, where*

- *W is a set of worlds.*
- *$\mathcal{D} : Agt \times W \times FLANG_0 \to [0,1]$ is a belief-based function.*
- *$\mathcal{N} : Agt \times W \to 2^W$ is a notional function.*
- *$V : W \times Atm \to [0,1]$ is a fuzzy valuation function.*

In this framework, $\mathcal{D}(i, w, \gamma)$ means the degree to which the agent i believes γ, while $V(w, p)$ is the truth value of p at w.

Definition 8. *The Gödel algebra is the structure $G = ([0,1], 0, 1, \otimes, \oplus, \to)$, where for any $c, d \in [0,1]$,*

$$c \otimes d \triangleq \min(c,d),$$
$$c \oplus d \triangleq \max(c,d),$$
$$c \to d \triangleq \begin{cases} 1 & \text{if } c \leq d \\ d & \text{otherwise} \end{cases}.$$

Definition 9. *Let $M = (W, \mathcal{D}, \mathcal{N}, V)$ be a fuzzy Kripke model. For any $w \in W$, the evaluation $[\![\cdot]\!]_{(M,w)} : FLANG \to [0,1]$ is given as follows:*

$[\![p]\!]_{(M,w)} \triangleq V(p,w)$ *if* $p \in Atm$.
$[\![t]\!]_{(M,w)} \triangleq t$ *if* $t \in [0,1]$.
$[\![\bot]\!]_{(M,w)} \triangleq 0$.
$[\![\varphi_1 \wedge \varphi_2]\!]_{(M,w)} \triangleq [\![\varphi_1]\!]_{(M,w)} \otimes [\![\varphi_2]\!]_{(M,w)}$.
$[\![\varphi_1 \vee \varphi_2]\!]_{(M,w)} \triangleq [\![\varphi_1]\!]_{(M,w)} \oplus [\![\varphi_2]\!]_{(M,w)}$.
$[\![\varphi_1 \to \varphi_2]\!]_{(M,w)} \triangleq [\![\varphi_1]\!]_{(M,w)} \to [\![\varphi_2]\!]_{(M,w)}$.
$[\![\triangle_i \gamma]\!]_{(M,w)} \triangleq \mathcal{D}(i,w,\gamma)$.
$[\![\Box_i \varphi]\!]_{(M,w)} \triangleq \inf\{[\![\varphi]\!]_{(M,y)} : y \in \mathcal{N}(i,w)\}$.
A formula φ holds at w if $[\![\varphi]\!]_{(M,w)} = 1$. A formula φ holds at M if it holds at every $w \in W$.

Definition 10. *A fuzzy Kripke model $M = (W, \mathcal{D}, \mathcal{N}, V)$ is said to be a fuzzy quasi-notional doxastic model (FQNDM) if and only if it satisfies the condition below:*

$$\text{(FC1}^*\text{)} \quad \text{if } v \in \mathcal{N}(i,w), \text{ then } \forall \gamma \in FLANG_0([\![\gamma]\!]_{(M,v)} \geq \mathcal{D}(i,w,\gamma)).$$

Obviously, when the qualified confidence such as $\mathcal{D}(i,w,\gamma)$ and truth values such as $V(p,w)$ in the model M are assigned 0 or 1, the condition FC1* degenerates into the condition C1*. The global consistency and belief correctness of FQNDMs can be defined similarly to Definitions 4 and 5. We write FQNDM$_Y$ for the set of all FQNDMs satisfying the conditions in Y for $Y \subseteq \{GC, BC\}$.

4. Soundness and Completeness

This section investigates the axiomatic results for the fuzzy logic of doxastic attitudes. To this end, a set of axioms and deduction rules are provided below.

Definition 11. *A subset $\mathcal{L}$ of $FLANG$ is the fuzzy logic of doxastic attitudes (FLDA) if it satisfies the following:*

(1) Every tautology belongs to $\mathcal{L}$.

(2) (Axioms) For any $\gamma, \gamma_1, \gamma_2 \in FLANG_0$; $\varphi_1, \varphi_2 \in FLANG$; and $i \in Agt$,

$$\Box_i(\varphi_1 \wedge \varphi_2) \longleftrightarrow \Box_i\varphi_1 \wedge \Box_i\varphi_2 \in \mathcal{L},$$
$$\triangle_i(\gamma_1 \wedge \gamma_2) \longleftrightarrow \triangle_i\gamma_1 \wedge \triangle_i\gamma_2 \in \mathcal{L},$$
$$(\Box_i\varphi_1 \wedge \Box_i(\varphi_1 \to \varphi_2)) \to \Box_i\varphi_2 \in \mathcal{L},$$
$$\triangle_i\gamma \to \Box_i\gamma \in \mathcal{L}.$$

(3) (Rules) For any $\varphi, \varphi_1, \varphi_2 \in FLANG$, substituting θ, $i \in Agt$, and $D \subseteq [0,1]$,
if $\varphi_1 \in \mathcal{L}$ and $\varphi_1 \to \varphi_2 \in \mathcal{L}$ then $\varphi_2 \in \mathcal{L}$.
if $\varphi \in \mathcal{L}$ and $\varphi\theta$ is well-defined then $\varphi\theta \in \mathcal{L}$.
if $\varphi \in \mathcal{L}$, then $\Box_i\varphi \in \mathcal{L}$.
if $t \to (\varphi_1 \to \varphi_2) \in \mathcal{L}$, then $t \to (\Box_i\varphi_1 \to \Box_i\varphi_2) \in \mathcal{L}$.
if $\forall l \in D(l \to \varphi_1 \in \mathcal{L})$, then $sup(D) \to \varphi_1 \in \mathcal{L}$.

