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Results on Linear Operators Associated with Pascal Distribution Series for a Certain Class of Normalized Analytic Functions

Saiful R. Mondal [1,*,†], Manas Kumar Giri [2,†] and Raghavendar Kondooru [2,†]

[1] Department of Mathematics and Statistics, College of Science, King Faisal University, Al-Hasa 31982, Saudi Arabia
[2] Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology Vellore, Vellore 632014, India; manaskumar.giri2020@vitstudent.ac.in (M.K.G.); raghavendar.k@vit.ac.in (R.K.)
* Correspondence: smondal@kfu.edu.sa
† These authors contributed equally to this work.

Abstract: In this study, we establish sufficient conditions for determining when linear operators related to the Pascal distribution series belong to various subclasses of univalent functions. Our findings are based on applying a coefficient bound from a specific class of analytic functions, providing valuable insights into the behavior of these functions under transformation.

Keywords: univalent functions; starlike functions; convex functions; spirallike functions; pascal distribution series

MSC: 30C45; 30C50; 33C90

1. Introduction

The Pascal distribution series [1], a fundamental component of probability theory, is extensively employed in statistics to construct generating functions that effectively model various probability and counting processes. This series is not only pivotal in these areas but also significantly contributes to the study of analytic functions, particularly in exploring essential properties such as univalence, starlikeness, and convexity.

Moreover, the Pascal distribution series serves as a foundation for deriving linear operators within geometric function theory, which enhances our understanding of the geometric behavior of functions. For instance, El-Deeb et al. [1] formulated a power series that links its coefficients to the Pascal distribution, allowing researchers to investigate the influence of these coefficients on the geometrical properties of univalent functions. This connection not only deepens our insight into the characteristics of such functions but also fosters further exploration in the field of geometric function theory. The Pascal distribution is described as follows to give additional context:

A variable y is said to follow a Pascal distribution [1] if it assumes values $0, 1, 2, 3, \cdots$ with specific associated probabilities,

$$(1-s)^d, \frac{sd(1-s)^d}{1!}, \frac{s^2d(d+1)(1-s)^d}{2!}, \frac{s^3d(d+1)(d+2)(1-s)^d}{3!}, \cdots$$

respectively, here s and d are termed as the parameters. So, then

$$P(y=j) = \binom{j+d-1}{d-1} s^j (1-s)^d, \; j = 0, 1, 2, \cdots .$$

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In [1], the Pascal distribution series derived as

$$\mathcal{P}_s^d(z) = z + \sum_{j=2}^{\infty} \binom{j+d-2}{d-1} s^{j-1}(1-s)^d z^j,$$

where $d \geq 1$ and $0 \leq s \leq 1$.

In complex analysis, researchers have applied the Pascal distribution (refer to [1–4] and reference therein) series to define subclasses of univalent functions, establishing sufficient conditions for these functions to be starlike or convex. This field is still evolving, with ongoing research focused on deriving new bounds. Its use deepens insights into transformation properties, conformal mappings, and harmonic structures, making it an essential tool for exploring complex structures in geometric contexts. Starlike and convex functions, which have applications in conformal mappings and geometric function theory, often have constraints on their series coefficients. By using the Pascal distribution series in generating the linear operators, researchers can derive necessary and sufficient conditions for different subclasses of univalent functions.

Let consider a function $\Psi(z)$ defined as

$$\Psi(z) = z + \sum_{j=2}^{\infty} \alpha_j z^j, \tag{1}$$

and $\mathcal{A}$ represent the class of all functions $\Psi(z)$ given in (1), that are analytic in the open unit disk $\mathbb{D}$, where $\mathbb{D} = \{z : |z| < 1\}$. Moreover, the classes $\mathcal{S}$, $\mathcal{S}^*$, and $\mathcal{C}$ (as described in [5]) represent the classes of univalent, starlike, and convex functions, respectively, and the analytic characterizations for these classes are given as follows:

$$\begin{aligned}
\mathcal{S} &:= \{\Psi \in \mathcal{A} : \Psi \text{ is univalent in } \mathbb{D}\},\\
\mathcal{S}^* &:= \{\Psi \in \mathcal{S} : \Psi \text{ is starlike in } \mathbb{D}\},\\
\mathcal{C} &:= \{\Psi \in \mathcal{S} : \Psi \text{ is convex in } \mathbb{D}\}.
\end{aligned}$$

Let $\Psi \in \mathcal{A}$ is said to be starlike of order σ $(0 \leq \sigma < 1)$ if it satisfies the following condition:

$$\Re\left\{\frac{z\Psi'}{\Psi}\right\} > \sigma, \quad z \in \mathbb{D},$$

and convex of order σ if it satisfies the following condition:

$$\Re\left\{1 + \frac{z\Psi''}{\Psi'}\right\} > \sigma, \quad z \in \mathbb{D}.$$

For $\sigma = 0$ the classes $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{C}(0) = \mathcal{C}$.

A function $\Psi \in S^*$ is said to be uniformly starlike in the unit disk $\mathbb{D}$ if for every circular arc γ, with center z_0 contained in $\mathbb{D}$ then $\Psi(\gamma)$ is also starlike with respect to $\Psi(z_0)$ in $\mathbb{D}$. We denote the class of uniformly starlike functions with $\mathcal{UST}$.

A function $\Psi \in C$ is said to be uniformly convex in $\mathbb{D}$ if for every circular arc γ with center z_0 in $\mathbb{D}$ then $\Psi(\gamma)$ is also convex in $\mathbb{D}$. We use $\mathcal{UCV}$ to denote the class of uniformly convex functions.

Additionally, consider a class of $k-$uniformly convex of order σ, denoted as $k-\mathcal{UCV}(\sigma)$, which is defined in [6] as follows.

Definition 1 ([6]). *Let $\Psi \in \mathcal{A}$, with $0 \leq k < \infty$ and $0 \leq \sigma < 1$, then*

$$\Psi \in k - \mathcal{UCV}(\sigma) \iff \Re\left(1 + \frac{z\Psi''}{\Psi'}\right) \geq k\left|\frac{z\Psi''}{\Psi'}\right| + \sigma.$$

Note that, if $\sigma = 0$ then $k - \mathcal{UCV}(\sigma) \equiv k - \mathcal{UCV}$, and if $k = 1$ then $1 - \mathcal{UCV} \equiv \mathcal{UCV}$, known as classes of uniformly convex functions (see [7–9]).

A related class, denoted by $k - \mathcal{S}_p(\sigma)$, is defined using the Alexander transform. Specifically, Ψ belongs to $k - \mathcal{UCV}(\sigma) \iff z\Psi' \in k - \mathcal{S}_p(\sigma)$. If $\sigma = 0$, then $k - \mathcal{S}_p(\sigma) \equiv k - \mathcal{ST}$ (see [10]).

Definition 2 ([11]). *Let $\Psi \in \mathcal{A}$ and $0 < \alpha < \infty$, then*

$$\Psi \in \mathcal{CP}(\alpha) \iff \Re\left(1 + \frac{z\Psi''}{\Psi'}\right) + \alpha \geq \left|\frac{z\Psi''}{\Psi'} + 1 - \alpha\right|.$$

Definition 3 ([12]). *When $\lambda > 0$, the classes $\mathcal{S}_\lambda^*$ and $\mathcal{C}_\lambda$ are defined in the following manner*

$$\mathcal{S}_\lambda^* = \left\{\Psi \in \mathcal{A} : \left|\frac{z\Psi'}{\Psi} - 1\right| < \lambda\right\},$$

and

$$\mathcal{C}_\lambda = \left\{\Psi \in \mathcal{A} : \left|\frac{z\Psi''}{\Psi'}\right| < \lambda\right\}.$$

Definition 4 ([13]). *A function $\Psi \in \mathcal{A}$ is in $k - \mathcal{ST}[X, Y] \iff$*

$$\Re\left(\frac{(Y-1)\frac{z\Psi'}{\Psi} - (X-1)}{(Y+1)\frac{z\Psi'}{\Psi} - (X+1)}\right) > k\left|\frac{(Y-1)\frac{z\Psi'}{\Psi} - (X-1)}{(Y+1)\frac{z\Psi'}{\Psi} - (X+1)} - 1\right|,$$

further, $\Psi \in \mathcal{A}$ is in $k - \mathcal{UCV}[X, Y] \iff$

$$\Re\left(\frac{(Y-1)\frac{(z\Psi')'}{\Psi'} - (X-1)}{(Y+1)\frac{(z\Psi')'}{\Psi'} - (X+1)}\right) > k\left|\frac{(Y-1)\frac{(z\Psi')'}{\Psi'} - (X-1)}{(Y+1)\frac{(z\Psi')'}{\Psi'} - (X+1)} - 1\right|,$$

where $k \geq 0$, $-1 \leq Y < X \leq 1$.

The subclasses $k - \mathcal{ST}[X, Y]$ and $k - \mathcal{UCV}[X, Y]$ extend various other subclasses of univalent functions by appropriately modifying the parameters. This leads us to observe that

i. $k - \mathcal{ST}[1 - 2\sigma, -1] \equiv k - \mathcal{S}_p(\sigma)$ and $k - \mathcal{UCV}[1 - 2\sigma, -1] \equiv k - \mathcal{UCV}(\sigma)$, $(0 \leq \sigma < 1)$ (see [6]).
ii. $k - \mathcal{ST}[1, -1] \equiv k - \mathcal{ST}$ and $k - \mathcal{UCV}[1, -1] \equiv k - \mathcal{UCV}$. (see [10,14]).
iii. $0 - \mathcal{ST}[1 - 2\sigma, -1] \equiv \mathcal{S}^*(\sigma)$ and $0 - \mathcal{UCV}[1 - 2\sigma, -1] \equiv \mathcal{C}(\sigma)$. (see [15]).
iv. $0 - \mathcal{ST}[X, Y] \equiv \mathcal{S}^*[X, Y]$ and $0 - \mathcal{UCV}[X, Y] \equiv \mathcal{C}[X, Y]$. (see [16]).

Definition 5 ([17]). *A function $\Psi \in \mathcal{S}$ is considered spirallike if there exists an angle ϑ such that $|\vartheta| < \frac{\pi}{2}$ and the following condition holds for all $z \in \mathbb{D}$*

$$\Re\left(e^{-i\vartheta}\frac{z\Psi'}{\Psi}\right) > 0.$$

Definition 6 ([18,19]). *For* $|\vartheta| < \frac{\pi}{2}$, $0 \leq \nu < 1$ *and* $0 \leq \varrho < 1$, *the classes* $\mathcal{S}(\vartheta, \nu, \varrho)$ *and* $\mathcal{K}(\vartheta, \nu, \varrho)$ *are defined accordingly:*

$$\mathcal{S}(\vartheta, \nu, \varrho) := \left\{ \Psi \in \mathcal{S} : \Re\left(e^{i\vartheta} \frac{z\Psi'}{(1-\varrho)\Psi + \varrho z\Psi'} \right) > \nu \cos\vartheta, \ \forall z \in \mathbb{D} \right\}.$$

$$\mathcal{K}(\vartheta, \nu, \varrho) := \left\{ h \in \mathcal{S} : \Re\left(e^{i\vartheta} \frac{z\Psi'' + \Psi'}{\Psi' + \varrho z\Psi''} \right) > \nu \cos\vartheta, \ \forall z \in \mathbb{D} \right\}.$$

If $\varrho = 0$, then (see [20])

$$\mathcal{S}(\vartheta, \nu, 0) \equiv \mathcal{SP}(\vartheta, \nu), \quad \text{and} \quad \mathcal{K}(\vartheta, \nu, 0) \equiv \mathcal{KP}(\vartheta, \nu).$$

Definition 7 ([21]). *For* $0 \leq \delta < 1$ *and* $\sigma \leq 1$, $\mathcal{UCSP}(\sigma, \delta)$ *and* $\mathcal{SP}_p(\sigma, \delta)$ *are defined as follows:*

$$\mathcal{UCSP}(\sigma, \delta) = \left\{ \Psi \in \mathcal{A} : \Re\left\{ e^{-i\sigma} \left(1 + \frac{z\Psi''}{\Psi'} \right) \right\} \geq \left| \frac{z\Psi''}{\Psi'} \right| + \delta \right\}$$

and

$$\mathcal{SP}_p(\sigma, \delta) = \left\{ \Psi \in \mathcal{A} : \Re\left\{ e^{-i\sigma} \frac{z\Psi'}{\Psi} \right\} \geq \left| \frac{z\Psi'}{\Psi} - 1 \right| + \delta \right\}.$$

If $\delta = 0$ *then* $\mathcal{UCSP}(\sigma, 0) \equiv \mathcal{UCSP}(\sigma)$ *and* $\mathcal{SP}_p(\sigma, 0) \equiv \mathcal{SP}_p(\sigma)$ *(see [22]).*

Definition 8 ([23,24]). *Let* $\Psi \in \mathcal{A}$, *if* $0 \leq \lambda < 1$ *and* $0 \leq \beta < 1$ *then* $\forall z \in \mathbb{D}$

- $\Psi \in \mathcal{S}(\lambda, \beta) \iff \Re\left(\frac{z\Psi' + \lambda z^2\Psi''}{\Psi} \right) > \beta,$
- $\Psi \in \mathcal{C}(\lambda, \beta) \iff \Re\left(\frac{z[z\Psi' + \lambda z^2\Psi'']'}{z\Psi'} \right) > \beta.$

Next, we are considering a subclass of $\mathcal{A}$, denoted by $\mathcal{M}^{\eta}_{\zeta,\chi}(\omega)$ which was introduced and studied in [25].

Definition 9 ([25]). *Let* $\Psi \in \mathcal{A}$, *with* $\eta \in \mathbb{C} \setminus \{0\}$ *and* $\omega < 1$, *if* $0 \leq \zeta < 1$ *and* $0 \leq \chi < 1$ *then*

$$\Psi \in \mathcal{M}^{\eta}_{\zeta,\chi}(\omega) \iff \left| \frac{\Psi' + \chi z\Psi'' + \zeta z^2\Psi''' - 1}{2\eta(1-\omega) + \Psi' + \chi z\Psi'' + \zeta z^2\Psi''' - 1} \right| < 1, \ \forall z \in \mathbb{D}. \tag{2}$$

The coefficient bound for the function $\Psi \in \mathcal{M}^{\eta}_{\zeta,\chi}(\omega)$ is:

$$|\alpha_j| \leq \frac{2|\eta|(1-\omega)}{j + j(2\zeta - \chi) + j^2(\chi - 3\zeta) + j^3\zeta}, \ j = 2, 3, \cdots. \tag{3}$$

Since,

$$\begin{aligned} j + j(2\zeta - \chi) + j^2(\chi - 3\zeta) + j^3\zeta &= j(1-\chi) + j\zeta(2 + j^2) + j^2(\chi - 3\zeta) \\ &\geq j^2(\chi - 3\zeta), \end{aligned}$$

then the inequality (3) can be written as (see [25]):

$$|\alpha_j| \leq \frac{2|\eta|(1-\omega)}{j^2(\chi - 3\zeta)}. \tag{4}$$

Similarly, if

$$\begin{aligned} j + j(2\zeta - \chi) + j^2(\chi - 3\zeta) + j^3\zeta &= j(1-\chi) + j\zeta(2+j^2) + j^2\chi - 3j^2\zeta \\ &= j(1-\chi) + j\zeta(2+j^2) + j^2\chi + 3j\zeta - 3j(j+1)\zeta \\ &\geq -j(j+1)3\zeta, \end{aligned}$$

then

$$|\alpha_j| \leq \frac{2|\eta|(1-\omega)}{j(j+1)3\zeta}. \tag{5}$$

If $\Psi, \psi \in \mathcal{A}$, then the Hadamard product [5] of Ψ and ψ be defined as

$$(\Psi * \psi)(z) = z + \sum_{j=2}^{\infty} \alpha_j \beta_j z^j,$$

where $\Psi(z) = z + \sum_{j=2}^{\infty} \alpha_j z^j$ and $\psi(z) = z + \sum_{j=2}^{\infty} \beta_j z^j$.

Now, let us define the convolution operator $\mathcal{P}_s^d(\Psi, z)$ using Hadamard product as

$$\mathcal{P}_s^d(\Psi, z) = \mathcal{P}_s^d(z) * \Psi(z) = z + \sum_{j=2}^{\infty} \binom{j+d-2}{d-1} s^{j-1}(1-s)^d \alpha_j z^j, \; z \in \mathbb{D}. \tag{6}$$

Recall the notion of the Pochhammer symbol, which is defined as

$$(a)_j := a(a+1)_{j-1}, \quad (a)_0 = 1, \quad j \in \mathbb{N}.$$

Using the Pochhammer symbol, the operator $\mathcal{P}_s^d(\Psi, z)$ in (6) can be rewritten as

$$\mathcal{P}_s^d(\Psi, z) = z + \sum_{j=2}^{\infty} X_j z^j, \tag{7}$$

where

$$X_j = \frac{(d)_{j-1}}{(1)_{j-1}} s^{j-1}(1-s)^d \alpha_j \tag{8}$$

The study of special functions like hypergeometric functions (refer to [25–29]), confluent hypergeometric functions (refer to [30,31]), Wright's function (refer to [32]), and generalized Bessel functions (refer to [33,34]), has garnered significant interest in Geometric Function Theory. Porwal ([3]) introduced the Poisson distribution series and demonstrated its intriguing applications within univalent functions. Following this, researchers expanded on this work by introducing series based on other distributions, such as the hypergeometric distribution (refer to [35]), confluent hypergeometric distribution (refer to [36]), Binomial distribution (refer to [37]), generalized distribution (refer to [4]) and Pascal distribution (refer to [1,2,19]). These studies established necessary and sufficient conditions for specific univalent function classes.

Motivated by previous research on the connections between distinct subclasses of analytic univalent functions and hypergeometric, Poisson, and binomial distribution series, we employ the convolution operator $\mathcal{P}_s^d(\Psi, z)$ and the integral operator $\mathcal{T}_s^d(\Psi, z)$ to ascertain specific inclusion relations between $\mathcal{M}_{\zeta,\chi}^{\eta}(\omega)$ and with different subclasses of analytic univalent functions. This is how the remainder of the article is organized: We present some preliminary findings related to the several subclasses mentioned above in Section 2. The inclusion properties of the linear operators $\mathcal{P}_s^d(\Psi, z)$ and $\mathcal{T}_s^d(\Psi, z)$ associated with the

Pascal distribution series function from the class $\mathcal{M}_{\zeta,\chi}^{\eta}(\omega)$ are discussed in Section 3. Our primary focus is on finding sufficient conditions for the operators $\mathcal{P}_s^d(\Psi, z)$ and $\mathcal{T}_s^d(\Psi, z)$. All of the findings are presented with proof in Section 3. As corollaries, several particular cases of the conclusions derived in Section 3 are described in Section 4. Section 5 contains the work's conclusion and potential next directions.

2. Preliminary Results

Lemma 1 ([18]). *Let $\Psi \in \mathcal{A}$ has the form specified in (1), a sufficient condition for Ψ to be in $\mathcal{S}(\vartheta, \nu, \varrho)$ is stated as follows*

$$\sum_{j=2}^{\infty} \left(((1-\varrho)\sec\vartheta + \varrho(1-\nu))j + (1-\varrho)(1-\nu-\sec\vartheta)\right)|\alpha_j| \leq 1-\nu, \tag{9}$$

where $|\vartheta| < \frac{\pi}{2}$, $0 \leq \nu < 1$ and $0 \leq \varrho < 1$.

Lemma 2 ([19]). *Let $\Psi \in \mathcal{A}$ has the form specified in (1), a sufficient condition for Ψ to be in $\mathcal{K}(\vartheta, \nu, \varrho)$ is stated as follows*

$$\sum_{j=2}^{\infty} j\left(((1-\varrho)\sec\vartheta + \varrho(1-\nu))j + (1-\varrho)(1-\nu-\sec\vartheta)\right)|\alpha_j| \leq 1-\nu, \tag{10}$$

where $|\vartheta| < \frac{\pi}{2}$, $0 \leq \nu < 1$ and $0 \leq \varrho < 1$.

Lemma 3 ([21]). *Assume $\Psi \in \mathcal{A}$ has the form specified in (1), then Ψ will be in $\mathcal{SP}_p(\sigma, \delta)$ if the following inequality holds*

$$\sum_{j=2}^{\infty} (2j - \cos\sigma - \delta)|\alpha_j| \leq \cos\sigma - \delta, \tag{11}$$

where $\sigma \leq 1$, $0 \leq \delta < 1$.

Lemma 4 ([21]). *Assume $\Psi \in \mathcal{A}$ has the form specified in (1), then Ψ will be in $\mathcal{UCSP}(\sigma, \delta)$ if the following inequality holds*

$$\sum_{j=2}^{\infty} j(2j - \cos\sigma - \delta)|\alpha_j| \leq \cos\sigma - \delta,\ z \in \mathbb{D} \tag{12}$$

where $\sigma \leq 1$, $0 \leq \delta < 1$.

Lemma 5 ([13]). *Let $\Psi \in \mathcal{A}$ has the form specified in (1), a sufficient condition for Ψ to be in $k - \mathcal{ST}[X, Y]$ is stated as follows*

$$\sum_{j=2}^{\infty} \left[2(k+1)(j-1) + |j(Y+1) - (X+1)|\right]|\alpha_j| \leq |Y - X|,$$

where $-1 \leq Y < X \leq 1$ and $k \geq 0$.

Lemma 6 ([13]). *Let $\Psi \in \mathcal{A}$ has the form specified in (1), a sufficient condition for Ψ to be in $k - \mathcal{UCV}[X, Y]$ is stated as follows*

$$\sum_{j=2}^{\infty} j\left[2(k+1)(j-1) + |j(Y+1) - (X+1)|\right]|\alpha_j| \leq |Y - X|,$$

where $-1 \leq Y < X \leq 1$ *and* $k \geq 0$.

Lemma 7 ([11]). *Assume* $\Psi \in \mathcal{A}$ *has the form specified in* (1)*, then* Ψ *will be in* $\mathcal{CP}(\alpha)$ *if the following inequality holds*

$$\sum_{j=2}^{\infty} \big(j + 2(\alpha - 1)\big) j|\alpha_j| \leq 2\alpha - 1, \tag{13}$$

where $0 < \alpha < \infty$.

Lemma 8 ([12]). *Assume* $\Psi \in \mathcal{A}$ *has the form specified in* (1)*, then* Ψ *will be in* $\mathcal{S}^*_\lambda$ *if the following inequality holds for* $\lambda > 0$

$$\sum_{j=2}^{\infty} (\lambda + j - 1)|\alpha_j| \leq \lambda. \tag{14}$$

Lemma 9 ([12]). *Assume* $\Psi \in \mathcal{A}$ *has the form specified in* (1)*, then* Ψ *will be in* $\mathcal{C}_\lambda$ *if the following inequality holds for* $\lambda > 0$

$$\sum_{j=2}^{\infty} j(\lambda + j - 1)|\alpha_j| \leq \lambda. \tag{15}$$

Lemma 10 ([23]). *Let* $\Psi \in \mathcal{A}$ *has the form specified in* (1)*, a sufficient condition for* Ψ *to be in* $\mathcal{S}(\lambda, \beta)$ *is stated as follows*

$$\sum_{j=2}^{\infty} \big(j + \lambda j(j-1) - \beta\big)|\alpha_j| \leq 1 - \beta, \tag{16}$$

where $0 \leq \lambda < 1$, $0 \leq \beta < 1$.

Lemma 11 ([24]). *Let* $\Psi \in \mathcal{A}$ *has the form specified in* (1)*, a sufficient condition for* Ψ *to be in* $\mathcal{C}(\lambda, \beta)$ *is stated as follows*

$$\sum_{j=2}^{\infty} j\big(j + \lambda j(j-1) - \beta\big)|\alpha_j| \leq 1 - \beta, \tag{17}$$

where $0 \leq \lambda < 1$, $0 \leq \beta < 1$.