For each $X \subseteq \{\mathbf{T}_{\Box_i}, \mathbf{D}_{\Box_i}\}$, $FLDA_X$ is the extension of the logic FLDA as a result of adding the corresponding axioms of X, for which $\mathbf{T}_{\Box_i}$ and $\mathbf{D}_{\Box_i}$ are defined below: for any $\varphi \in FLANG$ and $i \in Agt$,

$$\neg(\Box_i\varphi \wedge \Box_i\neg\varphi) \in \mathcal{L}. \tag{$\mathbf{T}_{\Box_i}$}$$

$$\Box_i\varphi \to \varphi \in \mathcal{L}, \tag{$\mathbf{D}_{\Box_i}$}$$

We denote the minimal (with respect to the inclusion) $FLDA_X$ with the symbol FK_X for $X \subseteq \{T_{\Box_i}, D_{\Box_i}\}$.

$\mathbf{T}_{\Box_i}$ and $\mathbf{D}_{\Box_i}$ represent global consistency (GC) and belief correctness (BC), respectively. Therefore, we define a mapping cf to capture this correspondence: $cf(\mathbf{T}_{\Box_i}) = GC$ and $cf(\mathbf{D}_{\Box_i}) = BC$. Furthermore, we set $CF(X) = \{cf(Ax) : Ax \in X\}$ for any $X \subseteq \{T_{\Box_i}, D_{\Box_i}\}$.

4.1. Soundness

This subsection is devoted to establishing the soundness of the axiom system of the FLDA. We only illustrate the proofs of axioms related to $\triangle_i$ and $\Box_i$. The proofs of other axioms and rules are similar to the ones in [26], and we leave them to the readers.

Lemma 1. *For any FQNDM $M = (W, \mathcal{D}, \mathcal{N}, V)$, $w \in W$, and $\varphi, \psi \in FLANG$,*

$$[\![(\Box_i\varphi \wedge \Box_i(\varphi \to \psi)) \to \Box_i\psi]\!]_{(M,w)} = 1.$$

Proof. Let $M = (W, \mathcal{D}, \mathcal{N}, V)$ be an FQNDM, $w \in W$, and $\varphi, \psi \in FLANG$. From Definitions 8 and 9, it suffices to show

$$[\![\Box_i\psi]\!]_{(M,w)} \geq [\![\Box_i\varphi \wedge \Box_i(\varphi \to \psi)]\!]_{(M,w)}.$$

For simplicity, suppose $[\![\Box_i\varphi]\!]_{(M,w)} = \delta_1$ and $[\![\Box_i(\varphi \to \psi)]\!]_{(M,w)} = \delta_2$. It follows from Definitions 8 and 9 that $[\![\Box_i\varphi \wedge \Box_i(\varphi \to \psi)]\!]_{(M,w)} = \min(\delta_1, \delta_2)$. From Definition 9, it is enough to show $[\![\psi]\!]_{(M,y)} \geq \min(\delta_1, \delta_2)$ for any $y \in \mathcal{N}(i, w)$. Let $y \in \mathcal{N}(i, w)$. Consider the following two cases.

Case 1: $[\![\varphi \to \psi]\!]_{(M,y)} = 1$. It follows from Definitions 8 and 9 that

$$[\![\psi]\!]_{(M,y)} \geq [\![\varphi]\!]_{(M,y)} \geq \delta_1 \geq \min(\delta_1, \delta_2).$$

Case 2: $[\![\varphi \to \psi]\!]_{(M,y)} \neq 1$. Thus, from Definitions 8 and 9, we obtain $[\![\psi]\!]_{(M,y)} = [\![\varphi \to \psi]\!]_{(M,y)} \geq [\![\Box_i(\varphi \to \psi)]\!]_{(M,w)} = \delta_2 \geq \min(\delta_1, \delta_2)$. □

Lemma 2. *For any FQNDM $M = (W, \mathcal{D}, \mathcal{N}, V)$, $w \in W$ and $\gamma \in FLANG_0$,*

$$[\![\triangle_i\gamma \to \Box_i\gamma]\!]_{(M,w)} = 1.$$

Proof. Let $M = (W, \mathcal{D}, \mathcal{N}, V)$ be an FQNDM, $w \in W$ and $\gamma \in FLANG_0$. It is enough to show that $[\![\Box_i\gamma]\!]_{(M,w)} \geq [\![\triangle_i\gamma]\!]_{(M,w)}$. From (FC1) in Definition 10, we have $[\![\gamma]\!]_{(M,y)} \geq \mathcal{D}(i, w, \gamma)$ for each $y \in \mathcal{N}(i, w)$. It follows from Definition 9 that $[\![\triangle_i\gamma]\!]_{(M,w)} = \mathcal{D}(i, w, \gamma)$ and $[\![\Box_i\gamma]\!]_{(M,w)} = \inf\{[\![\gamma]\!]_{(M,y)} : y \in \mathcal{N}(i, w)\}$. Therefore, we obtain $[\![\Box_i\gamma]\!]_{(M,w)} \geq [\![\triangle_i\gamma]\!]_{(M,w)}$. □

Lemma 3. *If $[\![\varphi]\!]_{(M,w)} = 1$ for any FQNDM $M = (W, \mathcal{D}, \mathcal{N}, V)$ and $w \in W$, then so does $\Box_i\varphi$.*

Proof. Suppose that $[\![\varphi]\!]_{(M,w)} = 1$ for any FQNDM $M = (W, \mathcal{D}, \mathcal{N}, V)$ and $w \in W$. Then, $[\![\varphi]\!]_{(M,y)} = 1$ for any FQNDM $M = (W, \mathcal{D}, \mathcal{N}, V)$, $w \in W$ and $y \in \mathcal{N}(i, w)$. From

Definition 9, we obtain that $[\![\Box_i\varphi]\!]_{(M,w)} = 1$ for any FQNDM $M = (W, \mathcal{D}, \mathcal{N}, V)$ and $w \in W$. □

Lemma 4. *For any FQNDM M, if M satisfies GC, then* $[\![\neg(\Box_i\varphi \wedge \Box_i\neg\varphi)]\!]_{(M,w)} = 1$ *for any formula* $\varphi \in FLANG$ *and any w in M.*

Proof. Suppose that FQNDM M satisfies GC, w is a state in M, and $\varphi \in FLANG$. From Definition 9, it suffices to show that $[\![\Box_i\varphi \wedge \Box_i\neg\varphi]\!]_{(M,w)} (= \min([\![\Box_i\varphi]\!]_{(M,w)}, [\![\Box_i\neg\varphi]\!]_{(M,w)})) = 0$.

Since FQNDM M satisfies GC, there exists $y \in \mathcal{N}(i,w)$. Consider the following two cases.

Case 1: $[\![\varphi]\!]_{(M,y)} = 0$. From Definition 9, we have $[\![\Box_i\varphi]\!]_{(M,w)} = 0$, and then

$$\min([\![\Box_i\varphi]\!]_{(M,w)}, [\![\Box_i\neg\varphi]\!]_{(M,w)}) = 0.$$

Case 2: $[\![\varphi]\!]_{(M,y)} \neq 0$. Then, from Definitions 8 and 9, we obtain $[\![\neg\varphi]\!]_{(M,y)} = 0$. It follows from Definition 9 and $y \in \mathcal{N}(i,w)$ that $[\![\Box_i\neg\varphi]\!]_{(M,w)} = 0$ and

$$\min([\![\Box_i\varphi]\!]_{(M,w)}, [\![\Box_i\neg\varphi]\!]_{(M,w)}) = 0.$$

□

Lemma 5. *For any FQNDM M, if M satisfies BC, then* $[\![\Box_i\varphi \to \varphi]\!]_{(M,w)} = 1$ *for any formula* $\varphi \in FLANG$ *and any w in M.*

Proof. Suppose that FQNDM M satisfies BC, w is a state in M, and $\varphi \in FLANG$. Then, it follows that $w \in \mathcal{N}(i,w)$. Further, it follows from Definition 9 that $[\![\varphi]\!]_{(M,w)} \geq [\![\Box_i\varphi]\!]_{(M,w)}$. Thus, we have $[\![\Box_i\varphi \to \varphi]\!]_{(M,w)} = 1$. □

Therefore, we have the following result.

Theorem 1. *(Soundness) For each* $X \subseteq \{D_{\Box_i}, T_{\Box_i}\}$, *if* $\psi \in FK_X$, *then* ψ *holds at every model in* $FQNDM_{CF(X)}$.

4.2. Completeness

This subsection intends to establish the completeness of the axiom system of the FLDA. As usual, the canonical model plays a central role. Thus, the notions of consistence and completion are provided first.

Definition 12. *An FLDA* $\mathcal{L}$ *is consistent if*

$$\forall p \in [0,1](p \in \mathcal{L} \Longrightarrow p = 1).$$

For any FLDA $\mathcal{L}$ *and* $\varphi \in FLANG$,

$$[\![\varphi]\!]_{\mathcal{L}} \triangleq \sup\{p \in [0,1] : p \to \varphi \in \mathcal{L}\}$$

Definition 13. *Let* $\mathcal{L}$ *be a consistent FLDA and* $u \subseteq FLANG$. *The set u is said to be* $\mathcal{L}$*-consistent if for any* $t \in [0,1]$ *and any limited subset of the set u, for example,*

$$\{p_1 \to \varphi_1, \ldots, p_n \to \varphi_n\}$$

where $p_1, \ldots, p_n \in [0,1]$ *and* $\varphi_1, \ldots, \varphi_n \in FLANG$,

$$\text{if } (\varphi_1 \wedge \cdots \wedge \varphi_n) \to t \in \mathcal{L}, \text{ then } (p_1 \otimes \cdots \otimes p_n) \leq t.$$

Definition 14. *For every pair* u_1, u_2 *of sets of* $FLANG$, $u_1 \leq u_2$ *if and only if for any* $p \to \varphi \in u_1$,

$$p = 0 \quad or \quad \exists t \geq p : t \to \varphi \in u_2.$$

Definition 15. *Let u be an $\mathcal{L}$-consistent set. For any $\varphi \in FLANG$,*

$$[\![\varphi]\!](u) \triangleq sup(D)$$

where D is the set of all elements of $t \in [0,1]$ such that $u \cup \{t \to \varphi\}$ is $\mathcal{L}$-consistent.

Definition 16. *Let u be a set of formulas of $FLANG$ and $\mathcal{L}$ the FLDA. Then, u is said to be $\mathcal{L}$-complete if u is $\mathcal{L}$-consistent and for any $\varphi \in FLANG$,*

$$[\![\varphi]\!](u) \to \varphi \in u.$$

Definition 17. *Let u be an $\mathcal{L}$-consistent set and x an $\mathcal{L}$-complete set. x is said to be a completion of u if*

$$u \leq x.$$

The following are the Lindenbaum's Lemma for our logic and some useful results. Their proofs are standard (cf. see [26]), and we omit them.

Theorem 2. *(1) For every $\mathcal{L}$-consistent set u, there is its completion x.*

(2) For any $\mathcal{L}$-complete set x and $\varphi, \psi \in FLANG$, $[\![\varphi \wedge \psi]\!](x) = [\![\varphi]\!](x) \otimes [\![\psi]\!](x)$, $[\![\varphi \vee \psi]\!](x) = [\![\varphi]\!](x) \oplus [\![\psi]\!](x)$, and $[\![\varphi \to \psi]\!](x) = [\![\varphi]\!](x) \to [\![\psi]\!](x)$.

The canonical model is a commonly employed method for demonstrating the completeness of a logical system. The canonical model of the FLDA is defined as follows.