3. Main Results

This section presents the main findings of our study. We derived conditions for the linear operators such that they would be in different subclasses of univalent functions. The results are analyzed in terms of accuracy which provides valuable insights into their theoretical and practical implications.

3.1. Results on Convolution Operator

In this subsection, we derived inclusion results for the linear operator $\mathcal{P}^d_s(\Psi, z)$.

Theorem 1. *Let* $\Psi \in \mathcal{A}$ *has the form specified in* (1)*. Suppose* Ψ *also in* $\mathcal{M}^{\eta}_{\zeta,\chi}(\omega)$ *and satisfies the following condition*

$$\begin{aligned}
&\frac{[(1-\varrho)\sec\vartheta+\varrho(1-\nu)](1-s)}{(\chi-3\zeta)s(d-1)}+\frac{(1-\varrho)(1-\nu-\sec\vartheta)(1-s)^2}{3\zeta s^2(d-2)_2}\\
&\quad-\frac{[(1-\varrho)\sec\vartheta+\varrho(1-\nu)](1+s(d-1))}{(\chi-3\zeta)s(d-1)}(1-s)^d\\
&\quad-\frac{(1-\varrho)(1-\nu-\sec\vartheta)\left(2+2s(d-2)+s^2(d-2)_2\right)}{6\zeta s^2(d-2)_2}(1-s)^d\\
&\leq\frac{(1-\nu)}{2|\eta|(1-\omega)},
\end{aligned}$$

then for $d>0$ *and* $0\leq s\leq 1$, *it follows that* $\mathcal{P}_s^d(\Psi,z)\in\mathcal{S}(\vartheta,\nu,\varrho)$.

Proof. Applying Equation (8) to the sufficient condition for the class $\mathcal{S}(\vartheta,\nu,\varrho)$ as presented in Lemma 1, which gives

$$\begin{aligned}
&\sum_{j=2}^{\infty}[(1-\varrho)\sec\vartheta+\varrho(1-\nu)]j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\\
&\quad+\sum_{j=2}^{\infty}(1-\varrho)(1-\nu-\sec\vartheta)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\leq 1-\nu.
\end{aligned}\tag{18}$$

Next, we substitute (4) into the first sum and (5) into the second sum of (18)

$$\begin{aligned}
&\sum_{j=2}^{\infty}[(1-\varrho)\sec\vartheta+\varrho(1-\nu)]j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\\
&\quad+\sum_{j=2}^{\infty}(1-\varrho)(1-\nu-\sec\vartheta)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\\
&\leq\sum_{j=2}^{\infty}[(1-\varrho)\sec\vartheta+\varrho(1-\nu)]j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j^2(\chi-3\zeta)}\\
&\quad+\sum_{j=2}^{\infty}(1-\varrho)(1-\nu-\sec\vartheta)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j(j+1)3\zeta}\\
&=2|\eta|(1-\omega)(1-s)^d\left(\frac{[(1-\varrho)\sec\vartheta+\varrho(1-\nu)]}{(\chi-3\zeta)s(d-1)}\sum_{j=2}^{\infty}\frac{(d-1)_j}{(1)_j}s^{(j)}\right.\\
&\quad\left.+\frac{(1-\varrho)(1-\nu-\sec\vartheta)}{3\zeta s^2(d-2)_2}\sum_{j=2}^{\infty}\frac{(d-2)_{j+1}}{(1)_{j+1}}s^{(j+1)}\right)\\
&=2|\eta|(1-\omega)(1-s)^d\left(\frac{[(1-\varrho)\sec\vartheta+\varrho(1-\nu)]}{(\chi-3\zeta)s(d-1)}\left[(1-s)^{-(d-1)}-(1+s(d-1))\right]\right.\\
&\quad\left.+\frac{(1-\varrho)(1-\nu-\sec\vartheta)}{3\zeta s^2(d-2)_2}\left[(1-s)^{-(d-2)}-\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)\right]\right).
\end{aligned}\tag{19}$$

Using the expression (19) and substituting it back into (18), we obtain

$$\begin{aligned}
&\frac{[(1-\varrho)\sec\vartheta+\varrho(1-\nu)](1-s)}{(\chi-3\zeta)s(d-1)}+\frac{(1-\varrho)(1-\nu-\sec\vartheta)(1-s)^2}{3\zeta s^2(d-2)_2}\\
&\quad-\frac{[(1-\varrho)\sec\vartheta+\varrho(1-\nu)](1+s(d-1))}{(\chi-3\zeta)s(d-1)}(1-s)^d\\
&\quad-\frac{(1-\varrho)(1-\nu-\sec\vartheta)}{3\zeta s^2(d-2)_2}\left(\frac{2+2s(d-2)+s^2(d-2)_2}{2}\right)(1-s)^d\\
&\leq\frac{(1-\nu)}{2|\eta|(1-\omega)}.
\end{aligned}\tag{20}$$

Using the hypothesis of the theorem along with Equation (20) completes the proof. □

Theorem 2. *Let $\Psi\in\mathcal{A}$ has the form specified in (1). Suppose Ψ also in $\mathcal{M}^{\eta}_{\zeta,\chi}(\omega)$ and satisfies the following condition*

$$\begin{aligned}
&[(1-\varrho)\sec\vartheta+\varrho(1-\nu)]s(d-1)+(1-\varrho)(1-\nu-\sec\vartheta)(1-s)\\
&\quad-\Big([(1-\varrho)\sec\vartheta+\varrho(1-\nu)]s(d-1)+(1-\varrho)(1-\nu-\sec\vartheta)(1+s(d-1))\Big)(1-s)^d\\
&\leq(1-\nu)\frac{s(d-1)(\chi-3\zeta)}{2|\eta|(1-\omega)},
\end{aligned}$$

then for $d>0$ and $0\leq s\leq 1$, it follows that $\mathcal{P}^d_s(\Psi,z)\in\mathcal{K}(\vartheta,\nu,\varrho)$.

Proof. Applying equation (8) to the sufficient condition for the class $\mathcal{K}(\vartheta,\nu,\varrho)$ in Lemma 2, which gives

$$\sum_{j=2}^{\infty}j\big[((1-\varrho)\sec\vartheta+\varrho(1-\nu))j+(1-\varrho)(1-\nu-\sec\vartheta)\big]\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\leq 1-\nu.\tag{21}$$

Using (4) on the left side of (21), we obtain

$$\begin{aligned}
&\sum_{j=2}^{\infty}j\big[((1-\varrho)\sec\vartheta+\varrho(1-\nu))j+(1-\varrho)(1-\nu-\sec\vartheta)\big]\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\\
&\leq\sum_{j=2}^{\infty}j\big[((1-\varrho)\sec\vartheta+\varrho(1-\nu))j\\
&\quad+(1-\varrho)(1-\nu-\sec\vartheta)\big]\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j^2(\chi-3\zeta)}\\
&=\frac{2|\eta|(1-\omega)}{(\chi-3\zeta)}(1-s)^d\sum_{j=2}^{\infty}[(1-\varrho)\sec\vartheta+\varrho(1-\nu)]\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}\\
&\quad+\frac{2|\eta|(1-\omega)}{(\chi-3\zeta)}(1-s)^d\sum_{j=2}^{\infty}(1-\varrho)(1-\nu-\sec\vartheta)\frac{(d)_{j-1}}{(1)_j}s^{(j-1)}\\
&=\frac{2|\eta|(1-\omega)}{(\chi-3\zeta)}(1-s)^d\Bigg([(1-\varrho)\sec\vartheta+\varrho(1-\nu)]\Big((1-s)^{-d}-1\Big)\\
&\quad+\frac{(1-\varrho)(1-\nu-\sec\vartheta)}{s(d-1)}\Big[(1-s)^{-(d-1)}-(1+s(d-1))\Big]\Bigg).
\end{aligned}\tag{22}$$

Using the expression (22) and substituting it back into (21), we obtain

$$\begin{aligned}(1-\varrho)\sec\vartheta+\varrho(1-\nu)-\frac{(1-\varrho)(1-\nu-\sec\vartheta)(1+s(d-1))}{s(d-1)}(1-s)^d\\ -((1-\varrho)\sec\vartheta+\varrho(1-\nu))(1-s)^d+\frac{(1-\varrho)(1-\nu-\sec\vartheta)(1-s)}{s(d-1)}\\ \leq\frac{(1-\nu)(\chi-3\zeta)}{2|\eta|(1-\omega)}.\end{aligned}\tag{23}$$

Using the hypothesis of the theorem along with Equation (23) completes the proof. □

Theorem 3. *Let $\Psi\in\mathcal{A}$ has the form specified in (1). Suppose Ψ also in $\mathcal{M}^{\eta}_{\zeta,\chi}(\omega)$ and satisfies the following condition*

$$\begin{aligned}\frac{2(1-s)}{(\chi-3\zeta)}-\frac{(\cos\sigma+\delta)(1-s)^2}{3\zeta s(d-2)}-\frac{2(1+s(d-1))(1-s)^d}{(\chi-3\zeta)}\\ +\frac{(\cos\sigma+\delta)}{3\zeta s(d-2)}\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)(1-s)^d\\ \leq\frac{s(d-1)(\cos\sigma-\delta)}{2|\eta|(1-\omega)},\end{aligned}$$

then for $d>0$ and $0\leq s\leq 1$, the operator $\mathcal{P}^d_s(\Psi,z)\in\mathcal{SP}_p(\sigma,\delta)$.

Proof. Applying Equation (8) in Lemma 3, which gives

$$\sum_{j=2}^{\infty}(2j-\cos\sigma-\delta)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\leq\cos\sigma-\delta.\tag{24}$$

Next, we substitute (4) and (5) into the left side of (24)

$$\begin{aligned}&\sum_{j=2}^{\infty}2j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|-\sum_{j=2}^{\infty}(\cos\sigma+\delta)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\\ &\leq\sum_{j=2}^{\infty}2j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j^2(\chi-3\zeta)}\\ &\quad-\sum_{j=2}^{\infty}(\cos\sigma+\delta)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j(j+1)3\zeta}\\ &=2|\eta|(1-\omega)(1-s)^d\left(\frac{2}{(\chi-3\zeta)s(d-1)}\sum_{j=2}^{\infty}\frac{(d-1)_j}{(1)_j}s^{(j)}\right.\\ &\quad\left.-\frac{(\cos\sigma+\delta)}{3\zeta s^2(d-2)_2}\sum_{j=2}^{\infty}\frac{(d-2)_{j+1}}{(1)_{j+1}}s^{(j+1)}\right)\\ &=2|\eta|(1-\omega)(1-s)^d\left(\frac{2}{(\chi-3\zeta)s(d-1)}\Big((1-s)^{-(d-1)}-(1+s(d-1))\Big)\right.\\ &\quad\left.-\frac{(\cos\sigma+\delta)}{3\zeta s^2(d-2)_2}\left[(1-s)^{-(d-2)}-\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)\right]\right).\end{aligned}\tag{25}$$

Using the expression (25) and substituting it back into (24), we obtain

$$\begin{aligned}&\frac{(\cos\sigma+\delta)}{3\zeta s^2(d-2)_2}\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)(1-s)^d\\&\quad-\frac{2(1+s(d-1))(1-s)^d}{(\chi-3\zeta)s(d-1)}+\frac{2(1-s)}{(\chi-3\zeta)s(d-1)}-\frac{(\cos\sigma+\delta)(1-s)^2}{3\zeta s^2(d-2)_2}\\&\leq\frac{(\cos\sigma-\delta)}{2|\eta|(1-\omega)}.\end{aligned}\tag{26}$$

Using the hypothesis of the theorem along with Equation (26) completes the proof. □

Theorem 4. *Let $\Psi\in\mathcal{A}$ has the form specified in (1). Suppose Ψ also in $\mathcal{M}^{\eta}_{\zeta,\chi}(\omega)$ and satisfies the following condition*

$$\begin{aligned}&2s(d-1)-(\cos\sigma+\delta)(1-s)\\&\quad-[2s(d-1)-(\cos\sigma+\delta)(1+s(d-1))](1-s)^d\\&\leq(\cos\sigma-\delta)\frac{s(d-1)(\upsilon-3\wp)}{2|\tau|(1-\gamma)},\end{aligned}$$

then for $d>0$ and $0\leq s\leq 1$, the operator $\mathcal{P}^d_s(\Psi,z)\in\mathcal{UCSP}(\sigma,\delta)$.