Definition 18. *A canonical model of the FLDA $\mathcal{L}$ is a fuzzy Kripke model*

$$M_{\mathcal{L}} \triangleq (W_{\mathcal{L}}, \mathcal{D}_{\mathcal{L}}, \mathcal{N}_{\mathcal{L}}, V_{\mathcal{L}}),$$

where

- *$W_{\mathcal{L}}$ consists of all $\mathcal{L}$-complete sets.*
- *$\mathcal{D}_{\mathcal{L}} : Agt \times W_{\mathcal{L}} \times FLANG_0 \longrightarrow [0,1]$ is defined as follows: for any $i \in Agt$, $x \in W_{\mathcal{L}}$, $\gamma \in FLANG_0$,*

$$\mathcal{D}_{\mathcal{L}}(i, x, \gamma) \triangleq [\![\triangle_i \gamma]\!](x).$$

- *$\mathcal{N}_{\mathcal{L}} : Agt \times W_{\mathcal{L}} \longrightarrow 2^{W_{\mathcal{L}}}$ is defined as follows: for any $x, y \in W_{\mathcal{L}}$, $i \in Agt$,*

$$y \in \mathcal{N}_{\mathcal{L}}(i, x) \text{ iff } [\![\Box_i \varphi]\!](x) \leq [\![\varphi]\!](y) \text{ for any } \varphi \in FLANG.$$

- *$V_{\mathcal{L}} : Atm \times W_{\mathcal{L}} \to [0,1]$ is defined as follows: for any $q \in Atm$, $x \in W_{\mathcal{L}}$,*

$$V_{\mathcal{L}}(q, x) \triangleq [\![q]\!](x).$$

Before demonstrating the completeness, the main property of the canonical model is needed.

Theorem 3. *Let $\mathcal{L}$ be a consistent FLDA.*

(1) For any $\varphi \in FLANG$ and $x \in W_{\mathcal{L}}$, $[\![\varphi]\!]_{(M_{\mathcal{L}}, x)} = [\![\varphi]\!](x)$.

(2) $M_{\mathcal{L}}$ satisfies FC1*, and then is an FQNDM.

Proof. (1) We prove this via induction with the complexity of φ.

Case 1: $\varphi = q \in Atm$. From Definitions 9 and 18, it follows that $[\![\varphi]\!]_{(M_{\mathcal{L}}, x)} = [\![\varphi]\!](x)$ for any $x \in W_{\mathcal{L}}$.

Case 2: $\varphi = t \in [0,1]$. From Definition 9, we obtain $[\![t]\!]_{(M_{\mathcal{L}}, x)} = t$ for any $x \in W_{\mathcal{L}}$. Moreover, for any $x \in W_{\mathcal{L}}$, since x is consistent, it follows from Definitions 15 and 8 that $t = [\![t]\!](x)$.

Case 3: $\varphi = \psi_1 \wedge \psi_2, \psi_1 \vee \psi_2$, or $\psi_1 \to \psi_2$. Let $x \in W_{\mathcal{L}}$. According to the induction hypothesis, we obtain

$$[\![\psi_1]\!]_{(M_{\mathcal{L}},x)} = [\![\psi_1]\!](x) \text{ and } [\![\psi_2]\!]_{(M_{\mathcal{L}},x)} = [\![\psi_2]\!](x).$$

Then, from Definition 9 and (2) in Theorem 2, we have

$$\begin{aligned}
[\![\varphi]\!]_{(M_{\mathcal{L}},x)} &= [\![\psi_1 \wedge \psi_2]\!]_{(M_{\mathcal{L}},x)} \\
&= [\![\psi_1]\!]_{(M_{\mathcal{L}},x)} \otimes [\![\psi_2]\!]_{(M_{\mathcal{L}},x)} \\
&= [\![\psi_1]\!](x) \otimes [\![\psi_2]\!](x) \\
&= [\![\psi_1 \wedge \psi_2]\!](x) \\
&= [\![\varphi]\!](x).
\end{aligned}$$

The other two subcases can be proved similarly.

Case 4: $\varphi = \triangle_i \gamma$. Due to Definitions 9 and 18, $[\![\varphi]\!]_{(M_{\mathcal{L}},x)} = [\![\varphi]\!](x)$ trivially holds for any $x \in W_{\mathcal{L}}$.

Case 5: $\varphi = \Box_i \psi$. Let $x \in W_{\mathcal{L}}$. According to the inductive hypothesis, for each $y \in W_{\mathcal{L}}$, $[\![\psi]\!]_{(M_{\mathcal{L}},y)} = [\![\psi]\!](y)$ holds. Thus, according to Definition 9, it suffices to prove $\inf\{[\![\psi]\!](y) : y \in \mathcal{N}_{\mathcal{L}}(i,x)\} = [\![\Box_i \psi]\!](x)$.

From Definition 18, we have $[\![\Box_i \psi]\!](x) \leq [\![\psi]\!](y)$ for any $y \in \mathcal{N}_{\mathcal{L}}(i,x)$. It follows that

$$[\![\Box_i \psi]\!](x) \leq \inf\{[\![\psi]\!](y) : y \in \mathcal{N}_{\mathcal{L}}(i,x)\}.$$

To prove $[\![\Box_i \psi]\!](x) \geq \inf\{[\![\psi]\!](y) : y \in \mathcal{N}_{\mathcal{L}}(i,x)\}$, we set

$$m \triangleq \{[\![\Box_i \varphi']\!](x) \to \varphi' : \varphi' \in FLANG\} \cup \{([\![\Box_i \psi]\!](x) \to 0) \to (\psi \to 0)\}.$$

We show the following claim first.

Claim. *m is $\mathcal{L}$-consistent.*

It is enough to prove that for every finite subset $\{p_1 \to \varphi_1, \ldots, p_n \to \varphi_n\} \subseteq m$ and every $t \in [0,1]$, if $(\varphi_1 \wedge \cdots \wedge \varphi_n) \to t \in \mathcal{L}$, then $(p_1 \otimes \cdots \otimes p_n) \leq t$.

Let $u = \{p_1 \to \varphi_1, \ldots, p_n \to \varphi_n\} \subseteq m$, $t \in [0,1]$ and $(\varphi_1 \wedge \cdots \wedge \varphi_n) \to t \in \mathcal{L}$. This claim is divided into the following two cases.

Case C-1: $([\![\Box_i \psi]\!](x) \to 0) \to (\psi \to 0) \in u$. Then, without loss of generality, assume that $p_1 \to \varphi_1 = ([\![\Box_i \psi]\!](x) \to 0) \to (\psi \to 0)$. Therefore, we obtain $p_j = [\![\Box_i \varphi_j]\!](x)$ for $j = 2, \ldots n$. Then, it follows from $(\varphi_1 \wedge \cdots \wedge \varphi_n) \to t \in \mathcal{L}$ that $((\psi \to 0) \wedge \varphi_2 \wedge \cdots \wedge \varphi_n) \to t \in \mathcal{L}$. Thus, we have $(t \to 0) \to ((\varphi_2 \wedge \cdots \wedge \varphi_n) \to \psi) \in \mathcal{L}$. According to the rule $t \to (\psi_1 \to \psi_2) \Rightarrow t \to (\Box_i \psi_1 \to \Box_i \psi_2)$ and the axiom $\Box_i(\psi_1 \wedge \psi_2) \leftrightarrow \Box_i \psi_1 \wedge \Box_i \psi_2$, we obtain

$$(t \to 0) \to ((\Box_i \varphi_2 \wedge \cdots \wedge \Box_i \varphi_n) \to \Box_i \psi) \in \mathcal{L}.$$

Therefore, it follows from Definition 15 that $(t \to 0) \leq [\![((\Box_i \varphi_2 \wedge \cdots \wedge \Box_i \varphi_n) \to \Box_i \psi)]\!](x)$. Thus, from (2) in Theorem 2, we have $(t \to 0) \leq ([\![\Box_i \varphi_2]\!](x) \otimes \cdots \otimes [\![\Box_i \varphi_n]\!](x)) \to [\![\Box_i \psi]\!](x)$. Then, it follows from Definition 8 that $([\![\Box_i \psi]\!](x) \to 0) \otimes [\![\Box_i \varphi_2]\!](x) \otimes \cdots \otimes [\![\Box_i \varphi_n]\!](x) \leq t$, i.e., $p_1 \otimes \cdots \otimes p_n \leq t$.

Case C-2: $([\![\Box_i \psi]\!](x) \to 0) \to (\psi \to 0) \notin u$. Therefore, for any $j \leq n$, we have $p_j = [\![\Box_i \varphi_j]\!](x)$. It follows from $(\varphi_1 \wedge \cdots \wedge \varphi_n) \to t \in \mathcal{L}$ that $(t \to 0) \to ((\varphi_1 \wedge \cdots \wedge \varphi_n) \to 0) \in \mathcal{L}$. Then, according to the rule $t \to (\psi_1 \to \psi_2) \Rightarrow t \to (\Box_i \psi_1 \to \Box_i \psi_2)$ and the axiom

$\Box_i(\psi_1 \wedge \psi_2) \leftrightarrow \Box_i\psi_1 \wedge \Box_i\psi_2$, we obtain $(t \to 0) \to ((\Box_i\varphi_1 \wedge \cdots \Box_i\varphi_n) \to 0) \in \mathcal{L}$. It follows from Definition 15 and Theorem 2 that

$$t \to 0 \leq [\![(\Box_i\varphi_1 \wedge \cdots \wedge \Box_i\varphi_n) \to 0)]\!](x) = ([\![\Box_i\varphi_1]\!](x) \otimes \cdots \otimes [\![\Box_i\varphi_n]\!](x)) \to 0.$$

Thus, we obtain $[\![\Box_i\varphi_1]\!](x) \otimes \cdots \otimes [\![\Box_i\varphi_n]\!](x) = p_1 \otimes \cdots \otimes p_n \leq t$, as desired.

Now, we return to the proof of Case 5. Let y be an completion of m. For any $\varphi' \in FLANG$, since $[\![\Box_i\varphi']\!](x) \to \varphi' \in y$, from Definition 15, we obtain $[\![\Box_i\varphi']\!](x) \leq [\![\varphi']\!](y)$ and then $y \in \mathcal{N}(i,x)$. Moreover, by $([\![\Box_i\psi]\!](x) \to 0) \to (\psi \to 0) \in m$, $m \leq y$, Definition 15 and Theorem 2, we have $[\![\Box_i\psi]\!](x) \to 0 \leq [\![\psi \to 0]\!](y) = [\![\psi]\!](y) \to 0$. Therefore, it follows that $[\![\Box_i\psi]\!](x) \geq \inf\{[\![\psi]\!](y) : y \in \mathcal{N}_{\mathcal{L}}(i,x)\}$.

(2) Let $y \in N_{\mathcal{L}}(i,x)$ and $\gamma \in FLANG_0$. It follows from Definition 11 that $\Delta_i\gamma \to \Box_i\gamma \in \mathcal{L}$. Then, since x is $\mathcal{L}$-consistent, we have $[\![\Delta_i\gamma]\!](x) \leq [\![\Box_i\gamma]\!](x)$. Thus, from $y \in N_{\mathcal{L}}(i,x)$, Definition 9 and (1), we obtain

$$[\![\Delta_i\gamma]\!](x) \leq [\![\Box_i\gamma]\!](x) = [\![\Box_i\gamma]\!]_{(\mathcal{L},x)} \leq [\![\gamma]\!]_{(\mathcal{L},y)}.$$

Further, it follows from Definition 18 that $\mathcal{D}_{\mathcal{L}}(i,x,y) = [\![\Delta_i\gamma]\!](x) \leq [\![\gamma]\!]_{(\mathcal{L},y)}$. Therefore, from Definition 10, $M_{\mathcal{L}}$ satisfies $FC1^*$ and then is an FQNDM. □

The completeness of the logic system is presented as follows.