Proof. Applying Equation (8) in Lemma 4, which gives

$$\sum_{j=2}^{\infty}j(2j-\cos\sigma-\delta)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\leq\cos\sigma-\delta.\tag{27}$$

Using (4) on the left side of (27), we obtain

$$\begin{aligned}&\sum_{j=2}^{\infty}j(2j-\cos\sigma-\delta)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\\&\leq\sum_{j=2}^{\infty}j(2j-\cos\sigma-\delta)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j^2(\chi-3\zeta)}\\&=\frac{2|\eta|(1-\omega)}{(\chi-3\zeta)}(1-s)^d\sum_{j=2}^{\infty}(2j-\cos\sigma-\delta)\frac{(d)_{j-1}}{(1)_j}s^{(j-1)}\\&=\frac{2|\eta|(1-\omega)}{(\chi-3\zeta)}(1-s)^d\left(2\sum_{j=1}^{\infty}\frac{(d)_j}{(1)_j}s^{(j)}-\frac{(\cos\sigma+\delta)}{s(d-1)}\sum_{j=2}^{\infty}\frac{(d-1)_j}{(1)_j}d^{(j)}\right)\\&=\frac{2|\eta|(1-\omega)}{(\chi-3\zeta)}(1-s)^d\Big(2((1-s)^{-d}-1)\\&\quad-\frac{(\cos\sigma+\delta)}{s(d-1)}[(1-s)^{-(d-1)}-(1+s(d-1))]\Big).\end{aligned}\tag{28}$$

Using the expression (28) and substituting it back into (27), we obtain

$$2(1-(1-s)^d)-\frac{(\cos\sigma+\delta)}{s(d-1)}\left[(1-s)-(1+s(d-1))(1-s)^d\right]\leq\frac{(\chi-3\zeta)(\cos\sigma-\delta)}{2|\eta|(1-\omega)}.\tag{29}$$

Using the hypothesis of the theorem along with Equation (29) completes the proof. □

Theorem 5. *Let $\Psi\in\mathcal{A}$ has the form specified in (1). Suppose Ψ also in $\mathcal{M}^{\eta}_{\zeta,\chi}(\omega)$ and satisfies the following condition*

$$\begin{aligned}
&\frac{[2(k+1)+|Y+1|]}{(\chi-3\zeta)s(d-1)}-\frac{[2(k+1)-|X+1|]}{3\zeta s^2(d-2)_2}(1-s)\\
&\quad-\frac{[2(k+1)+|Y+1|]}{(\chi-3\zeta)s(d-1)}(1+s(d-1))(1-s)^{d-1}\\
&\quad+\frac{[2(k+1)-|X+1|]}{3\zeta s^2(d-2)_2}\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)(1-s)^{d-1}\\
&\leq\frac{|Y-X|}{2|\eta|(1-\omega)(1-s)},
\end{aligned}$$

then for $d>0$ *and* $0\leq s\leq 1$, *it follows that* $\mathcal{P}_s^d(\Psi,z)\in k-\mathcal{ST}[X,Y]$.

Proof. Applying Equation (8) to the sufficient condition for the class $k-\mathcal{ST}[X,Y]$ in Lemma 5, which gives

$$\sum_{j=2}^{\infty}\left[2(k+1)(j-1)+|j(Y+1)-(X+1)|\right]|A_j|\leq|Y-X|. \tag{30}$$

Using (8) in the left side of (30), we obtain

$$\begin{aligned}
&\sum_{j=2}^{\infty}\left[2(k+1)(j-1)+|j(Y+1)-(X+1)|\right]\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\\
&\leq\sum_{j=2}^{\infty}\left[2(k+1)+|Y+1|\right]j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\\
&\quad-\sum_{j=2}^{\infty}\left[2(k+1)-|X+1|\right]\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|.
\end{aligned}\tag{31}$$

Next, we substitute (4) into the first sum and (5) into the second sum of (31)

$$\begin{aligned}
&\sum_{j=2}^{\infty}\left[2(k+1)+|Y+1|\right]j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\\
&\quad-\sum_{j=2}^{\infty}\left[2(k+1)-|X+1|\right]\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\\
&\leq\sum_{j=2}^{\infty}\left[2(k+1)+|Y+1|\right]j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j^2(\chi-3\zeta)}\\
&\quad-\sum_{j=2}^{\infty}\left[2(k+1)-|X+1|\right]\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j(j+1)3\zeta}\\
&=2|\eta|(1-\omega)\left(\frac{[2(k+1)+|Y+1|]}{(\chi-3\zeta)s(d-1)}\sum_{j=2}^{\infty}\frac{(d-1)_j}{(1)_j}s^{(j)}\right.\\
&\quad\left.-\frac{[2(k+1)-|X+1|]}{3\zeta s^2(d-2)_2}\sum_{j=2}^{\infty}\frac{(d-2)_{j+1}}{(1)_{j+1}}s^{(j+1)}\right)(1-s)^d\\
&=2|\eta|(1-\omega)\left(\frac{[2(k+1)+|Y+1|]}{(\chi-3\zeta)s(d-1)}\Big((1-s)^{-(d-1)}-(1+s(d-1))\Big)\right.\\
&\quad\left.-\frac{[2(k+1)-|X+1|]}{3\zeta s^2(d-2)_2}\left[(1-s)^{-(d-2)}-\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)\right]\right)(1-s)^d.
\end{aligned}\tag{32}$$

Using the expression (32) and substituting it back into (30), we obtain

$$\begin{aligned}&\frac{[2(k+1)-|X+1|]}{3\zeta s^2(d-2)_2}\left[\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)(1-s)^d-(1-s)^2\right]\\&\quad+\frac{[2(k+1)+|Y+1|]}{(\chi-3\zeta)s(d-1)}\left[(1-s)-(1+s(d-1))(1-s)^d\right]\\&\leq\frac{|Y-X|}{2|\eta|(1-\omega)}.\end{aligned}\tag{33}$$

We attain the desired outcome by employing Equation (33) together with the theorem's hypothesis. □

Theorem 6. *Let $\Psi\in\mathcal{A}$ has the form specified in (1). Suppose Ψ also in $\mathcal{M}^{\eta}_{\zeta,\chi}(\omega)$ and satisfies the following condition*

$$\begin{aligned}&(2(k+1)+|Y+1|)-\frac{(2(k+1)-|X+1|)}{s(d-1)}(1-s)-(2(k+1)+|Y+1|)(1-s)^d\\&\quad+\frac{(2(k+1)-|X+1|)}{s(d-1)}(1+s(d-1))(1-s)^d\\&\leq\frac{|Y-X|(\chi-3\zeta)}{2|\eta|(1-\omega)},\end{aligned}$$

then for $d>0$ and $0\leq s\leq 1$, it follows that $\mathcal{P}^d_s(\Psi,z)\in k-\mathcal{UCV}[X,Y]$.

Proof. Applying Equation (8) to the sufficient condition for the class $k-\mathcal{UCV}[X,Y]$ in Lemma 6, which gives

$$\sum_{j=2}^{\infty}j\left[2(k+1)(j-1)+|j(Y+1)-(X+1)|\right]\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\leq|Y-X|.\tag{34}$$

Using (4) on the left side of (34), we obtain

$$\begin{aligned}&\sum_{j=2}^{\infty}j\left[2(k+1)(j-1)+|j(Y+1)-(X+1)|\right]\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\\&\leq\sum_{j=2}^{\infty}\left[j^2(2(k+1)+|Y+1|)-j(2(k+1)-|X+1|)\right]\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j^2(\chi-3\zeta)}\\&=\frac{2|\eta|(1-\omega)}{(\chi-3\zeta)}(2(k+1)+|Y+1|)\sum_{j=2}^{\infty}\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\\&\quad-\frac{2|\eta|(1-\omega)}{(\chi-3\zeta)}(2(k+1)-|X+1|)\sum_{j=2}^{\infty}\frac{(d)_{j-1}}{(1)_j}s^{(j-1)}(1-s)^d\\&=\frac{2|\eta|(1-\omega)}{(\chi-3\zeta)}\Bigg((2(k+1)+|Y+1|)(1-(1-s)^d)\\&\quad-\frac{(2(k+1)-|X+1|)}{s(d-1)}[(1-s)-(1+s(d-1))(1-s)^d]\Bigg).\end{aligned}\tag{35}$$

Using the expression (35) and substituting it back into (34), we obtain

$$\begin{aligned}&(2(k+1)+|Y+1|)(1-(1-s)^d)\\&\quad-\frac{(2(k+1)-|X+1|)}{s(d-1)}[(1-s)-(1+s(d-1))(1-s)^d]\\&\leq\frac{|Y-X|(\chi-3\zeta)}{2|\eta|(1-\omega)}.\end{aligned}\tag{36}$$

We attain the desired outcome by employing Equation (36) together with the theorem's hypothesis. □

Theorem 7. *Let $\Psi \in \mathcal{A}$ has the form specified in (1) and assume it also belongs to the class $\mathcal{M}^{\eta}_{\zeta,\chi}(\omega)$. Suppose the following inequality holds*

$$\begin{aligned}&s(d-1)+2(\alpha-1)(1-s)\\&\quad-[s(d-1)+2(\alpha-1)(1+s(d-1))](1-s)^d\\&\leq\frac{s(d-1)(2\alpha-1)(\chi-3\zeta)}{2|\eta|(1-\omega)},\end{aligned}$$

then, for $d>0$ and $0\leq s\leq 1$ it follows that $\mathcal{P}^d_s(\Psi,z)\in\mathcal{CP}(\alpha)$.

Proof. Applying Equation (8) to the sufficient condition for the class $\mathcal{CP}(\alpha)$ in Lemma 7, which gives

$$\sum_{j=2}^{\infty}(j+2(\alpha-1))j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\leq 2\alpha-1.\tag{37}$$

Using (4) on the left side of (37), we obtain

$$\begin{aligned}&\sum_{j=2}^{\infty}(j+2(\alpha-1))j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\\&\leq\sum_{j=2}^{\infty}(j+2(\alpha-1))j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j^2(\chi-3\zeta)}\\&=\frac{2|\eta|(1-\omega)}{(\chi-3\zeta)}(1-s)^d\left(\sum_{j=2}^{\infty}\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}+2(\alpha-1)\sum_{j=2}^{\infty}\frac{(d)_{j-1}}{(1)_j}s^{(j-1)}\right)\\&=\frac{2|\eta|(1-\omega)}{(\chi-3\zeta)}\left((1-(1-s)^d)+\frac{2(\alpha-1)}{s(d-1)}[(1-s)-(1+s(d-1))(1-s)^d]\right).\end{aligned}\tag{38}$$

Using the expression (38) and substituting it back into (37), we obtain

$$1+\frac{2(\alpha-1)(1-s)}{s(d-1)}-\left(1+\frac{2(\alpha-1)}{s(d-1)}(1+s(d-1))\right)(1-s)^d\leq\frac{(2\alpha-1)(\chi-3\zeta)}{2|\eta|(1-\omega)}.\tag{39}$$

Using the hypothesis of the theorem along with Equation (39) completes the proof. □

Theorem 8. *Let $\Psi \in \mathcal{A}$ has the form specified in (1) and assume it also belongs to the class $\mathcal{M}^{\eta}_{\zeta,\chi}(\omega)$. Suppose the following inequality holds*

$$\begin{aligned}&\frac{1-s}{\chi-3\zeta}+\frac{(\lambda-1)(1-s)^2}{3\zeta s(d-2)}\\&\quad-\left[\frac{1+s(d-1)}{\chi-3\zeta}+\frac{\lambda-1}{3\zeta s(d-2)}\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)\right](1-s)^d\\&\leq\frac{s(d-1)\lambda}{2|\eta|(1-\omega)},\end{aligned}$$

then, for $d>0$ and $0\leq s\leq 1$ it follows that $\mathcal{P}^d_s(\Psi,z)\in\mathcal{S}^*_\lambda$.

Proof. Applying Equation (8) to the sufficient condition for the class $\mathcal{S}^*_\lambda$ in Lemma 8, which gives

$$\sum_{j=2}^{\infty} j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|+\sum_{j=2}^{\infty}(\lambda-1)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\leq\lambda. \tag{40}$$

Next, we substitute (4) into the first sum and (5) into the second sum of (40)

$$\begin{aligned}&\sum_{j=2}^{\infty} j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|+\sum_{j=2}^{\infty}(\lambda-1)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\\&\leq\sum_{j=2}^{\infty} j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j^2(\chi-3\zeta)}+\sum_{j=2}^{\infty}(\lambda-1)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j(j+1)3\zeta}\\&=2|\eta|(1-\omega)(1-s)^d\left(\frac{1}{(\chi-3\zeta)s(d-1)}\sum_{j=2}^{\infty}\frac{(d-1)_j}{(1)_j}s^{(j)}\right.\\&\quad\left.+\frac{(\lambda-1)}{3\zeta s^2(d-2)_2}\sum_{j=2}^{\infty}\frac{(d-2)_{j+1}}{(1)_{j+1}}s^{(j+1)}\right)\\&=2|\eta|(1-\omega)(1-s)^d\left(\frac{1}{(\chi-3\zeta)s(d-1)}\Big((1-s)^{-(d-1)}-(1+s(d-1))\Big)\right.\\&\quad\left.+\frac{(\lambda-1)}{3\zeta s^2(d-2)_2}\left[(1-s)^{-(d-2)}-\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)\right]\right)\\&=2|\eta|(1-\omega)\left(\frac{1}{(\chi-3\zeta)s(d-1)}\Big((1-s)-(1+s(d-1))(1-s)^d\Big)\right.\\&\quad\left.+\frac{(\lambda-1)}{3\zeta s^2(d-2)_2}\left[(1-s)^2-\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)(1-s)^d\right]\right).\end{aligned} \tag{41}$$

Using the expression (41) and substituting it back into (40), we obtain

$$\begin{aligned}&2|\eta|(1-\omega)\left[\frac{(1-s)}{(\chi-3\zeta)s(d-1)}+\frac{(\lambda-1)(1-s)^2}{3\zeta s^2(d-2)_2}\right.\\&\quad\left.-\left(\frac{(1+s(d-1))}{(\chi-3\zeta)s(d-1)}+\frac{(\lambda-1)}{3\zeta s^2(d-2)_2}\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)\right)(1-s)^d\right]\\&\leq\lambda.\end{aligned} \tag{42}$$

Using the hypothesis of the theorem along with Equation (42) completes the proof. □

Theorem 9. *Let $\Psi \in \mathcal{A}$ has the form specified in (1). Suppose Ψ also in $\mathcal{M}^{\eta}_{\zeta,\chi}(\omega)$ and satisfies the following condition*

$$\begin{aligned} & s(d-1)+(\lambda-1)(1-s) \\ & \quad -[s(d-1)+(\lambda-1)(1+s(d-1))](1-s)^d \\ & \leq \lambda \frac{s(d-1)(\chi-3\zeta)}{2|\eta|(1-\omega)}, \end{aligned}$$

then, for $d>0$ and $0 \leq s \leq 1$, it follows that $\mathcal{P}^d_s(\Psi,z) \in \mathcal{C}_\lambda$.

Proof. Applying Equation (8) in Lemma 9, which gives

$$\sum_{j=2}^{\infty} j(\lambda+j-1)\frac{(d)_{j-1}}{(1)_{j-1}} s^{(j-1)}(1-s)^d|\alpha_j| \leq \lambda. \tag{43}$$

Using (4) on the left side of (43), we obtain

$$\begin{aligned} & \sum_{j=2}^{\infty} j(\lambda+j-1)\frac{(d)_{j-1}}{(1)_{j-1}} s^{(j-1)}(1-s)^d|\alpha_j| \\ & \leq \sum_{j=2}^{\infty} j(\lambda+j-1)\frac{(d)_{j-1}}{(1)_{j-1}} s^{(j-1)}(1-s)^d \frac{2|\eta|(1-\omega)}{j^2(\chi-3\zeta)} \\ & = \frac{2|\eta|(1-\omega)}{(\chi-3\zeta)}(1-s)^d \sum_{j=2}^{\infty}(\lambda+j-1)\frac{(d)_{j-1}}{(1)_j} s^{(j-1)} \\ & = \frac{2|\eta|(1-\omega)}{(\chi-3\zeta)}(1-s)^d \left(\sum_{j=2}^{\infty} \frac{(d)_{j-1}}{(1)_{j-1}} s^{(j-1)} + (\lambda-1)\sum_{j=2}^{\infty}\frac{(d)_{j-1}}{(1)_j} s^{(j-1)} \right) \\ & = \frac{2|\eta|(1-\omega)}{(\chi-3\zeta)}(1-s)^d \left(((1-s)^{-d}-1) + \frac{(\lambda-1)}{s(d-1)}[(1-s)^{-(d-1)} - (1+s(d-1))] \right). \end{aligned} \tag{44}$$

Using the expression (44) and substituting it back into (43), we obtain

$$\frac{s(d-1)+(\lambda-1)(1-s)}{s(d-1)} - \frac{s(d-1)+(\lambda-1)(1+s(d-1))}{s(d-1)}(1-s)^d \leq \lambda \frac{(\chi-3\zeta)}{2|\eta|(1-\omega)}. \tag{45}$$

Using the hypothesis of the theorem along with Equation (45) completes the proof. □

Theorem 10. *Let $\Psi \in \mathcal{A}$ has the form specified in (1). Suppose Ψ also in $\mathcal{M}^{\eta}_{\zeta,\chi}(\omega)$ and satisfies the following condition*

$$\begin{aligned} & \lambda s d(1-s)^{-1} - (1-s)^d + 1 \\ & \quad - \frac{\beta}{(d-1)s}[(1-s)-(1+s(d-1))(1-s)^d] \\ & \leq (1-\beta)\frac{(\upsilon-3\wp)}{2|\tau|(1-\gamma)}, \end{aligned}$$

then, for $d>0$ and $0 \leq s \leq 1$, it follows that $\mathcal{P}^d_s(\Psi,z) \in \mathcal{S}(\lambda,\beta)$.

Proof. Applying Equation (8) to the sufficient condition for the class $\mathcal{S}(\lambda,\beta)$ in Lemma 10, which gives

$$\sum_{j=2}^{\infty} (\lambda j^2 + j(1-\lambda)-\beta)\frac{(d)_{j-1}}{(1)_{j-1}} s^{(j-1)}(1-s)^d|\alpha_j| \leq 1-\beta. \tag{46}$$

Substitute (4) and (5) in (46), we have

$$\begin{aligned}
&\sum_{j=2}^{\infty}\lambda j^2\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j| \\
&\quad+\sum_{j=2}^{\infty}j(1-\lambda)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|-\sum_{j=2}^{\infty}\beta\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j| \\
&\leq\sum_{j=2}^{\infty}\lambda j^2\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j^2(\chi-3\zeta)} \\
&\quad+\sum_{j=2}^{\infty}j(1-\lambda)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j^2(\chi-3\zeta)} \\
&\quad-\sum_{j=2}^{\infty}\beta\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j(j+1)3\zeta} \\
&=2|\eta|(1-\omega)(1-s)^d\left(\frac{\lambda}{(\chi-3\zeta)}\sum_{j=1}^{\infty}\frac{(d)_j}{(1)_j}s^{(j)}\right. \\
&\quad\left.+\frac{(1-\lambda)}{(\chi-3\zeta)s(d-1)}\sum_{j=2}^{\infty}\frac{(d-1)_j}{(1)_j}s^{(j)}-\frac{\beta}{3\zeta s^2(d-2)_2}\sum_{j=2}^{\infty}\frac{(d-2)_{j+1}}{(1)_{j+1}}t^{(j+1)}\right) \\
&=2|\eta|(1-\omega)(1-s)^d\left(\frac{\lambda}{(\chi-3\zeta)}((1-s)^{-d}-1)\right. \\
&\quad+\frac{(1-\lambda)}{(\chi-3\zeta)t(d-1)}[(1-s)^{-(d-1)}-(1+s(d-1))] \\
&\quad\left.-\frac{\beta}{3\zeta s^2(d-2)_2}\left[(1-s)^{-(d-2)}-\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)\right]\right).
\end{aligned}\tag{47}$$

Using the expression (47) and substituting it back into (46), we obtain

$$\begin{aligned}
&\frac{\beta(1-s)^d}{3\zeta s^2(d-2)_2}\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)-\frac{(1-\lambda)(1-s)^d}{(\chi-3\zeta)s(d-1)}(1+s(d-1)) \\
&\quad+\frac{\lambda}{(\chi-3\zeta)}+\frac{(1-\lambda)(1-s)}{(\chi-3\zeta)s(d-1)}-\frac{\beta(1-s)^2}{3\zeta s^2(d-2)_2}-\frac{\lambda}{(\chi-3\zeta)}(1-s)^d \\
&\leq\frac{(1-\beta)}{2|\eta|(1-\omega)}.
\end{aligned}\tag{48}$$

Using the hypothesis of the theorem along with Equation (48) completes the proof. □

Theorem 11. *Let $\Psi\in\mathcal{A}$ has the form specified in (1). Suppose Ψ also in $\mathcal{M}^{\eta}_{\zeta,\chi}(\omega)$ and satisfies the following condition*

$$\lambda sd(1-s)^{-1}+1-\frac{\beta(1-s)}{s(d-1)}-\frac{s(d-1)-\beta(1+s(d-1))}{s(d-1)}(1-s)^d\leq(1-\beta)\frac{(\chi-3\zeta)}{2|\eta|(1-\omega)},$$

then, for $d>0$ and $0\leq s\leq 1$, it follows that $\mathcal{P}^d_s(\Psi,z)\in\mathcal{C}(\lambda,\beta)$.