Theorem 4. *Let $X \subseteq \{\mathbf{D}_{\Box_i}, \mathbf{T}_{\Box_i}\}$. If a formula $\varphi \in FLANG$ holds at every $FQNDM_{CF(X)}$, then $\varphi \in FK_X$.*

Proof. Suppose that φ holds at every $\text{FQNDM}_{CF(X)}$. To complete the proof, assume $\varphi \notin FK_X$ and construct a contradiction below.

We show that the set $\{[\![(\varphi \to 0)]\!]_{FK_X} \to (\varphi \to 0)\}$ is FK_X-consistent first. Let $t \in [0,1]$ and $(\varphi \to 0) \to t \in FK_X$. Then, we have $(t \to 0) \to \varphi \in FK_X$. From Definition 12, we obtain $(t \to 0) \leq [\![\varphi]\!]_{FK_X}$ and then $[\![\varphi]\!]_{FK_X} \to 0 \leq t$. Therefore, it follows from Definition 13 that the set $\{([\![\varphi]\!]_{FK_X} \to 0) \to (\varphi \to 0)\}$ is FK_X-consistent. Further, from (1) in Theorems 2 and 3, there exists its completion $x \in W_{FK_X}$ such that

$$[\![\varphi]\!]_{FK_X} \to 0 \leq [\![\varphi \to 0]\!]_{(\mathcal{M}_{FK_X},x)} = [\![\varphi \to 0]\!](x). \tag{1}$$

As x is FK_X-complete, from (2) in Theorem 2, we have

$$[\![(\varphi \to 0)]\!](x) = [\![\varphi]\!](x) \to 0. \tag{2}$$

Then, from (1) and (2), we obtain $[\![\varphi]\!]_{FK_X} \to 0 \leq [\![\varphi]\!](x) \to 0$ and then $[\![\varphi]\!](x) \leq [\![\varphi]\!]_{FK_X}$. Since $\varphi \notin FK_X$, it follows that $[\![\varphi]\!]_{FK_X} < 1$. Thus, we obtain $[\![\varphi]\!]_{(\mathcal{M}_{FK_X},x)} = [\![\varphi]\!](x) < 1$, and φ does not hold at x. Therefore, a contradiction arises. □

Finally, we have

Theorem 5. *For any $X \subseteq \{\mathbf{D}_{\Box_i}, \mathbf{T}_{\Box_i}\}$, FK_X is sound and complete for the class $FQNDM_{CF(X)}$.*

5. Example

In this section, we give an application example of the FLDA which is referred to in [11]. We recall the story simply. An artificial companion HAL takes care of an elderly person called Bob. Bob prefers to stay at home watching TV or reading a book rather than to go out for a walk, although the latter is necessary to keep him in good health. Therefore, when the weather is pleasant, HAL suggests that Bob should take a walk outdoors:

"Hey Bob! It is a great sunny day. You should take advantage of it and go out for a walk before the end of the day". Bob replies that

"The last time I went out for a walk it was so cold. I did not like it at all. If I am sure that it is not cold outside, then I will follow your advice, otherwise I will not!"

HAL knows that the outside temperature is 11 °C and that it may not be cold outside. In the case described above, HAL persuades Bob to accept the suggestion to go out. We present how the FLDA captures this persuasion.

Assume that $Atm = \{Go-out, Temp, Sunny\}$, where Go-out means that Bob accepts the belief in going out, Temp describes the suitability of temperature, and Sunny shows the current weather. To operationalize the fuzzy concept of temperature, we map it to belief truth values using the following table (Table 2).

Table 2. Temperature correspondence.

Fuzzy Concept	Temperature Value (°C)	Belief Truth Value ($[\![Temp]\!]_A$)
Cold	0–10	0.2
Cool	10–15	0.8
Comfortable	15–18	1.0
Warm	18–22	0.8
Hot	>22	0.2

In initial state w_0, HAL's explicit belief is "It is a great sunny day" and "Bob should go out". Thus, we set $\mathcal{D}(HAL, w_0, Sunny) = 1$ and $\mathcal{D}(HAL, w_0, Go-out) = 1$. Similarly, Bob's explicit belief is that if the temperature is right, he can go out for a walk and then set $\mathcal{D}(Bob, w_0, Temp \rightarrow Go-out) = 1$. The above persuasion can be described by the evolution of the fuzzy quasi-notional doxastic model in the following table (Table 3). For simplicity, we only list explicit beliefs and used abbreviations, such as referring to $\mathcal{D}(HAL, w_0, Sunny) = 1$ as $Sunny = 1$.

From w_0 to w_1, HAL told Bob that the weather was sunny and that he should go out for a walk; Bob told HAL that he would only go out if the temperature was just right. From w_1 to w_2, HAL checked the temperature and calculated the temperature suitability to be 0.8 based on Table 2. From w_2 to w_3, HAL told Bob the current temperature. From w_3 to w_4, Bob resolved to go out for a walk based on the currently known beliefs.

Table 3. Evolution of model.

State	HAL's Beliefs	Bob's Beliefs
w_0	Go-out = 1, Sunny = 1	Temp→Go-out = 1
w_1	Go-out = 1, Sunny = 1, Temp→Go-out = 1	Temp→Go-out = 1, Sunny = 1
w_2	Go-out = 1, Sunny = 1, Temp→Go-out = 1, Temp = 0.8	Temp→Go-out = 1, Sunny = 1
w_3	Go-out = 1, Sunny = 1, Temp→Go-out = 1, Temp = 0.8	Temp→Go-out = 1, Sunny = 1, Temp = 0.8
w_4	Go-out = 1, Sunny = 1, Temp→Go-out = 1, Temp = 0.8	Temp→Go-out = 1, Sunny = 1, Temp = 0.8, Go-out = 0.8

6. Related Work

The representation and reasoning of knowledge with uncertainty is an important research direction in epistemic logic. A variety of technical approaches have been used in the literature to deal with uncertainty. The following is a comparison of these methods with the work in this paper.

The most similar work to this paper is fuzzy epistemic logic [27–30], which integrates many-valued modal logic with graded belief frameworks to formalize reasoning under conditions of vagueness and uncertainty. Fitting extends classical modal operators to

accommodate truth degrees and introduces many-valued modal logic [27]. Maruyama extended this paradigm to handle incomparable beliefs, where agents' fuzzy beliefs may not adhere to total orders [28]. Benevides et al. introduced graded epistemic logic, which quantifies belief strength using explicit numerical or symbolic grades [29,30]. The difference between these works and ours stems from the difference between epistemic logic and the logic of doxastic attitudes. Similar to the logic of doxastic attitudes, the framework in this study makes it easier to describe and distinguish between explicit and implicit beliefs than the above work, and will avoid the potential exponential growth of the original cognitive model as the length of the private declaration sequence increases.

Another class of work closely related to this study is probabilistic epistemic logic, which combines epistemic logic with probabilistic reasoning to formalize uncertainty in multi-agent systems. Halpern established the theoretical bedrock by merging Bayesian probability with epistemic modalities [12]. Kooi extended Halpern's framework to model information flow and probabilistic updates and provides probabilistic dynamic epistemic logic [13]. Pan and Guo explored the relationship between probability and belief based on the Lockean thesis and adopt neighborhood semantics that defines belief directly using probability [31]. The fuzzy logic of doxastic attitudes addresses vagueness by representing partial truth, where propositions can hold intermediate degrees of truth between 0 (false) and 1 (true). Probabilistic epistemic logic focuses on depicting the degree of confidence based on existing information. Its core lies in describing the absence of knowledge or randomness. The former is applicable to the reasoning of natural language concepts with unclear boundaries, while the latter is suitable for reasoning about random events and related risk decisions.

Intuitionistic epistemic logic diverges from classical epistemic logic by rejecting the law of excluded middle and requiring constructive proofs for knowledge claims [14,15]. For instance, knowledge operators (e.g., $K_i\psi$) may be interpreted through the Brouwer–Heyting–Kolmogorov (BHK) semantics: an agent i knows ψ only if a constructive justification for ψ exists. When an agent is unable to provide a proof for a formula nor a proof for its negation, the truth value of the formula is undetermined. This is different from our work and fuzzy epistemic logic, in which the truth value is given within the interval $[0, 1]$ but is definite.

Paraconsistent epistemic logic represents a significant advancement in a formal system that reconciles inconsistency-tolerant reasoning with epistemic notions such as belief, evidence, and knowledge [32,33]; in other words, it distinguishes between evidence (epistemic justification) and truth (ontological fact), such that true contradictions trivialize the system, while evidence for contradictions is tolerated as epistemic uncertainty. Such uncertainty differs from the ones in this study and within fuzzy epistemic logic.

7. Conclusions

This study introduced an extension of the logic of doxastic attitudes to a fuzzy setting, where beliefs are represented as continuous values within the interval [0,1] rather than binary Boolean values. This framework provides a more nuanced and realistic representation of uncertain and imprecise beliefs. We presented the axioms of the fuzzy logic of doxastic attitudes and established their theoretical foundations by proving soundness and completeness. Through a practical example in tourism scenarios, we demonstrated the applicability of this framework in modeling and reasoning with uncertain preferences and beliefs. This example highlights the potential of the proposed logic to enhance decision-making processes in real-world applications where uncertainty and ambiguity are inherent.

In our future research, we aim to explore the application of the fuzzy logic of doxastic attitudes in the financial field. Although traditional epistemic logic is useful, it has limitations in dealing with the inherent uncertainty and fuzziness that are prevalent in financial

data and decision-making processes. Through combining fuzzy logic with epistemic logic, we strive to construct a more detailed framework to better simulate the complex and often ambiguous nature of financial information. This hybrid approach will enable us to represent beliefs and knowledge as continuous values within the interval [0, 1], rather than being confined to binary true or false classifications. We believe this will more accurately reflect the ways in which financial entities reason and make decisions under conditions of uncertainty. Our research will focus on developing a set of axioms and rules for this fuzzy cognitive logic framework specifically tailored to financial contexts. We will investigate its potential in various financial applications, including risk assessment, market trend prediction, and portfolio optimization.

Author Contributions: Conceptualization, J.Z.; methodology, J.Z.; proof, J.Z.; validation, X.Z. and Y.Z.; formal analysis, J.Z.; investigation, X.Z.; resources, J.Z.; writing—original draft preparation, J.Z.; writing—review and editing, J.Z. and X.Z.; example L.T.; funding acquisition, J.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by National Natural Science Foundation (NNSF) of China grant number 61602249, and the Major Project of Natural Science Foundation of Jiangsu Education Department of Jiangsu Province of China grant number 21KJA630001.

Data Availability Statement: No new data were generated or analyzed in support of this research.

Conflicts of Interest: The authors declare that there are no known financial or personal conflicts that could have influenced the results of this study.