Proof. Applying Equation (8) in Lemma 11, which gives

$$\sum_{j=2}^{\infty}j(j+\lambda j(j-1)-\beta)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\leq 1-\beta.\tag{49}$$

Using (4) on the left side of (49), we obtain

$$\begin{aligned}
&\sum_{j=2}^{\infty} j(j+\lambda j(j-1)-\beta)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j| \\
&\leq \sum_{j=2}^{\infty} j(j+\lambda j(j-1)-\beta)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j^2(\chi-3\zeta)} \\
&= \frac{2|\eta|(1-\omega)}{(\chi-3\zeta)}(1-s)^d\left(\sum_{j=2}^{\infty}\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}+\sum_{j=2}^{\infty}\lambda\frac{(d)_{j-1}}{(1)_{j-2}}s^{(j-1)}-\sum_{j=2}^{\infty}\beta\frac{(d)_{j-1}}{(1)_j}s^{(j-1)}\right) \\
&= \frac{2|\eta|(1-\omega)}{(\chi-3\zeta)}\left(((1-s)^{-d}-1)+\lambda sd(1-s)^{-(d+1)}\right. \\
&\quad \left.-\frac{\beta}{s(d-1)}[(1-s)^{-(d-1)}-(1+s(d-1))]\right)(1-s)^d. \qquad (50)
\end{aligned}$$

Using the expression (50) and substituting it back into (49), we obtain

$$\begin{aligned}
&(1-(1-s)^d)+\lambda sd(1-s)^{-1} \\
&\quad -\frac{\beta}{s(d-1)}\left[(1-s)-(1+s(d-1))(1-s)^d\right] \\
&\leq \frac{(1-\beta)(\chi-3\zeta)}{2|\eta|(1-\omega)}. \qquad (51)
\end{aligned}$$

Using the hypothesis of the theorem along with Equation (51) completes the proof. □

Remark 1. *Note that in Theorem 1 if* $\vartheta = 0$, $\nu = 0$ *and* $\varrho = 0$ *with the condition* $2 + s(d-2) \leq \frac{(\chi-3\zeta)s(d-1)}{2|\eta|(1-\omega)(1-s)^d}$ *then the linear operator* $\mathcal{P}_s^d(\Psi, z)$ *is starlike function.*

3.2. Results on Integral Operator

Let us consider a integral operator $\mathcal{T}_s^d(z)$ defined as

$$\begin{aligned}
\mathcal{T}_s^d(z) &= \int_0^z \frac{\mathcal{P}_s^d(\theta)}{\theta}d\theta = \int_0^z \frac{\theta+\sum_{j=2}^{\infty}\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d}{\theta}\theta^j d\theta \\
&= \int_0^z 1+\sum_{j=2}^{\infty}\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\theta^{j-1}d\theta.
\end{aligned}$$

$$\mathcal{T}_s^d(z) = z+\sum_{j=2}^{\infty}\frac{(d)_{j-1}}{(1)_j}s^{(j-1)}(1-s)^d z^j.$$

Let $\Psi \in \mathcal{A}$ then

$$\mathcal{T}_s^d(\Psi, z) = z+\sum_{j=2}^{\infty}\frac{(d)_{j-1}}{(1)_j}s^{(j-1)}(1-s)^d\alpha_j z^j = z+\sum_{j=2}^{\infty}\Upsilon_j z^j,$$

here

$$\Upsilon_j = \frac{(d)_{j-1}}{(1)_j}s^{(j-1)}(1-s)^d\alpha_j. \qquad (52)$$

In this subsection, we derived the conditions for the integral operator $\mathcal{T}_s^d(\Psi, z)$ such that it would be in convex subclasses of univalent functions.

Theorem 12. *Let $\Psi(z) \in \mathcal{A}$ with the form (1), and assume it also belongs to the class $\mathcal{M}^{\eta}_{\zeta,\chi}(\omega)$. Suppose the following inequality holds*

$$\begin{aligned}
&\frac{[(1-\varrho)\sec\vartheta+\varrho(1-\nu)](1-s)}{(\chi-3\zeta)}+\frac{(1-\varrho)(1-\nu-\sec\vartheta)(1-s)^2}{3\zeta s(d-2)}\\
&\quad-\frac{[(1-\varrho)\sec\vartheta+\varrho(1-\nu)](1+s(d-1))}{(\chi-3\zeta)}(1-s)^d\\
&\quad+\frac{(1-\varrho)(1-\nu-\sec\vartheta)}{6\zeta s(d-2)}\Big(2+2s(d-2)+s^2(d-2)_2\Big)(1-s)^d\\
&\leq\frac{s(d-1)(1-\nu)}{2|\eta|(1-\omega)},
\end{aligned}$$

then, for $d>0$ and $0\leq s\leq 1$, it follows that $\mathcal{T}^d_s(\Psi,z)\in\mathcal{K}(\vartheta,\nu,\varrho)$.

Proof. Applying Equation (52) to the sufficient condition for the class $\mathcal{K}(\vartheta,\nu,\varrho)$ in Lemma 2, which gives

$$\begin{aligned}
&\sum_{j=2}^{\infty}[(1-\varrho)\sec\vartheta+\varrho(1-\nu)]j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\\
&\quad+\sum_{j=2}^{\infty}(1-\varrho)(1-\nu-\sec\vartheta)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\\
&\leq 1-\nu.
\end{aligned}\tag{53}$$

Next, we substitute (4) into the first sum and (5) into the second sum of (53)

$$\begin{aligned}
&\sum_{j=2}^{\infty}[(1-\varrho)\sec\vartheta+\varrho(1-\nu)]j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\\
&\quad+\sum_{j=2}^{\infty}(1-\varrho)(1-\nu-\sec\vartheta)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\\
&\leq\sum_{j=2}^{\infty}[(1-\varrho)\sec\vartheta+\varrho(1-\nu)]j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j^2(\chi-3\zeta)}\\
&\quad+\sum_{j=2}^{\infty}(1-\varrho)(1-\nu-\sec\vartheta)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j(j+1)3\zeta}\\
&=2|\eta|(1-\omega)(1-s)^d\left(\frac{[(1-\varrho)\sec\vartheta+\varrho(1-\nu)]}{(\chi-3\zeta)s(d-1)}\sum_{j=2}^{\infty}\frac{(d-1)_j}{(1)_j}s^{(j)}\right.\\
&\quad\left.+\frac{(1-\varrho)(1-\nu-\sec\vartheta)}{3\zeta s^2(d-2)_2}\sum_{j=2}^{\infty}\frac{(d-2)_{j+1}}{(1)_{j+1}}s^{(j+1)}\right)\\
&=2|\eta|(1-\omega)(1-s)^d\left(\frac{[(1-\varrho)\sec\vartheta+\varrho(1-\nu)]}{(\chi-3\zeta)s(d-1)}\Big[(1-s)^{-(d-1)}-(1+s(d-1))\Big]\right.\\
&\quad\left.+\frac{(1-\varrho)(1-\nu-\sec\vartheta)}{3\zeta s^2(d-2)_2}\left[(1-s)^{-(d-2)}-\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)\right]\right).
\end{aligned}\tag{54}$$

Using the expression (54) and substituting it back into (53), we obtain

$$\begin{aligned}&\frac{(1-\varrho)(1-\nu-\sec\vartheta)}{3\zeta s^2(d-2)_2}\left((1-s)^2-\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)(1-s)^d\right)\\&\quad+\frac{[(1-\varrho)\sec\vartheta+\varrho(1-\nu)]}{(\chi-3\zeta)s(d-1)}\left[1-s-(1+s(d-1))(1-s)^d\right]\\&\leq\frac{(1-\nu)}{2|\eta|(1-\omega)}.\end{aligned}\tag{55}$$

Using the hypothesis of the theorem along with Equation (55) completes the proof. □

Theorem 13. *Let $\Psi(z)\in\mathcal{A}$ with the form (1), and assume it also belongs to the class $\mathcal{M}^{\eta}_{\zeta,\chi}(\omega)$. Suppose the following inequality holds*

$$\begin{aligned}&\frac{2(1-s)}{(\chi-3\zeta)}-\frac{(\cos\sigma+\delta)(1-s)^2}{3\zeta s(d-2)}-\frac{2(1+s(d-1))(1-s)^d}{(\chi-3\zeta)}\\&\quad+\frac{(\cos\sigma+\delta)}{6\zeta s(d-2)}\Big(2+2s(d-2)+s^2(d-2)_2\Big)(1-s)^d\\&\leq\frac{s(d-1)(\cos\sigma-\delta)}{2|\eta|(1-\omega)},\end{aligned}$$

then, for $d>0$ and $0\leq s\leq 1$, it follows that $\mathcal{T}^d_s(\Psi,z)\in\mathcal{UCSP}(\sigma,\delta)$.

Proof. Applying Equation (52) in Lemma 4, which gives

$$\sum_{j=2}^{\infty}2j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|-\sum_{j=2}^{\infty}(\cos\sigma+\delta)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\leq\cos\sigma-\delta.\tag{56}$$

Next, we substitute (4) into the first sum and (5) into the second sum of (56)

$$\begin{aligned}&\sum_{j=2}^{\infty}2j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|-\sum_{j=2}^{\infty}(\cos\sigma+\delta)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\\&\leq\sum_{j=2}^{\infty}2j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j^2(\chi-3\zeta)}\\&\quad-\sum_{j=2}^{\infty}(\cos\sigma+\delta)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j(j+1)3\zeta}\\&=2|\eta|(1-\omega)(1-s)^d\left(\frac{2}{(\chi-3\zeta)s(d-1)}\sum_{j=2}^{\infty}\frac{(d-1)_j}{(1)_j}s^{(j)}\right.\\&\quad\left.-\frac{(\cos\sigma+\delta)}{3\zeta s^2(d-2)_2}\sum_{j=2}^{\infty}\frac{(d-2)_{j+1}}{(1)_{j+1}}s^{(j+1)}\right)\\&=2|\eta|(1-\omega)(1-s)^d\left(\frac{2}{(\chi-3\zeta)s(d-1)}\Big((1-s)^{-(d-1)}-(1+s(d-1))\Big)\right.\\&\quad\left.-\frac{(\cos\sigma+\delta)}{3\zeta s^2(d-2)_2}\left[(1-s)^{-(d-2)}-\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)\right]\right).\end{aligned}\tag{57}$$

Using the expression (57) and substituting it back into (56), we obtain

$$\begin{aligned}
&\frac{2\Big[(1-s)-(1+s(d-1))(1-s)^d\Big]}{(\chi-3\zeta)s(d-1)}\\
&\quad-\frac{(\cos\sigma+\delta)}{3\zeta s^2(d-2)_2}\left((1-s)^2-\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)(1-s)^d\right)\\
&\leq\frac{(\cos\sigma-\delta)}{2|\eta|(1-\omega)}.
\end{aligned}\tag{58}$$

Using the hypothesis of the theorem along with Equation (58) completes the proof. □

Theorem 14. *Let $\Psi(z)\in\mathcal{A}$ with the form (1), and assume it also belongs to the class $\mathcal{M}^{\eta}_{\zeta,\chi}(\omega)$. Suppose the following inequality holds*

$$\begin{aligned}
&\frac{\big[2(k+1)+|Y+1|\big]}{(\chi-3\zeta)s(d-1)}-\frac{\big[2(k+1)-|X+1|\big]}{3\zeta s^2(d-2)_2}(1-s)\\
&\quad-\frac{\big[2(k+1)+|Y+1|\big]}{(\chi-3\zeta)s(d-1)}(1+s(d-1))(1-s)^{d-1}\\
&\quad+\frac{\big[2(k+1)-|X+1|\big]}{3\zeta s^2(d-2)_2}\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)(1-s)^{d-1}\\
&\leq\frac{|Y-X|}{2|\eta|(1-\omega)(1-s)},
\end{aligned}$$

then, for $d>0$ and $0\leq s\leq 1$, it follows that $\mathcal{T}^d_s(\Psi,z)\in k-\mathcal{UCV}[X,Y]$.

Proof. The sufficient condition for the class $k-\mathcal{UCV}[X,Y]$ in Lemma 6 gives

$$\sum_{j=2}^{\infty}j\big[2(k+1)(j-1)+|j(Y+1)-(X+1)|\big]|B_j|\leq|Y-X|.\tag{59}$$

Using (52) on the left side of (59), we obtain

$$\begin{aligned}
&\sum_{j=2}^{\infty}\big[2(k+1)(j-1)+|j(Y+1)-(X+1)|\big]\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\\
&\leq\sum_{j=2}^{\infty}\big[2(k+1)+|Y+1|\big]j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\\
&\quad-\sum_{j=2}^{\infty}\big[2(k+1)-|X+1|\big]\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|.
\end{aligned}\tag{60}$$

Next, we substitute (4) into the first sum and (5) into the second sum of (60)

$$\begin{aligned}
&\sum_{j=2}^{\infty}\left[2(k+1)+|Y+1|\right]j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^{d}|\alpha_j| \\
&\quad-\sum_{j=2}^{\infty}\left[2(k+1)-|X+1|\right]\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^{d}|\alpha_j| \\
&\leq\sum_{j=2}^{\infty}\left[2(k+1)+|Y+1|\right]j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^{d}\frac{2|\eta|(1-\omega)}{j^2(\chi-3\zeta)} \\
&\quad-\sum_{j=2}^{\infty}\left[2(k+1)-|X+1|\right]\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^{d}\frac{2|\eta|(1-\omega)}{j(j+1)3\zeta} \\
&=2|\eta|(1-\omega)\left(\frac{\left[2(k+1)+|Y+1|\right]}{(\chi-3\zeta)}\sum_{j=2}^{\infty}\frac{(d)_{j-1}}{(1)_j}s^{(j-1)}\right. \\
&\quad\left.-\frac{\left[2(k+1)-|X+1|\right]}{3\zeta}\sum_{j=2}^{\infty}\frac{(d)_{j-1}}{(1)_{j+1}}s^{(j-1)}\right)(1-s)^{d} \\
&=2|\eta|(1-\omega)\left(\frac{\left[2(k+1)+|Y+1|\right]}{(\chi-3\zeta)s(d-1)}\Big((1-s)^{-(d-1)}-(1+s(d-1))\Big)\right. \\
&\quad\left.-\frac{\left[2(k+1)-|X+1|\right]}{3\zeta s^2(d-2)_2}\left[(1-s)^{-(d-2)}-\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)\right]\right)(1-s)^{d}. \qquad (61)
\end{aligned}$$

Using the expression (61) and substituting it back into (59), we obtain

$$\begin{aligned}
&\frac{\left[2(k+1)+|Y+1|\right]}{(\chi-3\zeta)s(d-1)}\left[(1-s)-(1+s(d-1))(1-s)^{d}\right] \\
&-\frac{\left[2(k+1)-|X+1|\right]}{3\zeta s^2(d-2)_2}\left((1-s)^2-\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)(1-s)^{d}\right)\leq\frac{|Y-X|}{2|\eta|(1-\omega)}. \qquad (62)
\end{aligned}$$

Using the hypothesis of the theorem along with Equation (62) completes the proof. □

Theorem 15. *Let $\Psi(z)\in\mathcal{A}$ with the form (1), and assume it also belongs to the class $\mathcal{M}^{\eta}_{\zeta,\chi}(\omega)$. Suppose the following inequality holds*

$$\begin{aligned}
&\frac{1-s}{\chi-3\zeta}+\frac{(\lambda-1)(1-s)^2}{3\zeta s(d-2)} \\
&\quad-\left(\frac{1+s(d-1)}{\chi-3\zeta}+\frac{\lambda-1}{6\zeta s(d-2)}\Big(2+2s(d-2)+s^2(d-2)_2\Big)\right)(1-s)^{d} \\
&\leq\frac{s(d-1)\lambda}{2|\eta|(1-\omega)},
\end{aligned}$$

then, for $d>0$ and $0\leq s\leq 1$, it follows that $\mathcal{T}_s^d(\Psi,z)\in\mathcal{C}_\lambda$.

Proof. Applying Equation (52) to the sufficient condition for the class $\mathcal{C}_\lambda$ in Lemma 9, which gives

$$\sum_{j=2}^{\infty}j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^{d}|\alpha_j|+\sum_{j=2}^{\infty}(\lambda-1)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^{d}|\alpha_j|\leq\lambda. \qquad (63)$$

Next, we substitute (4) into the first sum and (5) into the second sum of (63)

$$\begin{aligned}
&\sum_{j=2}^{\infty} j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j| + \sum_{j=2}^{\infty}(\lambda-1)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j| \\
&\leq \sum_{j=2}^{\infty} j\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j^2(\chi-3\zeta)} + \sum_{j=2}^{\infty}(\lambda-1)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j(j+1)3\zeta} \\
&= 2|\eta|(1-\omega)(1-s)^d\left(\frac{1}{(\chi-3\zeta)s(d-1)}\sum_{j=2}^{\infty}\frac{(d-1)_j}{(1)_j}s^{(j)}\right. \\
&\quad \left.+\frac{(\lambda-1)}{3\zeta s^2(d-2)_2}\sum_{j=2}^{\infty}\frac{(d-2)_{j+1}}{(1)_{j+1}}s^{(j+1)}\right) \\
&= 2|\eta|(1-\omega)(1-s)^d\left(\frac{1}{(\chi-3\zeta)s(d-1)}\Big((1-s)^{-(d-1)}-(1+s(d-1))\Big)\right. \\
&\quad \left.+\frac{(\lambda-1)}{3\zeta s^2(d-2)_2}\left[(1-s)^{-(d-2)}-\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)\right]\right). \qquad (64)
\end{aligned}$$

Using the expression (64) and substituting it back into (63), we obtain

$$\begin{aligned}
&\frac{1-s}{\chi-3\zeta}+\frac{(\lambda-1)(1-s)^2}{3\zeta s(d-2)} \\
&\quad -\left(\frac{1+s(d-1)}{\chi-3\zeta}+\frac{\lambda-1}{3\zeta s(d-2)}\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)\right)(1-s)^d \\
&\quad \leq \frac{s(d-1)\lambda}{2|\eta|(1-\omega)}. \qquad (65)
\end{aligned}$$

Using the hypothesis of the theorem along with Equation (65) completes the proof. □

Theorem 16. *Let $\Psi(z) \in \mathcal{A}$ with the form (1), and assume it also belongs to the class $\mathcal{M}^{\eta}_{\zeta,\chi}(\omega)$. Suppose the following inequality holds*

$$\begin{aligned}
&\left(\frac{2+2s(d-2)+s^2(d-2)_2}{6\zeta s(d-2)}\beta-\frac{\lambda s(d-1)}{(\chi-3\zeta)}+\frac{(1-\lambda)}{(\chi-3\zeta)}(1+s(d-1))\right)(1-s)^d \\
&\quad +\frac{\lambda s(d-1)}{(\chi-3\zeta)}+\frac{(1-\lambda)(1-s)}{(\chi-3\zeta)}-\frac{\beta(1-s)^2}{3\zeta s(d-2)} \\
&\quad \leq \frac{s(d-1)(1-\beta)}{2|\eta|(1-\omega)},
\end{aligned}$$

then, for $d > 0$ and $0 \leq s \leq 1$, it follows that $\mathcal{T}^d_s(\Psi, z) \in \mathcal{C}(\lambda, \beta)$.

Proof. Applying Equation (52) in Lemma 11, which gives

$$\sum_{j=2}^{\infty}\left(j+\lambda j(j-1)-\beta\right)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j| \leq 1-\beta. \qquad (66)$$

Next, we substitute (4) into the first sum and (5) into the second sum of (66)

$$\begin{aligned}
&\sum_{j=2}^{\infty}\lambda j^2\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|+\sum_{j=2}^{\infty}j(1-\lambda)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\\
&\quad-\sum_{j=2}^{\infty}\beta\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d|\alpha_j|\\
&\leq\sum_{j=2}^{\infty}\lambda j^2\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j^2(\chi-3\zeta)}\\
&\quad+\sum_{j=2}^{\infty}j(1-\lambda)\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j^2(\chi-3\zeta)}\\
&\quad-\sum_{j=2}^{\infty}\beta\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}(1-s)^d\frac{2|\eta|(1-\omega)}{j(j+1)3\zeta}\\
&=2|\eta|(1-\omega)(1-s)^d\left(\frac{\lambda}{(\chi-3\zeta)}\sum_{j=2}^{\infty}\frac{(d)_{j-1}}{(1)_{j-1}}s^{(j-1)}\right.\\
&\quad\left.+\frac{(1-\lambda)}{(\chi-3\zeta)}\sum_{j=2}^{\infty}\frac{(d)_{j-1}}{(1)_j}s^{(j-1)}-\frac{\beta}{3\zeta}\sum_{j=2}^{\infty}\frac{(d)_{j-1}}{(1)_{j+1}}s^{(j-1)}\right)\\
&=2|\eta|(1-\omega)(1-s)^d\left(+\frac{(1-\lambda)}{(\chi-3\zeta)s(d-1)}[(1-s)^{-(d-1)}-(1+s(d-1))]\right.\\
&\quad\left.+\frac{\lambda((1-s)^{-d}-1)}{(\chi-3\zeta)}-\frac{\beta}{3\zeta s^2(d-2)_2}\left[(1-s)^{-(d-2)}-\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)\right]\right).
\end{aligned}\tag{67}$$

Using the expression (67) and substituting it back into (66), we obtain

$$\begin{aligned}
&\frac{\lambda}{(\chi-3\zeta)}(1-(1-s)^d)+\frac{(1-\lambda)}{(\chi-3\zeta)s(d-1)}[(1-s)-(1+s(d-1))(1-s)^d]\\
&\quad-\frac{\beta}{3\zeta s^2(d-2)_2}\left((1-s)^2-\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)(1-s)^d\right)\\
&\leq\frac{(1-\beta)}{2|\eta|(1-\omega)}.
\end{aligned}\tag{68}$$

Using the hypothesis of the theorem along with Equation (68) completes the proof. □

Remark 2. *Observe that the sufficient conditions for the integral operator $\mathcal{T}_s^d(\Psi,z)$, as outlined in Theorem 12, 13, 14, 15 and 16, align perfectly with those specified in Theorem 1, 3, 5, 8 and 10, respectively, thus ensuring the corresponding validity of the convolution operator $\mathcal{P}_s^d(\Psi,z)$.*

4. Special Cases

In this section, we present some corollaries, which are derived directly from the theorems in Section 3.

4.1. Corollaries for Convolution Operator

In this subsection we derived corollaries for the linear operator $\mathcal{P}_s^d(\Psi,z)$.

If we take $\varrho=0$ in Theorem 1 and Theorem 2, we obtain the following results directly.

Corollary 1. *Let $\Psi \in \mathcal{A}$ with the form (1), and assume it also belongs to the class $\mathcal{M}^{\eta}_{\zeta,\chi}(\omega)$. Suppose the following inequality holds*

$$\begin{aligned}&\frac{(1-s)\sec\vartheta}{(\chi-3\zeta)s(d-1)}+\frac{(1-\nu-\sec\vartheta)(1-s)^2}{3\zeta s^2(d-2)_2}-\frac{(1+s(d-1))(1-s)^d\sec\vartheta}{(\chi-3\zeta)s(d-1)}\\&\quad-\frac{(1-\nu-\sec\vartheta)\big(2+2s(d-2)+s^2(d-2)_2\big)}{6\zeta s^2(d-2)_2}(1-s)^d\\&\leq\frac{(1-\nu)}{2|\eta|(1-\omega)},\end{aligned}$$

then for $d>0$ and $0\leq s\leq 1$, it follows that $\mathcal{P}^d_s(\Psi,z)\in\mathcal{SP}(\vartheta,\nu)$.

Corollary 2. *Let $\Psi \in \mathcal{A}$ has the form specified in (1) and assume it also belongs to the class $\mathcal{M}^{\eta}_{\zeta,\chi}(\omega)$. Suppose the following inequality holds*

$$\begin{aligned}&s(d-1)\sec\vartheta+(1-\nu-\sec\vartheta)(1-s)-s(d-1)(1-s)^d\sec\vartheta\\&\quad-(1-\nu-\sec\vartheta)(1+s(d-1))(1-s)^d\\&\leq(1-\nu)\frac{s(d-1)(\chi-3\zeta)}{2|\eta|(1-\omega)},\end{aligned}$$

then for $d>0$ and $0\leq s\leq 1$, it follows that $\mathcal{P}^d_s(\Psi,z)\in\mathcal{KP}(\vartheta,\nu)$.

Corollary 3. *Let $\Psi \in \mathcal{A}$ has the form specified in (1) and assume it also belongs to the class $\mathcal{M}^{\eta}_{\zeta,\chi}(\omega)$. Suppose the following inequality holds*

$$\begin{aligned}&\frac{2(1-s)}{(\chi-3\zeta)}-\frac{(1-s)^2\cos\sigma}{3\zeta s(d-2)}-\frac{2(1+s(d-1))(1-s)^d}{(\chi-3\zeta)}\\&\quad+\frac{\cos\sigma}{3\zeta s(d-2)}\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)(1-s)^d\\&\leq\frac{s(d-1)\cos\sigma}{2|\eta|(1-\omega)},\end{aligned}$$

then for $d>0$ and $0\leq s\leq 1$, it follows that $\mathcal{P}^d_s(\Psi,z)\in\mathcal{SP}_p(\sigma)$.