References

1. Hintikka, J. *Knowledge and Belief: An Introduction to the Logic of the Two Notions*; King's College Publications: London, UK, 1962.
2. Benthem, J.V. Epistemic Logic and Epistemology: The State of Their Affairs. *Philos. Stud. Int. J. Philos. Anal. Tradit.* **2006**, *128*, 49–76. [CrossRef]
3. Ditmarsch, H.V.; Hoek, W.V.D.; Kooi, B.P. *Dynamic Epistemic Logic*; Springer: Berlin/Heidelberg, Germany, 2008.
4. Meyer, J.J.C.; Meyer, J.J.C.; van der Hoek, W. *Epistemic Logic for AI and Computer Science*; Cambridge University Press: Cambridge, UK, 1995.
5. Kaneko, M. Epistemic logics and their game theoretic applications: Introduction. *Econ. Theory* **2002**, *19*, 7–62. [CrossRef]
6. Rosser, J.B. Logic and Epistemology in Behavioral Economics. In *Artificial Intelligence, Learning and Computation in Economics and Finance*; Springer: Cham, Switzerland, 2023; pp. 27–40.
7. Harvey, A.G.; Watkins, E.; Mansell, W. *Cognitive Behavioural Processes Across Psychological Disorders: A Transdiagnostic Approach to Research and Treatment*; Oxford University Press: Oxford, UK, 2004.
8. Holliday, W.H. Epistemic Logic and Epistemology. In *Introduction to Formal Philosophy*; Springer: Cham, Switzerland, 2018; pp. 351–369.
9. Lorini, E. In Praise of Belief Bases: Doing Epistemic Logic Without Possible Worlds. In Proceedings of the AAAI Conference on Artificial Intelligence, New Orleans, LA, USA, 2–7 February 2018; Volume 32.
10. Lorini, E.; Jimenez, B.F.R. Decision Procedures for Epistemic Logic Exploiting Belief Bases. In Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems, Montreal, QC, Canada, 13–17 May 2019; pp. 944–952.
11. Lorini, E. Rethinking Epistemic Logic with Belief Bases. *Artif. Intell.* **2020**, *282*, 103–233. [CrossRef]
12. Halpern, J.Y. *Reasoning about Uncertainty*; The MIT Press: Cambridge, MA, USA, 2017. [CrossRef]
13. Kooi, B.P. Probabilistic Dynamic Epistemic Logic. *J. Logic Lang. Inf.* **2003**, *12*, 381–408. [CrossRef]
14. Artemov, S.; Protopopescu, T. Intuitionistic Epistemic Logic. *Rev. Symb. Log.* **2016**, *9*, 266–298. [CrossRef]
15. Su, Y.; Sano, K. First-Order Intuitionistic Epistemic Logic. In Proceedings of the Logic, Rationality, and Interaction, Chongqing, China, 18–21 October 2019.
16. Zadeh, L.A. Fuzzy Sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
17. Valdez, F.; Castillo, O.; Melin, P. A Review on Type-2 Fuzzy Systems in Robotics and Prospects for Type-3 Fuzzy. In Proceedings of the Applied Mathematics and Computational Intelligence, Tripura, India, 23–24 December 2023; pp. 211–223.
18. Chou, C.H.; Teng, J.C. A fuzzy logic controller for traffic junction signals. *Inf. Sci.* **2002**, *143*, 73–97. [CrossRef]
19. Hadrani, A.; Guennoun, K.; Saadane, R.; Wahbi, M. Fuzzy Rough Sets: Survey and Proposal of an Enhanced Knowledge Representation Model Based on Automatic Noisy Sample Detection. *Cogn. Syst. Res.* **2020**, *64*, 37–56. [CrossRef]

20. Dang, T.T.; Nguyen, N.A.T.; Nguyen, V.T.T.; Dang, L.T.H. A Two-Stage Multi-Criteria Supplier Selection Model for Sustainable Automotive Supply Chain under Uncertainty. *Axioms* **2022**, *11*, 228. [CrossRef]
21. Mitra, S.; Pal, S.K. Fuzzy sets in pattern recognition and machine intelligence. *Fuzzy Sets Syst.* **2005**, *156*, 381–386. [CrossRef]
22. Marín Díaz, G.; Gómez Medina, R.; Aijón Jiménez, J.A. Integrating Fuzzy C-Means Clustering and Explainable AI for Robust Galaxy Classification. *Mathematics* **2024**, *12*, 2797. [CrossRef]
23. Nik-Khorasani, A.; Mehrizi, A.; Sadoghi-Yazdi, H. Robust hybrid learning approach for adaptive neuro-fuzzy inference systems. *Fuzzy Sets Syst.* **2024**, *481*, 108890. [CrossRef]
24. Lima-Junior, F.R. Advances in Fuzzy Logic and Artificial Neural Networks. *Mathematics* **2024**, *12*, 3949. [CrossRef]
25. Kripke, S. Semantical Considerations on Modal Logic. *Acta Philos. Fenn.* **1963**, *16*, 83–94.
26. Mironov, A.M. Fuzzy modal logics. *J. Math. Sci.* **2005**, *128*, 3461–3483.
27. Fitting, M. Many-Valued Modal Logics. *Fundam. Informaticae* **1991**, *15*, 235–254.
28. Maruyama, Y. Reasoning about Fuzzy Belief and Common Belief: With Emphasis on Incomparable Beliefs. In Proceedings of the 22nd International Joint Conference on Artificial Intelligence, Barcelona, Spain, 16–22 July 2011; pp. 1008–1013.
29. Benevides, M.R.F.; Madeira, A.; Martins, M.A. A Family of Graded Epistemic Logics. *Electron. Notes Theor. Comput. Sci.* **2018**, *338*, 45–59.
30. Benevides, M.; Madeira, A.; Martins, M.A. Graded Epistemic Logic with Public Announcement. *J. Log. Algebr. Methods Program.* **2022**, *125*, 100732.
31. Pan, Y.; Guo, M. Probabilistic epistemic logic based on neighborhood semantics. *Synthese* **2024**, *203*, 135.
32. Walter, C.; Abilio, R. On epistemic and ontological interpretations of intuitionistic and paraconsistent paradigms. *Log. J. IGPL* **2021**, *29*, 569–584.
33. Carnielli, W.; Rodrigues, A. An epistemic approach to paraconsistency: A logic of evidence and truth. *Synthese* **2019**, *196*, 3789–3813.

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