Corollary 4. *Let $\Psi \in \mathcal{A}$ has the form specified in (1) and assume it also belongs to the class $\mathcal{M}^{\eta}_{\zeta,\chi}(\omega)$. Suppose the following inequality holds*

$$\begin{aligned}&2s(d-1)-\cos\sigma(1-s)-2s(d-1)(1-s)^d\\&\quad+\cos\sigma(1+s(d-1))(1-s)^d\\&\leq\cos\sigma\frac{s(d-1)(\chi-3\zeta)}{2|\tau|(1-\gamma)},\end{aligned}$$

then for $d>0$ and $0\leq s\leq 1$, it follows that $\mathcal{P}^d_s(\Psi,z)\in\mathcal{UCSP}(\sigma)$.

Corollary 5. *Let $\Psi \in \mathcal{A}$ has the form specified in (1) and assume it also belongs to the class $\mathcal{M}^{\eta}_{\zeta,\chi}(\omega)$. Suppose the following inequality holds*

$$\frac{(1+k)(1-s)}{(\chi-3\zeta)}-\frac{(k+\sigma)(1-s)^2}{3\zeta s(d-2)}-\frac{(1+k)(1+s(d-1))(1-s)^d}{(\chi-3\zeta)}+\frac{(k+\sigma)}{3\zeta s(d-2)}\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)(1-s)^d\leq\frac{s(d-1)(1-\sigma)}{2|\eta|(1-\omega)},$$

then for $d>0$ *and* $0\leq s\leq 1$, *it follows that* $\mathcal{P}_s^d(\Psi,z)\in k-\mathcal{S}_p(\sigma)$.

Corollary 6. *Let* $\Psi\in\mathcal{A}$ *has the form specified in* (1) *and assume it also belongs to the class* $\mathcal{M}_{\zeta,\chi}^{\eta}(\omega)$. *Suppose the following inequality holds*

$$s(1+k)(d-1)-s(1+k)(d-1)(1-s)^d-(k+\sigma)(1-s)+(k+\sigma)(1+s(d-1))(1-s)^d\leq(1-\sigma)\frac{s(d-1)(\chi-3\zeta)}{2|\eta|(1-\omega)},$$

then for $d>0$ *and* $0\leq s\leq 1$, *it follows that* $\mathcal{P}_s^d(\Psi,z)\in k-\mathcal{UCV}(\sigma)$.

Corollary 7. *Let* $\Psi\in\mathcal{A}$ *has the form specified in* (1) *and assume it also belongs to the class* $\mathcal{M}_{\zeta,\chi}^{\eta}(\omega)$. *Suppose the following inequality holds*

$$\frac{(1+k)(1-s)}{(\chi-3\zeta)}-\frac{(1+k)(1+s(d-1))}{(\chi-3\zeta)}(1-s)^d+\frac{k(1-s)^d}{3\zeta s(d-2)}\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)-\frac{k(1-s)^2}{3\zeta s(d-2)}\leq\frac{s(d-1)}{2|\eta|(1-\omega)},$$

then for $d>0$ *and* $0\leq s\leq 1$, *it follows that* $\mathcal{P}_s^d(\Psi,z)\in k-\mathcal{ST}$.

Corollary 8. *Let* $\Psi\in\mathcal{A}$ *has the form specified in* (1) *and assume it also belongs to the class* $\mathcal{M}_{\zeta,\chi}^{\eta}(\omega)$. *Suppose the following inequality holds*

$$[k(1+s(d-1))-s(1+k)(d-1)](1-s)^d+s(1+k)(d-1)-k(1-s)\leq\frac{s(d-1)(\chi-3\zeta)}{2|\eta|(1-\omega)},$$

then for $d>0$ *and* $0\leq s\leq 1$, *it follows that* $\mathcal{P}_s^d(\Psi,z)\in k-\mathcal{UCV}$.

Corollary 9. *Let* $\Psi\in\mathcal{A}$ *has the form specified in* (1) *and assume it also belongs to the class* $\mathcal{M}_{\zeta,\chi}^{\eta}(\omega)$. *Suppose the following inequality holds*

$$\frac{(2-|X+1|)(1-s)^{d-1}}{3\zeta s^2(d-2)_2}\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)-\frac{(2-|X+1|)(1-s)}{3\zeta s^2(d-2)_2}+\frac{2+|Y+1|}{(\chi-3\zeta)s(d-1)}-\frac{2+|Y+1|}{(\chi-3\zeta)s(d-1)}(1+s(d-1))(1-s)^{d-1}\leq\frac{|Y-X|}{2|\eta|(1-\omega)(1-s)},$$

then for $d > 0$ and $0 \leq s \leq 1$, it follows that $\mathcal{P}_s^d(\Psi, z) \in \mathcal{S}^*[X, Y]$.

Corollary 10. *Let $\Psi(z) \in \mathcal{A}$ with the form (1), and assume it also belongs to the class $\mathcal{M}_{\zeta,\chi}^{\eta}(\omega)$. Suppose the following inequality holds*

$$\begin{aligned}&\frac{(2-|X+1|)(1-s)^d}{s(d-1)}(1+s(d-1))+2+|Y+1|\\&\quad-\frac{(2-|X+1|)}{s(d-1)}(1-s)-(2+|Y+1|)(1-s)^d\\&\leq\frac{|Y-X|(\chi-3\zeta)}{2|\eta|(1-\omega)},\end{aligned}$$

then for $d > 0$ and $0 \leq s \leq 1$, it follows that $\mathcal{P}_s^d(\Psi, z) \in \mathcal{C}[X, Y]$.

4.2. Corollaries for Integral Operator

In this subsection we derived corollaries for the linear operator $\mathcal{T}_s^d(\Psi, z)$. If we take $\varrho = 0$ in Theorem 12, we obtain the following result directly.

Corollary 11. *Let $\Psi \in \mathcal{A}$ has the form specified in (1) and assume it also belongs to the class $\mathcal{M}_{\zeta,\chi}^{\eta}(\omega)$. Suppose the following inequality holds*

$$\begin{aligned}&\frac{(1-s)\sec\vartheta}{(\chi-3\zeta)s(d-1)}+\frac{(1-\nu-\sec\vartheta)(1-s)^2}{3\zeta s^2(d-2)_2}-\frac{(1+s(d-1))(1-s)^d\sec\vartheta}{(\chi-3\zeta)s(d-1)}\\&\quad-\frac{(1-\nu-\sec\vartheta)\left(2+2s(d-2)+s^2(d-2)_2\right)}{6\zeta s^2(d-2)_2}(1-s)^d\\&\leq\frac{(1-\nu)}{2|\eta|(1-\omega)},\end{aligned}$$

then for $d > 0$ and $0 \leq s \leq 1$, it follows that $\mathcal{T}_s^d(\Psi, z) \in \mathcal{KP}(\vartheta, \nu)$.

Corollary 12. *Let $\Psi \in \mathcal{A}$ has the form specified in (1) and assume it also belongs to the class $\mathcal{M}_{\zeta,\chi}^{\eta}(\omega)$. Suppose the following inequality holds*

$$\begin{aligned}&\frac{2(1-s)}{(\chi-3\zeta)}-\frac{(1-s)^2\cos\sigma}{3\zeta s(d-2)}-\frac{2(1+s(d-1))(1-s)^d}{(\chi-3\zeta)}\\&\quad+\frac{\cos\sigma}{3\zeta s(d-2)}\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)(1-s)^d\\&\leq\frac{s(d-1)\cos\sigma}{2|\eta|(1-\omega)},\end{aligned}$$

then for $d > 0$ and $0 \leq s \leq 1$, it follows that $\mathcal{T}_s^d(\Psi, z) \in \mathcal{UCSP}(\sigma)$.

Corollary 13. *Let $\Psi \in \mathcal{A}$ has the form specified in (1) and assume it also belongs to the class $\mathcal{M}_{\zeta,\chi}^{\eta}(\omega)$. Suppose the following inequality holds*

$$\begin{aligned}&\frac{(1+k)(1-s)}{(\chi-3\zeta)}+\frac{(k+\sigma)}{3\zeta s(d-2)}\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right)(1-s)^d\\&\quad-\frac{(1+k)(1+s(d-1))(1-s)^d}{(\chi-3\zeta)}-\frac{(k+\sigma)(1-s)^2}{3\zeta s(d-2)}\\&\leq\frac{s(d-1)(1-\sigma)}{2|\eta|(1-\omega)},\end{aligned}$$

then for $d > 0$ and $0 \leq s \leq 1$, it follows that $\mathcal{T}_s^d(\Psi, z) \in k - \mathcal{UCV}(\sigma)$.

Corollary 14. *Let $\Psi \in \mathcal{A}$ has the form specified in (1) and assume it also belongs to the class $\mathcal{M}_{\zeta,\chi}^{\eta}(\omega)$. Suppose the following inequality holds*

$$\begin{aligned}&\frac{(1+k)(1-s)}{(\chi-3\zeta)} - \frac{(1+k)(1+s(d-1))}{(\chi-3\zeta)}(1-s)^d \\ &\quad + \frac{k(1-s)^d}{3\zeta s(d-2)}\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right) - \frac{k(1-s)^2}{3\zeta s(d-2)} \\ &\leq \frac{s(d-1)}{2|\eta|(1-\omega)},\end{aligned}$$

then for $d > 0$ and $0 \leq s \leq 1$, it follows that $\mathcal{T}_s^d(\Psi, z) \in k - \mathcal{UCV}$.

Corollary 15. *Let $\Psi(z) \in \mathcal{A}$ with the form (1), and assume it also belongs to the class $\mathcal{M}_{\zeta,\chi}^{\eta}(\omega)$. Suppose the following inequality holds*

$$\begin{aligned}&\frac{2+|Y+1|}{(\chi-3\zeta)s(d-1)} + \frac{(2-|X+1|)(1-s)^{d-1}}{3\zeta s^2(d-2)_2}\left(1+s(d-2)+\frac{s^2(d-2)_2}{2}\right) \\ &\quad - \frac{(2-|X+1|)(1-s)}{3\zeta s^2(d-2)_2} - \frac{(2+|Y+1|)(1-s)^{d-1}}{(\chi-3\zeta)s(d-1)}(1+s(d-1)) \\ &\leq \frac{|Y-X|}{2|\eta|(1-\omega)(1-s)},\end{aligned}$$

then for $d > 0$ and $0 \leq s \leq 1$, it follows that $\mathcal{T}_s^d(\Psi, z) \in \mathcal{C}[X, Y]$.

Remark 3. *Observe that the conditions for the linear operator $\mathcal{T}_s^d(\Psi, z)$, as outlined in Corollary 11, 12, 13, 14 and 15, align perfectly with those specified in Corollary 1, 3, 5, 7 and 9, respectively, thus ensuring the corresponding validity of the linear operator $\mathcal{P}_s^d(\Psi, z)$.*

5. Conclusions

This study presents significant results on the inclusion properties of $\mathcal{P}_s^d(\Psi, z)$ and $\mathcal{T}_s^d(\Psi, z)$ for a certain class of normalized analytic functions. Using the coefficient bounds of the functions in the class $\mathcal{M}_{\zeta,\chi}^{\eta}(\omega)$, we determined the sufficient conditions for the linear operators. These findings help to determine the geometric properties of linear operators generated by functions from subclasses of analytic functions, which play a significant role in geometric function theory.

Future research is on finding sharp conditions for these results and exploring different linear operators generated by special functions and functions from subclasses of analytic functions. This study is the foundation for further studies that offer new possibilities for theoretical and practical applications.

Problem 1. *Determine sufficient conditions for the integral operator $\mathcal{T}_s^d(\Psi, z)$ to belong to the classes $\mathcal{S}(\vartheta, \nu, \varrho)$, $\mathcal{SP}_p(\sigma, \delta)$, $k - \mathcal{ST}[X, Y]$, $\mathcal{S}_\lambda^*$ and $\mathcal{S}(\lambda, \beta)$.*

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